AN INTRODUCTION TO ALGEBRAIC STATISTICS

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Introduction

Statistics and Algebraic Statistics

At the beginning of a book on Algebraic Statistics it is undoubtedly a good idea to give the reader some idea of the goals of the discipline.

A reader who is already familiar with the basics of Statistics and Probability is probably curious about what the prefix "Algebraic" might mean. As we will see, Algebraic Statistics has its own problems which are somewhat different from the problems that classical Statistics studies.

We will illustrate this point of view with some examples which consider well known statistical models and problems. At the same time we will point out the difference in the two approaches to these examples.

The treatment of random variables

The initial concern of classical Statistics is the behavior of one random variable X. Usually X is identified with a function whose domain is the real numbers. This is clearly an approximation. For example, if one records the height of the members of a population, it is unlikely that the measure goes much further than the second decimal digit (assume that the unit is 1 meter). So, the corresponding graph is a histogram, with basic interval 0.01 meters. This is translated to a continuous variable, by taking the (experimental) limit as the basic interval is shortened (and the size of the population increases).

For random variables of this type, the first natural distribution that one expects is the celebrated Gaussian distribution, which corresponds to the

function

$$X(t) = \frac{1}{\sigma\sqrt{1\pi}} e^{\frac{-(t-\mu)^2}{2\sigma^2}}$$

where μ and σ are parameters which describe the *shape* of the curve. (Of course, other types of distributions are possible, in connection with special behaviors of the random variable $X(t)$.)

The first goal of classical Statistics is the study of the shape of the function $X(t)$, together with the determination of its numerical parameters.

When two or more variables are considered in the framework of classical Statistic, their interplay can be studied with several techniques. For instance, if we consider both the heights and the weights of the members of a population and our goal is a proof of the (obvious) fact that the two variables are deeply connected, then we can consider the distribution over pairs (height, weight), which is represented by a bivariate Gaussian, in order to detect the existence of the connection.

The starting point of Algebraic Statistic is quite different. Instead of considering variables as continuous functions, Algebraic Statistics prefers to deal with a finite (and possibly small) range of values for the variable X. So, Algebraic Statistic emphasizes the discrete nature of the starting histogram, and tends to group together values in wider ranges, instead of splitting them. A distribution over the variable X is thus identified with a discrete function (to begin with, over the integers).

Algebraic Statistics is rarely interested in situations where just one random variable is concerned.

Instead, networks containing several random variables are considered and some relevant questions, raised in this perspective, are:

- are there connections between the two or more random variables of the network?

- which kind of connection is suggested by a set of data?

- can one measure the complexity of the connections in a given network of interacting variables?

Since, from the new point of view, we are interested in determining the relations between discrete functions, in Algebraic Statistics a distribution over a set of variables is usually represented by matrices, when two variables are involved, or multidimensional matrices (i.e. tensors), as the number of variables increases.

It is a natural consequence of the previous discussion, that while the main mathematical tools for classical Statistics are based on Multivariate Analysis and Measure Theory, the underlying mathematical machinery for Algebraic Statistics is principally based on the Linear and Multilinear Algebra of tensors (over the integers, at the start, but quickly one considers both real and complex tensors).

Relations among variables

Just to give an example, let us consider the behavior of a population after the introduction of a new medicine.

Assume that a population is affected by a disease, which dangerously alters the value of a glycemic indicator in the blood. This dangerous condition is partially treated with the new drug. Assume that the purpose of the experiment is to detect the existence of a substantial improvement in the health of the patients.

In classical Statistics, one considers the distribution of the random variable X_1 = the value of the glycemic indicator over a selected population of patients *before* the delivery of the drug, and the random variable X_2 = the value of the glycemic indicator of patients *after* the delivery of the drug. Both distributions are likely to be represented by Gaussians, the first one centered at an abnormally high value of the glycemic indicator, the second one centered at a (hopefully) lower value. The comparison between the two distributions aims to detect if (and how far) the descent of the recorded

values of the glycemic indicator is statistically meaningful, i.e. if it can be distinguished from the natural underlying ground noise. The celebrated *Student's t-test* is the world-accepted tool for comparing the two Gaussians and for determining the existence of a statistically significant response.

The approach of Algebraic Statistics to the same problem is usually along the following lines: The population is divided into two subsets, one of which is treated with the drug, while the other one is treated with traditional methods. Then, the values of the glycemic indicator are divided in classes (in the roughest case just two classes, i.e. a threshold which separates two classes is established). After some passage of time, one records the distribution of the population in the four resulting categories (treated + under-threshold, treated + over-threshold ...) which determines a 2×2 matrix, whose properties encode the existence of a relation between the new treatment and an improved normalization of the value of the glycemic indicator (this is just to give an example: in the real world, a much more sophisticated analysis is recommended!)

Bernoulli binary models

Another celebrated model, which is different from the Gaussian distribution, and is often introduced at the beginning of a course in Statistics, is the so-called Bernoulli model over one binary variable.

Assume we are given an object that can assume only two states. A coin, with the two traditional states H (heads) and T (tails), is a good representation. One has to bear in mind, however, that in the real world, binary objects usually correspond to *biased* coins, i.e. coins for which the expected distribution over the two states is not even.

If p is the *probability* of obtaining a result (say H) by throwing the coin, then one can roughly estimate p by throwing the coin several times and determining the ratio

$$\frac{\text{number of throws giving } H}{\text{total number of throws}}$$

but this is usually considered too naïve. Instead, one divides the total set of throws into several packages, each consisting of r throws, and determines for how many packages, denoted $q(t)$, one obtained H exactly t times. The value of the constant p is thus determined by Bernoulli's formula

$$q(t) = p^t(1-p)^{r-t}.$$

By increasing the number of total throws (and thus increasing the number of packages and the number of throws r in each packages) the function $q(t)$ tends to a real function, which can be treated with the usual analytic methods.

Notice that in this way, at the end of the process, the discrete variable *Coin* is substituted by a continuous variable $q(t)$. Usually one even goes one step further, by substituting the variable q with its logarithm, ending up with a linear description.

Algebraic Statistics is scarcely interested in knowing how a single given coin is biased. Instead, the main goal of Algebraic Statistics is to understand the connections between the behavior of *two* coins. Or, better, the connections between the behavior of a collection of coins.

Consequently, in Algebraic Statistics one defines a collection of variables, one for each coin, and defines a distribution by counting the records in which the variables $X_1, X_2, \ldots, X_n$ have a fixed combination of states. The distribution is transformed into a tensor of type $2 \times 2 \times \cdots \times 2$. All coins can be biased, with different loads: this does not matter too much. In fact, the main questions that one expects to solve are:

- are there connections between the outputs of two or more coins?
- which kind of connection is suggested by the distribution?
- can one divide the collection of coins in clusters, such that the behavior of coins of the same cluster are similar?

Answers are expected from an analysis of the associated tensor, i.e. in the framework of Multilinear Algebra.

The importance of this last question can be better understood if one replaces coins with positions in a composite digital signal. Each position has, again, two possible states, 0 and 1. If the signal is the result of the superposition of many elementary signals, coming from different sources, and digits coming from the same source behave similarly, then the division of the signal in clusters yields the reconstruction of the original message that each source issued.

Splitting in types

Of course, the separation of several phenomena that are mixed together in a given distribution is also possible using methods of classical Statistics.

In a famous analysis of 1984, the biologist Karl Pearson made a statistical study of the shape of a population of crabs. He constructed the histogram for the ratio between the "forehead" breadth and the body length for 1000 crabs, sampled in Naples, Italy by W.F.R. Weldon. The resulting approximating curve was quite different from a Gaussian and presented a clear asymmetry around the average value. The shape of the function suggested the existence of two distinct types of crabs, each determining its own Gaussian, that were mixed together in the observed histogram. Pearson succeeded in separating the two Gaussians with the method of *moments*. Roughly speaking he introduced new statistical variables, induced by the same collection of data, and separated the types by studying the interactions between the Gaussians of these new variables.

This is a first instance of a computation which takes care of *several* parameters of the population under analysis, though the variables are derived from the same set of data. Understanding the interplay between the variables provides the fundamental step for a qualitative description of the population of crabs.

From the point of view of Algebraic Statistics, one could obtain the same

description of the two types which compose the population, by adding variables representing other ratios between lengths in the body of crabs, and analyzing the resulting tensor.

Mixture models

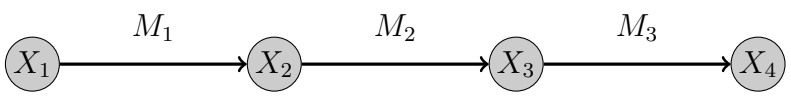
Summarizing, Algebraic Statistics becomes useful when the existence and the nature of the relations between several random variables is explored.

We stress that knowing the shape of the interaction between random variables is a central problem for the description of phenomena in Biology, Chemistry, Social Sciences etc. Models for the description of the interactions are often referred to as *Mixture Models*. Thus, mixture models are a fundamental object of study in Algebraic Statistics.

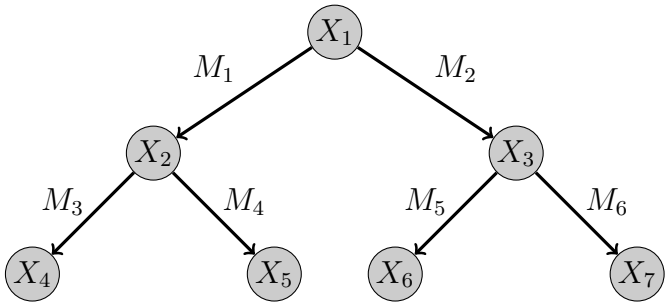
Perhaps, the most famous and easily described mixture models are the *Markov chains*, in which the set of variables is organized in a totally ordered chain, and the behavior of the variable X_i is only influenced by the behavior of the variable X_{i-1} (usually this interaction depends on a given matrix).

Of course, much more complicated types of networks are expected when the complexity of the collection of variables under analysis increases. So, when one studies composite signals in the real world, or pieces of a DNA chain, or regions in a neural tissue, higher level models are likely to be necessary for an accurate description of the phenomenon.

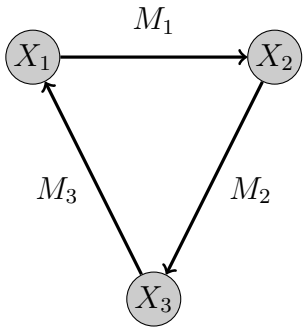
One thus moves from the study of Markov chains



to the study of Markov trees



and the study of neural nets



In classical Statistics, the structure of the connections among variables is often a postulate. In Algebraic Statistics, determining the Combinatorics and the Topology of the network is a fundamental task. On the other hand, the time-depending activating functions that transfer information from one variable to the next ones, deeply studied by classical Statistics, are of no immediate interest for Algebraic Statistics which, at first, considers steady states of the configuration of variables.

The Multilinear Algebra behind the aforementioned models is not completely understood. It requires a deep analysis of subsets of linear spaces described by parametric or implicit *polynomial* equations. This is the reason why, at a certain point, methods of Algebraic Geometry are invoked to push the analysis further.

Conclusion

Algebraic Statistics is focused on aspects of the theory of random variables which are different from the targets of classical Statistics.

The discipline is currently living in a rapidly expanding net of new insights and new areas of application. Our knowledge of what we can do in this area is constantly increasing and it is reasonable to hope that many of the problems introduced in this book will soon be solved and, if not solved soon, without doubt will be better understood. We feel that the time is right to provide a systematic foundation for a field that promises to act as a stimulus for mathematical research in Statistics, and also as a source of suggestions for further developments in Multilinear Algebra and Algebraic Geometry.

Contents

Part I

Algebraic Statistics

Chapter 1

Systems of Random Variables and Distributions

1.1 Systems of random variables

This section contains the basic definitions with which we will construct our statistical theory.

It is important to point out right away that in the field of Algebraic Statistics, a still rapidly developing area of study, the basic definitions are not yet standardized. Therefore, the definitions which we shall use in this text can differ significantly (more in form than in substance) from those of other texts.

Definition 1.1.1. A *random variable* is a variable x taking values in a finite non-empty set of symbols, denoted $A(x)$. The set $A(x)$ is called the *alphabet* of x or the *set of states* of x. We will say that every element of $A(x)$ is a *state* of the variable x.

A *system of random variables* S is a finite set of random variables.

The condition of finiteness, required both for the alphabet of a random variable and the number of variables of a system, is typical of *Algebraic* Statistics. In other statistical situations this hypothesis is often not present.

Definition 1.1.2. A *subsystem* of a system S of random variables is a system defined by a subset $S' \subset S$.

Example 1.1.3. The simplest examples of a system of random variables are those containing a single random variable. A typical example is obtained by thinking of a die x as a random variable, i.e. as the unique element of S. Its alphabet is $A(x) = \{1, 2, 3, 4, 5, 6\}$.

Another familiar example comes by thinking of the only element of S as a coin c with alphabet $A(c) = \{H, T\}$ (heads and tails).

Example 1.1.4. On internet sites about soccer betting one finds systems in which each random variable has three states. More precisely the set S of random variables are (say) all the professional soccer games in a given country. For each random variable x, (i.e. game), its alphabet is $A(x) = \{1, 2, T\}$. The random variable takes value "1" if the game was won by the home team, value "2" if the game was won by the visiting team and value "T" if the game was a tie.

Example 1.1.5. a) We can, similar to Example 1.1.3, construct a system S with two random variables, namely with two dice $\{x_1, x_2\}$, both having alphabet $A(x_i) = \{1, 2, 3, 4, 5, 6\}$.

b) An example of another system of random variables T, closely related to the previous one but **different**, is given by taking a single random variable as the *ordered pair* of dice $x = (x_1, x_2)$ and, as alphabet $A(x)$, all possible values obtained by throwing the dice simultaneously: $\{(1, 1), \ldots (1, 6), \ldots, (6, 1), (6, 2), \ldots, (6, 6)\}$.

c) Another example W, still related to the two above (but different), is given by taking as system the unique random variable the set consisting of two dice $z = \{x_1, x_2\}$ and as alphabet, $A(z)$, the sum of the values of the two dice after throwing them simultaneously: $A(z) = \{2, 3, 4, \ldots, 12\}$.

Remark 1.1.6. The random variables of the systems S, T and W might seem, at first glance, to be the same, but it is important to make clear that

they are **very different**. In S there are two random variables while in b) and c) there is only one. Also notice that in T we have chosen an ordering of the two dice, while in W the random variable is an unordered set of two dice. With example a) there is nothing stopping us from throwing the die x_1, say, 20 times and the die x_2 ten times. However, in both b) and c) the dice are each thrown the same number of times.

Example 1.1.7. There are many naturally occurring examples of systems with many random variables. In fact, some of the most significant ones come from applications in economics and biology and have an astronomical number of variables.

For example, in economics and in market analysis, there are systems with one random variable for each company which trades in a particular market. It is easy to see that, in this case, we can have thousands, even tens of thousands, of variables.

In biology, very important examples come from studying systems in which the random variables represent hundreds (or thousands) of positions in the DNA sequence of one or several species. The alphabet of each variable consists of the four basic ingredients of DNA: Adenine, Cytosine, Guanine and Thymine. As a shorthand notation, one usually denotes the alphabet of such random variables as $\{A, C, G, T\}$).

In this book, we will refer to the systems arising from DNA sequences, as *DNA-systems*.

Example 1.1.8. For cultural reasons (one of the authors was born and lives in Siena!), we will have several examples in the text of systems describing probabilistic events related to the famous and colorful Sienese horse race called the Palio di Siena. Horses which run in the Palio represent the various medieval neighborhoods of the city (called *contrade*) and the Palio is a substitute for the deadly feuds which existed between the various sections of the city.

The names of the neighborhoods are listed below with a shorthand letter abbreviation for each of them:

Aquila	(eagle)	(symbol: A)	Bruco	(catepillar)	(symbol: B)
Chiocciola	(snail)	(symbol: H)	Civetta	(little owl)	(symbol: C)
Drago	(dragon)	(symbol: D)	Giraffa	(giraffe)	(symbol: G)
Istrice	(crested porcupine)	(symbol: I)	Leocorno	(unicorn)	(symbol: E)
Lupa	(she-wolf)	(symbol: L)	Nicchio	(conch)	(symbol: N)
Oca	(goose)	(symbol: O)	Onda	(wave)	(symbol: Q)
Pantera	(panther)	(symbol: P)	Selva	(forest)	(symbol: S)
Tartuca	(tortoise)	(symbol: R)	Torre	(tower)	(symbol: T)
Valdimontone	(valley of the ram)	(symbol: M).			

Definition 1.1.9. A random variable x of a system S is called a *boolean variable* if its alphabet has cardinality 2. A system is *boolean* if all its random variables are boolean.

Remark 1.1.10. The states of a boolean random variable can be thought of as the pair of conditions *(true, false)*. As a matter of fact the standard alphabet of a boolean random variable can be thought of as the elements of the finite field $\mathbb{Z}_2$, where 1 = true and 0 = false (this is our convention; be careful: in some texts this notation is reversed!). Other alphabets, such as heads-tails or even-ood, are also often used for the alphabets of boolean random variables.

Definition 1.1.11. A *map* or *morphism* between systems S and T of random variables is a pair $f = (F, G)$ where F is a function $F : S \to T$ and, for all $x \in S$, G defines a function between alphabets $G(x) : A(x) \to A(F(x))$.

The terminology used for functions can be transferred to maps of system of random variables. Thus we can have injective maps (in which case both F and each of the $G(x)$ are injective), surjective maps (in which case both F and each of the $G(x)$ are surjective), isomorphism (in which case both F and all the maps $G(x)$ are 1-1 correspondences). With respect to these definitions, the systems of random variables form a category.

Example 1.1.12. If S' is a subsystem of S, the inclusion function $S' \to S$ defines, in a obvious way, an injective map of systems. In this case, the maps between alphabets are always represented by the identity map.

Example 1.1.13. Let $S = \{x\}$ be the system defined by a die as in Example 1.1.3 with alphabet $\{1, 2, 3, 4, 5, 6\}$. Let T be the system defined by $T = \{y\}$, with $A(y) = \{E, O\}$ (E =even, O =odd). The function $F : S \to T$, defined by $F(x) = y$, and the function $G : A(x) \to A(y)$ defined by $G(1) = G(3) = G(5) = O$ and $G(2) = G(4) = G(6) = E$, define a surjective map from S to T of systems of random variables.

The following definition will be of fundamental importance for the study of the relationship between systems of random variables.

Definition 1.1.14. The *(total) correlation* of a system S of random variables $\{x_1, \ldots, x_n\}$ is the system $\Pi S = \{x\}$, with a unique random variable $x = (x_1, \ldots, x_n)$ (the cartesian product of the elements $x_1, \ldots, x_n$ of S). Its alphabet is given by $A(x_1) \times \cdots \times A(x_n)$, the cartesian product of the alphabets of the individual random variables.

Remark 1.1.15. It is very important to notice that the definition of the total correlation uses the idea of a cartesian product. Moreover the idea of the cartesian product requires that we *fix an ordering of the variables in S.*

Thus, the total correlation of a system is not uniquely determined, but it changes as the chosen ordering of the random variables changes.

It is easy to see, however, that all the possible total correlations of the system S are isomorphic.

Example 1.1.16. If S is a system with *two* coins c_1, c_2, each having alphabet $\{H, T\}$ then the only random variable in its total correlation, has an alphabet with *four* elements $\{(T, T), (T, H), (H, T), (H, H)\}$. I.e. we have to distinguish between the states (H, T) and (T, H). This is how the ordering of the coins enters into the definition of the random variable (c_1, c_2) of the total correlation.

Example 1.1.17. Let S be the system of random variables consisting of two dice, D_1 and D_2 each having alphabet the set $\{1, 2, 3, 4, 5, 6\}$. The total

correlation of this system, ΠS, is the system with a unique random variable $D = (D_1, D_2)$ and alphabet the set $\{(i,j) \mid 1 \leq i,j \leq 6\}$. So, the alphabet consists of 36 elements.

Now let T be the system whose unique random variable is the set $x = \{D_1, D_2\}$ and whose alphabet consists of the eleven numbers $\{2, 3, 4, ..., 11, 12\}$.

We can consider the surjective morphism of systems $\phi : \Pi S \to T$ which takes the unique random variable of ΠS to the unique random variable of T and takes the element (i,j) of the alphabet of the unique variable of ΠS to $i + j$ in the alphabet of the unique variable of T.

Undoubtedly this morphism is familiar to us all!

Clearly if S is a system containing a single random variable, then X coincides with its total correlation.

Definition 1.1.18. Let $f : S \to T$ be a map of systems of random variables, defined by $F : S \to T$ and $G(x) : A(x) \to A(F(x))$ for all random variables $x \in S$, and suppose that F is bijective i.e. S and T have the same number of random variables.

Then f defines, in a natural way, a map $\Pi f : \Pi S \to \Pi T$ between the total correlations as follows: for each state $s = (s_1, \ldots, s_n)$ of the unique variable $(x_1, \ldots, x_n)$ of ΠS,

$$(\Pi f)(s) = (G(x_1)(s_1), \ldots, G(x_n)(s_n)).$$

1.2 Distributions

One of the basic notions in the study of systems of random variables is the idea of a *distribution*. Making the definition of a distribution precise will permit us to explain clearly the idea of an *observation* on the random variables of a system. This latter concept is extremely useful for the description of real phenomena.

Definition 1.2.1. Let K be any set. A *K-distribution* on a system S with random variables $x_1, \ldots, x_n$, is a set of functions $D = \{D_1, \ldots, D_n\}$, where for $1 \leq i \leq n$, D_i is a function from $A(x_i)$ to K.

Remark 1.2.2. In most concrete examples, K will be a numerical set, i.e. some subset of $\mathbb{C}$ (the complex numbers).

The usual use of the idea of a distribution is to associate to each state of a variable x_i in the system S, the number of times (or the percentage of times) such a state is observed in a sequence of observations.

Example 1.2.3. Let S be the system having as unique random variable a coin c, with alphabet $A(c) = \{T, H\}$ (the coin need not be honest!).
Suppose we throw the coin n times and observe the state T exactly d_T times and the state H exactly d_H times ($d_T + d_H = n$). We can use those observations to get an $\mathbb{N}$-distribution ($\mathbb{N}$ the natural numbers), denoted D_c, where $D_c : \{T, H\} \to \mathbb{N}$ by

$$D_c(T) = d_T, D_c(H) = d_H \ .$$

One can identify this distribution with the element $(d_T, d_H) \in \mathbb{N}^2$.
We can define a different distribution, D'_c on S (using the same series of observations) as follows:

$$D'_c : \{T, H\} \to \mathbb{Q} \text{ (the rational numbers)},$$

$$\text{where } D'_c(T) = d_T/n \text{ and } D'_c(H) = d_H/n.$$

Example 1.2.4. Now consider the system S with *two* coins c_1, c_2, and with alphabets $A(c_i) = \{T, H\}$.
Again, suppose we simultaneously throw both coins n times and observe that the first coin comes up with state T exactly d_1 times and with state H exactly e_1 times, while the second coin comes up T exactly d_2 times and comes up H exactly e_2 times.

From these observations we can define an $\mathbb{N}$-distribution, $D = (D_1, D_2)$, on S defined by the functions

$$D_1 : \{T, H\} \to \mathbb{N}, \quad D_1(T) = d_1,\ D_1(H) = e_1,$$
$$D_2 : \{T, H\} \to \mathbb{N}, \quad D_2(T) = d_2,\ D_2(H) = e_2$$

It is also possible to identify this distribution with the element

$$((d_1, e_1), (d_2, e_2)) \in \mathbb{N}^2 \times \mathbb{N}^2.$$

It is also possible to use this series of observations to define a distribution on the total correlation ΠS.

That system has a unique variable $c = c_1 \times c_2 = (c_1, c_2)$ with alphabet $A(c) = \{TT, TH, HT, HH\}$. The $\mathbb{N}$-distribution on ΠS we have in mind associates to each of the four states how often that state appeared in the series of throws.

Notice that if we only had the first distribution we could not calculate the second one since we would not have known (solely from the functions D_1 and D_2) how often each of the states in the second system were observed.

Definition 1.2.5. The set of K-distributions of a system S of random variables forms the *space of distributions* $\mathcal{D}_K(S)$.

Example 1.2.6. Consider the DNA-system S with random variables precisely 100 fixed positions (or sites) $p_1, \ldots, p_{100}$ on the DNA strand of a given organism. As usual, each variable has alphabet $\{A, C, G, T\}$. Since each alphabet has exactly four members, the space of $\mathbb{Z}$-distributions on S is $\mathcal{D}(S) = \mathbb{Z}^4 \times \cdots \times \mathbb{Z}^4 (100 \text{ times}) = \mathbb{Z}^{400}$.

Suppose we now collect $1,000$ organisms and observe which DNA component occurs in site i. With the data so obtained we can construct a $\mathbb{Z}$-distribution $D = \{D_1, \ldots, D_{100}\}$ on S where D_i associates to each of the members of the alphabet $A(p_i) = \{A, C, G, T\}$ the number of occurrences of

the corresponding component in the $i-$th position. Note that for each D_i we have

$$D_i(A) + D_i(C) + D_i(G) + D_i(T) = 1,000.$$

Remark 1.2.7. Suppose that S is a system with n random variables $x_1, \ldots, x_n$ and that the cardinality of each alphabet $A(x_i)$ is exactly a_i. As we have said before, a_i is simply the number of states that the random variable X_i can assume.
With this notation, the K-distributions on S can be seen as points in the space

$$K^{a_1} \times \cdots \times K^{a_n} \ .$$

We will often identify $\mathcal{D}_K(S)$ with this space.

It is also certainly true that $K^{a_1} \times \cdots \times K^{a_n} = K^{a_1 + \cdots + a_n}$, and so it might seem reasonable to say that this last is the set of distributions on S. However, since there are so many ways to make this last identification we could easily lose track of what a particular distribution did on a member of the alphabet of one of the variables in S.

Remark 1.2.8. If S is a system with two variables x_1, x_2, whose alphabets have cardinality (respectively) a_1 and a_2, then the unique random variable in the total correlation ΠS has $a_1 a_2$ states. Hence, as we said above, the space of K-distributions on ΠS could be identified with $K^{a_1} \times K^{a_2}$.

Since we also wish to remember that the unique variable of ΠS arises as the cartesian product of the variables of S, it is even more convenient to think of $\mathcal{D}_K(\Pi S) = K^{a_1} \times K^{a_2}$ as the set of $a_1 \times a_2$ matrices with coefficients in K.

Thus, for a distribution D on ΠS, we denote by D_{ij} the value associated to the state (i, j) of the unique variable, which corresponds to the states i of x_1 and j of x_2.

For system with a bigger number of variables, we need to use *multidimensional matrices*, commonly called *tensors* (see Definition 6.1.3).

The study of tensors is thus strongly connected to the study of systems of random variables when we want to fix relationships among the variables (i.e. look at distributions on the system). In fact, the algebra (and geometry) of spaces of tensors represents the point of connection between the study of statistics on discrete sets and other disciplines, such as Algebraic Geometry. The exploration of this connection is our main goal in this book. We will take up that connection in another chapter.

Definition 1.2.9. Let S and T be two systems of random variables and $f = (F, G) : S \to T$ a map of systems where F is a surjection. Let D be a distribution on S, and D' a distribution on T.

The *induced distribution* f^D_* on T (called *the image distribution*) is defined as follows: for t a state of the variable $y \in T$:

$$(f^D_*)_y(t) = \sum_{x \in F^{-1}(y), s \in G(x)^{-1}(t)} D_x(s)$$

.

The *induced distribution* $f^*_{D'}$ on S (called *the preimage distribution*) is defined as follows: for s a state of the variable x in S:

$$(f^*_{D'})_x(s) = D'_{F(x)}(G(x)(s)).$$

We want to emphasize that distributions on a system of random variables should, from a certain point of view, be considered as data on a problem. Data from which one hopes to deduce other distributions or infer certain physical, biological or economic facts about the system. We illustrate this idea with the following example.

Example 1.2.10. In the city of Siena (Italy) two spectacular horse races have been run every year since the seventeeth century, with a few interruptions caused by the World Wars. Each race is called a *Palio*, and the Palio takes place in the main square of the city. In addition there have been some additional extraordinary Palios run from time to time. From the last interruption, which ended in 1945, up to now (2014), a total number of 152

Palios have taken place. Since the main square is large, but not enormous, not every contrada can participate in every Palio. There is a method, partly based on chance, that decides whether or not a contrada can participate in a particular Palio.

Let's build a system with 17 boolean random variables, one for each contrada. For each variable we consider the alphabet $\{0, 1\}$. The space of $\mathbb{Z}$-distributions of this system is $\mathbb{Z}^2 \times \cdots \times \mathbb{Z}^2 = \mathbb{Z}^{34}$.

Let us define a distribution by indicating, for each contrada x, $D_x(1) =$ number of Palios where contrada x took part and $D_x(0) =$ number of Palios where contrada x did not participate. Thus we must always have $D_x(0) + D_x(1) = 152$.

The data are given in the following table

x	*name*	$D_x(1)$	$D_x(0)$	x	*name*	$D_x(1)$	$D_x(0)$
A	*Aquila*	88	64	B	*Bruco*	92	60
H	*Chiocciola*	84	68	C	*Civetta*	90	62
D	*Drago*	95	57	G	*Giraffa*	89	63
I	*Istrice*	84	68	E	*Leocorno*	99	52
L	*Lupa*	89	63	N	*Nicchio*	84	68
O	*Oca*	87	65	Q	*Onda*	84	68
P	*Pantera*	96	56	S	*Selva*	89	63
R	*Tartuca*	91	61	T	*Torre*	90	62
M	*Valdimontone*	89	63				

We see that the Leocorno (unicorn) contrada participated in the most Palios while the contrada Istrice (crested porcupine), Nicchio (conch), Onda (wave), Chiocciola (snail) participated in the fewest.

On the same system, we can consider another distribution E, where $E_x(1) =$ number of Palios that contrada x won and $E_x(0) =$ number of Palios that contrada x lost (non-participation is considered a loss). The Win-Loss table is given below:

x	*name*	$E_x(0)$	$E_x(1)$	x	*name*	$E_x(0)$	$E_x(1)$
A	*Aquila*	8	144	B	*Bruco*	5	147
H	*Chiocciola*	9	143	C	*Civetta*	8	144
D	*Drago*	11	141	G	*Giraffa*	12	140
I	*Istrice*	8	144	E	*Leocorno*	9	143
L	*Lupa*	5	147	N	*Nicchio*	9	143
O	*Oca*	14	138	Q	*Onda*	9	143
P	*Pantera*	8	144	S	*Selva*	15	137
R	*Tartuca*	10	142	T	*Torre*	3	149
M	*Valdimontone*	9	143				

From the two tables we see that more participation in the Palios does not necessarily imply more victories.

1.3 Measurements on a distribution

We now introduce the concepts of *sampling* and *scaling* on a distribution for a system of random variables.

Definition 1.3.1. Let K be a numerical set and let $D = (D_1, \ldots, D_n)$ be a distribution on the system of random variables $S = \{x_1, \ldots, x_n\}$. The number

$$c_D(x_i) = \sum_{s \in A(x_i)} D_i(s).$$

is called the *sampling* of the variable x_i in D. We will say that D has *constant sampling* if all variables in S have the same sampling in D.
A K-distribution D on S is called *probabilistic* if each $x_i \in S$ has sampling equal to 1.

Remark 1.3.2. Let S be a system with random variables $\{x_1, \ldots, x_n\}$ and let $D = (D_1, \ldots, D_n)$ be a K-distribution on S, where K is a numerical field.

If every variable x_i has sampling $c_D(x_i) \neq 0$, we can obtain from D an *associated probabilistic distribution* $\tilde{D} = (\tilde{D}_1, \dots \tilde{D}_n)$ defined as follows:

$$\text{for all } i \text{ and for all states } s \in A(x_i) \text{ set } \tilde{D}_i(s) = \frac{D_i(s)}{c_D(x_i)}.$$

Remark 1.3.3. In Example 1.2.3, the distribution D' is exactly the probabilistic distribution associated to D (seen as a $\mathbb{Q}$-distribution)

Convention. To simplify the notation in what follows and since we will always be thinking of the set K as some set of numbers, usually clear from the context, we won't mention K again but will speak simply of a *distribution* on a system S of random variables.

Warning. We want to remind the reader again that the basic notation in Algebraic Statistics is far from being standardized. In particular, the notation for a distribution is quite varied in the literature and in other texts.

E.g. if s_{ij} is the $j-$th state of the $i-$th variable x_i of the system S, and D is a distribution on S, we will denote this by writing $D_i(s_{ij})$ as the value of D on that state.

You will also find this number $D_i(s_{ij})$ denoted by $D_{x_i=s_{ij}}$.

Example 1.3.4. Suppose we have a tennis tournament with 8 players where a player is eliminated as soon as that player loses a match. So, in the first set of matches four players are eliminated and in the second two more are eliminated and then we have the final match between the remaining two players.

We can associate to this tournament a system with 8 boolean random variables, one variable for each player. We denote by D the distribution that, for each player x_i, is defined as:

$$D_i(0) = \text{ number of matches won ;}$$

$$D_i(1) = \text{ number of matches lost.}$$

Clearly the sampling $c(x_i)$ of every player x_i represents the number of matches played. For example, $c(x_i) = 3$ iff x_i is a finalist, while $c(x_i) = 1$ for the four players eliminated at the end of the first match. Hence D is not a distribution with constant sampling.

Notice that this distribution doesn't have any variable with sampling equal to 0 and hence there is an associated probabilistic distribution $\tilde{D}$, which represents the statistics of victories. For example, for the winner x_k, one has

$$\tilde{D}_k(0) = 1, \quad \tilde{D}_k(1) = 0.$$

Instead, for a player x_j eliminated in the semi-final,

$$\tilde{D}_j(0) = \tilde{D}_j(1) = \frac{1}{2}.$$

While for a player x_i eliminated after the first round we have

$$\tilde{D}_i(0) = 0, \quad \tilde{D}_i(1) = 1.$$

The concept of an *associated probabilistic distribution* to a distribution D is quite important in texts concerned with the analytic Theory of Probability. This is true to such an extent that those texts work directly only with probabilistic distributions.

This is not the path we have chosen in this text. For us the concept that will be more important than a probabilistic distribution is the concept of *scaling*. This latter idea is more useful in connecting the space of distributions with the usual spaces in which Algebraic Geometry is done.

Definition 1.3.5. Let $D = (D_1, \ldots, D_n)$ be a distribution on a system S with random variables $\{x_1, \ldots, x_n\}$. A distribution $D' = (D'_1, \ldots, D'_n)$ is a *scaling* of D if, for any $x = x_i \in S$, there exists a constant $\lambda_x \in K \setminus \{0\}$ such that, for all states $s \in A(x)$, $D'_x(s) = \lambda_x D_x(s)$.

Remark 1.3.6. Notice that the probabilistic distribution, D', associated to a distribution D is an example of a scaling of D, where $\lambda_x = 1/c(x)$.

Note moreover that, give a scaling D' of D, if D, D' have the same sampling, then they must coincide.

Remark 1.3.7. In the next chapters we will see that scaling doesn't substantially change a distribution. Using a projectivization method, we will consider two distributions "equal" if they differ only by a scaling.

Proposition 1.3.8. *Let $f : S \to T$ be a map of systems which is a bijection on the sets of variables. Let D be a distribution on S and D' a scaling of D. Then $f_*^{D'}$ is a scaling of f_*^D.*

Proof. Let y be a variable of T and let $t \in A(y)$. Since f is a bijection there is a unique $x \in S$ for which $f(x) = y$. Then by definition we have

$$(f_*^{D'})_y(t) = \sum_{s \in A(x)} D'(s) = \sum_{s \in A(x)} \lambda_i D(s) = \lambda_x (f_*^D)_y(t).$$

□

1.4 Exercises for Chapter 1

Exercise 1. Let us consider the random system associated with the tennis tournament, see Example 1.3.4.

Compute the probabilistic distribution for the finalist who did not win the tournament.

Compute the probabilistic distribution for a tournament with 16 participants.

Exercise 2. Let S be a random system with variables $x_1, \ldots, x_n$ and assume that all the variables have the same alphabet $A = \{a_1, \ldots, a_m\}$. Then one can create the *dual* system S' by taking $a_1, \ldots, a_m$ as variables, each a_i with alphabet $X = \{x_1, \ldots, x_n\}$.

Determine the relation between the dimension of the spaces of K-distributions of S and S'.

Exercise 3. Let S be a random system and let S' be a subsystem of S.

Determine the relation between the spaces of K-distributions of the correlations of S and S'.

Exercise 4. Let $f : S \to S'$ be a surjective map of random systems.

Prove that if a distribution D on S' has constant sampling, then the same is true for f^*_D.

Exercise 5. One can define a *partial correlation* over a system S, by connecting only *some* of the variables.

For instance, if S has variables $x_1, \dots, x_n$ and $m < n$, one can consider the partia correlation on the variables $x_1, \dots, x_m$ as a system T whose variables are $Y, x_{m+1}, \dots x_n$, where Y stands for the variable $x_1 \times \cdots \times x_m$, with alphabet the product $A(x_1) \times \cdots \times A(x_m)$

If S has variables c_1, c_2, c_3, all of them with alphabet $\{T, H\}$ (see Example 1.1.3), determine the space of K-distributions of the partial correlation T with random variables $c_1 \times c_2$ and c_3

Chapter 2

Models

Chapter 3

Complex projective algebraic statistics

Chapter 4

Conditional independence

Chapter 5

Maximum likelihood estimate

Part II

Multilinear Algebra

Chapter 6

Tensors

6.1 Basic definitions

The main objects of multilinear algebra that we will use in the study of Algebraic Statistics are multidimensional matrices, that we will call *tensors*.

One begins by observing that matrices are very versatile objects! One can use them for keeping track of information in a systematic way. In this case the entries in the matrix are "place holders" for the information. Any elementary book on Matrix Theory will be filled with examples (ranging from uses in Accounting, Biology and Combinatorics to uses in Zoology) which illustrate how thinking of matrices in this way gives a very important perspective for certain types of applied problems.

On the other hand, from a first course in Linear Algebra we know that matrices can be used to describe important mathematical objects. For example, one can use matrices to describe linear transformations between vector spaces or to represent quadratic forms. Coupled with the calculus these ideas form the backbone of much of mathematical thinking.

We want to now mention yet another way that matrices can be used: namely to describe bilinear forms. To see this let M be an $m \times n$ matrix with entries from the field K. Consider the two vector spaces K^m and K^n and suppose they have the standard bases. If $v \in K^m$ and $w \in K^n$ we will

represent them as $1 \times m$ and $1 \times n$ matrices respectively, where the entries in the matrices are the coordinates of v and w with respect to the chosen basis. So, let

$$v = \begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix}$$

and

$$w = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} .$$

The matrix M above can be used to define a function

$$K^m \times K^n \to K$$

described by

$$(v, w) \to vMw^t$$

where the expression on the right is simply the multiplication of three matrices (^{t} denoting matrix transpose). Notice that this function is linear both in K^m and in K^n , and hence is called a bilinear form.

On the other hand, given any bilinear form $B : K^m \times K^n \to K$ i.e. a function which is linear in both arguments, and choosing a basis for both K^m and K^n, we can associate to that bilinear form an $m \times n$, matrix, N, as follows: if $\{v_1, \ldots, v_m\}$ is the basis chosen for K^m and $\{w_1, \ldots, w_n\}$ is the basis chosen for K^n then we form the $m \times n$ matrix $N = (n_{i,j})$ where $n_{i,j} := B(v_i, w_j)$.

It is easy to see that if $v \in K^m$, $v = \sum_{i=1}^m a_i v_i$ and $w \in K^n$, $w = \sum_{j=1}^n b_j w_j$ Then

$$B(v, w) = \begin{pmatrix} a_1 & \cdots & a_m \end{pmatrix} N \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} .$$

Thus, bilinear forms mapping $K^m \times K^n \to K$ (and a choice of basis for both K^m and for K^n) are in 1-1 correspondence with $m \times n$ matrices with entries from K.

Remark 6.1.1. One should note that although $K^m \times K^n$ is a vector space of dimension $m+n$, the bilinear map defined above from that vector space to K is **not** a linear map. In fact, any vector in the cartesian product of the form $(v, \overline{0})$ or $(\overline{0}, w)$ (where $\overline{0}$ is the zero vector) is sent to 0 under the bilinear form, but the sum of those two vectors is (v, w) which does not necessarily get sent to 0 by the bilinear form.

Example 6.1.2. Recall that if S is a system with two random variables, say x and y, where $A(x)$ contains m elements and $A(y)$ contains n elements, then we used an $m \times n$ matrix M to encode all the information of a distribution on the total correlation ΠS. The (i, j) entry in M was the value of the distribution on the $(i, j)^{th}$ element in the alphabet of the unique random variable (x, y) of the system ΠS (see .....). This is an example where we used a matrix as a convenient place to store the information of a distribution on ΠS.

However, if we consider the i^{th} element of the alphabet of the random variable x as corresponding to the matrix

$$v = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

(where the 1 occurs in the i^{th} place in this $1 \times m$ matrix) and

$$w = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

(where this time the 1 occurs in the j^{th} place in this $1 \times n$ matrix) then the product vMw^t is precisely the (i, j) entry in the matrix M. But, as we noted above, this is the value of the distribution on the (i, j) element in the alphabet of the unique random variable in the total correlation we described above.

So, although the matrix M started out being considered simply as a place holder for information, we see that considering it as a bilinear form on an appropriate pair of vector spaces it can also be used to give us information about the original distribution.

Tensors will give us a way to generalize what we have just seen for two random variables to any finite number of random variables. So, tensors will encode information about the connections between distinct variables in a random system. As the study of the properties of such connections is a fundamental goal in Algebraic Statistics, it is clear that the role of tensors is ubiquitous in this book.

From the discussion above concerning bilinear forms and matrices, we see that we have a choice as to how to proceed. We can define tensors as multidimensional arrays or we can define tensors as multilinear functions on a cartesian product of a finite number of vector spaces. Both points of view are equally valid and will eventually bring us to the same place. The two ways are equivalent, as we saw above for bilinear forms, although sometimes one point of view is preferable to the other. We will continue with both points of view but, for low dimensional tensors, we will usually prefer to deal with the multidimensional arrays.

Before we get too involved in studying tensors, this is probably a good time to forewarn the reader that although matrices are very familiar objects for which there are well understood tools to aid in their study, that is far from the case for multidimensional matrices i.e. tensors. The search for appropriate tools to study tensors is part of ongoing research. The abundance of research on tensors (research being carried out by mathematicians, computer scientists, statisticians and engineers as well as by people in other scientific fields) attests to the importance that these objects have nowadays in real life applications.

Notation. For every positive integer i, we will denote by $[i]$ the set $\{1, \ldots, i\}$.

For the rest of the section K can indicate any set, but in practice K will always be a set of numbers (like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$).

Definition 6.1.3. A *tensor* T over K, of dimension n and type

$$d_1 \times \cdots \times d_n$$

is a multidimensional table of elements of K, in which any element is determined by a multiindex $(i_1, \ldots, i_n)$, where i_j ranges between 1 and d_j.

In more formal terms, a tensor T as above is a map:

$$T : [d_1] \times \cdots \times [d_n] \to K.$$

Equivalently (when K is a field), such a tensor T is a multilinear map

$$T : K^{d_1} \times \cdots \times K^{d_n} \to K$$

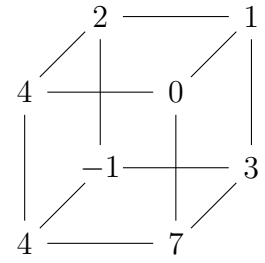
where we consider the standard bases for each of the K^{d_i}.

Remark 6.1.4. If we think of T as a multilinear map and suppose that for each $1 \leq i \leq n$, $\{e^i_j \mid 1 \leq j \leq d_i\}$ is the standard basis for K^{d_i} then the entry in the multidimensional array representation of T corresponding to the multiindex $(i_1, \ldots, i_n)$ is

$$T(e^1_{i_1}, e^2_{i_2}, \ldots, e^n_{i_n}) \ .$$

Tensors are a natural generalization of matrices. Indeed matrices of real numbers and of type $m \times n$ correspond exactly to tensors over $\mathbb{R}$ of dimension 2 and type $m \times n$.

Example 6.1.5. An example of a tensor over $\mathbb{R}$, of dimension 3 and type $2 \times 2 \times 2$ is:

$T =$ 

Notation. Although we have written a $2 \times 2 \times 2$ tensor above, we have not made clear which place in that array corresponds to $T(e^1_{i_1}, e^2_{i_2}, e^3_{i_3})$. We will

have to make a convention about that. Again, the conventions in the case of three dimensional tensors are not uniform across all books on multilinear algebra, but we will attempt to motivate the notation that we use, and is most common, by looking at the cases in which there is widespread agreement. I.e. the cases of 1 dimensional and 2 dimensional tensors.

Let's start by recalling the conventions for how to represent a 1 dimensional tensor, i.e. a linear function

$$T : \mathbb{R}^n \to \mathbb{R} \ .$$

Recall that such a tensor can be represented by a $1 \times n$ matrix as follows: let $e_1, \ldots, e_n$ be the standard basis for K^n and suppose that $T(e_i) = a_i$ then the matrix for this linear map is:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \ .$$

So, if $v = \sum_{i=1}^{n} \alpha_i e_i$ is any vector in K^n then

$$T(v) = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Now suppose that we have a 2-dimensional tensor T of type $m \times n$, i.e. a bilinear form

$$T : K^m \times K^n \to K \ .$$

Recall that such a tensor is represented by an $m \times n$ matrix, $\mathcal{A}$, as follows: let

$$\{e_j^1 \mid 1 \leq j \leq m\} \ \text{ be the standard basis for } K^m;$$

$$\{e_j^2 \mid 1 \leq j \leq n\} \ \text{ be the standard basis for } K^n$$

then

$$\mathcal{A} = (a_{i,j}) \ \text{ where } a_{i,j} := T(e_i^1, e_j^2) \ .$$

So

$$\mathcal{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

Now suppose we have a 3-dimensional tensor T of type $m \times n \times r$ i.e. a trilinear form

$$T : K^m \times K^n \times K^r \to K \ .$$

This tensor is represented by an $m \times n \times r$ box, $\mathcal{A}$, as follows: let

$$\{e_j^1 \mid 1 \leq j \leq m\} \text{ and } \{e_j^2 \mid 1 \leq j \leq n\}$$

be the standard basis for K^m and K^n respectively (as above) and let

$$\{e_j^3 \mid 1 \leq j \leq r\} \text{ be the standard basis for } K^r \ .$$

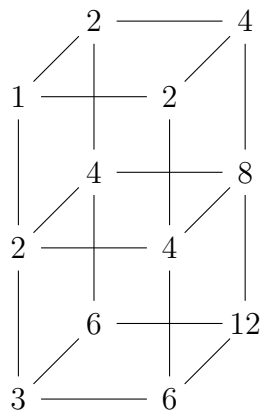
Then

$$\mathcal{A} = (a_{i,j,k}) \text{ where } a_{(i,j,k)} := T(e_i^1, e_j^2, e_k^3) \ .$$

How will we arrange these values in a rectangular box? We let the front (or *first*) face of the box be the $m \times n$ matrix whose (i, j) entry is $T(e_i^1, e_j^2, e_1^3)$. The second face, parallel to the first face, is the $m \times n$ matrix whose (i, j) entry is $T(e_i^1, e_j^2, e_2^3)$. We continue in this way so that the back face (the r^{th} face), parallel to the first face is the $m \times n$ matrix whose (i, j) entry is $T(e_i^1, e_j^2, e_r^3)$.

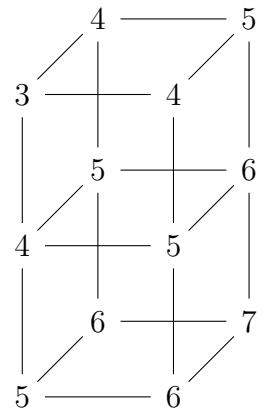
Example 6.1.6. Let T be the three dimensional tensor of type $3 \times 2 \times 2$ whose (i, j, k) entry is equal to ijk (the product of the three numbers). Then the $3 \times 2 \times 2$ rectangle has first face a 3×2 matrix whose (i, j) entry is $(ij) \cdot 1$. The second (or back) face is a 3×2 matrix whose (i, j) entry is $(ij) \cdot 2$. We put this all together to get our $3 \times 2 \times 2$ tensor.

$T_1 =$



To be assured that you have the conventions straight for trilinear forms, verify that the three dimension tensor of type $3\times2\times2$ whose multidimensional matrix representation has entries $(i, j, k) = i + j + k$, looks like

$T_1 =$



Remark 6.1.7. We saw above that elements of K^n can be considered as tensors of dimension 1 and type n. Notice that they can also be considered as tensors of dimension 2 and type $1 \times n$, or tensors of dimension 3 and type $1 \times 1 \times n$, etc.

Similarly, $n \times m$ matrices are tensors of dimension 2 but they can also be seen as tensors of dimension 3 and type $1 \times n \times m$, etc.

Elements of K can be seen as tensors of dimension 0.

As a generalization of what we can do with matrices, we mention the following easy fact.

Proposition 6.1.8. *When K is a field, the set of all tensors of fixed dimension n and type $d_1 \times \cdots \times d_n$ is a vector space where the operations are defined over elements with corresponding multiindices.*

This space, whose dimension is the product $d_1 \dots d_n$, will be denoted by $K^{d_1,\dots,d_n}$. One basis for this vector space is obtained by considering all the multidimensional matrices with a 1 in precisely one place and a zero in every other place. If that unique 1 is in the position $(i_1, \dots, i_n)$, we refer to that basis vector as $e_{(i_1,\dots,i_n)}$.

The null element of a space of tensors is the tensor having all entries equal to 0.

Now that we have established our convention about how the entries in a multidimensional array can be thought of it remains to be precise about how a multidimensional array gives us a multilinear map.

So, suppose we have a tensor T which is a tensor of dimension n and type $d_1 \times \cdots \times d_n$. Let $\mathcal{A} = (a_{i_1,i_2,\dots,i_n})$, where $1 \leq i_j \leq d_j$, be the multidimensional array which represents this tensor. We want to use $\mathcal{A}$ to define a multilinear map

$$\mathcal{T} : K^{d_1} \times \cdots \times K^{d_n} \to K$$

whose multidimensional matrix representation is precisely $\mathcal{A}$. Let $v_j \in K^{d_j}$, where v_j has coordinates $(\alpha_{j,1}, \dots, \alpha_{j,d_j})$ with respect to the standard basis for K^{d_j}. Then define

$$\mathcal{T}(v_1, v_2, \dots, v_n) = \sum (a_{i_1,i_2,\dots,i_n})(\alpha_{1,i_1} \cdot \alpha_{2,i_2} \cdot \cdots \cdot \alpha_{n,i_n}) \ .$$

Now if $\{e_i^{[j]} \mid 1 \leq i \leq d_j, \ 1 \leq j \leq n\}$ is the standard basis for K^{d_j} then it is easy to see that

$$\mathcal{T}(e_{i_1}^{[1]}, \dots, e_{i_n}^{[n]}) = a_{i_1,i_2,\dots,i_n}$$

Because the $(e_{i_1}^{[1]}, \ldots, e_{i_n}^{[n]})$ form a basis for the space $K^{d_1} \times \ldots K^{d_n}$ and $\mathcal{T}$ is the unique multilinear map with values equal to the entries in the multidimensional matrix $\mathcal{A}$ (**see Exercise ????**) we are done.

6.2 The tensor product

Besides the natural operations (addition and scalar multiplication) between tensors of the same type, there is another operation, the *tensor product*, which combines tensors of any type. This tensor product is fundamental for our analysis of the properties of tensors.

The simplest way to define the tensor product is to think of tensors as multilinear maps. With that in mind we make the following definition.

Definition 6.2.1. Let $T \in K^{d_1,\ldots,d_n}$, $U \in K^{d'_1,\ldots,d'_m}$ be tensors. We define the *tensor product* $T \otimes U$ as the tensor $W \in K^{d_1,\ldots,d_n,d'_1,\ldots,d'_m}$ such that:

$$\text{if } a_i \in K^{d_i}, b_j \in K^{d'_j} \text{ then } W(a_1, \ldots, a_n, b_1, \ldots, b_m) = T(a_1, \ldots, a_n)U(b_1, \ldots, b_m).$$

We extend this definition to consider more factors. So, for any finite collection of tensors $T_j \in K^{d_{j1},\ldots,d_{jn_j}}$, $j = 1, \ldots, m$, one can define their tensor product as the tensor

$$W = T_1 \otimes \cdots \otimes T_m \in K^{d_{11},\ldots,d_{1n_1},\ldots,d_{m1},\ldots,d_{mn_m}}$$

such that

$$W(i_{11}, \ldots, i_{1n_1}, \ldots, i_{m1}, \ldots, i_{mn_m}) = T_1(i_{11}, \ldots, i_{1n_1}) \cdots T_m(i_{m1}, \ldots, i_{mn_m}).$$

Definition. This innocent looking definition actually contains some new and wonderful ideas. The following examples will illustrate some of the things that come from the definition. The reader should keep in mind how different this multiplication is from the usual multiplication that we know for matrices.

Example 6.2.2. Given two one dimensional tensors v and w of type m and n respectively we write $v = (a_1, \ldots, a_m) \in K^m$ and $w = (b_1, \ldots, b_n) \in K^n$. Then v defines a linear map (which we'll also call v)

$$v : K^m \to K \text{ defined by:} \quad v(x_1, \ldots, x_m) = \sum_{i=1}^{m} a_i x_i$$

and w a linear map (again abusively denoted w)

$$w : K^n \to K \text{ defined by:} \quad w(y_1, \ldots, y_n) = \sum_{i=1}^{n} b_i y_i \ .$$

By definition, the tensor product $v \otimes w$ is the bilinear map:

$$v \otimes w : K^m \times K^n \to K$$

defined by

$$v \otimes w : ((x_1, \ldots, x_m), (y_1, \ldots, y_n)) \to (\sum_{i=1}^{m} a_i x_i)(\sum_{i=1}^{n} b_i y_i).$$

If we let $\{e_1, \ldots, e_m\}$ be the standard basis for K^m and $\{e'_1, \ldots, e'_n\}$ be the standard basis for K^n then

$$v \otimes w : (e_i, e'_j) \to a_i b_j$$

and so the matrix for this bilinear form is $v^t w$.

To give a very specific example of this let $v = (1, 2) \in \mathbb{R}^2$ and $w = (2, -1, 3) \in \mathbb{R}^3$. Then:

$$v \otimes w = v^t w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \end{pmatrix}$$

We could just as well have considered the tensor $w \otimes v$. In the specific example we just considered, notice that

$$w \otimes v = w^t v = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & -2 \\ 3 & 6 \end{pmatrix} = (v^t w)^t.$$

We see here that the tensor product is not commutative. In fact, the two multiplications did not even give us tensors of the same type.

Example 6.2.3. Let's now consider a slightly more complicated example. This time we will take the tensor product of v, a 1-dimensional tensor of type 2, and multiply it by w, a 2-dimensional tensor of type 2×2. We can represent v by a 1×2 matrix and w by a 2×2 matrix. So, let

$$v = (2, -3) \in \mathbb{R}^2 \text{ and } w = \begin{pmatrix} 2 & -1 \\ 4 & 3. \end{pmatrix}$$

Then v defines a linear map

$$v : K^2 \to K \quad \text{given by} \quad v(x_1, x_2) = 2x_1 - 3x_2$$

and w defines a bilinear map

$$w : K^2 \times K^2 \to K \quad \text{given by} \quad w : ((y_1, y_2), (z_1, z_2)) = \begin{pmatrix} y_1 & y_2 \end{pmatrix} w \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} =$$

$$= 2y_1z_1 + 4y_2z_1 - y_1z_2 + 3y_2z_2.$$

Putting these all together we have a trilinear form,

$$v \otimes w : (K^2) \times (K^2 \times K^2) \to K$$

defined by

$$v \otimes w((x_1, x_2), (y_1, y_2), (z_1, z_2)) = (2x_1 - 3x_2)(2y_1z_1 + 4y_2z_1 - y_1z_2 + 3y_2z_2) =$$

$$= 4x_1y_1z_1 + 8x_1y_2z_1 - 2x_1y_1z_2 + 6x_1y_2z_2 - 6x_2y_1z_1 - 12x_2y_2z_1 + 3x_2y_1z_2 - 9x_2y_2z_2.$$

From this we express $v \otimes w$ as a $2 \times 2 \times 2$ multidimensional matrix, namely

$$v \otimes w = \begin{array}{ccc} & -2 & \text{———} & 6 \\ & | & & | \\ 4 & \text{———} & 8 & \\ | & & | & \\ | & & | & \\ & 3 & \text{———} & -9 \\ | & & & | \\ -6 & \text{———} & (-12) & \end{array}$$

On the other hand,

$$w \otimes v((x_1, x_2), (y_1, y_2), (z_1, z_2)) = w((x_1, x_2), (y_1, y_2))v(z_1, z_2) =$$

$$(2x_1y_1 + 4x_2y_1 - x_1y_2 + 3x_2y_2)(2z_1 - 3z_2) =$$

$$= 4x_1y_1z_1+8x_2y_1z_1-2x_1y_2z_1+6x_2y_2z_1-6x_1y_1z_2-12x_2y_1z_2+3x_1y_2z_2-9x_2y_2z_2 \, .$$

So, the multidimensional array for $w \otimes v$ is:

$$w \otimes v = \begin{array}{ccc} -6 & & 3 \\ & \diagdown & \\ 4 & & -2 \\ | & & | \\ & -12 & \\ & \diagdown & \\ 8 & & 6 \end{array}$$

Example 6.2.4. Observe that if T, U are $n \times n$ matrices, the tensor product $T \otimes U$ *does not* coincide with their row-by-column product. The tensor product of these two matrices is a tensor of dimension 4, of type $n \times n \times n \times n$.

Definition. As we just noted, the tensor product does not define an internal operation in the spaces of tensors of the same dimension and same type. It is possible, however, to define something called the *tensor algebra* on which the tensor product behaves like a product. We will just give the definition of the tensor algebra, but won't have occasion to use it in this text.

Definition 6.2.5. Let K be a field. The *tensor algebra over the space* K^n is the direct sum

$$\mathscr{T}(n) = K \oplus K^n \oplus K^{n,n} \oplus \cdots \oplus K^{n,\ldots,n} \oplus \cdots$$

The tensor product defines a homogeneous operation inside $\mathscr{T}(n)$.

Remark 6.2.6. It is an easy (but messy) consequence of our definition that the tensor product is an associative product, i.e. if T, U, V are tensors, then

$$T \otimes (U \otimes V) = (T \otimes U) \otimes V.$$

Notice that the tensor product is *not*, in general, a commutative product (see Example 6.2.3 above). Indeed, in that example we saw that even the spaces in which $T \otimes U$ and $U \otimes T$ lie can be different.

Remark 6.2.7. The tensor product of tensors has the following properties: for any $T, T' \in K^{d_1,\dots,d_n}$, $U, U' \in K^{d'_1,\dots,d'_m}$ and $a \in K$, one has

- $T \otimes (U + U') = T \otimes U + T \otimes U'$;
- $(T + T') \otimes U = T \otimes U + T' \otimes U$;
- $(aT) \otimes U = T \otimes (aU) = a(T \otimes U)$.

This can be expressed by saying that the tensor product is linear over the two factors.

More generally, the tensor product defines a map

$$K^{d_{11},\dots,d_{1n_1}} \times \cdots \times K^{d_{m1},\dots,d_{mn_m}} \longrightarrow K^{d_{11},\dots,d_{1n_1},\dots,d_{m1},\dots,d_{mn_m}}$$

which is linear in any factor. For this reason we say that the tensor product is a multilinear product in its factors.

The following useful proposition holds for the tensor product.

Proposition 6.2.8. *(Vanishing Law) Let T, U be tensors. Then:*
- If $T = 0$ or $U = 0$, then $T \otimes U = 0$.
- Conversely, if $T \otimes U = 0$ then either $T = 0$ or $U = 0$.

Proof. Assume $T' \in K^{d_1,\dots,d_n}$, $U \in K^{e_1,\dots,e_m}$.

If $T = 0$ then for any choice of the indices $i_1, \dots, i_n, j_1, \dots, j_m$ one has:

$$(T \otimes U)_{i_1,\dots,i_n,j_1,\dots,j_m} = T_{i_1,\dots,i_n} \cdot U_{j_1,\dots,j_m} = 0 \cdot U_{j_1,\dots,j_m} = 0.$$

A similar computation holds when $U = 0$.

Conversely, if $T \neq 0$ and $U \neq 0$, then there exist two sets of indices, $i_1, \ldots, i_n$ and $j_1, \ldots, j_m$, such that $T_{i_1,\ldots,i_n} \neq 0$ and $U_{j_1,\ldots,j_m} \neq 0$. Thus

$$(T \otimes U)_{i_1,\ldots,i_n,j_1,\ldots,j_m} = T_{i_1,\ldots,i_n} \cdot U_{j_1,\ldots,j_m} \neq 0.$$

□

The bilinear map

$$K^{d_1,\ldots,d_n} \times K^{e_1,\ldots,e_m} \to K^{d_1,\ldots,d_n,e_1,\ldots,e_m}$$

determined by the tensor product, is not injective (as the Vanishing Law clearly shows). However we can characterize tensors $T, T' \in K^{d_1,\ldots,d_n}$ and $U, U' \in K^{e_1,\ldots,e_m}$ such that $T \otimes U = T' \otimes U'$.

Proposition 6.2.9. *Let $T, T' \in K^{d_1,\ldots,d_n}$ and $U, U' \in K^{e_1,\ldots,e_m}$ satisfy*

$$T \otimes U = T' \otimes U' \neq 0.$$

Then there exists a non-zero scalar $a \in K$ such that $T' = aT$ and $U' = \frac{1}{a}U$.

In particular if $U = U'$ then $T = T'$ (and conversely).

Proof. Put $Z = T \otimes U = T' \otimes U'$. Since $Z \neq 0$, there exists a choice of indices such that

$$Z_{i_1,\ldots,i_n,j_1,\ldots,j_m} = T_{i_1,\ldots,i_n} \cdot U_{j_1,\ldots,j_m} = T'_{i_1,\ldots,i_n} \cdot U'_{j_1,\ldots,j_m} \neq 0.$$

Thus $T_{i_1,\ldots,i_n} \neq 0$.

Let

$$b = T'_{i_1,\ldots,i_n} / T_{i_1,\ldots,i_n} \ .$$

Since $b \neq 0$, it is easy to show that

$$U'_{k_1,\ldots,k_m} = \frac{T'_{i_1,\ldots,i_n}}{T_{i_1,\ldots,i_n}} U_{k_1,\ldots,k_m} = bU_{k_1,\ldots,k_m},$$

for all $k_1, \dots, k_m$, i.e. $U' = bU$.

Similarly, since $U_{j_1,\dots,j_m} \neq 0$, we can let $a \neq 0$ be the quotient $U'_{j_1,\dots,j_m}/U_{j_1,\dots,j_m}$. As above one shows that $T' = aT$.

Finally, by multilinearity, we get $Z = T' \otimes U' = (ab)(T \otimes U)$. Hence $ab = 1$, i.e. $b = \frac{1}{a}$.

The final statement in the Proposition is clear from the preceeding since a must equal 1. □

Using the associativity of the tensor product and slightly modifying the proof of the preceeding proposition one can prove, by induction on the number of factors, the following result:

Proposition 6.2.10. *Let* $T_1, U_1 \in K^{d_{11},\dots,d_{1n_1}}, \dots, T_s, U_s \in K^{d_{s1},\dots,d_{sn_s}}$ *satisfy*

$$T_1 \otimes T_2 \otimes \cdots \otimes T_s = U_1 \otimes U_2 \otimes \cdots \otimes U_s \neq 0.$$

Then there exist non-zero scalars $a_1, \dots, a_s \in K$ *such that* $U_i = a_i T_i$ *for all* i*, and moreover* $a_1 \cdots a_s = 1$.

Remark 6.2.11. We mentioned above that the tensor product of two bilinear forms, represented by matrices M and N respectively, doesn't correspond to the product of the two matrices M and N. Indeed, in most cases we cannot even take the product of the two matrices!

However, when M is an $n \times m$ matrix and N is an $m \times s$ matrix we can form their product as matrices and also form their tensor product. It turns out that there is a relation between these two objects.

The tensor product is an element of the vector space $K^n \times K^m \times K^m \times K^s$ while the matrix product can be considered as an element of $K^n \times K^s$. How can we recover the regular product from the tensor product?

Now the tensor product is the tensor Q of dimension 4 and type (n, m, m, s), such that $Q(i, j, k, l) = M(i, j)N(k, l)$. The row-by column product of M, N is obtained by sending Q to the matrix $Z \in K^{n,s}$ defined by:

$$Z(i, l) = \sum_j T(i, j, j, l).$$

So, the ordinary matrix product is obtained, in this case, by taking the tensor product and following that by a projection onto the space $K^n \times K^s = K^{n,s}$.

6.3 Rank of tensors

In the next two sections we generalize, to tensors of any dimension,a definition which is basic in the theory of matrices, namely the notion of the *rank* of a matrix.

To find the appropriate generalization to tensors we will have to choose among the many equivalent ways one can define the rank of a matrix. It turns out that it is not convenient to choose, as the definition of rank, its characterization as the dimension of either the row space or the column space of a matrix. We will use, instead, a characterization of the rank of a matrix which is probably less familiar to the reader, but which turns out to be perfect for a generalization to arbitrary tensors. The starting point is a simple characterization of matrices of rank 1.

Proposition 6.3.1. *Let $A = (a_{ij})$ be a non-zero $m \times n$ matrix with coefficients in a field K. A has rank 1 if and only if there are non − zero vectors $v \in K^m$, $w \in K^n$ such that,*

$$A = v \otimes w = v^t w.$$

Proof. Assume that v and w exist. Since $A = v^t w$ every row of A is a multiple of w and so the row space of A has dimension 1 and hence the rank of A is 1.

Conversely, if the rank of A is 1 then every row of A is a multiple of some non-zero vector, which we will call w. I.e. the i^{th} row of A is $c_i w$. If we set $v = (c_1, \ldots, c_m)$ then clearly $A = v^t w = v \otimes w$. □

Thus, one can define matrices of rank 1 in terms of the tensor product of vectors.

Although the rank of a matrix M is usually defined as the dimension of either the row space or column space of M, we now give a neat characterization of rank(M) in terms of matrices of rank 1.

Proposition 6.3.2. *Let $M \neq 0$ be an $m \times n$ matrix. Then the rank of M is equal to the smallest integer r such that M is a sum of r matrices of rank* 1.

Proof. Assume $M = M_1 + \cdots + M_r$, where every M_i has rank 1. Then we may write $M_i = (v_i)^t w_i$ where $v_i \in K^m$, $w_i \in K^n$. Form the matrix A whose columns are the vectors v_i^t, and the matrix B whose rows are the vectors w_i. It is easy to see that

$$M = AB$$

and so the rows of M are linear combinations of the rows of B. Since B has only r rows we obtain that $rank(M) \leq r$.

Conversely, assume that M has rank r. Then we can find r linearly independent vectors in K^n which generate the row space of M. Call those vectors $w_1, \ldots, w_r$. Suppose that the i^{th} row of M is $c_{i,1}w_1 + \cdots + c_{i,r}w_r$. Form the vector $v_i = (c_{i,1}, \ldots, c_{i,r})$ and construct a matrix A whose i^{th} column is v_i^t. If B is the matrix whose j^{th} row is w_j then $M = AB = \sum_{i=1}^{r} v_i^t w_i$ is a sum of r matrices of rank 1 and we are done. □

The two previous results on matrices allow us to extend the definition of *rank* to tensors of any type.

Definition 6.3.3. A non-zero tensor $T \in K^{d_1,\ldots,d_n}$ has *rank* 1 if there are vectors $v_i \in K^{d_i}$ such that $T = v_1 \otimes \cdots \otimes v_n$. (since the tensor product is associative, there is no need to specify the order in which the tensor products in the formula are performed).

We define the *rank* of a non-zero tensor T to be the minimum r such that there exist r tensors $T_1, \ldots, T_r$ of rank 1 with

$$T = T_1 + \cdots + T_r. \tag{6.3.1}$$

Remark 6.3.4. A tensor of rank 1 is also called a *simple* or *decomposable* tensor.

For any tensor T of rank r, the expression 6.3.1 is called a *(decomposable) decomposition* of T.

We will sometimes just refer to the decomposable decomposition of T as a *decomposition* of T or a *rank decomposition* of T..

By convention we say that null tensors, i.e. tensors whose entries are all 0, have rank 0.

Remark 6.3.5. Let T be a tensor of rank 1 and let $a \neq 0, a \in K$. Then, using the multilinearity of the tensor product, we see that aT also has rank 1. More generally, if T has rank r then aT also has rank r. Then (exactly as for matrices), the union of the null tensor with all the tensors in $K^{d_1,\ldots,d_n}$ of rank r is closed under scalar multiplication.

Subsets of vector spaces that are closed under scalar multiplication are called *cones*. Thus the set of tensors in $K^{d_1,\ldots,d_n}$ of fixed rank (plus 0) is a cone.

On the other hand (again exactly as happens for matrices), in general the sum of two tensors in $K^{d_1,\ldots,d_n}$ of rank r need not have rank r. Thus the set of tensors in $K^{d_1,\ldots,d_n}$ having fixed rank (union the null tensor) is *not* a subspace of $K^{d_1,\ldots,d_n}$.

6.4 Tensors of rank 1

In this section we give a useful characterization of tensors of rank 1. There exists a generalization for *matrices* of higher rank but, unfortunately, there does not exist a similar characterization for tensors of higher rank and having dimension ≥ 3.

Recall that we are using the notation $[i] = \{1, 2, \ldots, i-1, i\}$.

Definition 6.4.1. Let $0 < m \leq n$ be integers. An injective *non-decreasing* function

$$f : [m] \to [n]$$

is a function with the property that

$$\text{whenever } \ a, b \in [m] \text{ and } \ a < b \text{ then } \ f(a) < f(b) \ .$$

With this technical definition made we are now able to define the notion of a *subtensor* of a given tensor.

Definition 6.4.2. Let T be a tensor in $K^{d_1,\dots,d_n}$. We consider T as a map

$$T : [d_1] \times \cdots \times [d_n] \to K \ .$$

For any choice of positive integers $d'_j \leq d_j$ ($1 \leq j \leq n$) and for any choice of injective, non-decreasing maps $f_j : [d'_j] \to [d_j]$ we define the tensor $T' \in K^{d'_1,\dots,d'_n}$ as follows:

$$T' : [d'_1] \times \cdots \times [d'_n] \to K$$

where

$$T'_{i_1 \dots i_n} = T_{i_{f_1(i_1)} \dots i_{f_n(i_n)}} \ .$$

Remark 6.4.3. This is a formal (and perhaps a bit odd) way to say that we are fixing a few values for the indices $i_1, \dots, i_n$ and forgetting the elements of T whose k-th index is not in the range of the map f_k.

Since we usually think of a tensor of type $1 \times d_2 \times \cdots \times d_n$ as a tensor of type $d_2 \times \cdots \times d_n$, whenever a $d'_k = 1$, we simply *forget* the k-th index in T. In this case the dimension of T' is $n - m$, where m is the number of indices for which $d'_k = 1$.

Example 6.4.4. A $3 \times 2 \times 2$ tensors T can be denoted as follows

$$T = \begin{array}{ccc} T_{112} & - & T_{122} \\ | & \diagdown & | \\ T_{111} & - & T_{121} \\ | & & | \\ T_{212} & - & T_{222} \\ | & \diagdown & | \\ T_{211} & - & T_{221} \\ | & & | \\ T_{312} & - & T_{322} \\ | & \diagdown & | \\ T_{311} & - & T_{321} \end{array}$$

and an instance is:

$$T = \begin{array}{ccc} 1 & - & 0 \\ | & \diagdown & | \\ -2 & - & 4 \\ | & & | \\ 2 & - & 3 \\ | & \diagdown & | \\ 0 & - & 1 \\ | & & | \\ -3 & - & 4 \\ | & \diagdown & | \\ 2 & - & 1 \end{array}$$

If one takes the maps $f_2 = f_3 =$ identity, $f_1 : [2] \to [3]$ defined as $f_3(1) = 1, f_3(2) = 3$, the the corresponding subtensor is:

$$T' = \begin{array}{ccc} 1 & - & 0 \\ | & \diagdown & | \\ -2 & - & 4 \\ | & & | \\ -3 & - & 4 \\ | & \diagdown & | \\ 2 & - & 1 \end{array}$$

i.e. one just cancels the layer corresponding to the elements whose first index is 2.

If, instead, one takes $f_2 = f_3 =$ identity, $f_1 : [1] \to [3]$ defined as $f_1(1) = 1$, then one gets the top face (matrix) of elements whose first index is 1:

$$T' = \begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$$

Definition 6.4.5. A subtensor of T of dimension 2 is called a *submatrix* of T. Note that any submatrix of T is a 2×2 matrix inside T (considered as a multidimensional array) *which is parallel to one of the faces of the array.* So, for instance, in the example 1.4.4 above, the array

$$\begin{pmatrix} T_{112} & T_{122} \\ T_{211} & T_{221} \end{pmatrix}$$

is not a submatrix of T.

Proposition 6.4.6. *If T has rank 1, and T' is a subtensor of T then either T' is the null tensor or T' has rank 1.*

In particular, if T has rank 1, then the determinant of any 2×2 submatrix of T vanishes.

Proof. Assume that $T \in K^{d_1,\dots,d_n}$ has rank 1. Then there exist vectors $v_i \in K^{d_i}$ such that $T = v_1 \otimes \cdots \otimes v_n$. Eliminating from T the elements whose k-th index has some value q corresponds to eliminating the q-th component in the vector v_k. Thus, the corresponding subtensor T' is the tensor product of the vectors $v_1', \dots, v_n'$, where $v_i' = v_i$ if $i \neq k$, and v_k' is the vector obtained from v_k by eliminating the q-th component. Thus T' has rank ≤ 1 (it has rank 0 if $v_k' = 0$). For a general subtensor $T' \in K^{e_1,\dots,e_n}$ of T, we obtain the result arguing step by step, by deleting each time one value for one index of T, i.e. arguing by induction on $(d_1 + \cdots + d_n) - (e_1 + \cdots + e_n)$.

The second claim in the statement of the theorem is immediate from what we have just said and the fact that a 2×2 matrix of rank 1 has determinant 0. □

Corollary 6.4.7. *The rank of a subtensor of T cannot be bigger than the rank of T.*

Proof. If T has rank 1, the claim follows from Proposition 6.4.6. For tensors T of higher rank r, the claim follows since if $T = T_1 + \cdots + T_k$, with T_i of rank 1, then a subtensor T' of T is equal to $T_1' + \cdots + T_k'$, where T_i' is the subtensor

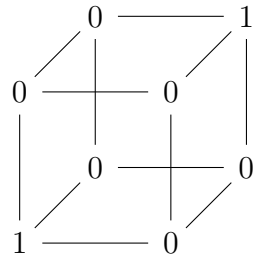
of T_i obtained by eliminating all the elements corresponding to elements of T eliminated in the passage from T to T'. Thus, by Proposition 6.4.6 each T'_i is either 0 or it has rank 1, and the claim follows. □

Example 6.4.8. Recall that a non-zero matrix has rank 1 if and only if all of its 2×2 submatrices have determinant equal to zero. This is not true for tensors of dimension greater than 2, as the following example shows. Recall our earlier warning about the subtle differences between matrices and tensors of dimension greater than 2.

Consider the $2 \times 2 \times 2$ tensor T, defined by:

$$\begin{array}{llll} T_{1,1,1} = 0 & T_{1,2,1} = 0 & T_{2,1,1} = 1 & T_{2,2,1} = 0 \quad (\text{ front face }) \\ T_{1,1,2} = 0 & T_{1,2,2} = 1 & T_{2,1,2} = 0 & T_{2,2,2} = 0 \quad (\text{back face}) \ . \end{array}$$

$T =$



It is clear that all the 2×2 submatrices of T have determinant equal to 0. On the other hand, if T has rank 1, i.e. $T = (a_1, a_2) \otimes (b_1, b_2) \otimes (c_1, c_2)$, then $T_{2,1,1} = a_2 b_1 c_1 \neq 0$ which implies $a_2, b_1, c_1 \neq 0$. However, $T_{2,1,2} = T_{1,1,1} = T_{2,2,1} = 0$ implies $a_1 = b_2 = c_2 = 0$. But then $T_{1,2,2} = a_1 b_2 c_2 = 1 \neq 0$ yields a contradiction.

We want to find a set of conditions which describe the set of all tensors of rank 1. To this aim, we need to introduce some new piece of notation.

Notation. Recall that we denote by $[n]$ the set $\{1, \ldots, n\}$.

Fix a subset $J \subset [i]$. Then for any fixed pair of multi-indexes $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, we denote by $J(a, b)$ the multi-index $(c_1, \ldots, c_n)$

where

$$c_j = \begin{cases} a_j & \text{if } j \in J, \\ b_j & \text{otherwise.} \end{cases}$$

Example 6.4.9. Let $n = 4$ and set $J = \{2, 3\} \subset [4]$. Consider the two multiindices $a = (1, 3, 3, 2)$ and $b = (2, 1, 3, 4)$. Then $J(a, b) = (2, 3, 3, 4)$. Notice that if $J' = [n] \setminus J = \{1, 4\}$ then $J'(a, b) = (1, 1, 3, 2)$.

Remark 6.4.10. If T has rank 1, then for any pair of multi-indexes $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ and for any subset $J \subset [n]$, the entries of T satisfy:

$$T_a T_b = T_{J(a,b)} T_{J'(a,b)} \tag{6.4.1}$$

where $J' = [n] \setminus J$.

To see why this is so recall that since T has rank 1 we can write $T = v_1 \otimes \cdots \otimes v_n$, with $v_i = (v_{i1}, v_{i2}, \dots)$. In this case both of the products in (6.4.1) are equal

$$v_{1a_1} v_{1b_1} \cdots v_{na_n} v_{nb_n}$$

and so the result is obvious.

Remark 6.4.11. When a, b differ only in two indices, the equality 6.4.1 simply says that the determinant of a 2×2 submatrix of T is 0.

Example 6.4.12. Look back to Example 6.4.8, and notice that if one takes $a = (1, 1, 1)$, $b = (2, 2, 2)$ and $J = \{1\} \subset [3]$, then $J(a, b) = (1, 2, 2)$ and $J'(a, b) = (2, 1, 1)$ so that formula (6.4.1) does not hold, since

$$T_a T_b = 0 \neq 1 = T_{J(a,b)} T_{J'(a,b)}.$$

Theorem 6.4.13. *A tensor $T \neq 0$ of dimension n has rank 1 if and only if it satisfies all the equalities* (6.4.1)*, for any choice of multiindices a, b and $J \subset [n]$.*

Proof. Thanks to Remark 6.4.10, we need only prove that if all the equalities (6.4.1) hold, then T has rank 1.

Let us argue by induction on the dimension n of $T \in K^{d_1,\ldots,d_n}$. The case $n = 2$ is well known: a matrix has rank 1 if and only if all its 2×2 minors vanish.

For $n > 2$, pick an entry $T_a = T_{a_1,\ldots,a_n} \neq 0$ in T.

Let $J_1 \subset [d_1]$ where $J_1 = \{1\}$ and let $f_1 : J_1 \to [d_i]$ be defined by $f_1(1) = a_1$. For $2 \leq i \leq n$, let $f_i = \textit{identity}$. Let T' be the subtensor corresponding to this data. T' is a tensor of dimension $n-1$ and hence satisfies the equalities (6.4.1). By induction, we obtain that $\text{rank}(T') = 1$, so there are vectors $v_2, \ldots v_n$ such that, for any choice of $i_2, \ldots, i_n$, one gets

$$T_{a_1,i_2,\ldots,i_n} = T'_{i_2,\ldots,i_n} = v_{2i_2} \cdots v_{ni_n}. \tag{a}$$

For all $m \in [d_1]$ define the number

$$p_m = \frac{T_{m,a_2,\ldots,a_n}}{T_{a_1,a_2,\ldots,a_n}}. \tag{b}$$

We use those numbers to define the vector $v_1 = (p_1, \ldots, p_{d_1})$.

We now claim that $T = v_1 \otimes v_2 \otimes \cdots \otimes v_n$.

Indeed for any $b = (b_1, \ldots, b_n)$, by setting $J = \{1\}$, and hence $J' = \{2, \ldots, n\}$, one obtains from the equalities (6.4.1) that:

$$T_a T_b = T_{J(a,b)} T_{J'(ab)} = T_{a_1,b_2,\ldots,b_n} T_{b_1,a_2,\ldots,a_n} = T_{a_1,b_2,\ldots,b_n} \cdot p_{b_1} T_{a_1,a_2,\ldots,a_n} \ .$$

Using the terms at the beginning and end of this string of equalities and also taking into account (a) and (b) above, we obtain:

$$v_{2a_2} \cdots v_{na_n} T_b = v_{2b_2} \cdots v_{nb_n} \cdot v_{1b_1} \cdot v_{2a_2} \cdots v_{na_n} \ .$$

Since $T_a \neq 0$, and hence $v_{2a_2}, \ldots, v_{na_n} \neq 0$, we can divide both sides of this equality by $v_{2a_2}, \ldots, v_{na_n}$ and finally get

$$T_b = v_{2b_2} \cdots v_{nb_n} \cdot v_{1b_1},$$

which proves the claim. □

The equations corresponding to the equalities (6.4.1) determine a set of polynomial (quadratic) equations, in the space of tensors $K^{d_1,\dots,d_n}$, which describe the locus of decomposable tensors. (Interestingly enough, it turns out that in many cases this set of equations is not minimal.)

In any event, Proposition 6.4.13 provides a finite procedure which allows us to decide if a given tensor has rank 1 or not. We simply plug the coordinates of the given tensor into the equations we just described and see if all the equations vanish or not.

Unfortunately, as the dimension grows, the number of operations required in the algorithm rapidly becomes quite large!

Recall that for matrices there is a much simpler method for calculating the rank of the matrix: one uses Gaussian reduction to find out how many non-zero rows that reduction has. That number is the rank. We really don't have to calculate the determinants of all the 2×2 submatrices of the original matrix.

There is nothing like the simple and well known Gaussian reduction algorithm (which incidentally calculates the rank for a tensor of dimension 2) for calculating the rank of tensors of dimension greater than 2. All known procedures for calculating the rank of such a tensor quickly become not effective.

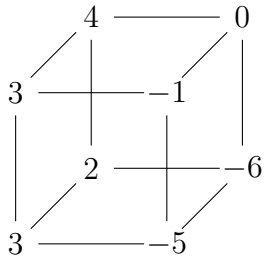
There are many other ways in which the behavior of rank for tensors having dimension greater than 2 differs considerably from the behavior of rank for matrices (tensors of dimension exactly 2). E.g. although a matrix of size $m \times n$ (a 2 dimensional tensor of type (m, n)) cannot have rank which exceeds the minimum of m and n, tensors of type $d_1 \times \cdots \times d_n$ (for $n > 2$) may have rank bigger than $\max\{d_i\}$. Although the *general* matrix of size $m \times n$ has rank $= min\{m, n\}$ (the maximum possible rank) there are often special tensors of a given dimension and type whose rank is bigger than the rank of a general tensor of that dimension and type.

The attempt to get a clearer picture of how rank behaves for tensors of a

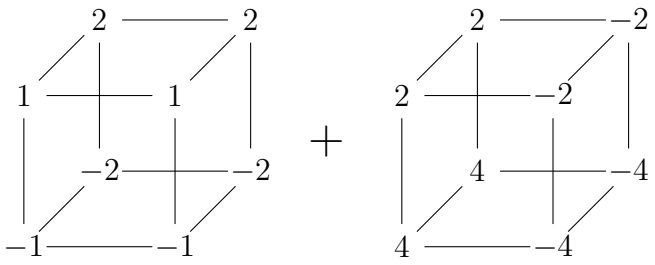
given dimension and type has many difficult problems associated to it. E.g. is there some nice geometric structure for the set of tensors having a given rank? when are there no tensors of a given rank? what is the maximum rank for a tensor of given dimension and type? These questions, and several variants of them, are the subject of research for many mathematicians and other scientists today.

We conclude this section with some examples which illustrate that although there is no algorithm for finding the rank of a given tensor, one can sometimes decide, using ad-hoc methods, exactly what is the rank of the tensor.

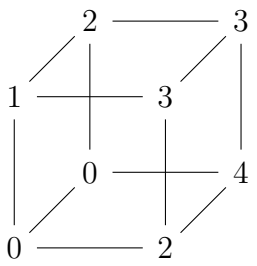
Example 6.4.14. The following tensor of type $2 \times 2 \times 2$ has rank 2:



Indeed it cannot have rank 1, because some of its 2×2 submatrices have determinant different from 0. T has rank 2 because it is the sum of two tensors of rank 1 (one can check, using the algorithm, that the summands have rank 1)



Example 6.4.15. The tensor

$D =$ 

has rank 3 i.e. one cannot write D as a sum of two tensors of rank 1. Let us see why.

Let's assume that D is the sum of two tensors $T = (T_{ijk})$ e $T' = (T'_{ijk})$ of rank 1 and let's try to derive a contradiction from that assumption.

Notice that the vector $(D_{211}, D_{212}) = (0, 0)$ would have to be equal to the sum of the vectors $(T_{211}, T_{212}) + (T'_{211}, T'_{212})$, Consequently the two vectors (T_{211}, T_{212}) and (T'_{211}, T'_{212}) are negatives of each other and hence span a subspace $W \subset K^2$ of dimension ≤ 1.

If one (hence both) of these vectors is non-zero, then also the vectors (T_{111}, T_{112}), (T_{221}, T_{222}),and (T'_{111}, T'_{112}), (T'_{221}, T'_{222}), would also have to belong to W because all the 2×2 determinants of T and T' vanish. But notice that (T_{121}, T_{122}) and (T'_{121}, T'_{122}) must also belong to W by Remark 6.4.10 (take $J = \{3\} \subset [3]$).

It follows that both vectors $(D_{111}, D_{112}) = (1, 2)$ and $(D_{121}, D_{122}) = (3, 3)$, must belong to W. This is a contradiction, since $\dim(W) = 1$ and $(1, 2), (3, 3)$ are linearly independent.

So, we are forced to the conclusion that $(T_{211}, T_{212}) = (T'_{211}, T'_{212}) = (0, 0)$. Since the sum of (T_{111}, T_{112}) and (T'_{111}, T'_{112}) is $(1, 2) \neq (0, 0)$ we may assume that one of them, say (T_{111}, T_{112}), is non-zero. As T has rank 1, there exists $a \in K$ such that $(T_{221}, T_{222}) = a(T_{111}, T_{112})$ (we are again using Remark 6.4.10).

Now, the determinant of the front face of the tensor T is 0, i.e.

$$0 = T_{111}T_{221} - T_{121}T_{211}.$$

Since $T_{211} = 0$ and $T_{221} = aT_{111}$ we get $0 = aT_{111}^2$. Doing the same argument on the back face of the tensor T we get $0 = aT_{112}^2$. It follows that $a = 0$ and so the bottom face of the tensor T consists only of zeroes.

It follows that $(T'_{221}, T'_{222}) = (2, 4)$. Since the tensor T' has rank 1, it follows that the vector (T'_{111}, T'_{112}) is also a multiple of $(2, 4)$, as is the vector (T'_{121}, T'_{122}).

Since $(T_{111}, T_{112}) = (1, 2) - (T'_{111}, T'_{112})$, and both $(1, 2)$ and (T'_{111}, T'_{112}) are multiples of $(2, 4)$, it follows that the vector (T_{111}, T_{112}) (which we assumed was not 0) is also a multiple of $(2, 4)$. Thus, since the tensor T has rank 1, the vector (T_{121}, T_{122}) is also a multiple of $(2, 4)$. Since we already noted that (T'_{121}, T'_{122}) is a multiple of $(2, 4)$ it follows that the vector $(3, 3)$ is a multiple of $(2, 4)$, which is the final contradiction.

Chapter 7

Symmetric tensors

In this chapter we make a specific analysis of the behavior of symmetric tensors, with respect to the rank and the decomposition.

We will see, indeed, that besides their utility to understand some models of random systems, symmetric tensors have a relevant role in the study of the algebra and the computational complexity of polynomials.

7.1 Generalities and examples

Definition 7.1.1. A *cubic* tensor is a tensor of type $d_1 \times \cdots \times d_n$ where all the d_i's are equal, i.e. a tensor of type $d \times \cdots \times d$ (n times).

We say that a *cubic* tensor T is *symmetric* if for any multiindex $(i_1, \ldots, i_n)$ and for any permutation σ on the set $\{i_1, \ldots, i_n\}$, it satisfies

$$T_{i_1,\ldots,i_n} = T_{i_{\sigma(1)},\ldots,i_{\sigma(n)}}.$$

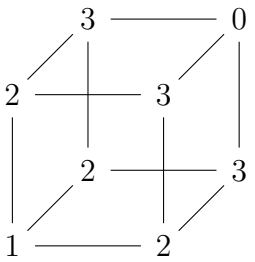
Example 7.1.2. When T is a square matrix, then the condition for the symmetry of T simply requires that $T_{i,j} = T_{j,i}$ for any choice of the indices. In other words, our definition of *symmetric tensor* coincides with the plain old definition of symmetric matrix, when T has dimension 2.

If T is a cubic tensor of type $2 \times 2 \times 2$, then T is symmetric if and only if

the following equalities hold:

$$\begin{cases} T_{1,1,2} &= T_{1,2,1} = T_{2,1,1} \\ T_{2,2,1} &= T_{2,1,2} = T_{1,2,2} \end{cases}$$

An example of a $2 \times 2 \times 2$ symmetric tensor is the following:



Remark 7.1.3. The set of symmetric tensors is a linear subspace of $K^{(d,\dots,d)}$. Namely it is defined by a set of linear equations:

$$T_{i_1,\dots,i_n} = T_{\sigma(i_1),\dots,\sigma(i_n)}$$

in the coordinates of $K^{(d,\dots,d)}$.

As a vector space itself, the space of symmetric tensors of type $d \times \cdots \times d$, n times, is usually denoted by $Sym^n(K^d)$.

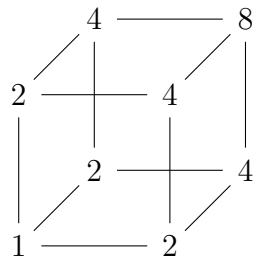
Of course, from the point of view of multilinear forms, $Sym^n(K^d)$ coincides with the space of *symmetric* multilinear maps $(K^d)^n \to K$.

As we will see later (see Proposition 7.3.8), the dimension of $Sym^n(K^d)$ is

$$\dim(Sym^n(K^d)) = \binom{n+d-1}{n} = \binom{n+d-1}{d-1}.$$

7.2 The rank of a symmetric tensor

Next step is the study of the behavior of symmetric tensors with respect to the rank. It is easy to realize that there are *symmetric tensors of rank* 1, i.e. the space $Sym^n(K^d)$ intersects the set of decomposable tensors. Just to give an instance, look at:

$D =$ 

The following proposition determines how one construct decomposable, symmetric tensors.

Proposition 7.2.1. *Let T be a cubic tensor of type $d \times \cdots \times d$, n times. Then T is symmetric, of rank 1, if and only if there exists a scalar $\alpha \in K$ and a non-zero vector $v \in K^d$ with:*

$$T = \alpha(v \otimes v \otimes \cdots \otimes v).$$

If moreover K is an algebraically closed field (as the complex field $\mathbb{C}$), then we may assume $\alpha = 1$.

Proof. If $T = \alpha(v \otimes v \otimes \cdots \otimes v)$, $v \neq 0$, then T cannot be zero by Proposition **??**, thus it has rank 1. Moreover if $v = (a_1, \ldots, a_d)$, then for any multiindex $(i_1, \ldots, i_n)$ and for any permutation σ:

$$T_{i_1,\ldots,i_n} = a_{i_1} \cdots a_{i_n} = T_{\sigma(i_1),\ldots,\sigma(i_n)}$$

thus T is symmetric.

Conversely, assume that T is symmetric of rank 1, say $T = v_1 \otimes \cdots \otimes v_n$, where no $v_i \in K^d$ can be 0, by Proposition **??**. Write $v_i = (v_{i,1}, \ldots, v_{i,d})$ and fix a multiindex $(i_1, \ldots, i_n)$ such that $v_{1,i_1} \neq 0$, $\ldots$, $v_{n,i_n} \neq 0$. Then $T_{i_1,\ldots,i_n} = v_{1,i_1} \cdots v_{n,i_n}$ cannot vanish. Define $b_2 = v_{2,i_1}/v_{1,i_1}$. Then we claim that $v_2 = b_2 v_1$. Namely, for all j we have, by symmetry:

$$v_{1,i_1} v_{2,j} v_{3,i_3} \cdots v_{n,i_n} = T_{i_1,j,i_3,\ldots,i_n} = T_{j,i_1,i_3,\ldots,i_n} = v_{1,j} v_{2,i_1} v_{3,i_3} \cdots v_{n,i_n},$$

which means that $v_{1,i_1}v_{2,j} = v_{1,j}v_{2,i_1}$, so that $v_{2,j} = b_2 v_{1,j}$. Similarly we can define $b_3 = v_{3,i_1}/v_{1,i_1}, \ldots$, $b_d = v_{d,i_1}/v_{1,i_1}$, and obtain that $v_3 = b_3 v_1$, $\ldots$, $v_d = b_d v_1$. Thus, if $\alpha = b_2 \cdot b_3 \cdot \cdots \cdot b_d$, then

$$T = v_1 \otimes v_2 \otimes \cdots \otimes v_n = v_1 \otimes (b_2 v_1) \otimes \cdots \otimes (b_n v_1) = \alpha(v_1 \otimes v_1 \otimes \cdots \otimes v_1).$$

When K is algebraically close, then take a d-th root β of $\alpha \in K$ and define $v = \beta v_1$. Then $T = \beta^d(v_1 \otimes v_1 \otimes \cdots \otimes v_1) = v \otimes v \otimes \cdots \otimes v$. □

Notice that purely algebraic properties of K can be relevant in determining the shape of a decomposition.

Remark 7.2.2. In the sequel, we will often write $v^{\otimes d}$ for $v \otimes v \otimes \cdots \otimes v$, d times.

If K is algebraically closed, then a symmetric tensor $T \in Sym^n(K^d)$ of rank 1 has a finite number (exactly: d) decompositions as a product $T = v^{\otimes d}$.

Namely if $w \otimes \cdots \otimes w = v \otimes \cdots \otimes v$, then by Proposition 6.2.9 there exists a scalar β such that $w = \beta v$ and moreover $\beta^d = 1$, thus w is equal to v multiplied by a d-th root of unity.

Passing from rank 1 to higher ranks, the situation becomes suddenly more involved.

The definition itself of *rank* of a symmetric tensors is not completely trivial, as we have two natural choices for it:

- **First choice**. The rank of a symmetric tensor $T \in Sym^n(K^d)$ is simply its rank as a tensor in $K^{(d,\ldots,d)}$, i.e. it is the minimum r for which one has r decomposable tensors $T_1,\ldots,T_r$ with
$$T = T_1 + \cdots + T_r.$$

- **Second choice**. The rank of a symmetric tensor $T \in Sym^n(K^d)$ is the minimum r for which one has r *symmetric* decomposable tensors $T_1,\ldots,T_r$ with
$$T = T_1 + \cdots + T_r.$$

Then, the natural question is about which choice gives the *correct definition*. Here *correct* definition means the definition which proves to be most useful, for the applications to Multilinear Algebra and random systems.

The reader could be disappointed in knowing that there is no clear preference between the two options: each can be preferable, depending on the point of view.

Thus, we will leave the word *rank* for the minimum r for which one has a decomposition $T = T_1 + \cdots + T_r$, with the T_i's not necessarily symmetric (i.e. the first choice above).

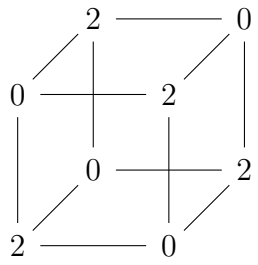
Then, we give the following:

Definition 7.2.3. The *symmetric rank* $\operatorname{srank}(T)$ of a symmetric tensor $T \in Sym^n(K^d)$ is the minimum r for which one has r *symmetric* decomposable tensors $T_1,\ldots,T_r$ with

$$T = T_1 + \cdots + T_r.$$

Example 7.2.4. The symmetric tensor

$T =$



has not rank 1, as one can compute by taking the determinant of some face.

T has rank 2, because it is expressible as the sum of two decomposable tensors $T = T_1 + T_2$, where:

$$T_1 = \begin{array}{ccc} 1 & \text{---} & 1 \\ | & & | \\ 1 & \text{---} & 1 \\ | & & | \\ 1 & \text{---} & 1 \end{array}$$

$$T_2 = \begin{array}{ccc} 1 & \text{---} & (-1) \\ | & & | \\ (-1) & \text{---} & 1 \\ | & & | \\ 1 & \text{---} & (-1) \end{array}$$

and $T_1 = (1,1)^{\otimes 3}$, $T_2 = (-1,1)^{\otimes 3}$.

Example 7.2.5. The tensor (over $\mathbb{C}$):

$$T = \begin{array}{ccc} 0 & \text{---} & 7 \\ | & & | \\ 1 & \text{---} & 0 \\ | & & | \\ 7 & \text{---} & 8 \\ | & & | \\ 0 & \text{---} & 7 \end{array}$$

is not decomposable. Let us prove that the symmetric rank is bigger than 2.

Assume that $T = (a,b)^{\otimes 3} + (c,d)^{\otimes 3}$. Then we have

$$\begin{cases} a^3c^3 & = 1 \\ a^2b + c^2d & = 0 \\ ab^2 + cd^2 & = 7 \\ b^3d^3 & = 8 \end{cases}$$

Notices that none of a, b, c, d can be 0. Moreover we have $ac = \epsilon$ and $bd = 2\epsilon'$, where ϵ, ϵ' are two cubic roots of unit, not necessarily equal. But then $c = \epsilon/a$

and $d = \epsilon'/b$, so that $a^2b + c^2d = 0$ yields $1 + \epsilon^2\epsilon' = 0$, which cannot hold, because -1 is not a cubic root of unit.

Remark 7.2.6. Proposition 7.2.1 shows in particular that any symmetric tensor of (general) rank 1 has also the symmetric rank equal to 1.

The relations between the rank and the symmetric rank of a tensor are not obvious at all, when the ranks are bigger than 1. It is clear that

$$\operatorname{srank}(T) \geq \operatorname{rank}(T),$$

but we have no description of situations where the strict inequality holds. Namely *no examples* are known of symmetric tensors over the complex field $\mathbb{C}$, for which the two ranks are different!

The difficulty in finding examples where the two ranks are different, despite the large number of concrete tensors tested, suggested to the French mathematician Pierre Comon to launch the following:

Conjecture 7.2.7. *(***Comon,2000***) Over the complex field, the rank and the symmetric rank of a symmetric tensor always coincide.*

In other words, for any $T = Sym^n(\mathbb{C}^d)$*, if there exists a decomposition* $T = T_1 + \cdots + T_r$ *in terms of tensors of rank* 1*, then there exists a decomposition with the same number of summands, in which each* T_i *is* symmetric*, of rank* 1*.*

A lot of partial results are known about Comon's Conjecture, in some particular case. For instance, it is easy to prove that the Conjecture is true when T is a symmetric matrix (and this is left as an exercise at the end of the chapter). Experimental evidence proves that the Conjecture is true, as far as modern computers can produce an answer.

However, nobody has been able, since now, to prove the complete result (or produce a counterexample).

The reader could wonder that such a question, which seems rather elementary in its formulation, could yield a problem which is still open, after being studied by many mathematicians, with modern techniques.

We remark that Comon's Conjecture is among the truly open questions in modern Mathematics, in the sense that the general consensus is towards its validity, mainly because in any test made since now the two ranks proved to be equal. Yet, no clear heuristic argument in favor of the Conjecture is presently available.

This explains a reason why, at the beginning of the chapter, we warned the reader that problems that are simple for Linear Algebra and matrices, can suddenly become prohibitive, as the dimension of the tensors grows.

7.3 Symmetric tensors and polynomials

Homogeneous polynomials and symmetric tensors are two apparently rather different mathematical objects, that indeed have a strict interaction, so that one can skip from each other, translating properties of tensors to properties of polynomials, and viceversa.

The main construction behind this interaction is probably well known to the reader, for the case of polynomials of degree 2. It is a standard fact that one can associate a symmetric matrix to quadratic homogeneous polynomial, in a one-to-one correspondence, so that properties of the quadratic form (as well as properties of quadratic hypersurfaces) can be read as properties of the associated matrix.

The aim of this section is to point out that a similar correspondence holds, more generally, between homogeneous forms of any degree and symmetric tensors of higher dimension.

Definition 7.3.1. There is a natural map between a space $K^{n,\dots,n}$ of cubic tensors of dimension d and the space of homogeneous polynomials of degree d in n variables (i.e. the d-th graded piece R_d of the ring of polynomials

$R = K[x_1, \ldots, x_n]$), defined by sending a tensor T to the polynomial F_T such that

$$F = \sum_{i_1,\ldots,i_n} T_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n}.$$

It is clear that the previous correspondence is not one-to-one, as soon as general tensors are considered. Namely, for the case $n, d = 2$, one immediately sees that the two matrices

$$\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

define the same polynomial of degree 2 in two variables $F = 2x_1^2 + 2x_1x_2 + x_2^2$.

The correspondence becomes one-to-one (and onto) when restricted to symmetric tensors. To see this, we need to introduce a piece of notation.

Definition 7.3.2. For any multiindex $(i_1, \ldots, i_d)$ we will define the multiplicity $m(i_1, \ldots, i_d)$ as the number of different permutations of the multiindex.

Definition 7.3.3. Let $R = K[x_1, \ldots, x_n]$ be the ring of polynomials, with coefficients in K, with n variables. Then there are linear isomorphisms

$$p : Sym^d(K^n) \to R_d \qquad t : R_d \to Sym^d(K^n)$$

defined as follows. The map p is the restriction to $Sym^d(K^n)$ of the previous map

$$p(T) = \sum_{i_1,\ldots,i_n} T_{i_1,\ldots,i_n} x_{i_1} \cdots x_{i_n}.$$

The map t is defined by sending the polynomial F to the tensor $t(F)$ such that

$$t(F)_{i_1,\ldots,i_n} = \frac{1}{m(i_1, \ldots, i_d)} (\text{the coefficient of } x_{i_1} \cdots x_{i_n} \text{ in } F).$$

Example 7.3.4. If G is a quadratic homogeneous polynomial in 3 variables $G = Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2$, then $t(G)$ is the symmetric matrix

$$t(G) = \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix}$$

which the usual matrix of the bilinear form associated to G.

Example 7.3.5. Consider the homogeneous cubic polynomial in two variables

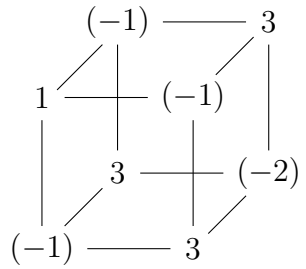
$$F(x_1, x_2) = x_1^3 - 3x_1^2x_2 + 9x_1x_2^2 - 2x_2^3.$$

Since one easily computes that

$$m(1,1,1) = 1, \quad m(1,1,2) = m(2,1,1) = 3, \quad m(2,2,2) = 1,$$

then the symmetric tensor $t(F)$ is:

$T =$



It is an easy exercise to prove that the two maps p and t defined above are inverse each other.

Once the correspondence is settled, one can easily speak about the rank or the the symmetric rank of a polynomial.

Definition 7.3.6. For any homogeneous polynomial $G \in K[x_1, \ldots, x_n]$ we define the *rank* (respectively the *symmetric rank*) of G as the *rank* (respectively the *symmetric rank*) of the associated tensor $t(G)$.

Example 7.3.7. The polynomial $G = x_1^3 + 21x_1x_2^2 + 8x_2^3$ has rank 3, since the associated tensor $t(G)$ is exactly the $2 \times 2 \times 2$ symmetric tensor of Example 7.5.1.

Proposition 7.3.8. *The linear space $Sym^d(K^n)$ has dimension*

$$\dim(Sym^d(K^n)) = \binom{n+d-1}{d} = \binom{n+d-1}{n-1}.$$

Proof. This is obvious once one knows that $\binom{n+d-1}{d}$ is the dimension of the space of homogeneous polynomials R_d of degree d in n variables. We prove it for the sake of completeness.

Since monomials of degree d in n variables are a basis for R_d, it is enough to count the number of such monomials.

The proof goes by induction on n. For $n = 2$ the statement is easy: we have $d + 1$ monomials, namely $x_1^d, x_1^{d-1}x_2, \ldots, x_2^d$.

Assume the formula holds for $n - 1$ variables $x_2, \ldots, x_n$. Every monomial of degree d in n variables is obtained by multiplying x_1^a by any monomial of degree $d - a$ in $x_2, \ldots, x_2$. Thus we have 1 monomial with x_1^d, n monomials with x_1^{d-1},..., $\binom{n+d-a-2}{d-a}$ monomials with x_1^a, and so on. Summing up

$$\dim(Sym^d(K^n)) = \sum_{a=0}^{d} \binom{n+d-a-2}{d-a},$$

and the sum is $\binom{n+d-1}{d}$, by standard facts on binomials. □

7.4 The complexity of polynomials

In this section, we rephrase the results on the rank of symmetric tensors in terms of the associated polynomials.

It will turn out that the rank decomposition of a polynomial is the analogue of a long standing series of problems in Number Theory, for the espression of integers as a sum of powers.

In principle, from the point of view of Algebraic Statistic, the complexity of a polynomial is the complexity of the associated symmetric tensor. So, the most elementary case of polynomials correspond to symmetric tensor of rank 1. We start with a description of polynomials of this type.

Remark 7.4.1. Before we proceed, we need to came back to the *multiplicity* of a multiindex $J = (i_1, \ldots, i_d)$, introduced in Definition 7.3.2.

In the correspondence between polynomials and tensors, the element $T_{i_1,\dots,i_d}$ is a linked with the coefficient of the monomial $x_{i_1}\cdots x_{i_d}$. Notice that $i_1,\dots,i_d$ need not be distinct, so the monomial $x_{i_1}\cdots x_{i_d}$ could be written unproperly. The usual way in which $x_{i_1}\cdots x_{i_d}$ is written is:

$$x_{i_1}\cdots x_{i_d} = x_1^{m_J(1)}x_2^{m_J(2)}\cdots x_n^{m_J(n)},$$

where $m_J(i)$ indicates the times in which i occurs in the multiindex J.

With the notation just introduce, one can describe the multiplicity $m(i_1,\dots,i_n)$. Indeed a permutation changes the multiindex, unless it simply switches indices i_a, i_b which are equal. Since the number of permutations over a set with m elements is $m!$, then one finds that:

$$m(J) = m(i_1,\dots,i_d) = \frac{d!}{m_J(1)!\cdots m_J(n)!}.$$

Proposition 7.4.2. *Let G be a homogeneous polynomial of degree d in n variables, so that $t(G) \in Sym^d(K^n)$.*

Then $t(G)$ has rank 1 if and only if there exists a homogeneous linear *polynomial $L \in K[x_1,\dots,x_n]$, such that $G = L^d$.*

Proof. It is sufficient to prove that $t(G) = v^{\otimes d}$, where $v = (a_1,\dots,a_n) \in K^n$, if and only if $G = (a_1x_1 + \cdots + a_nx_n)^d$.

To this aim, just notice that the coefficient of the monomial $x_1^{m_1}\cdots x_n^{m_n}$ in $p(v^{\otimes d})$ is the sum of the entries of the tensors $v^{\otimes d}$ whose multiindex J satisfies $m_J(1) = m_1,\dots,m_J(n) = m_n$. These entries are all equal to $a_1^{m_1}\cdots a_n^{m_n}$ and their number is $d!/m(J)$, which is equal to $m_1!\cdots m_n!$, by the previous Remark. On the other hand, by the well known Newton formula, $(m_1!\cdots m_n!)(a_1^{m_1}\cdots a_n^{m_n})$ is exactly the coefficient of the monomial $x_1^{m_1}\cdots x_n^{m_n}$ in the power $(a_1x_1 + \cdots + a_nx_n)^d$. □

Corollary 7.4.3. *The symmetric rank of a homogeneous polynomial $G \in K[x_1,\dots,x_n]_d$ is the minimum r for which there are r linear homogeneous forms $L_1,\dots,L_r \in K[x_1,\dots,x_n]$, with*

$$G = L_1^d + \cdots + L_r^d. \tag{7.4.1}$$

The symmetric rank is the number that computes the complexity of symmetric tensors, hence the complexity of homogeneous polynomials, from the point of view of Algebraic Statistics. Hence it turns out that the simplest polynomials, in this sense, are powers of linear forms. We guess that nobody will object to the statement that powers are rather simple!

We should mention, however, that sometimes the behavior of polynomials with respect to the complexity can be much less intuitive.

For instance, the rank of monomials is usually very high, so that the complexity of general monomials is over the average (and we expect that most people will be surprised). Even worse, efficient formulas for the rank of monomials were obtained only very recently by Enrico Carlini, Maria Virginia Catalisano and the third author. For other famous polynomials, as the determinant of a matrix of indeterminates, we do not even know the rank. All we have are lower and upper bounds, not matching.

We finish the chapter by mentioning that the problem of finding the rank of polynomials reflects a well known problem in Number Theory. Solving a question posed by Diophantus, the Italian mathematician Giuseppe Lagrange proved that any positive integer N can be written as a sum of 4 squares, i.e. for any positive integer G there are integers L_1, L_2, L_3, L_4 such that $G = L_1^2 + L_2^2 + L_3^2 + L_4^2$. The problem has been generalized by the English mathematician Edward Waring, who asked in 1770 for the minimum integer $r(k)$ such that any positive integer G can be written as a sum of $r(k)$ powers L_i^k. In other words, find the minimum $r(k)$ such that any positive integers is of the form

$$G = L_1^k + \cdots L_{r(k)}^k.$$

The analogy with the decomposition (7.4.1) that computes the symmetric rank of a polynomial is evident.

The determinantion of $r(k)$ is called, from then, the *Waring problem* for integers. Because of the analogy, the symmetric rank of a polynomial is also called the *Waring rank*.

For integers, few values of $r(k)$ are known, e.g. $r(2) = 4$, $r(3) = 9$, $r(4) = 19$. There are also variations on the Waring problem, as asking for the minimum $r'(k)$ such that all positive integers, except for a finite subset, are the sum of $r'(k)$ k-th powers (the *little* Waring problem).

Going back to the polynomial case, as for integers, a complete description of the maximal complexity that a homogeneous polynomial of given degree in a given number of variables can have, is not known. We have only upper bounds for the maximal rank. On the other hand, we know the solution of an analogue to the little Waring problem, for polynomials over the complex field.

Theorem 7.4.4. *(***Alexander-Hirschowitz, 1995***) Over the complex field, the symmetric rank of a* general *homogeneous polynomial of degree d in n variables (here* general *means: all polynomials outside a set of measure* 0 *in $\mathbb{C}[x_1, \dots, x_n]_d$; or also: all polynomials outside a Zariski closed subset of the space $\mathbb{C}[x_1, \dots, x_n]_d$, see* **??***) is:*

$$r = \lceil \frac{\binom{n+d-1}{d}}{n} \rceil$$

except for the following cases:

- *$d = 2$, any n, where $r = n$;*
- *$d = 3$, $n = 5$, where $r = 8$;*
- *$d = 4$, $n = 3$, where $r = 6$.*

For specific tensors, an efficient way to compute the rank requires the use of *inverse systems*, which will be explained in a next chapter.

7.5 Exercises

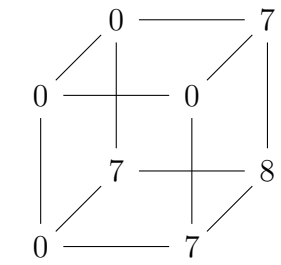
Exercise 6. Prove that the two maps p and t introduced in Definition 7.3.3 are linear and inverse each other.

Exercise 7. Prove the Comon's Conjecture for matrices: a symmetric matrix M has rank r if and only if there are r *symmetric* matrices of rank 1, $M_1,\ldots,$ M_r, such that $M = M_1 + \cdots + M_r$.

Exercise 8. Prove that the tensor T of Example 7.5.1 cannot have rank 2.

Exercise 9. Prove that the tensor T of Example 7.5.1 has symmetric rank $\operatorname{srank}(T) = 3$.

Example 7.5.1. The tensor (over $\mathbb{C}$):

$T =$ 

is not decomposable. Let us prove that the symmetric rank is bigger than 2.

Assume that $T = (a,b)^{\otimes 3} + (c,d)^{\otimes 3}$. Then we have

$$\begin{cases} a^3 + c^3 & = 0 \\ a^2b + c^2d & = 0 \\ ab^2 + cd^2 & = 7 \\ b^3 + d^3 & = 8 \end{cases}$$

Notices that if $a = 0$, then $c = 0$ and we get a contradiction with the third equality. A similar argument shows that $c \neq 0$. If $b = 0$, then $d \neq 0$ thus $c = 0$ from the second equality, a contradiction. A similar argument excludes $d = 0$, so that none of a, b, c, d can be 0.

We have $a = -\epsilon c$ where ϵ is a cubic root of unit. Since $c \neq 0$, the second equality yields $d = -\epsilon^2 b$ so that $d^3 = -b^3$. Thus the last equality yields a contradiction.

Chapter 8

Flattening and marginalisation

Chapter 9

Antisymmetric tensors and Grassmannians

Part III

Commutative Algebra and Algebraic Geometry

Chapter 10

Elements of Algebraic Geometry

The aim of this section, together with the following ones, is a quick introduction to the main tools of Algebraic Geometry (of projective saces) that we need to understand some aspects of algebraic Models in Statistics.

The material we collect here is far from being self-contained. For many technical results, as the Nullstellensatz, we refer to specific texts on the subject.

We assume in the sequel that the reader knows the basic definitions of algebraic structures, as rings, ideals, homomorphisms, etc. as well as the main properties of polynomial rings.

This part of the book could also be utilized for a short course or a cutway through the results of Algebraic and Projective Geometry which are relevant in the study of Statistics.

10.1 Projective varieties

The first step is a definition of the ambient space. We will do that in more generality than usual, because sometimes the more general ambient is simpler than specific ones.

So, since we are going to deal with subsets defined by polynomial equations, the starting point is the polynomial rings over the *complex* field, the field where solutions of polynomial equations live properly. Several claims that we are going to illustrate would work also on any algebraically closed field. We deal only with the complex field in order to avoid details on the structure of fields of arbitrary characteristic.

Next, we are going to consider the coordinates of a point *up to scaling*. This leads us directly into the theory of complex projective spaces.

Definition 10.1.1. Let V be a linear space over $\mathbb{C}$. Define on $V \setminus \{0\}$ an equivalence relation $\sim$ which associates v, v' if and only if there exists $\alpha \in \mathbb{C}$ with $v' = \alpha v$. The quotient $\mathbb{P}(V) = V \setminus \{0\}/\sim$ is the *projective space associated to* V.

The *projective dimension* of $\mathbb{P}(V)$ is the number $\dim(V) - 1$ (notice that it decreases by 1).

When $V = \mathbb{C}^{n+1}$, we will denote the projective space $\mathbb{P}(V)$ also with $\mathbb{P}^n$

Points of the projective space are thus equivalent classes of vectors, in the relation $\sim$, hence are formed by a vector $v \neq 0$ together with all its multiples.

Thus, $P \in \mathbb{P}^n$ is an equivalence class of $(n+1)$-tuples of complex numbers. The *homogeneous coordinates* are any representative of the equivalence class.

Notice that the coordinates, in a projective space, are no longer uniquely defined, but only defined modulo a (non-zero) scalar multiplication.

By a small abuse, we will also write $P = [a_0 : \cdots : a_n]$ when $(a_0, \ldots, a_n)$ is a representative for the homogeneous coordinates of P.

Remark 10.1.2. $\mathbb{P}^n$ contains several subsets in natural one-to-one correspondence with $\mathbb{C}^n$.

Indeed, take the subset U_i of points with homogeneous coordinates $[a_0 : \cdots : a_n]$ whose i-th coordinate a_i is non-zero. The condition is clearly independent from the representative of the class that we choose. There is a

one-to-one correspondence $U_i \leftrightarrow \mathbb{C}^n$, obtained as follows:

$$[a_0 : \cdots : a_n] \mapsto (\frac{a_0}{a_i}, \frac{a_1}{a_i}, \ldots, \frac{\hat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i})$$

U_i is called the *i-th affine subspace*.

Notice that when $P = [a_0 : \cdots : a_n] \in U_i$, hence $a_i \neq 0$, then there exists a unique representative of P with $a_i = 1$. The previous process identifies $P \in U_i$ with the point of $\mathbb{C}^n$ whose coordinates correspond to such representative of P, excludind the i-th coordinate.

Definition 10.1.3. A subset C of a linear space V is a *cone* if for any $v \in C$ and $a \in \mathbb{C}$ then $av \in C$.

Remark 10.1.4. Cones of a linear space V are the fundamental subsets that define subsets in the associated projective space.

Indeed there is an obvious map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ that sends $(n+1)$-tuples to their equivalence classes. If $W \subset \mathbb{P}^n$ is any subset, then $\pi^{-1}(W)(\cup\{0\})$ is a cone in V.

Conversely, every cone in $\mathbb{C}^{n+1}$ is the inverse image in π of a subset of $\mathbb{P}^n$ (one must add $\cup\{0\}$).

In general, one cannot expect that a polynomial $p \in \mathbb{C}[x_0, \ldots, x_n]$ defines a cone in $\mathbb{C}^{n+1}$. This turns out to be true when p is *homogeneous* Indeed, if p is homogeneous of degree d and $a \in \mathbb{C}$ is any scalar, then

$$p(ax_1, \ldots, ax_n) = a^d p(x_1, \ldots, x_n),$$

thus for $a \neq 0$ the vanishing of $p(ax_1, \ldots, ax_n)$ is equivalent to the vanishing of $p(x_1, \ldots, x_n)$.

The observation can be reversed, as follows.

Lemma 10.1.5. *Let $p = p(t)$ be a polynomial in $\mathbb{C}[t_0, \ldots, t_n]$, of degree bigger than 0.*

Then there exists a point $x = (x_0, \ldots, x_n) \in \mathbb{C}^n$ with $p(x) = 0$ and infinitely many points $y = (y_0, \ldots, y_n) \in \mathbb{C}^n$ with $p(y) \neq 0$.

Proof. Make induction on the number of variables.

When p has only one variable, then the first claim is exactly the definition of *algebraically closed field*. The second claim holds because every non-zero polynomial of degree d has at most d roots.

If we know both claims for n variables, then write $p \in \mathbb{C}[t_0, \dots, t_n]$ as a polynomial in t_0, with coefficients in $\mathbb{C}[t_1, \dots, t_n]$:

$$p = p_d x_0^d + p_{d-1} x_0^{d-1} + \cdots + p_0$$

where each p_i is a polynomial in $x_0, \dots, x_n$. We may assume that $p_d \neq 0$, otherwise p has only $n-1$ variables, and the claim holds by induction. By induction, there are infinitely many points $(y_1, \dots, y_n) \in \mathbb{C}^n$ which are not a solution of p_d (notice that such points exist trivially if $p_d \neq 0$ is constant). Then for any $(y_1, \dots, y_n)$ the polynomial $p' = p(t_0, y_1, \dots, y_n)$ has just one variable and degree $d > 0$, hence there are infinitely many $y_0 \in \mathbb{C}$ with $p'(y_0) = p(y_0, \dots, y_n) = 0$. □

For the next proposition, recall that any polynomial $p(t) \in \mathbb{C}[t_0, \dots, t_n]$ can be written uniquely as a sum of homogeneous polynomials

$$p(t) = p_d + p_{d-1} + \cdots + p_0,$$

with p_i homogeneous of degree i for all i. The previous sum is called the *homogeneous decomposition of* $p(t)$.

Proposition 10.1.6. *Let $p = p(t)$ be a polynomial in $\mathbb{C}[t_0, \dots, t_n]$ of degree $d > 0$. Assume that $p(t)$ is not homogeneous.*

Then there exists $x = (x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ and a scalar $\alpha \in \mathbb{C} \setminus \{0\}$ such that $p(x) = 0$ and $p(\alpha x) \neq 0$.

Proof. Take the homogeneous decomposition of $p(t)$

$$p(t) = p_d + p_{d-1} + \cdots + p_0.$$

Since $p(t)$ is not homogeneous, we may assume $p_d, p_i \neq 0$ for some $i < d$. Take the minimal i with $p_i \neq 0$. Choose $y = (y_0, \ldots, y_n) \in \mathbb{C}^n$ with $p_d(y) \neq 0$ (it exists by the previous Lemma). Then $p(ay) = a^d p_d(y) + a^{d-1} p_{d-1}(y) + \cdots + a^i p_i(y)$ is a polynomial of degree $d > 0$ in the unique variable a, which can be divided by a^i, i.e. $p(ay) = a^i q(ay)$ with $q(ay)$ polynomial of degree $d - i > 0$ in a, whose constant term is non-zero. By the previous Lemma, there exist $a_1, a_2 \in \mathbb{C}$ with $q(a_1 y) = 0$ and $q(a_2 y) \neq 0$. Notice that $a_1 \neq 0$, since the constant term of $q(ay)$ does not vanish. The claim holds by taking $x = a_1 y$ and $\alpha = a_2/a_1$. □

The previous Proposition shows that the vanishing of a polynomial in $\mathbb{C}[t_0, \ldots, t_n]$ is not defined over a cone, hence over a subset of $\mathbb{P}^n$, unless the polynomial is homogeneous. Conversely, if $p \in \mathbb{C}[t_0, \ldots, t_n]$ is homogeneous, then the vanishing of p at a set of projective coordinates $[a_0 : \cdots : a_n]$ of $P \in \mathbb{P}^n$ implies the vanishing of p at *any* set of homogeneous coordinates of p, because the vanishing set of p is a cone.

Consequently, we give the following, basic:

Definition 10.1.7. We call *projective variety* of $\mathbb{P}^n$ every subset of $\mathbb{P}^n_K$ defined by the vanishing of a family $J = \{p_j\}$ of homogeneous polynomials $p_j \in \mathbb{C}[t_0, \ldots, t_n]$.

In other words, projective varieties are subsets of $\mathbb{P}^n$ whose equivalence classes are the solutions of a system of *homogeneous* polynomial equations.

When V is any linear space of dimension d, we define the projective varieties in $\mathbb{P}(V)$ by taking an identification $V \sim \mathbb{C}^d$ (hence taking a basis of V).

We will denote with $X(J)$ the projective variety defined by the family J of homogeneous polynomials.

Example 10.1.8. let $\{p_1, \ldots, p_m\}$ be a family of *linear* homogeneous polynomials in $\mathbb{C}[t_0, \ldots, t_n]$. The projective variety X defined by $\{p_1, \ldots, p_m\}$ is called a *linear* projective variety.

The polynomials $p_1, \dots, p_m$ define also a linear subspace $W \subset \mathbb{C}^{n+1}$. One can easily prove that there is a canonical identification of X with the projective space $\mathbb{P}(W)$.

Remark 10.1.9. Let X be a projective variety defined by a set J of homogeneous polynomials and take a subset $J' \subset J$. Then the projective variety $X' = X(J')$ defined by J' *contains* X.
One can easily find examples (even of linear varieties) with $X' = X$ even if J' is properly contained in J.

Remark 10.1.10. Projective varieties provide a system of *closed* sets for a topology, called the *Zariski topology* on $\mathbb{P}^n$.

Namely $\emptyset$ and $\mathbb{P}^n$ are both projective varieties, defined respectively by the families of polynomials $\{1\}$ and $\{0\}$. Se $\{W_i\}$ is a family of projective varieties, with $W_i = X(J_i)$, then $\bigcap\{W_i\}$ is the projective variety defined by the family $J = \bigcup\{J_i\}$ of homogeneous polynomials. Finally, if $W_1 = X(J_1)$ and $W_2 = X(J_2)$ are projective varieties, then $W_1 \cup W_2$ is the projective variety defined by the family of homogeneous polynomials:

$$J_1 J_2 = \{pq : p \in J_1, q \in J_2\}.$$

Example 10.1.11. Every singleton in $\mathbb{P}^n$ is a projective variety
Namely, if $[a_0 : \cdots : a_n]$ are homogeneous coordinates of a point P, with $a_i \neq 0$, then the set of linear homogeneous polynomials

$$I = \{a_0 x_i - a_i x_0, \dots, a_n x_i - a_i x_n\}$$

defines $\{P\} \subset \mathbb{P}^n$
In particular, the Zariski topology satisfies the first separation axiom T_1.

Example 10.1.12. Every Zariski closed subset of $\mathbb{P}^1$ is finite, except $\mathbb{P}^1$ itself.

Namely let p is any non zero homogeneous polynomial of degree d in $\mathbb{C}[x_0, x_1]$. Then setting $x_1 = 1$, we get a polynomial $\bar{p} \in \mathbb{C}[x_0]$ which, by the Fundamental Theorem of Algebra, can be uniquely decomposed in a products $\bar{p} = e(x_0 - a_0)^{m_0} \cdots (x_0 - a_k)^{m_k}$, where $a_1, \ldots, a_k$ are the roots of $\bar{p}$ and $e \in \mathbb{C}$. Going back to p, we see that there exists a power x_1^β (maybe $\beta = 0$) such that

$$p = ex_1^\beta (x_0 - a_0 x_1)^{m_0} \cdots (x_0 - a_k x_1)^{m_k}.$$

It follows immediately that p vanishes only at the points $[a_0 : 1], \ldots, [a_k : 1]$, with the addition of $[1 : 0]$ if $\beta > 0$.
Thus, the open sets in the Zariski topology on $\mathbb{P}^1$ are $\emptyset$ and the *cofinite* sets, i.e. sets whose complement is finite. In other words, the Zariski topology on ¶^{1} coincides with the *cofinite* topology.

In higher projective spaces there are non-trivial closed subsets which are infinite. Thus the Zariski topology on $\mathbb{P}^n$, $n > 1$, is not the cofinite topology.

Corollary 10.1.13. *Let $p \neq 0$ be a homogeneous polynomial in $\mathbb{C}[t_0, \ldots, t_n]$, of degree bigger than 0. Assume $n > 1$.*
Then the variety $V(p)$, which is not $\mathbb{P}^n$ by Lemma **??***, has infinitely many points.*

Proof. If all points $Q = [q_0 : q_1 : \cdots : q_n]$ with $q_0 \neq 0$ belong to $V(p)$, then we are done. So we may assume that we may take $Q = [1 : q_1 : \cdots : q_n] \notin V(p)$ For any choice of $m = (m_2, \ldots, m_n) \in \mathbb{C}n - 2$ consider the line L_m, passing through Q, defined by the vanishing of the linear polynomials

$$t_2 - m_2(t_1 - q_1 t_0) - q_2 t_0, \ldots, t_n - m_n(t_1 - q_1 t_0) - q_n t_0.$$

Define the polynomial

$$p_m = p(t_0, t_1, m_2(t_1 - q_1 t_0) + q_2 t_0, \ldots, m_n(t_1 - q_1 t_0) + q_n t_0).$$

If (a_0, a_1) is a solution of the equation $p_m = 0$, then the intersection $V(p) \cap L_m$ contains the point

$$[a_0 : a_1 : m_2(a_1 - q_1 a_0) + q_2 a_0, \ldots, m_n(a_1 - q_1 a_0) + q_n a_0].$$

Since the polynomial p_m is homogeneous of the same degree than p, then it vanishes at some point, so that $V(p) \cap L_m \neq \emptyset$. Since two different lines $L_m, L_{m'}$ meet only at $Q \notin V(p)$, the claim follows. □

Next, we define the ideals associated with a projective variety.

Definition 10.1.14. Let I be an ideal of a polynomial ring $R = \mathbb{C}[t_0, \ldots, t_n]$. We say that I *is generated by* $J \subset R$, and write $I =< J >$, when

$$I = \{f_1 p_1 + \cdots + f_m p_m : f_1, \ldots, f_m \in R, \quad p_1, \ldots, p_m \in J\}.$$

We say that I is a *homogeeous* ideal if there is a set of homogeneus elements $J \subset R$ such that $I =< J >$.

Notice that not every element of a homogeneous ideal is homogeneous. for instance, in $\mathbb{C}[x]$ the homogeneous ideal $I =< x >$ contains the non homogeneous element $x + x^2$.

Proposition 10.1.15. *The ideal I is homogeneous if and only if for any polynomial $p \in I$, whose homogeneous components are $p_d, \ldots, p_0$, then every p_i belongs to I.*

Proof. Assume that I is generated by a set of homogeneous elements J and take $p \in I$. Consider the decomposition in homogeneous components $p = p_d + \cdots + p_0$. There are homogeneous polynomials $q_1, \ldots, q_m \in J$ such that $p = f_1 q_1 + \cdots + f_m q_m$, for some polynomials $f_j \in R$. Denote by d_j the degree of q_j and denote by f_{ij} the homogeneous component of degree i in f_j (with $f_{ij} = 0$ whenever $i < 0$). Then, comparing the homogeneous components, one has for every degree i

$$p_i = f_{1\ i-d_1} q_1 + \cdots + f_{m\ i-d_m} q_m$$

and thus $p_i \in < J >= I$ for every i.

Conversely, I is always contained in the ideal generated by the homogeneous components of its elements. Thus, when these components are also in I for all $p \in I$, then I is generated by homogeneous polynomials. □

Remark 10.1.16. If W is a projective variety defined by the vanishing of a set J of homogeneous polynomials, then W is also defined by the vanishing of all the polynomials in the ideal $I =< J >$.

Indeed if P is a point of W, then for all $p \in I$, write $p = f_1 p_1 + \cdots + f_m p_m$ for some p_i's in J. We have

$$p(P) = f_1(P)p_1(P) + \cdots + f_m(P)p_m(P) = 0.$$

It follows that every projective variety can be defined as the vanishing locus of a homogemneous ideal.

For any ideal $I \subset R$, define the *radical* of I as the set

$$\sqrt{I} = \{p : p^m \in I \text{ for some exponent } m\}.$$

$\sqrt{I}$ is an ideal of R and contains I.

When I is a homogeneous ideal, then also $\sqrt{I}$ is homogeneous.

For any ideal I, the projective varieties $X(I)$ and $X(\sqrt{I})$ are equal. Indeed p^m vanishes at P if and only if p does.

Before stating the basic result in the correspondence between projective varieties and homogeneous ideals (i.e. the homogeneous version of the celebrated Hilbert's Nullstellensatz), we need some more piece of notation.

Definition 10.1.17. We say that an ideal I in R is *radical* if $I = \sqrt{I}$. For any ideal I, $\sqrt{I}$ is a radical ideal, since $\sqrt{\sqrt{I}} = \sqrt{I}$.

We call *irrelevant ideal* the ideal of $R = \mathbb{C}[t_0, \ldots, t_n]$ generated by the indeterminates $t_0, \ldots, t_n$.

The irrelevant ideal is a radical ideal that defines the empty set in $\mathbb{P}^n$. Indeed, no points of $\mathbb{P}^n$ can annihilate all the variables, as no points in $\mathbb{P}^n$ have all the homogeneous coordinates equal to 0.

Example 10.1.18. In $\mathbb{C}[x, y]$ consider the homogeneous element x^2. One computes soon that the radical of the ideal $I =< x^2 >$ is the ideal generated by x.

The three sets $\{x^2\}$, $< x^2 >$, $\sqrt{< x^2 >} =< x >$ all define the same projective subvariety of $\mathbb{P}^1$: the point of homogeneous coordinates $[0:1]$.

Theorem 10.1.19 (**homogeneous Nullstellensatz**). *Two homogeneous ideals I_1, I_2 in the polynomial ring $R = \mathbb{C}[t_0, \ldots, t_n]$ define the same projective variety X if and only if*

$$\sqrt{I_1} = \sqrt{I_2},$$

with the unique exception $I_1 = R$, $I_2 =$ the irrelevant ideal.

Thus, if two radical homogeneous ideals I_1, I_2 define the same projective variety X, and none of them is the whole ring R, then $I_1 = I_2$.

A proof of the homogeneous Nullstellesatz can be found in ZS.

We should notice that the Theorem works because $\mathbb{C}$ is algebraically closed. The statement is not true over a non algebraically closed field, as the real field $\mathbb{R}$. This is a good reason to start the study of Projective Geometry from varieties defined on an algebraically closed field, as $\mathbb{C}$.

A consequence of the homogeneous Nullstellensatz is that two sets J_1, J_2 of homogeneous elements define the same projective variety X if and only if $\sqrt{< J_1 >} = \sqrt{< J_2 >}$.

We list below other consequences.

Corollary 10.1.20. *Let $J \in \mathbb{C}[t_0 \ldots, t_n]$ be a set of homogeneous polynomials which define a projective variety $X \neq \emptyset$. Then the set*

$$J(X) = \{p \in \mathbb{C}[t_0 \ldots, t_n] : p \text{ is homogeneous and } p(P) = 0 \text{ for all } P \in X\}$$

coincides with the radical of the ideal generated by J.

We will call $J(X)$ the homogeneous ideal *associated with X.*

Corollary 10.1.21. *Let $I \in R = \mathbb{C}[t_0 \ldots, t_n]$ be a homogeneous ideal. I defines the empty set in $\mathbb{P}^n$ if and only if, for some m, all the powers t_i^m belong to I.*

Proof. Indeed we get from the homogeneous Nullstellensatz that $\sqrt{I}$ is either R or the irrelevant ideal. In the former case, the claim is obvious. In the latter, for every i there exists m_i such that $x_i^{m_i} \in I$, and one can take m as the maximum of the m_i's. □

Another fundamental fact in the study of projective varieties is encoded in the following algebraic result (for the proof, see [ZS]):

Theorem 10.1.22 (**Basis Theorem).** *Let J be a set of polynomials and let I be the ideal generated by J. Then there exists a* finite *subset $J' \subset J$ that generate the ideal I.*

As a first consequence, the Basis Theorem tells us that any projective variety can be defined by the vanishing of a *finite* set of homogeneous polynomials.

Let us list other consequences.

Definition 10.1.23. Call *hypersuperface* any projective variety defined by the vanishing of a single homogeneous polynomial. By abuse we will write $X(p)$ to indicate $X(\{p\})$.

When p has degree 1, then $X(p)$ is called a *hyperplane.*

Corollary 10.1.24. *Every projective variety is the intersection of a finite number of hypersurfaces. Equivalently, every open set in the Zariski topology is a finite union of complements of hypersurfaces.*

Proof. Se $X = X(J)$ be a projective variety, defined by the set J of homogeneous polynomials. Find a finite subset $J' \subset J$ such that $< J >=< J' >$. Then:

$$X = X(J) = X(< J >) = X(< J' >) = X(J').$$

If $J = \{p_1, \ldots, p_m\}$ then, by remark 10.1.10:

$$X = X(p_1) \cap \cdots \cap X(p_m).$$

□

Example 10.1.25. If $L \subset \mathbb{P}^n$ is a linear variety which corresponds to a linear subspace of dimension $m+1$ in $\mathbb{C}^{n+1}$, then L can be defined by $n-m$ linear homogeneous polynomials, i.e. L is the intersection of $n-m$ hyperplanes.

Remark 10.1.26. One could think that every projective variety in $\mathbb{P}^n$ can be defined with a finite set of homogeneous polynomials of cardinality bounded by a function of n.

F.S. Macaualay proved that this guess is false.

Indeed, in [Mac] he found that for every integer m there exists a subset (curve) in $\mathbb{P}^3$ which cannot be defined by a set of less than m homogeneous polynomials.

Definition 10.1.27. A topological space Y is *irreducible* when any pair of non-empty open subsets have a non-empty intersection.

Equivalently, Y is irreducible if it is not the union of two proper closed subsets.

Equivalently, Y is irreducible if the closure of every non-empty open subset A is Y itself, i.e. every non-empty open subset is dense in Y.

The following Proposition is easy and left to the reader as an exercise.

Proposition 10.1.28. *(i) Every singleton is irreducible.*

(ii) If Y is an irreducible subset, then the closure of Y is irreducible.

(iii) If an irreducible subset Y is contained in a finite union of closed subsets $X_1 \cup \cdots \cup X_m$, then Y is contained in some X_i.

(iv) If $Y_1 \subset \ldots Y_i \subset \ldots$ is an ascending chain of irreducible subsets, then $\bigcup Y_i$ is irreducible.

Corollary 10.1.29. *Any projective space $\mathbb{P}^n$ is irreducible and compact in the Zariski topology.*

Proof. Let A_1, A_2 be non-empty open subsets, in the Zariski topology, and assume that A_i is the complement of the projective variety $X_i = X(J_i)$, where

J_1, J_2 are two subsets of homogeneous polynomials in $\mathbb{C}[t_0, \ldots, t_n]$. We may assume, by the Basis Theorem, that bot J_1, J_2 are finite. Notice that none of $X(J_1), X(J_2)$ can coincide with $\mathbb{P}^n$, thus both J_1, J_2 contain a non-zero element. To prove that $\mathbb{P}^n$ is irreducible, we must show that $A_1 \cap A_2$ cannot be empty, i.e. that $X_1 \cup X_2$ cannot coincide with $\mathbb{P}^n$. By Remark 10.1.10, $X_1 \cup X_2$ is the projective variety defined by the set of products $J_1 J_2$. If we take $p_1 \neq 0$ in J_1 and $p_2 \neq 0$ in J_2, then $p = p_1 p_2$ is a non-zero element in $J_1 J_2$. By Lemma 10.1.5 there exist points $P \in \mathbb{P}^n$ such that $p(P) \neq 0$. Thus P does not belong to $X_1 \cup X_2$, and the irreducibility of $\mathbb{P}^n$ is settled.

For the compactness, we prove that $\mathbb{P}^n$ enjoys the Finite Intersection Property. Let $\{X_i\}$ be any family of closed subsets such that $\bigcap X_i = \emptyset$. Assume $X_i = X(J_i)$ and define $J = \bigcup J_i$. By Remark 10.1.10, $\bigcap X_i = X(J)$, thus also $\bigcap X_i = X(< J >)$. By the Basis Theorem, there exists a finite subset J' of J such that $< J' >=< J >$. Thus there exists a finite subfamily $J_{i_1}, \ldots, J_{i_k}$ such that $< J_{i_1} \cup \cdots \cup J_{i_k} >=< J >$. Thus

$$\emptyset = X(< J >) = X(< J_{i_1} \cup \cdots \cup J_{i_k} >) = X(J_{i_1} \cup \cdots \cup J_{i_k}) = X_{i_1} \cap \cdots \cap X_{i_k}$$

and the Finite Intersection Property holds. □

Closed subsets in a compact space are compact. Thus any projective variety $X \subset \mathbb{P}^n$ is compact in the topology induced by the Zariski topology of $\mathbb{P}^n$.

Notice that irreducible topological spaces are far from being Hausdorff spaces. Thus no non-trivial projective space satisfies the Hausdorff separation axiom T_2.

Another important consequence of the Basis Theorem is the following.

Theorem 10.1.30. *Any non-empty family $\mathscr{F}$ of closed subsets of $\mathbb{P}^n$ (i.e. of projective varieties), partially ordered by inclusion, has a minimal element.*

Proof. Let the claim fail. Then one can find an infinite chain of elements of $\mathscr{F}$,

$$X_0 \supset X_1 \supset \cdots \supset X_i \supset \ldots$$

where all the inclusions are strict. Consider for all i the ideal $I(X_i)$ generated by the homogeneous polynomials which vanish at X_i. Then one gets an *ascending* chian of ideals

$$I(X_0) \supset I(X_1) \supset \cdots \supset I(X_i) \supset \ldots$$

where again all the inclusions are strict. Let $I = \bigcup I(X_i)$. It is immediate to see that I is a homogeneous ideal. By the Basis Theorem, there exists a finite set of homogeneous generators $g_1, \ldots, g_k$ for I. Since every g_j belongs to $\bigcup I(X_i)$, for i_0 sufficiently large we have $g_j \in I(X_{i_0})$ for all j. Thus $I = I(X_{i_0})$, so that $I(X_{i_0}) = I(X_{i_0+1})$, a contradiction. □

Definition 10.1.31. For any projective varietiy X, a subset X' of X is an *irreducible component* of X if it is closed (in the Zariski topology, thus X' is a projective variety itself), irreducible and X' maximal with respect to these two properties.

It is clear that X is irreducible if and only if X itself is the unique irreducible component of X.

Theorem 10.1.32. *Let X be any projective variety. Then the irreducible components of X exist and their number is finite.*

Moreover there exists a unique decomposition of X as the union

$$X = X_1 \cup \cdots \cup X_k$$

where $X_1, \ldots, X_k$ are precisely the irreducible components of X.

Proof. First, let us prove that irreducible components exist. To do that, consider the family $\mathscr{F}_p$ of closed irreducible subsets containing a point P. $\mathscr{F}_P$ is not empty, since it contains $\{P\}$. If $X_1 \subset \ldots X_i \subset \ldots$ is an ascending chain of elements of $\mathscr{F}_p$, then the union $Y = \bigcup X_i$ is irreducible by 10.1.28 (iv), thus the closure of Y sits in $\mathscr{F}_p$ (by 10.1.28 (ii)) and it is an upper bound for the chain. Then the family $\mathscr{F}_p$ has maximal elements, by Zorn's Lemma. These elements are irreducible components of X.

Notice that we also proved that every point of X sits in some irreducible component, i.e. X is the union of its irreducible components. If Y is an irreducible component, by 10.1.28 (ii) also the closure of Y is irreducible. Thus, by maximality, Y must be closed.

Next, we prove that X is a finite union of irreducible closed subsets. For, assume this is false. Call $\mathscr{F}$ the family of closed subsets of X which are *not* a finite union of irreducible subsets. $\mathscr{F}$ is non-empty, since it contains X. By Theorem 10.1.30, $\mathscr{F}$ has some minimal element X'. As $X' \in \mathscr{F}$, then X' cannot be irreducible. Thus there are two closed subsets X_1, X_2, properly contained in X', whose union is X'. Since X' is minimal in $\mathscr{F}$, none of X_1, X_2 is in $\mathscr{F}$, thus both X_1, X_2 are union of a finite number of irreducible closed subsets. But then also X' would be a finite union of closed irreducible subsets. As $X' \in \mathscr{F}$, this is a contradiction.

Thus, there are irreducible closed subsets $X_1, \ldots, X_k$, whose union is X. Then, if Y is any irreducible component of X, we have $Y \subset X = X_1 \cup \cdots \cup X_k$. By 10.1.28 (iii), Y is contained in some X_i. By maximality, we get that Y coincides with some X_i. This proves that the number of irreducible components of X is finite.

We just proved that X decomposes in the union of its irreducible components $Y_1, \ldots, Y_m$. By 10.1.28 (iii), none of the Y_i can be contained in the union of the remaining components. Thus the decomposition is unique. □

If X is a finite projective variety, then the irreducible components of X are its singletons.

Example 10.1.33. Let X be the variety in $\mathbb{P}^2$ defined by the vanishing of the homogeneous polynomial $g = t_0 t_2 - t_1^2$. Then X is irreducible.

Proving the irreducibility of a projective variety, in general, is not an easy task. We do that, in this case, introducing a method that will be refined later.

Assume that X is the union of two proper closed subsets X_1, X_2, where X_i is defined by the vanishing of homogeneous polynomials in the set J_i.

We consider the map $f : \mathbb{P}^1 \to \mathbb{P}^2$ defined by sending each point $P = [y_0 : y_1]$ to the point $f(P) = [y_0^2 : y_0y_1 : y_1^2]$ of $\mathbb{P}^2$. It is immediate to control, indeed, that the point $f(P)$ does not depend on the choice of a particular pair of homogeneous coordinates for P. Here f is simply a set-theoretic map. We will see, later, that f has relevant geometric properties.

The image of f is contained in X, for any point with homogeneous coordinates $[x_0 : x_1 : x_2] = [y_0^2 : y_0y_1 : y_1^2]$ annihilates g. Moreover the image of f is exactly X. Indeed let $Q = [a_0 : a_1 : a_2]$ be a point of X. Fix elements $b, c \in \mathbb{C}$ such that $b^2 = a_0$ and $c^2 = a_2$. Then we cannot have both b, c equal to 0, for in this case $a_0 = a_2 = 0$ and also $a_1 = 0$, because $g(Q) = 0$, a contradiction. Moreover $(bc)^2 = a_0a_2 = a_1^2$. Thus, after possibly the change of the sign of one between b and c, we may also assume $bc = a_1$. Then:

$$f([b : c]) = [b^2 : bc : c^2] = [a_0 : a_1 : a_2] = Q.$$

The map f is one-to-one. To see this, assume $f([b : c]) = f([b' : c'])$. Then $(b'^2, b'c', c'^2)$ is equal to (b^2, bc, c^2) multiplied by some non-zero scalar $z \in \mathbb{C}$. Taking a suitable square root w of Z, we may assume $b' = wb$. We have $c' = \pm wc$, but if $c' \neq wc$ then $b'c' = -zbc \neq zbc$, a contradiction. Thus also $c' = wc$ and (b', c'), (b, c) define the same point in $\mathbb{P}^1$.

In conclusion, f is a bijective map $f : \mathbb{P}^1 \to X$.

Next, we prove that $Z_1 = f^{-1}(X_1)$ is closed in $\mathbb{P}^1$. Indeed for any polynomial $p = p(y_0, y_1, y_2) \in J_1$ consider the polynomial $p' = p(x_0^2, x_0x_1, x_1^2) \in \mathbb{C}[x_0, x_1]$. It is immediate to control that any $P \in \mathbb{P}^1$ satisfies $p'(P) = 0$ if and only if $f(P)$ satisfies $p(f(P)) = 0$. Thus Z_1 is the projective variety in $\mathbb{P}^1$ associated to the set of homogeneous polynomials

$$J' = \{p(x_0^2, x_0x_1, x_1^2) : p \in J_1\} \subset \mathbb{C}[x_0, x_1].$$

Similarly $Z_2 = f^{-1}(X_2)$ is closed in $\mathbb{P}^1$.

Since f is bijective, then Z_1, Z_2 are proper closed subset of $\mathbb{P}^1$, whose union is $\mathbb{P}^1$. This contradicts the irreducibility of $\mathbb{P}^1$.

We will see below (Example 10.1.38) that any linear variety is irreducible.

Example 10.1.34. Let X be the variety in $\mathbb{P}^2$ defined by the set of homogeneous polynomials $J = \{t_0t_1, t_0(t_0 - t_2)\}$. Then X is the union of the sets $L_1 = \{P = [x_0 : x_1 : x_2] : x_0 = 0\}$ and $L_2 = \{P = [x_0 : x_1 : x_2] : x_1 = 0, x_0 = x_2\}$. These are both linear varieties, hence they are irreducible (L_2 is indeed a singleton). Moreover $L_1 \cap L_2 = \emptyset$. It follows that X is not irreducible: L_1, L_2 are its irreducible components.

Definition 10.1.35. We say that a polynomial $p \in \mathbb{C}[x_0, \ldots, x_n]$ is *irreducible* when $p = q_1q_2$ implies that either q_1 or q_2 are constant.

Example 10.1.36. Every linear polynomial is irreducible.

Theorem 10.1.37 (**Unique factorization).** *Any polynomial f can be written as a product $f = f_1f_2 \cdots f_h$ where the f_i's are irreducible polynomials. The f_i's are called* irreducible factors *of f and are unique up to scalar, in the sense that if $f = g_1 \cdots g_s$, with each g_j irreducible, then $h = s$ and, after a possible permutation, there are scalars $c_1, \ldots, c_h \in \mathbb{C}$ with $g_i = c_if_i$ for all i.*

If f is homogeneous, also the irreducible factors of f are homogeneous.

Notice that the irreducible factors of f need not to be distinct. In any event, the irreducible factors of a product fg are the union of the irreducible factors of f and the irreducible factors of g.

Example 10.1.38. Let $X = X(f)$ be a hypersurface an take a decomposition $f = f_1 \cdots f_h$ of f into irreducible factors. Then the $X(f_i)$'s are presisely the irreducible components of X.

To prove this, first notice that when f is irreducible, then X is irreducible. Indeed assume that $X = X_1 \cup X_2$, where X_1, X_2 are closed an none of them contains X. Then take f_1 (respectively f_2) in the radical ideal of X_1 (respectively X_2) and such that X is not contained in $X(f_1)$ (respectively in $X(f_2)$).

We have $X_1 \subset X(f_1)$ and $X_2 \subset X(f_2)$, thus:

$$X(f) \subset X(f_1) \cup X(f_2) = X(f_1 f_2).$$

It follows that $f_1 f_2$ belongs to the radical of the ideal generated by f, thus some power of $f_1 f_2$ belongs to the ideal generated by f, i.e. there is an equality

$$(f_1 f_2)^n = fh$$

for some exponent r and some polynomial h. It follows that f is either an irreducible factor of either f_1 or f_2. In the former case $f_1 = fh_1$ hence $X(f_1)$ contains X. In the latter, $X(f_2)$ contains X.

In particular, X is irreducible if and only if f has a unique irreducible factor. This clearly happens when f is irreducible, but also when f is a power of an irreducible polynomial.

10.2 Multiprojective varieties

Let us move to consider *products* of projective spaces, which we will call also *multiprojective spaces*.

The non-expert reader would be surprised, at first, by knowing that a product of projective spaces is *not* trivially a projective space itself.

For instance, consider the product $\mathbb{P}^1 \times \mathbb{P}^1$, whose points have a pair of homogeneous coordinates $([x_0 : x_1], [y_0, y_1])$. These pairs can be multiplied separately by two different scalars. Thus, $([1 : 1], [1 : 2])$ and $([2 : 2], [1 : 2])$ represent the same point of the product. On the other hand, the most naïve association with a point in a projective space yields to relate $([x_0 : x_1], [y_0, y_1])$ with $[x_0 : x_1 : y_0 : y_1]$ (which, by the way, sits in $\mathbb{P}^3$), but $([1 : 1 : 1 : 2])$ and $([2 : 2 : 1 : 2])$ are different points in $\mathbb{P}^3$.

In the next chapter, we will see how a product can be identified with a subset (indeed, with a projective variety) of a higher dimensional projective space.

By now, we develope independently a theory for products of projective spaces and their relevant subsets: multiprojective varieties.

Remark 10.2.1. Consider a product $\mathbb{P} = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$. A point $P \in \mathbb{P}$ corresponds to an equivalence class whose elements are n-tuples

$$((a_{1,0}, \ldots, a_{1,m_1}), \ldots, (a_{n,0}, \ldots, a_{n,m_n}))$$

where, for all i, $(a_{i,1}, \ldots, a_{i,m_i}) \neq 0$. Two such elements

$$a = ((a_{1,0}, \ldots, a_{1,m_1}), \ldots, (a_{n,0}, \ldots, a_{n,m_n}))$$
$$b = ((b_{1,0}, \ldots, b_{1,m_1}), \ldots, b_{n,0}, \ldots, b_{n,m_n}))$$

belong to the same class when there are scalars $k_1, \ldots k_n \in \mathbb{C}$ (all of them necessarily non-zero) such that, for all i, j, $b_{ij} = k_i a_{ij}$.

We will denote the elements of the equivalence class that define P as sets of multihomogeneous coordinates for P, writing

$$P = ([a_{1,0} : \cdots : a_{1,m_1}], \ldots, [a_{n,0} : \cdots : a_{n,m_n}]).$$

Since we want to construct a projective geometry for multiprojective spaces, we need to define the vanishing of a polynomial

$$p \in \mathbb{C}[t_{1,0}, \ldots, t_{1,m_1}, \ldots, t_{n,0}, \ldots, t_{n,m_n}]$$

at a point P of the product above. This time, it is not sufficient that p is homogeneous, for subsets of coordinates referring to factors of the product can be scaled independently.

Definition 10.2.2. A polynomial $p \in \mathbb{C}[t_{1,0}, \ldots, t_{1,m_1}, \ldots, t_{n,0}, \ldots, t_{n,m_n}]$ is *multihomogeneos* of multidegree $(d_1, \ldots, d_n)$ if p, considered as a polynomial in the variables $t_{i,0}, \ldots, t_{i,m_i}$, is homogeneos of degree d_i, for every i.

Strictly speaking, the definition of a multihomogeneous polynomial in a polynomial ring $\mathbb{C}[x_0, \ldots, x_N]$ makes sense only after we defined a partition

in the set of variables. Moreover, if we change the partition, the notion of multihomogeneous polynomial also changes.

Notice however that a partition is canonically determined when we consider the polynomial ring $\mathbb{C}[t_{1,0},\dots,t_{1,m_1},\dots,t_{n,,0},\dots,t_{nm_n}]$ associated to the multiprojective space $\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_n}$.

Multihomogeneous polynomials are homogeneous, but the converse is false.

Example 10.2.3. Consider the polynomial ring $\mathbb{C}[x_0,x_1,y_0,y_1]$, with the partition $\{x_0,x_1\},\{y_0,y_1\}$, and consider the two homogneous polynomials

$$p_1 = x_0^2y_0 + 2x_0x_1y_1 - 3x_1^2y_0 \qquad p_2 = x_0^3 - 2x_1y_0y_1 + x_0y_1^2.$$

Then p_1 is multihomogeneous (of multidegree $(2,1)$) while p_2 is not multihomogeneous.

Example 10.2.4. In $\mathbb{C}[t_{1,0},\dots,t_{1,m_1},\dots,t_{n,0},\dots,t_{n,m_n}]$ the homogeneous linear polynomial $t_{1,0}+\cdots+t_{1,m_1}+\cdots+t_{n,0}+\cdots+t_{n,m_n}$ is never multihomogeneous, except for the trivial partition.

For the trivial partition, homogeneous and multihomogeneous polynomials coincide.

If for any i one takes a homogeneous polynomial $p_i\in\mathbb{C}[t_{i,0},\dots,t_{i,m_1}]$ of degree d_i, then the product $p_1\cdots p_n$ is multihomogeneous of multidegree $(d_1,\dots,d_n)$, in the ring $\mathbb{C}[t_{1,0},\dots,t_{1,m_1},\dots,t_{n,0},\dots,t_{n,m_n}]$ with the natural partition.

It is immediate to verify that given two representatives of the same class in $\mathbb{P}^{m_1}\times\cdots\times\mathbb{P}^{m_n}$:

$$a = ((a_{1,0},\dots,a_{1,m_1}),\dots,(a_{n,0},\dots,a_{nm_n}))$$
$$b = ((b_{1,0},\dots,b_{1,m_1}),\dots,b_{n,,0},\dots,b_{n,m_n})),$$

when $p\in\mathbb{C}[t_{1,0},\dots,t_{1,m_1},\dots,t_{n,0},\dots,t_{n,m_n}]$ is multihomogeneous of any multidegree, then $p(a)=0$ if and only if $p(b)=0$.

Thus one can define the vanishing of p at a point $P = ([a_{1,0} : \cdots : a_{1,m_1}], \ldots, [a_{n,0} : \cdots : a_{n,m_n}])$ of the product, as the vanishing of p at any set of multihomogeneous coordinates.

Definition 10.2.5. We call *multiprojective variety* every subset $X \subset \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$ defined by the vanishing of a family J of multihomogeneous polynomials

$$J \subset \mathbb{C}[t_{1,0}, \ldots, t_{1,m_1}, \ldots, t_{n,0}, \ldots, t_{n,m_n}].$$

We will also write $X(J)$ to denote the multiprojective variety defined by J.

Example 10.2.6. Consider the product $\mathbb{P} = \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$ and consider, for all i, a projective variety X_i in $\mathbb{P}^{m_i}$. Then the product $X_1 \times \cdots \times X_n$ is a multiprojective variety in $\mathbb{P}$.

Indeed, assume that X_i is defined by a subset J_i of homogeneous polynomials in the variables $t_{i,0}, \ldots, t_{i,n_i}$. Then $X_1 \times \cdots \times X_n$ is defined by the set of products:

$$J = \{f_1 \cdots f_n : f_i \in J_i\}.$$

Example 10.2.7. There are multiprojective varieties that are not a product of projective varieties.

For instance, consider the multiprojective variety X defined by $x_0y_1 - x_1y_0$ in the product $\mathbb{P}^1 \times \mathbb{P}^1$. X does not coincide with $\mathbb{P}^1 \times \mathbb{P}^1$, but for each point $P \in \mathbb{P}^1$, the point (P, P) of the product sits in X. Thus X cannot be the product of two subsets of $\mathbb{P}^1$, one of which is a proper subset.

Most properties introduced in the previous section for projective varieties also hold for multiprojective varieties. We give here a short survey.

Remark 10.2.8. Let $X(J)$ be a multiprojective variety, defined by a set J of multihomogeneous polynomials. Then for any $J' \subset J$, the variety $X(J')$ contains $X(J)$.

Beware that one can have $X(J') = X(J)$ even if J' is properly contained in J.

For instance, $X(J)$ is also defined by the ideal $< J >$ and by its radical $\sqrt{< J >}$.

Remark 10.2.9. Multiprojective varieties define a family of closed subset for a topology on a product $\mathbb{P} = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$. We call this topology the *Zariski topology* on $\mathbb{P}$.

$\mathbb{P}$ is irreducible and compact, in the Zariski topology. Thus every multiprojective variety is itself compact, in the induced topology.

Remark 10.2.10. Theorem 10.1.22 guarantees that every multihomogeneous variety is the zero-locus of a *finite* set of multihomogeneous polynomials

Every multihomogeneous variety is indeed the intersection of a finite number of *hypersurfaces* in $\mathbb{P} = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$, where a hypersurface is defined as a multiprojective variety $X(J)$, where J is a singleton.

Theorem 10.2.11. *Let X be a multiprojective variety. Then there exists a unique decomposition of X as the union*

$$X = X_1 \cup \cdots \cup X_k$$

where $X_1, \ldots, X_k$ are irreducible multiprojective varieties: the irreducible components of X.

10.3 Projective and multiprojective maps

A theory of algebraic objects in Mathematics cannot be considered complete unless one introduces also the notion of *good* maps between the objects.

We define in this section a class of maps between projective and multiprojective varieties, that are good for our purposes. We will call them *projective or multiprojective maps*.

In principle, a projective map is a map which is described by polynomials. Unfortunately, one cannot take this as a global definition, because it is too restrictive and would introduce some undesirable phenomenon.

Instead, we define projective maps in terms of a *local* description by polynomials.

Definition 10.3.1. let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be projective varieties. We say that a map $f : X \to Y$ is *projective* if the following property holds:
for any $P \in X$ there exist an open set U of X (in the Zariski topology), containing P, and $m+1$ polynomials $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$, homogeneous *of the same degree*, such that for all $Q = [a_0 : \cdots : a_n] \in U$:

$$f(Q) = [f_0(a_0, \ldots, a_n) : \cdots : f_m(a_0, \ldots, a_n)].$$

We will also write that, over U, the map f is given parametrically by the system:

$$\begin{cases} y_0 & = f_0(x_0, \ldots, x_n) \\ \cdots & \quad \cdots \\ y_m & = f_m(x_0, \ldots, x_m) \end{cases}$$

Thus X is covered by open subsets on which f is defined by polynomials. Since X is compact, we can always assume that the cover is finite.

Notice that if we take another set of homogeneous coordinates for the point $Q \in U$, i.e. we write $Q = [ca_0 : \cdots : ca_n]$, where c is a non-zero scalar, then since the polynomials are homogeneous of the same degree, say degree d, we get $f_i(ca_0, \ldots, ca_n) = c^d f_i(a_0, \ldots, a_n)$ for all i. Thus $f(Q)$ is independent on the choice of a specific set of homogeneous coordinates for Q.

We may always consider a projective map $f : X \to Y \subset \mathbb{P}^m$ as a map from X to the projective space $\mathbb{P}^m$. Next proposition shows that when the domain X of f is a projective space itself, then the localization to open sets, in the definition of projective maps, is useless.

Proposition 10.3.2. *Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a projective map. Then there exists a set of $m+1$ homogeneous polynomials $f_0, \ldots, f_m \in \mathbb{C}[x_0, \ldots, x_n]$ of the same degree, such that $f(Q)$ is defined by $f_0, \ldots, f_m$ for all $Q \in \mathbb{P}^n$.*

In other words, in the definition we can always take just one open subset $U = \mathbb{P}^n$.

Proof. Take two open subsets U_1, U_2 where f is defined, respectively, by homogeneous polynomials $g_0, \dots, g_m$ and $h_0, \dots, h_m$ of the same degrees. Since $\mathbb{P}^n$ is irreducible, then $U = U_1 \cap U_2$ is a non-empty, dense open subset. For any point $P \in U$ there exists a scalar $\alpha_P \in \mathbb{C} - \{0\}$ such that, if $P = [a_0 : \cdots : a_n]$, then

$$(g_0(a_0, \dots, a_n), \dots, g_m(a_0, \dots, a_n)) = \alpha_P(h_0(a_0, \dots, a_n), \dots, h_m(a_0, \dots, a_n)).$$

It follows that the homogeneous polynomials

$$g_j h_i - g_i h_j$$

vanish at all the points of U. Thus they must vanish at all the points of $\mathbb{P}^n$, since U is dense. In particular they vanish in all the points of $U_1 \cup U_2$. It follows immediately that for any $P \in U_1 \cup U_2$, $P = [a_0 : \cdots : a_n]$, the sets of coordinates $[g_0(a_0, \dots, a_n) : \cdots : g_m(a_0, \dots, a_n)]$ and $[h_0(a_0, \dots, a_n) : \cdots : h_m(a_0, \dots, a_n)]$ determine the same, well defined point of $\mathbb{P}^m$. The claim follows. □

Remark 10.3.3. After Proposition 10.3.2 one may wonder if the local definition of projective maps is really necessary. Well, *it is, as illustrated in the following Example 10.3.5.*

The fundamental point is that necessarily the polynomials $f_0, \dots, f_m$ that define the projective map f over U, cannot have a common zero $Q \in U$, otherwise the map would not be defined in Q. Sometimes this property cannot be obtained globally by a unique set of polynomials. It is necessary to use an open cover and vary the polynomials, passing from one open subset to another one.

Example 10.3.4. Assume $n \leq m$ and consider the map between projective spaces $f : \mathbb{P}^n \to \mathbb{P}^m$, defined globally by polynomials $p_0(x_0, \dots, x_n)$, …, $p_m(x_0, \dots, x_n)$ where

$$p_i(x_0, \dots, x_n) = \begin{cases} x_i & \text{if } i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Then f is a projective map. It is easy to prove that f is injective.

Be careful that the map f would not exist for $n > m$!

Indeed, if for instance $n = m + 1$, then the image of the point $P \in \mathbb{P}^n$, with coordinates $[0 : \cdots : 0 : 1]$, would be the point of coordinates $[0 : \cdots : 0]$, which does not exist in $\mathbb{P}^m$.

Example 10.3.5. Let X be the hypersurface of the projective plane $\mathbb{P}^2$, defined by $g(x_0, x_1, x_2) = x_0^2 + x_1^2 - x_2^2$. People can immediately realize that X corresponds to a usual circle of Analytic Geometry.

We define a projective map (stereographic projection) PICTURE??? $f : X \to \mathbb{P}^1$.

Consider the two hypersurfaces $X(h_1), X(h_2)$, where h_1, h_2 are respectively equal to $x_1 - x_2$ and $x_1 - x_2$. Notice that $X(h_1) \cap X(h_2)$ is just the point of coordinates $[1 : 0 : 0]$, which does not belong to X. Thus the open subsets of the plane $(X(h_1))^c, (X(h_2))^c$ cover X. Define two open subsets of X by $U_1 = X \cap (X(h_1))^c$, $U_2 = X \cap (X(h_2))^c$.

Then defined the map f as follows:

$$\text{on } U_1, f = \begin{cases} y_0 & = x_0 \\ y_1 & = x_1 - x_2 \end{cases}, \qquad \text{on } U_2, f = \begin{cases} y_0 & = x_1 + x_2 \\ y_1 & = -x_0 \end{cases}.$$

We need to prove that the definition is consistent in the intersection $U_1 \cap U_2$. Notice that if $Q = [0 : a_1 : a_2]$ belongs to X, then $a_1^2 - a_2^2 = 0$, so that Q belongs either to U_1 or to U_2. Thus any point $Q = [a_0 : a_1 : a_2] \in U_1 \cap U_2$ satisfies $a_0 \neq 0$. Since clearly Q also satisfies $a_1 + a_2 \neq 0$, then

$$\begin{aligned} [a_0 : a_1 - a_2] &= [a_0(a_1 + a_2) : (a_1 - a_2)(a_1 + a_2)] = \\ &= [a_0(a_1 + a_2) : a_1^2 - a_2^2] = [a_0(a_1 + a_2) : -a_0^2] = [a_1 + a_2 : -a_0]. \end{aligned}$$

Thus f is a well defined projective map.

Notice that the two polynomials that define the map on U_1 cannot define the map globally, because $X \setminus U_1$ contains the point $[0 : 1 : 1]$, where both x_0 and $x_1 - x_2$ vanish.

The map f is one-to-one and onto. Indeed if $B = [b_0 : b_1] \in \mathbb{P}^1$, then $f^{-1}(B)$ consists of the unique point $[2b_0b_1 : -b_0^2 + b_1^2 : -b_0^2 - b_1^2]$, as one can easily compute.

Remark 10.3.6. The composition of two projective maps is a projective map.

The identity from a projective variety to itself is a projective map.

Proposition 10.3.7. *Projective maps are continuous in the Zariski topology.*

Proof. Cosider $X \subset \mathbb{P}^n$ and a projective map $f : X \to Y \subset \mathbb{P}^m$. We may assume indeed $Y = \mathbb{P}^m$, to prove the continuity. Let U be an open subset of $\mathbb{P}^m$, which is the complement of a hypersurface $X(g)$, for some $g \in \mathbb{C}[y_0, \ldots, y_m]$. Let Ψ be an open subset of X where f is defined by the polynomials $f_0, \ldots, f_m$. Then $f^{-1}(U) \cap \Psi$ is the intersection of Ψ with the complement of the hypersurface defined by the homogeneous polynomial $g(f_0, \ldots, f_m) \in \mathbb{C}[x_0, \ldots, x_n]$. It follows that $f^{-1}(U)$ is a union of open sets, hence it is open.

Since every open subset of $\mathbb{P}^m$ is a (finite) union of complements of hypersurfaces (by 10.1.24), the claim follows. □

Definition 10.3.8. We will say that a projective map $f : X \to Y$ is an *isomorphism* if there is a projective map $g : Y \to X$ such that $g \circ f =$ identity on X and $f \circ g =$ identity on Y.

Equivalently, a projective map f is an isomorphism if it is one-to-one and onto, and the set-theoretic inverse g is itself a projective map.

Example 10.3.9. Let us prove that the map f defined in Example 10.3.5, from the hypersurface $X \subset \mathbb{P}^2$ defined by the polynomial $x_0^2 + x_1^2 - x_2^2$ to $\mathbb{P}^1$ is an isomorphism.

We already know, indeed, what is the inverse of f: it is the map $g : \mathbb{P}^1 \to X$

defined parametrically by

$$\begin{cases} x_0 &= 2y_0y_1 \\ x_1 &= -y_0^2 + y_1^2 \\ x_2 &= -y_0^2 - y_1^2 \end{cases}$$

It is immediate to check, indeed, that both $g \circ f$ and $f \circ g$ are the identity on the respective spaces.

Remark 10.3.10. We are now able to prove that the map f of Example 10.3.5 cannot be defined globally by a pair of polynomials:

$$\begin{cases} y_0 &= p_0(x_0, x_1, x_2) \\ y_1 &= p_1(x_0, x_1, x_2) \end{cases}$$

Otherwise, since the map g defined in the previous example is the inverse of f, we would have that for any choice of $Q = (b_0, b_1) \neq (0, 0)$, the homogeneous polynomials

$$h_{b_0,b_1} = b_1 p_0(y_0y_1, y_1^2 - y_2^2, -y_1^2 - y_2^2) - b_0 p_1(y_0y_1, y_1^2 - y_2^2, -y_1^2 - y_2^2),$$

whose vanishing defines $f(g(Q))$, vanishes at a single points of $\mathbb{P}^1$. Notice that the degree d of any h_{b_0,b_1} is at least 2.

By the Fundamental Theorem of Algebra, a homogeneous polynomial in two variables that vanishes at a single point is a power of linear form. Thus any polynomial h_{b_0,b_1} is a d-th power of a linear form. In particular there are scalars a_0, a_1, c_0, c_1 such that:

$$h_{1,0} = p_0(y_0y_1, y_1^2 - y_2^2, -y_1^2 - y_2^2) = (a_0y_0 - a_1y_1)^d$$
$$h_{0,1} = p_1(y_0y_1, y_1^2 - y_2^2, -y_1^2 - y_2^2) = (c_0y_0 - c_1y_1)^d$$

Notice that the point $Q' = [a_1 : a_0]$ cannot be equal to $[c_1 : c_0]$, otherwise both p_0, p_1 would vanish at $g(Q') \in X$. Then $h_{1,-1} = (a_0y_0 - a_1y_1)^d - (c_0y_0 - c_1y_1)^d$ vanishes at two different points, namely $[a_1 + c_1 : a_0 + c_0]$ and $[ea_1 + c_1 : ea_0 + c_0]$, where e is any d-root of unit, different from 1.

In the case of multiprojective varieties, most definitions and properties above can be rephrased and proved straightforwardly.

Definition 10.3.11. Let $X \subset \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ be a multiprojective variety. A map $f : X \to \mathbb{P}^m$ is a *projective map* if there bexists a open cover $\{U_i\}$ of the domain X such that f is defined over each U_i by multihomogeneous polynomials, all of the same multidegree.

In other words f is multiprojective if for any U_i of a cover there are multihomogeneous polynomials $p_0, \ldots, p_m$ in $\mathbb{C}[x_{0,1}, \ldots, x_{n,a_n}]$, of the same multidegree, such that for all $P \in X$, $P = ([c_{0,1} : \cdots : c_{0,a_1}), \ldots, [c_{n,1} : \cdots : c_{n,a_n}))$, then $f(P)$ has coordinates

$$f(P) = [p_0((c_{0,1}, \ldots, c_{0,a_1}), \ldots, (c_{n,1}, \ldots, c_{n,a_n})) : \ldots \\ \cdots : p_m((c_{0,1}, \ldots, c_{0,a_1}), \ldots, (c_{n,1}, \ldots, c_{n,a_n}))).$$

We will write in parametric form:

$$\begin{cases} y_0 & = f_0(x_{0,1}, \ldots, x_{n,a_n}) \\ \cdots & \quad \cdots \\ y_m & = f_m(x_{0,1}, \ldots, x_{n,a_n}) \end{cases}$$

We will say that

$$f : X \to \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_m}$$

is a *multiprojective map* if all of its components are.

Remark 10.3.12. The composition of two multiprojective maps is a multiprojective map.

The identity from a multiprojective variety to itself is a multiprojective map.

Multiprojective maps are continuous in the Zariski topology.

10.4 Exercises

Exercise 10. Prove that any open subset of a projective variety X is covered by a finite union of open subsets which are the intersection of X with the complement of a hypersurface.

Chapter 11

Chow's Theorem

We collect in this chapter some fundamental facts on projective maps induced by linear maps.

The non trivial case concerns linear maps which are surjective but not injective. After a change of coordinates, such maps induce maps between projective varieties that can be described as *projections*.

Despite of the fact that the words *projective* and *projection* have a common origin (in the paintings of Renaissance, indeed) projections not always give rise to projective maps.

The description of the image of a projective variety under projections requires indeed some non trivial algebraic tool: the rudiments of the *elimination theory*.

11.1 Linear maps and change of coordinates

The following definition generalizes Example 1.3.4.

Definition 11.1.1. Consider a linear map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ *which is injective*.

Then f defines a projective map (which, by abuse, we will still denote by f) between the projective spaces $\mathbb{P}^n \to \mathbb{P}^m$, as follows:

for all $P \in \mathbb{P}^n$, consider a set of homogeneous coordinates $[a_0 : \cdots : a_n]$ and send P to the point $f(P) \in \mathbb{P}^m$ with homogeneous coordinates $[f(x_0, \ldots, x_n)]$.

Such maps are called *linear projective maps*.

It is plain that the point $f(P)$ does not depend on the choice of a set of coordinates for P, since f is linear.

Also, notice that the map cannot be defined in the same way when f is not injective, for in this case the image of a point P whose coordinates lie in the Kernel of f would be indeterminate.

Since any linear map $\mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ is defined by linear homogeneous polynomials, then it is clear that the induced map between projective spaces is indeed a projective map.

Example 11.1.2. If the map f is an isomorphism of $\mathbb{C}^{n+1}$, then the corresponding linear projective map is called a *change of coordinates*.

Indeed f corresponds to a change of basis inside $\mathbb{C}^{n+1}$.

Such a projective map is an isomorphism, since clearly the inverse isomorphism f^{-1} determines a projective map which is the inverse of f.

In particular, any change of coordinates in a projective space is a homeomorphism of the corresponding topological space, in the Zariski topology. So, the image of a projective variety under a change of coordinates is still a projective variety.

This remark generalizes to any linear projective map.

Example 11.1.3. Every injective map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$, $m \geq n$, can be factored as a composition $f = g \circ f'$ where f' is the inclusion which sends $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_n, 0, \ldots, 0)$ (with $m-n$ zeroes), followed by a change of coordinates.

As such, up to a change of coordinates, any linear projective map can be reduced to the map that embeds $\mathbb{P}^n$ into $\mathbb{P}^m$ as the linear space defined by equations $x_{n+1} = \cdots = x_m = 0$.

It follows that any linear projective map is *closed* in the Zariski topology: the image of the projective subvariety $X \subset \mathbb{P}^n$ defined by homogeneous polynomials $p_1 = \cdots = p_s = 0$ is exactly the image, under a change of coordinates, of the projective subvariety defined in $\mathbb{P}^m$ by equations $p_1 = \cdots = p_s = x_{n+1} = \cdots = x_m = 0$.

Remark 11.1.4. Let V be a linear space of dimension $n+1$. The choice of a basis for V corresponds to fixing an isomorphism between V and $\mathbb{C}^{n+1}$. Thus we can identify, after a choice of the basis, the projective space $\mathbb{P}(V)$ with $\mathbb{P}^n$.

A base change on V corresponds to a change of coordinates $\mathbb{P}^n \to \mathbb{P}^n$.

From now on, we will make use of an identification between V and C^{n+1}, induced by the choice of a basis, to introduce and study projective varieties into $\mathbb{P}(V)$.

The simple situation described above becomes much more involved if we analyze what happens for non-injective linear maps $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$.

Let us now consider a linear map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ which is *not* injective.

In this case, we cannot define through f a projective map $\mathbb{P}^n \to \mathbb{P}^m$ as above, since for any vector $(a_0, \ldots, a_n)$ in the kernel of f, the image of the point $[a_0 : \cdots : a_n]$ is unefined, because $f(a_0, \ldots, a_n)$ vanishes.

On the other hand, the kernel of f defines a projective linear subspace of $\mathbb{P}^n$, the *projective kernel*, which will be denoted by K_f.

If $X \subset \mathbb{P}^n$ is a subvariety *which does not meet* K_f, then the restriction of f to the coordinates of the points of X determines a well defined map from X to $\mathbb{P}^m$.

Example 11.1.5. Consider the subspace $M \subset \mathbb{C}^{n+1}$ of dimension $m+1 < n+1$, formed by the elements of type $(x_0, \ldots, x_m, 0, \ldots, 0)$. Let $f : \mathbb{C}^{n+1} \to M$ be the linear surjective map which sends any $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_m, 0, \ldots, 0)$.

Notice that M defines a linear projective subspace $\mathbb{P}(M) \subset \mathbb{P}^n$, of projective dimension m. Let P be a point of $\mathbb{P}^n$, outside the projective kernel K_f (this means exactly that $P = [a_0 : \cdots : a_n]$, with $a_i \neq 0$ for some index i between 0 and m). Then the image of P under f corresponds to the projection of P to $\mathbb{P}(M) = \mathbb{P}^m$, from the center of projection K_f.

If $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ is any surjective map, then we can always find an isomorphism of vector spaces g from $\mathbb{C}^{m+1}$ to the subspace M of $\mathbb{P}^n$ of all vectors with $x_{m+1} = \cdots = x_n = 0$, so that the composition $g \circ f$ sends $(x_0, \ldots, x_n)$ to $(x_0, \ldots, x_m, 0, \ldots, 0)$.

Thus, after a projective isomorphism, f acts on points of $\mathbb{P}^n \setminus K_f$ as a geometric projection. The observation explains the following definition.

Definition 11.1.6. Given a linear surjective map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$ and a subvariety $X \subset \mathbb{P}^n$ which does not meet K_f, the restriction map $f_{|X} : X \to \mathbb{P}^m$ is a well defined projective map, which will be denoted as a *projection of X with center K_f*.

Next, we will show that for any choice of f and X as above, the image of X in $\mathbb{P}^m$ is an algebraic subvariety of $\mathbb{P}^m$. To do that, we will need at least the rudiments of Elimination Theory.

11.2 Elimination Theory

Let us consider with the following problem.

Assume we are given two (not necessarily homogeneous) polynomials $p, q \in \mathbb{C}[t]$. Clearly both p and q factorise in a product of linear factors.

Which algebraic condition do p, q satisfy when they have a common factor?

The answer to this question relies in the notion of *resultant* of p and q.

Proposition 11.2.1. *There exists a polynomial $R(p, q)$ in the coefficients*

of p and q, such that $R(p,q) = 0$ exactly when p, q share a common linear factor. The polynomial $R(p,q)$ is called then resultant *of p and qt.*

Proof. Write $p = a_0 + a_1 t + \cdots + a_r t^r$ and $q = b_0 + b_1 t + \dots b_s t^s$ ($a_r, b_s \neq 0$). The existence of a common root y for p, q, which is also a solution for $tp, t^2 p, \dots, t^s p$ and $tq, t^2 q, \dots, t^r q$, implies that the linear system associated to the $(r+s) \times (r+s)$ matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_r & 0 & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{r-1} & a_r & 0 & \dots & 0 \\ \dots & & & & & & & & \\ 0 & \dots & 0 & a_0 & a_1 & a_2 & 0 & \dots & a_r \\ b_0 & b_1 & b_2 & \dots & b_s & 0 & 0 & \dots & 0 \\ \dots & & & & & & & & \\ 0 & \dots & 0 & b_0 & b_1 & b_2 & 0 & \dots & b_s \end{pmatrix}$$

has a non trivial solution: the powers $(1, y, y^2, \dots)$.
Conversely, if the system above has a non-trivial solution, then either a_0, b_0 are both 0 (in which case p, q both vanish at the origin) or, if say $a_0 \neq 0$, then $a_0^r b_0^s \neq 0$, so the rank of the matrix is $r + s - 1$ and the solutions of the system are all proportional. Substituting in the rows, one sees immediately that there exists a solution of type $(1, y, y^2, \dots)$.
Thus, the determinant of the matrix is a polynomial $R(p,q)$ in the coefficients of p, q, that vanishes exactly when p, q have a common root. □

When two polynomials p, q have many variables $t_0, \dots, t_n$, then we can consider them as polynomials in x_0, with coefficients in $\mathbb{C}[t_1, \dots, t_n]$. The corresponding resultant $R_0(p,q)$ is a polynomial in $\mathbb{C}[t_1, \dots, t_n]$ which vanishes at $(y_1, \dots, y_n)$ exactly when there exists y_0 with

$$p(y_0, y_1, \dots, y_n) = q(y_0, y_1, \dots, y_n) = 0.$$

We define similarly the resultant $R_i(p,q)$ by isolating the variable t_i in the two polynomials.

Remark 11.2.2. When p, q are homogeneous polynomials in $\mathbb{C}[t_0, \dots, t_n]$, then for any i the resultant $R_i(p, q)$ is a homogeneous polynomial in $\mathbb{C}[t_1, \dots, t_n]$

Proposition 11.2.3. *Let X be a projective variety, with homogeneous ideal $I(X)$. For any pair $p, q \in I(X)$ and for any index i, the resultant $R_i(p, q)$ belongs to $I(X)$.*

Proof. Assume $i = 0$ and let $P = [y_0 : \cdots : y_n]$ be any point in X. Then $p(y_0, y_1, \dots, y_n) = q(y_0, y_1, \dots, y_n) = 0$. Thus $R_0(p, q)(P)$, which is clearly equal to $R_0(p, q)(y_1, \dots, y_n)$ since $R_0(p, q)$ does not have the variable x_0, vanishes. So $R_0(p, q)$ vanishes at the points of X, thus it belongs to $I(X)$.

The same argument works for any index. □

The resultant is a tool that shows that the image of a projective variety in a projection is a projective variety.

Proposition 11.2.4. *Given a linear surjective map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$, $m < n$, and a subvariety $X \subset \mathbb{P}^n$ which does not meet K_f, the image of the restriction $f_{|X} : X \to \mathbb{P}^m$ is a projective subvariety 0f $\mathbb{P}^m$.*

Proof. After a change of coordinates we may assume that the map f sends $(x_0, \dots, x_n)$ to $(x_0, \dots, x_m)$. Then we may consider f as a sequence of maps, each killing one variable. Thus we may restrict ourselves to prove the claim in the case $m = n - 1$.

Consider the homogeneous ideal $I(X)$ of X and let $I' \subset \mathbb{C}[x_0, \dots, x_{n-1}]$ be the ideal generated by the homogeneous elements of $I(X)$ belonging to $\mathbb{C}[x_0, \dots, x_{n-1}]$. We will prove that $f_{|X}(X)$ is exactly the projective variety defined by I'.

Indeed if a point $Q = [y_0 : \cdots : y_{n-1}]$ belongs to $f_{|X}(X)$, then there exists y_n such that $P = [y_0 : \cdots : y_n]$ sits in X. Since $I' \subset I(X)$, then for every $p \in I'$, we have

$$p(y_0, \dots, y_{n-1}) = p(y_0, \dots, y_n) = 0.$$

Conversely assume that $Q \in \mathbb{P}^m \setminus f_{|X}(X)$, $Q = [y_0 : \cdots : y_{n-1}]$. Then for any choice of y_n, there exists a polinomial $p \in I(X)$ such that $\bar{p} = p(y_0, \ldots, y_{n-1}, x_n) \in \mathbb{C}[x_n]$ does not vanish at $x_n = y_n$. In particular $\bar{p}$ is not identically 0. Call $u_1, \ldots, u_k$ the roots of $\bar{p}$. For each j there exists $p_j \in I(X)$ such that $\bar{p}_j = p_j(y_0, \ldots, y_{n-1}, x_n)$ does not vanish at u_j. Thus, a general linear combination of the p_j's determine a polynomial $q \in I(X)$ such that $\bar{q} = p(y_0, \ldots, y_{n-1}, x_n)$ does not vanish at any u_j, otherwise $[y_0 : \cdots : y_{n-1}, : u_j] \in X$, so that $Q = [y_0 : \cdots : y_{n-1}] \in f_{|X}(X)$, a contradiction. We claim that the resultant $R_n(p, q)$ is a homogeneous polynomial in I' which does not vanish at Q. Indeed $R_n(p, q) \in I'$ by Proposition 11.2.3. If $R_n(p, q)(Q) = R_n(p, q)(y_0, \ldots, y_{n-1}) = 0$, then there exists y_n such that $P' = [y_0 : \cdots : y_{n-1} : y_n]$ annihilates both p and q. This is excluded by construction.

Summarizing, we proved that $p(Q) = 0$ for all $Q \in f_{|X}(X)$ and $p \in I'$, while if $Q \in \mathbb{P}^m \setminus f_{|X}(X)$ then $p(Q) \neq 0$ for at least one $p \in I'$. It follows that $f_{|X}(X)$ is the projective variety defined by I'. □

11.3 The projective and multiprojective Chow's Theorem

We prove in this section that the images of all projective and multiprojective maps are projective (or multiprojective) varieties.

The first step is the proof that any projective or multiprojective map factorizes through change of coordinates, projections and two special projective maps: the Veronese map and the Segre map. These two maps owe their names to two famous Italian mathematicians of the XIX century: Giuseppe Veronese and Corrado Segre, who developed a systematic geometric study of projective spaces.

Both maps are the projective counterpart of fundamental operations in multilinear algebra: the symmetric and the general tensor products.

Definition 11.3.1. Fix n, d and set $N = \binom{n+d}{d} - 1$. There are exactly $N + 1$

monic monomials of degree d in $n+1$ variables $x_0, \dots, x_n$. Let us choose an ordering and call $M_0, \dots, M_N$ these monomials.

The *Veronese map* of degree d in $\mathbb{P}^n$ is the map $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ which sends a point $[a_0 : \cdots : a_n]$ to $[M_0(a_0, \dots, a_n) : \cdots : M_N(a_0, \dots, a_n)]$.

The map is well defined, since for any $P = [a_0 : \cdots : a_n] \in \mathbb{P}^n$ there exists $a_i \neq 0$, and among the monomials one has $M = x_i^d$, which satisfies $M(a_0, \dots, a_n) = a_i^d \neq 0$.

The Veronese map is injective. Indeed if $P = [a_0 : \cdots : a_n]$ and $Q = [b_0 : \cdots : b_n]$, have the same image, then the powers of the a_i's and the b_i's are equal, up to a scalar multiplication. Thus, up to a scalar multiplication, one may assume $a_i^d = b_i^d$ for all i, so that $b_i = e_i a_i$, for some choice of a d-root of unit e_i. If the e_i's are not all equal to 1, then there exists a monic monomial M such that $M(e_0, e_1, \dots, e_n) \neq 1$, thus $M(a_0, \dots, a_n) \neq M(b_0, \dots, b_n)$, which contradicts $v_{n,d}(P) = v_{n,d}(Q)$.

Because of its injectivity, sometimes we will refer to the Veronese map as the *Veronese embedding*.

Proposition 11.3.2. *Every projective map $f : \mathbb{P}^n \to \mathbb{P}^m$ factors through a Veronese map, a change of coordinates and a projection.*

Proof. By proposition **??** there are homogeneous polynomials $f_0, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$ of the same degree d, which do not vanish simultaneously at any point $P \in \mathbb{P}^n$, and such that f is defined by the f_j's. Each f_j is a linear combination of monic monomials of degree d. Hence there exists a change of coordiantes g in the target space $\mathbb{P}^N$ of $v_{n,d}$ such that f is equal to $v_{n,d}$ followed by g and by the projection to the first $m+1$ coordinates. Notice that since $(f_0(P), \dots, f_m(P)) \neq 0$ for all $P \in \mathbb{P}^n$, the projection is well defined on the image of $g \circ v_{n,d}$. □

A similar procedure holds to describe a canonical decomposition of multiprojective maps.

Definition 11.3.3. Fix $a_1, \dots, a_n$ and set $N = (a_1 + 1) \cdot (a_2 + 1) \cdots (a_n + 1) - 1$. There are exactly $N + 1$ monic monomials of multidegree $(1, \dots, 1)$ in the variables $x_{1,0}, \dots, x_{1,a_1}, x_{2,0}, \dots, x_{2,a_2}, \dots, x_{n,0} \dots x_{n,a_n}$. Let us choose an ordering and call $M_0, \dots, M_N$ these monomials.

The *Segre map* of degree d in $\mathbb{P}^n$ is the map $s_{a_1,\dots,a_n} : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n} \to \mathbb{P}^N$ which sends a point $P = ([p_{10} : \cdots : p_{1a_1}], \dots, [p_{n0} : \cdots : p_{na_n}])$ to $[M_0(P) : \cdots : M_N(P)]$.

The map is well defined, since for any $i = 1, \dots, n$ there exists $p_{ij_i} \neq 0$, and among the monomials one has $M = x_{1,j_1} \cdots x_{n,j_n}$, which satisfies $M(P) = p_{1j_1} \cdots p_{nj_n} \neq 0$.

Notice that when $n = 1$, then the Segre map is the identity.

Proposition 11.3.4. *The Segre map is injective.*

Proof. Make induction on n, the case $n = 1$ being trivial.
For the general case, assume that

$$P = ([p_{10} : \cdots : p_{1a_1}], \dots, [p_{n,0} : \cdots : p_{n,a_n}]),$$
$$Q = ([q_{10} : \cdots : q_{1a_1}], \dots, [q_{n,0} : \cdots : q_{n,a_n}])$$

have the same image. Fix indexes such that $p_{1j_1}, \dots, p_{nj_n} \neq 0$. The monomial $M = x_{1,j_1} \cdots x_{n,j_n}$ does not vanish at P, hence also $q_{1j_1}, \dots, q_{nj_n} \neq 0$.
Call $\alpha = q_{1j_1}/p_{1j_1}$. Our first task is to show that $\alpha = q_{1i}/p_{1i}$ for $i = 1, \dots, a_1$, so that $[p_{11} : \cdots : p_{1a_1}] = [q_{11} : \cdots : q_{1a_1}]$. Define $\beta = (q_{2j_2} \cdots q_{nj_n})/(p_{2j_1} \cdots p_{nj_n})$. Then $\beta \neq 0$ and

$$\alpha\beta = (q_{1j_1} \cdots q_{nj_n})/(p_{1j_1} \cdots p_{nj_n}).$$

Since P, Q have the same image in the Segre map, then for all $i = 1, \dots, a_1$, the monomials $N_i = x_{1,i}x_{2,j_2} \cdots x_{n,j_n}$ satisfy

$$\alpha\beta N_i(P) = N_i(Q).$$

It follows immediately $\alpha\beta(p_{1i} \cdots p_{nj_n}) = (q_{1i} \cdots q_{nj_n})$ so that $\alpha p_{1i} = q_{1i}$ for all i. Thus $[p_{10} : \cdots : p_{1a_1}] = [q_{10} : \cdots : q_{1a_1}]$.

We can repeat the argument for the remaining factors of P, Q, obtaining $P = Q$. □

Because of its injectivity, sometimes we will refer to the Segre map as the *Segre embedding.*

Proposition 11.3.5. *Every multiprojective map $f : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n} \to \mathbb{P}^N$ factors through Veronese maps, aSegre map, a change of coordinates and a projection.*

Proof. By proposition **??** there are multihomogeneous polynomials f_j's in the ring $\mathbb{C}[x_{1,0}, \ldots, x_{1,a_1}, \ldots, x_{n,0} \ldots x_{n,a_n}]$ of the same multidegrees $(d_1, \ldots, d_n)$, which do not vanish simultaneously at any point $P \in \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$, and such that f is defined by the f_j's. Each f_j is a linear combination of products of monic monomials of degrees $d_1, \ldots, d_n$ respectively in the set of coordinates $(x_{1,0}, \ldots, x_{1,a_1}), \ldots, (x_{n,0} \ldots x_{n,a_n})$. If v_{a_i,d_i} denotes the Veronese embedding of degree d_i of $\mathbb{P}^{a_i}$ into the corresponding space $\mathbb{P}^{A_i}$, then f factors through $v_{a_1,d1} \times \cdots \times v_{a_n,d_n}$ followed by a multilinear map $F : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n} \to \mathbb{P}^N$ defined by multihomogeneous polynomials F_j of multidegree $(1, \ldots, 1)$. Each F_j is a linear combination of products of n coordinates in the sets $(x_{1,0}, \ldots, x_{1,a_1}), \ldots, (x_{n,0} \ldots x_{n,a_n})$ respectively. Hence F factors through a Segre map $s_{A_1,\ldots,A_n}$, followed by a change of coordinates in $\mathbb{P}^M$, $M = (A_1 + 1) \cdots (A_n + 1) - 1$, and then followed by a projection. □

Now we are ready to state and prove the Chow's Theorem.

Theorem 11.3.6. Chow's Theorem *Every projective map $f : \mathbb{P}^n \to \mathbb{P}^N$ is Zariski-closed, i.e. the image of a projective subvariety is a projective subvariety.*
Every multiprojective map $f : \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n} \to \mathbb{P}^M$ is Zariski-closed.

Proof. In view of the previous propositions, it is enough to prove that the Veronese map and the Segre map are Zariski-closed.

Let us consider a Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^N$ and let Y be a projective subvariety in $\mathbb{P}^n$. Then Y is a finite intersection of hypersurfaces. Thus, if we prove that the image of any hypersurface of $\mathbb{P}^n$ is Zariski-closed in $\mathbb{P}^N$, then the claim holds for any Y, because $v_{n,d}$ is injective. Tus we may reduce ourselves to the case where Y is a hypersurface of degree e in $\mathbb{P}^n$, defined by the homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$.

Fix an integer m such that $u = md - e$ is non-negative. The polynomials $g_0 = fx_0^u, \ldots, g_n = fx_n^u$ are homogeneous of degree md. So each g_i is a linear combinations of products of m monic monomials of degree d in the x_j's. Thus, following the notation of the definition of the Veronese maps, every g_i corresponds to some polinomial G_i of degree m in the variables M_j's of $\mathbb{P}^N$. We want to prove that the image of Y is the vanishing locus of the G_i's.

Indeed if $P = [p_0 : \cdots : p_n] \in Y$, then for all i, $g_i(p_0, \ldots, p_n) = 0$. Thus $G_i(M_0(p_0, \ldots, p_n), \ldots, M_N(p_0, \ldots, p_n)) = 0$, i.e. each G_i vanishes at the point $[M_0(p_0, \ldots, p_n) : \cdots : M_N(p_0, \ldots, p_n)] = v_{n,d(P)}$.

Conversely, assume that $P \notin Y$, so that $f(P) \neq 0$. Let i be the index of a non-vanishing coordinate of P. Then $g_i(P) \neq 0$. It follows soon $G_i(M_0(p_0, \ldots, p_n), \ldots, M_N(p_0, \ldots, p_n)) \neq 0$.

In the multiprojective case, we can as above reduce ourselves to varieties Y defines by a single polynomial f. So, let Y be defined by the multihomogeneous polynomial f, of multidegree $(d_1, \ldots, d_n)$. Define $d = \max\{d_i\}$ and $u_i = d - d_i$. Consider the polynomials $fx_{1,i_1}^{u_1} \cdots x_{n,i_n}^{u_n}$, for any choice of indexes $(i_1, \ldots, i_n)$. These polynomials are multihomogeneous of multidegree $(d, \ldots, d)$, so they split in a product of d monomials of multidegree $(1, \ldots, 1)$. In other words, by the definition of the Segre map, each $fx_{1,i_1}^{u_1} \cdots x_{n,i_n}^{u_n}$ determines a homogeneous polynomial $G_{i_1,\ldots,i_n}$ in the coordinates of the target space of the Segre embedding. As above it is quite straightforward that these polynomials vanish exactly on the image of Y. □

Example 11.3.7. Let us consider the projective map $f : \mathbb{P}^1 \to \mathbb{P}^2$ defined

by

$$f(x_1, x_2) = (x_1^3, x_1^2x_2 - x_1x_2^2, x_2^3).$$

We can decompose f as the Veronese map $v_{1,3}$, followed by the linear isomorphism $g(a, b, c) = (a, b - c, c - d, d)$ and then followed by the projection π to the first, second and fourth coordinate.
Namely:

$$(\pi \circ g \circ v_{1,3})(x_1, x_2) = (\pi \circ g)(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3) = \\ = \pi(x_1^3, x_1^2x_2 - x_1x_2^2, x_1x_2^2 - x_2^3, x_2^3) = (x_1^3, x_1^2x_2 - x_1x_2^2, x_2^3).$$

The image of π is a projectve curve in $\mathbb{P}^3$, whose equation can be obtained by elimination theory. One can see that, in the coordinates z_0, z_1, z_2 of $\mathbb{P}^2$, $f(\mathbb{P}^1)$ is the zero locus of

$$z_1^3 - z_0z_2(z_0 - 3z_1 - z_2).$$

Example 11.3.8. Let us consider the subvariety Y of $\mathbb{P}^1 \times \mathbb{P}^1$, defined by the multihomogeneous polynomial $f = x_0 - x_1$, of multidegree $(1, 0)$ in the coordinates $(x_0, x_1), (y_0, y_1)$ of $\mathbb{P}^1 \times \mathbb{P}^1$. Y corresponds to $[1 : 1] \times \mathbb{P}^1$.
Take the Segre embedding $s : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$,

$$(x_0, x_1), (y_0, y_1) = (x_0y_0, x_0y_1, x_1y_0, x_1y_1).$$

Then the image $s(\mathbb{P}^1 \times \mathbb{P}^1)$ corresponds to the quadric Q in $\mathbb{P}^3$ defined by the vanishing of the homogeneous polynomial $g = z_0z_3 - z_1z_2$.
The image of Y is a projective subvariety of $\mathbb{P}^3$, which is constained in Q, but it is no longer defined by g and another polynomial: we need two polynomials, other than g.
Namely, Y is defined also by the two multihomogeneous polynomials, of multidegree $(1, 1)$, $f_0 = fy_0 = x_0y_0 - x_1y_0$ and $f_1 = fy_1 = x_0y_1 - x_1y_1$. Thus $s(Y)$ is defined in $\mathbb{P}^3$ by $g, g_0 = z_0 - z_1, g_1 = z_2 - z_3$. (Indeed, in this case, g_0, g_1 alone are sufficient to determine $s(Y)$, which is a line).

Chapter 12

Dimension Theory

The concept of *dimension* of a projective variety is a fairly intuitive but surprisingly delicate invariant, from an algebraic point of view.

If one considers projective varieties over $\mathbb{C}$ with their natural structure of complex or holomorphic varieties, then the algebraic definition of dimension coincides with the usual (complex) dimension.

On the other hand, for many purposes, it is necessary to deal with the concept from a completely algebraic point of view, so the definition of dimension that we give below is fundamental for our analysis.

The starting point that we take is simply concerned with geometric, projective definitions, though at a certain point, for the sake of completeness, we cannot avoid to invoke some algebraic result, in order to make the theory self-contained.

12.1 Complements on Irreducible varieties

For many purposes it is natural to

The first step is rather technical, from an algebraic point of view: we need some algebraic properties of irreducible varieties. We recall that the definition of irreducible topological spaces, together with examples, can be found in Definition 1.1.25**??** of the first Chapter **??**.

So, from now on, dealing with projective varieties, we will always refer to reducibility or irreducibility with respect to the induced Zariski topology.

Let us start with a characterization of irreducible varieties, in terms of the associated homogeneous ideal (see Corollary **??**1.1.17).

Definition 12.1.1. An ideal J of a polynomial ring $R = \mathbb{C}[x_0, \dots, x_n]$ is a *prime ideal* if $f_1 f_2 \in J$ implies that either $f_1 \in J$ or $f_2 \in J$.

Equivalently, J is prime if and only if the quotient ring R/J is a domain, i.e. $a, b \in R/J$, $ab = 0$ implies that either $a = 0$ or $b = 0$.

Proposition 12.1.2. *Let $Y \subset \mathbb{P}^n$ be a projective variety and call J the homogeneous ideal defined by Y. Then Y is irreducible if and only if J is a prime ideal.*

Proof. Assume $Y = Y_1 \cup Y_2$, where the Y_i's are proper closed subsets. Then there exist polynomials f_1, f_2 such that f_i vanishes on Y_i but not on Y. Thus $f_1, f_2 \notin J$, while $f_1 f_2$ vanishes at any point of Y, i.e. $f_1 f_2 \in J$.
The previous argument can be inverted to show that the existence of $f_1, f_2 \notin J$ such that $f_1 f_2 \in J$ implies that Y is reducible. □

Definition 12.1.3. Let $Y \subset \mathbb{P}^n$ be an *irreducible* projective variety and let $J \subset \mathbb{C}[x_0, \dots, x_n]$ be its homogeneous ideal. Then J is a prime ideal and $R_Y = \mathbb{C}[x_0, \dots, x_n]/J$ is a domain. So, one can construct the *quotient field* $k(R_Y)$ as the field of all quotients $\{\frac{a}{b} : a, b \in R_y, b \neq 0\}$, where $\frac{a}{b} = \frac{a'}{b'}$ if and only if $ab' = a'b$.

We call $k(R_Y)$ the *function field* of Y.

The following topological property is elementary, and we leave it to the reader as an exercise.

Proposition 12.1.4. *The image of an irreducible subset under a continuous map, is irreducible.*

X is infinite iff K_X is not C.

12.2 Dimension

There are several definitions of *dimension* of an irreducible varieties. All of them have some difficult aspect. In some case it is laborious even to prove that the definition itself makes sense. For more technical approach, it is not immediate to see that the geometric naïve notion of dimension corresponds to the algebraic notion.

Our choice is to make use, as far as possible, of the geometric approach, entering deeply in the algebraic background just to justify some computational aspect.

The final target is the theorem on the dimension of general fibers (Theorem **??**), which allows to manage the notion of dimension quite completely.

Definition 12.2.1. Given a projective map $f : X \to Y$ we call *fibers* of f the inverse images of points $f^{-1}(P)$, $P \in Y$.

Remark 12.2.2. Since projective maps are continuous in the Zariski topology, and singletons are projective varieties, then the fiber over any point $P \in Y$ is closed in the Zariski topology, hence it is a projective variety.

Now we arrive to the starting definition of *dimension*

Definition 12.2.3. We say that an irreducible projective variety X has *dimension* n if there exists a projective surjective map $X \to \mathbb{P}^n$, whose fibers are all finite sets.
We assign dimension -1 to the emptyset.

The identity makes it obvious that a projective space $\mathbb{P}^n$ has dimension n.

Example 12.2.4. Since $\mathbb{P}^0$ has just one point, clearly singletons have a surjective map to $\mathbb{P}^0$, with finite fibers. So singletons have dimension 0.
Finite projective varieties are reducible, unless they are singleton. Thus, by definition, singletons are the only projective irreducible varieties of dimension 0.

Example 12.2.5. A linear subspace $L \subset \mathbb{P}^N$ is the projectivization of a vector subspace $V \subset \mathbb{C}^{n+1}$, thus it has a well defined projective dimension. The linear subspace L_n in $\mathbb{P}^N$ defined by $t_{n+1} = \cdots = t_N = 0$ is isomorphic to $\mathbb{P}^n$. Base changes provide isomorphisms between any linear variety of projective dimension n and L_n. Thus, for linear subspaces, the projective dimension is a value for the dimension, as defined above.

Example 12.2.6. Let C be a projective variety in $\mathbb{P}^2$, defined by the vanishing of one irreducible homogeneous polynomial $g \neq 0$. Then C is an irreducible variety (Example **??**1.1.36) and it contains infinitely many points (Example **??**).
By Lemma **??**1.1.5, there exists a point $P_0 \in \mathbb{P}^2$ which does not belong to C. The projection π from P_0 maps C to $\mathbb{P}^1$. Every fiber of π is a *proper* projective subvariety of a line, since it cannot contain P_0. Since the Zariski topology on a line is the co-finite topology (Example 10.1.12), then every fiber of π is finite. The image of π, which is a projective subvariety of $\mathbb{P}^1$ (Theorem **??**) cannot be finite, so it coincides with $\mathbb{P}^1$, hence π is surjective. We have just proved that C has dimension 1.

By now, the dimension of an irreducible projective variety is not uniquely defined, since we did not exclude the existence of two different maps $X \to \mathbb{P}^n$ and $X \to \mathbb{P}^m$, $m \neq n$, both with finite fibers.

It is not easy to face the problem directly. Instead, we show that the existence of a map $X \to \mathbb{P}^n$ with finite fibers is related with a numerical invariant of the irreducible variety X.

Proposition 12.2.7. *For every projective variety $X \in \mathbb{P}^r$ there exists a projective surjective map $f : X \to \mathbb{P}^m$ whose fibers are finite.*

Proof. We make induction on r. If $r = 1$ the claim follows immediately since in this case either $X = \mathbb{P}^1$ of X is a finite set.
For $R > 1$, if $X = \mathbb{P}^r$ then the claim is obvious. Otherwise fix a point $P \notin X$. The projection π from P maps X to a subvariety of $\mathbb{P}^{r-1}$. Since the fibers of

π are closed subvarieties of a line and do not contain the point P of the line, the the fibers of π are finite. The claim follows by induction on $\pi(X)$. □

The invariant which defines the dimension is connected with a notion in the algebraic theory of field extensions: the *trascendence degree*. We recall some basis in the following remark. For the proofs, we refer to [ZS][].

Remark 12.2.8. Let K_1, K_2 be fields, with a non-zero homomorphism $\phi : K_1 \to K_2$. Then ϕ is injective, since the kernel is an ideal of K_1, hence it is trivial. So we can consider ϕ as a inclusion which realizes K_2 as an extension of K_1.
The extension is *algebraic* when K_2 is finitely generated as a vector space over K_1. Otherwise the extension is *trascendent*. If K_2 is al algebraic extension of K_1, then for any $e \in K_2$ there exists a polynomial $p(x)$, with coefficient in K_1, such that $p(e) = 0$.
For any $e \in K_2$ define $K_1(e)$ as the minimal subfield of K_2 which contains K_1 and e. We say that e is an *algebraic element* over K_1 if $K_1(e)$ is an algebraic extension of K_1, otherwise e is a *trascendent element*.
Since $\mathbb{C}$ is algebraically closed, then every element of an extension of $\mathbb{C}$ either belongs to $\mathbb{C}$ or it is trascendent over $\mathbb{C}$.
If $K_2 = K_1(x)$ is the field of fractions of the polynomial ring $K[x]$, then K_2 is a trascendent extension. Conversely, if e is any trascendent element over K_1, then $K_1(e)$ is isomorphic to $K_1(x)$.
A set of elements $e_1, \ldots, e_n \in K_2$ such that for all i e_i is trascendent over $K_1(e_1, \ldots, e_{i-1})$ and K_2 is an algebraic extension of $K_1(e_1, \ldots, e_n)$ is a *trascendence* basis of the extension. All trascendence basis have the same number of elements, which is called the *trascendence degree* of the extension.

Theorem 12.2.9. *A projective map $f : X \to \mathbb{P}^n$ from the irreducible variety X to a projective space has finite fibers, if and only if the quotient field K_X is an algebraic extension of $\mathbb{C}(x_1, \ldots, x_n)$.*

Proof. Assume in the proof that $X \subset \mathbb{P}^N$.
If $n = 0$, then the map exists if and only if X is one point, and the claim follows by **??**. So we can make induction on n.
Assume there exists an element $e \in K_X$ which is trascendent over $K_{\mathbb{P}^n} = \mathbb{C}(x_1, \ldots, x_n)$. Then $e \neq 0$ is the quotient of two equivalence classes of homogeneous polynomials $f, g \in \mathbb{C}[y_0, \ldots, y_N]$, both not vanishing on the whole X. Fix a point $P \in X$ where f, g does not vanish and consider the fiber $X' = f^{-1}(f(P))$. The restriction $f_{|X'} : X' \to f(p)$ determines an inclusion

Consider the point $P \in \mathbb{P}^n$, with coordinates $[a_0 : \cdots : a_n]$, where we may assume $a_0 \neq 0$ without loss of generality. □

Example 12.2.10. Hypersurfaces X in $\mathbb{P}^n$ have dimension $n - 1$.
Indeed take a points $P \notin X$ and consider the projection π of X from P to $\mathbb{P}^{n-1}$. The fibers of the projection are closed proper subvarieties of lines, hence they are finite. Moreover, the projection is surjective, by **??**. The claim follows.

For reducible varieties, the definition of dimension is straightforward.

Definition 12.2.11. Let $X_1, \ldots, X_m$ be the irreducible components of a variety X. Then we define:

$$\dim(X) = \max\{\dim(X_i)\}.$$

Theorem 12.2.12. *Let $X \subset \mathbb{P}^N$ be an irreducible variety and let $g \in \mathbb{C}[y_0, \ldots, y_N]$ be a homogeneous polynomial which does not belong to the ideal of X. Then* $\dim(X \cap V(g)) = \dim(X) - 1$.

Proof. We prove the theorem in several steps.
First if $n = 0$ then X is a point and $X \cap V(g) = \emptyset$, so the claim follows.

□

If the resultant of f,g is trivial, then one divides the other.

Dim n-1 only for hypersurfaces.

Definition of inductivity.

12.3 Fibers of maps

Chapter 13

Dimension Theory

Chapter 14

Secant varieties

Chapter 15

Groebner basis