Probabilistic couplings for cryptography and privacy

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Relational properties

Properties about two runs of the same program

- Assume inputs are related by Ψ
- Want to prove the outputs are related by Φ

Examples

Monotonicity

- $\Psi : in_1 \leq in_2$
- $\Phi : out_1 \leq out_2$
- "Bigger inputs give bigger outputs"

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Non-interference

- $\Psi : lowinp_1 = lowinp_2$
- $\Phi : lowout_1 = lowout_2$
- "If low inputs are equal, then low outputs are equal"

Probabilistic relational properties

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Richer properties

- Indistinguishability, differential privacy

Probabilistic couplings

- Used by mathematicians for proving relational properties
- Applications: Markov chains, probabilistic processes

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- Place two processes in the same probability space
- Coordinate the sampling

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Why is this interesting?

- Proving relational probabilistic properties reduced to proving non-relational non-probabilistic properties
- Compositional

Introducing probabilistic couplings

Basic ingredients

- Given: two distributions X_1, X_2 over set A
- Produce: joint distribution Y over $A \times A$
  - Projection over the first component is X_1
  - Projection over the second component is X_2

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Definition

Given two distributions X_1, X_2 over a set A, a coupling Y is a distribution over $A \times A$ such that $\pi_1(Y) = X_1$ and $\pi_2(Y) = X_2$

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$$\pi_1(Y)(a_1) = \sum_{a_2} Y(a_1, a_2)$$

Fair coin toss

- One way to coordinate: require $x_1 = x_2$
- A different way: require $x_1 = \neg x_2$
- Yet another way: product distribution
- Choice of coupling depends on application
- Couplings always exist

Couplings vs liftings

Let $\mu_1, \mu_2 \in \mathsf{Distr}(A)$, $\mu \in \mathsf{Distr}(A \times A)$ and $R \subseteq A \times A$. Then

$$\mu \blacktriangleleft_R \langle \mu_1 \ \& \ \mu_2 \rangle \triangleq \pi_1(\mu) = \mu_1 \wedge \pi_2(\mu) = \mu_2 \wedge \Pr_{y \leftarrow \mu}[y \in R] = 1$$

Different couplings yield liftings for different relations

Convergence of random walks

Simple random walk on integers

- Start at some position p
- Each step, flip coin $x \xleftarrow{\$} \mathit{flip}$
- Heads: $p \leftarrow p + 1$
- Tails: $p \leftarrow p - 1$

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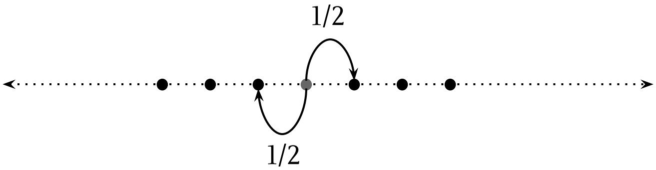

Coupling the walks to meet

Case $p_1 = p_2$: Walks have met

- Arrange samplings $x_1 = x_2$
- Continue to have $p_1 = p_2$

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Under coupling, if walks meet, they move together

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Memorylessness

Positions converge as we take more steps

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Theorem

If Y is a coupling of two distributions (X_1, X_2), then

$$\|X_1 - X_2\|_{TV} \triangleq \sum_{a \in A} |X_1(a) - X_2(a)| \leq \Pr_{(y_1, y_2) \sim Y}[y_1 \neq y_2].$$

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probabilistic Relational Hoare Logic

$\vdash \{P\}c_1 \sim c_2\{Q\}$ iff there exists μ such that

$$P(m_1 \uplus m_2) \Rightarrow \mu \blacktriangleleft_Q \langle [\![c_1]\!]\ m_1\ \&\ [\![c_2]\!]\ m_2 \rangle$$

where

$$\mu \blacktriangleleft_R \langle \mu_1\ \&\ \mu_2 \rangle \triangleq \pi_1(\mu) = \mu_1 \wedge \pi_2(\mu) = \mu_2 \wedge \mathrm{supp}(\mu) \subseteq R$$

Fundamental lemma of pRHL

If $Q \triangleq E_1 \Rightarrow E_2$ then $\Pr_{([\![c_1]\!]m_1)}[E_1] \leq \Pr_{([\![c_2]\!]m_2)}[E_2]$

Core rules

$$\frac{\{\Phi\}c_1 \sim c_2\{\Theta\} \quad \{\Theta\}c_1' \sim c_2'\{\Psi\}}{\{\Phi\}c_1; c_1' \sim c_2; c_2'\{\Psi\}}$$

$$\frac{\{\Phi \wedge b_1 \wedge b_2\}c_1 \sim c_2\{\Psi\} \quad \{\Phi \wedge \neg b_1 \wedge \neg b_2\}c_1' \sim c_2'\{\Psi\}}{\{\Phi \wedge b_1 = b_2\}\text{if } b_1 \text{ then } c_1 \text{ else } c_1' \sim \text{if } b_2 \text{ then } c_2 \text{ else } c_2'\{\Psi\}}$$

$$\frac{\{\Phi \wedge b_1 \wedge b_2\}c_1 \sim c_2\{\Phi \wedge b_1 = b_2\}}{\{\Phi \wedge b_1 = b_2\}\text{while } b_1 \text{ do } c_1 \sim \text{while } b_2 \text{ do } c_2\{\Phi \wedge \neg b_1 \wedge \neg b_2\}}$$

Loops

- Benton: same number of iterations
- EasyCrypt ($\leq$ 2015): one-sided rules
- EasyCrypt (2016): asynchronous loop rule
  $\Longrightarrow$ relatively complete, subsumes 1-sided rules

$$\frac{\begin{array}{c} \Psi \implies p_0 \oplus p_1 \oplus p_2 \\ \Psi \wedge p_0 \implies e_1 \wedge e_2 \qquad \Psi \wedge p_1 \implies e_1 \qquad \Psi \wedge p_2 \implies e_2 \\ \text{while } e_1 \wedge p_1 \text{ do } c_1 \Downarrow \text{while } e_2 \wedge p_2 \text{ do } c_2 \\ \{\Psi \wedge p_1\} c_1 \sim \text{skip} \{\Psi\} \qquad \{\Psi \wedge p_2\} \text{skip} \sim c_2 \{\Psi\} \\ \{\Psi \wedge p_0\} c_1 \sim c_2 \{\Psi\} \end{array}}{\{\Psi\} \text{while } e_1 \text{ do } c_1 \sim \text{while } e_2 \text{ do } c_2 \{\Psi \wedge \neg e_1 \wedge \neg e_2\}}$$

Example

$x \leftarrow 0; i \leftarrow 0; \text{while } i \leq N \text{ do } (x \mathrel{+}= i; i{+}{+})$

$y \leftarrow 0; j \leftarrow 1; \text{while } j \leq N \text{ do } (y \mathrel{+}= j; j{+}{+})$

Rule for random assignment

$$\frac{\mu \blacktriangleleft_Q \langle \mu_1 \,\&\, \mu_2 \rangle}{\vdash \{\top\} x_1 \xleftarrow{\$} \mu_1 \sim x_2 \xleftarrow{\$} \mu_2 \{Q\}}$$

Specialized rule

$$\frac{f \in T \xrightarrow{1-1} T \qquad \forall v \in T.\, d_1(v) = d_2(f\ v)}{\vdash \{\forall v, Q[v/x_1, f\ v/x_2]\} x_1 \xleftarrow{\$} \mu_1 \sim x_2 \xleftarrow{\$} \mu_2 \{Q\}}$$

Notes

- Bijection f: specifies how to coordinate the samples
- Side condition: marginals are preserved under f
- Assume: samples coupled when proving postcondition Φ

Proofs as (products) programs: xpRHL

- Every pRHL derivation yields a product program
- Different derivations yield different programs
- Can be modelled by a proof system

$$\vdash \{\Phi\} c_1 \sim c_2 \{\Psi\} \rightsquigarrow c$$

Fundamental lemma of xpRHL

- $\vdash \{\Phi\} c_1 \sim c_2 \{\Psi \implies x_1 = x_2\} \rightsquigarrow c$
- $\{\Box\Phi\}\ c\ \{\Pr[\neg\Psi] \leq \epsilon\}$

implies

$$m_1\ \Phi\ m_2 \Rightarrow \left|\Pr_{(\llbracket c_1 \rrbracket\ m_1)}[E(x_1)] - \Pr_{(\llbracket c_2 \rrbracket\ m_2)}[E(x_2)]\right| \leq \epsilon$$

Dynkin's card trick (shift coupling)

$p \leftarrow s$; $l \leftarrow [p]$;
while $p < N$ **do**
  $n \xleftarrow{\$} [1, 10]$;
  $p \leftarrow p + n$;
  $l \leftarrow p :: l$;
return p

$p_1 \leftarrow s_1$; $p_2 \leftarrow s_2$;
$l_1 \leftarrow [p_1]$; $l_2 \leftarrow [p_2]$;
while $n_1 < N \vee n_2 < N$ **do**
  if $p_1 = p_2$ **then**
    $n \xleftarrow{\$} ([1, 10])$;
    $p_1 \leftarrow p_1 + n$; $p_2 \leftarrow p_2 + n$;
    $l_1 \leftarrow p_1 :: l_1$; $l_2 \leftarrow p_2 :: l_2$;
  else
    if $p_1 < p_2$ **then**
      $n_1 \xleftarrow{\$} [1, 10]$;
      $p_1 \leftarrow p_1 + n_1$;
      $l_1 \leftarrow p_1 :: l_1$;
    else
      $n_2 \xleftarrow{\$} [1, 10]$;
      $p_2 \leftarrow p_2 + n_2$;
      $l_2 \leftarrow p_2 :: l_2$;
return (p_1, p_2)

Convergence

If $s_1, s_2 \in [1, 10]$, and $N > 10$, then $\Delta(p_1^{\text{final}}, p_2^{\text{final}}) \leq (9/10)^{N/5-2}$

Applications to cryptography

Experiment G_1

- Cryptosystem
- Adversary $\mathcal{A}$
- Winning condition E

Experiment G_2

- Hardness assumption
- Adversary $\mathcal{B}$
- Winning condition F

For all adversary $\mathcal{A}$, there exists adversary $\mathcal{B}$ s.t. $t_{\mathcal{A}} \approx t_{\mathcal{B}}$ and

$$\Pr_{G_1}[E] \leq q \cdot \Pr_{G_2}[F] + \delta$$

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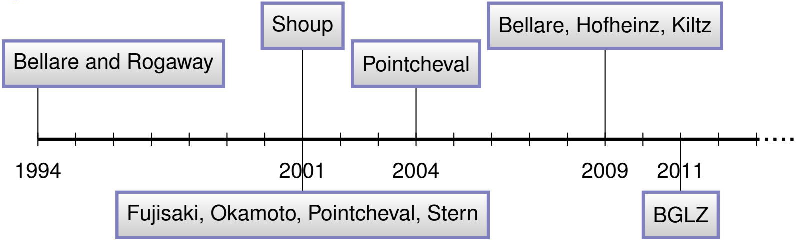
For all adversary $\mathcal{A}$, there exists adversary $\mathcal{B}$ s.t. $t_{\mathcal{A}} \approx t_{\mathcal{B}}$ and

- $\vdash \{\top\} G_1 \sim G_2 \{E \Rightarrow (F' \vee F_{bad})\}$
- $\Pr_{G_2}[F'] \leq q \cdot \Pr_{G_2}[F]$ and $\Pr_{G_2}[F_{bad}] \leq \delta$

Formalizing cryptographic proofs?

- *In our opinion, many proofs in cryptography have become essentially unverifiable. Our field may be approaching a crisis of rigor*. Bellare and Rogaway, 2004-2006
- *Do we have a problem with cryptographic proofs? Yes, we do [...] We generate more proofs than we carefully verify (and as a consequence some of our published proofs are incorrect)*. Halevi, 2005

OAEP



Provable security of OAEP

Game INDCCA($\mathcal{A}$) :	Encryption	Game sPDOW($\mathcal{I}$)
$(sk, pk) \leftarrow \mathcal{K}(\,)$;	$\mathcal{E}_{\mathrm{OAEP}(pk)}(m)$:	$(sk, pk) \leftarrow \mathcal{K}()$;
$(m_0, m_1) \leftarrow \mathcal{A}_1^{\mathcal{G},\mathcal{H},\mathcal{D}}(pk)$;	$r \stackrel{\$}{\leftarrow} \{0,1\}^{k_0}$;	$y_0 \stackrel{\$}{\leftarrow} \{0,1\}^{n_0}$;
$b \stackrel{\$}{\leftarrow} \{0,1\}$;	$s \leftarrow G(r) \oplus (m \parallel 0^{k_1})$;	$y_1 \stackrel{\$}{\leftarrow} \{0,1\}^{n_1}$;
$c^\star \leftarrow \mathcal{E}_{pk}(m_b)$;	$t \leftarrow H(s) \oplus r$;	$x^\star \leftarrow \mathrm{f}_{pk}(y_0 \parallel y_1)$;
$\overline{b} \leftarrow \mathcal{A}_2^{\mathcal{G},\mathcal{H},\mathcal{D}}(c^\star)$;	return $\mathrm{f}_{pk}(s \parallel t)$	$\overline{Y} \leftarrow \mathcal{I}(x^\star)$;
return $(\overline{b} = b)$	**Decryption** ...	return $(y_0 \in \overline{Y})$

FOR ALL IND-CCA adversary $\mathcal{A}$ against $(\mathcal{K}, \mathcal{E}_{\mathrm{OAEP}}, \mathcal{D}_{\mathrm{OAEP}})$,
THERE EXISTS a sPDOW adversary $\mathcal{I}$ against $(\mathcal{K}, \mathrm{f}, \mathrm{f}^{-1})$ st

$$\left|\Pr_{\text{IND-CCA}(\mathcal{A})}\left[\overline{b} = b\right] - \frac{1}{2}\right| \leq \Pr_{\text{PDOW}(\mathcal{I})}\left[y_0 \in \overline{Y}\right] + \frac{3q_D q_G + q_D^2 + 4q_D + q_G}{2^{k_0}} + \frac{2q_D}{2^{k_1}}$$

and

$$t_{\mathcal{I}} \leq t_{\mathcal{A}} + q_D\, q_G\, q_H\, T_{\mathrm{f}}$$

The code-based game-playing approach

- Everything is a probabilistic program
- Decompose the proof in sequence of transitions
- Prove each transition using pRHL
- Bound prob. of events w/ non-relational logic

Typical couplings

- Bridging step: $\mu_1 \; =^{\#} \mu_2$, then for every event X,

$$\Pr_{z \leftarrow \mu_1}[X] = \Pr_{z \leftarrow \mu_2}[X]$$

- Failure Event: If $x \; R \; y$ iff $F(x) \Rightarrow x = y$ and $F(x) \Leftrightarrow F(y)$, then for every event X,

$$|\Pr_{z \leftarrow \mu_1}[X] - \Pr_{z \leftarrow \mu_2}[X]| \leq \max\left(\Pr_{z \leftarrow \mu_1}[\neg F], \Pr_{z \leftarrow \mu_2}[\neg F]\right)$$

- Reduction: If $x \; R \; y$ iff $F(x) \Rightarrow G(y)$, then

$$\Pr_{x \leftarrow \mu_2}[G] \leq \Pr_{y \leftarrow \mu_1}[F]$$

EasyCrypt

- Interactive proof assistant
  - backend to SMT solvers, CAS, etc.
  - encryption, signatures, hash designs, key exchange protocols, zero knowledge protocols, garbled circuits. . .
  - SHA3, e-voting
- Back-end for automated tools
- Front-end for certified compilers

approximate probabilistic Relational Hoare Logic

- Quantitative generalization of pRHL $\vdash_{\epsilon,\delta} \{P\} c_1 \sim c_2 \{Q\}$
- Valid if there exists μ_L, μ_R such that

$$P(m_1 \uplus m_2) \implies \mu_L, \mu_R \blacktriangleleft_Q^{\epsilon,\delta} \langle [\![c_1]\!]\ m_1\ \&\ [\![c_2]\!]\ m_2 \rangle$$

where

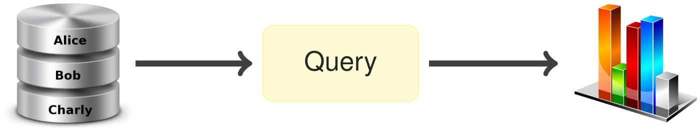
$$\mu_L, \mu_R \blacktriangleleft_Q^{\epsilon,\delta} \langle \mu_1\ \&\ \mu_2 \rangle \triangleq \begin{cases} \pi_1(\mu_L) = \mu_1 \wedge \pi_2(\mu_R) = \mu_2 \\ \text{supp}(\mu_L), \text{supp}(\mu_R) \subseteq Q \\ \Delta_\epsilon(\mu_1, \mu_2) \leq \delta \end{cases}$$

- Fundamental theorem of apRHL: if $Q \triangleq E_1 \Rightarrow E_2$ then

$$\Pr_{([\![c_1]\!]\ m_1)}[E_1] \leq \exp(\epsilon) \Pr_{([\![c_2]\!]\ m_2)}[E_2] + \delta$$

- Extends to f-divergences

Application: differential privacy



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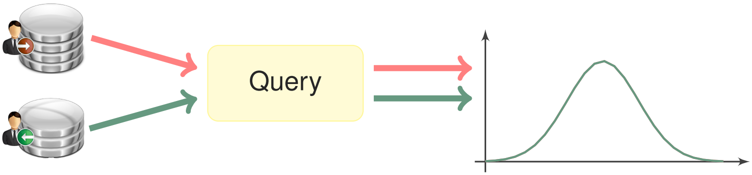

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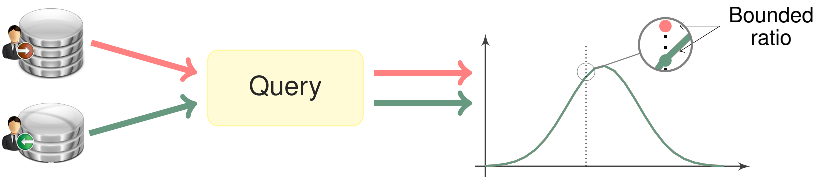

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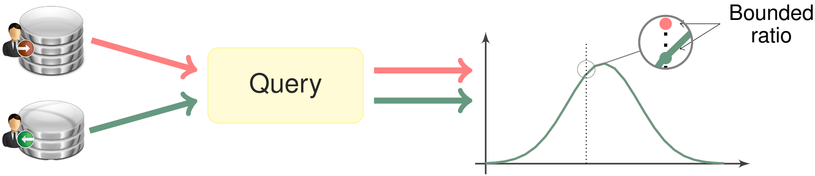

A randomized algorithm $\mathcal{K}$ is (ϵ, δ)-differentially private w.r.t. Φ iff for all databases D_1 and D_2 s.t. $\Phi(D_1, D_2)$

$$\forall S.\ \Pr[\mathcal{K}(D_1) \in S] \leq \exp(\epsilon) \cdot \Pr[\mathcal{K}(D_2) \in S] + \delta$$

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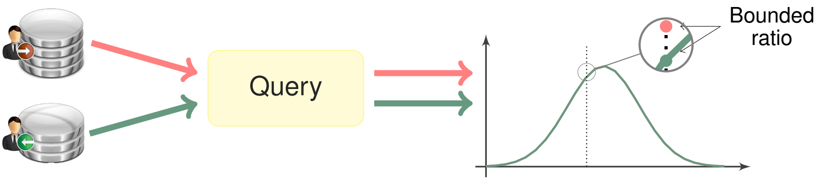

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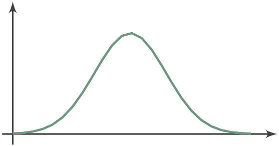
Privacy as approximate couplings

$\mathcal{K}$ is (ϵ, δ)-differentially private wrt Φ iff $\vdash_{\epsilon,\delta} \{\Phi\} \mathcal{K}_1 \sim \mathcal{K}_2 \{\equiv\}$

Differential privacy via output perturbation

Let f be k-sensitive w.r.t. Φ:

$$\Phi(a, a') \implies |f\ a - f\ a'| \leq k$$



Then $a \mapsto \mathrm{Lap}_\epsilon(f(a))$ is $(k \cdot \epsilon, 0)$-differentially private w.r.t. Φ

Proof principles for Laplace mechanism

Making different things look equal

$$\frac{\Phi \triangleq |e_1 - e_2| \leq k'}{\vdash_{k'\cdot\epsilon,0} \{\Phi\} y_1 \xleftarrow{\$} \mathcal{L}_\epsilon(e_1) \sim y_2 \xleftarrow{\$} \mathcal{L}_\epsilon(e_2) \{y_1 = y_2\}}$$

Making equal things look different

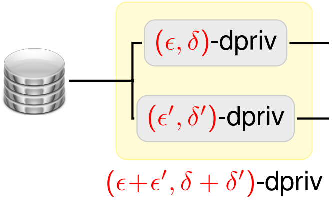
$$\frac{\Phi \triangleq e_1 = e_2}{\vdash_{k\cdot\epsilon,0} \{\Phi\} y_1 \xleftarrow{\$} \mathcal{L}_\epsilon(e_1) \sim y_2 \xleftarrow{\$} \mathcal{L}_\epsilon(e_2) \{y_1 + k = y_2\}}$$

Pointwise equality

$$\frac{\forall i.\ \vdash_{\epsilon,0} \{\Phi\} c_1 \sim c_2 \{x_1 = i \Rightarrow x_2 = i\}}{\vdash_{\epsilon,0} \{\Phi\} c_1 \sim c_2 \{x_1 = x_2\}}$$

Differential privacy by sequential composition

- If $\mathcal{K}$ is (ϵ, δ)-differentially private, and
- $\lambda a.\ \mathcal{K}'(a, b)$ is (ϵ', δ')-differentially private for every $b \in B$,
- then $\lambda a.\ \mathcal{K}'(a, \mathcal{K}(a))$ is $(\epsilon + \epsilon', \delta + \delta')$-differentially private



Beyond composition: Sparse Vector Technique

$\mathsf{SparseVector}_{bt}(a, b, M, N, d) :=$
$i \leftarrow 0; l \leftarrow []; u \xleftarrow{\$} \mathcal{L}_\epsilon(0); A \leftarrow a - u; B \leftarrow b + u;$
while $i < N$ do
  $i \leftarrow i + 1; q \leftarrow \mathcal{A}(l); S \xleftarrow{\$} \mathcal{L}_\epsilon(q(d));$
  if $(A \le S \le B \wedge |l| < M)$ then $l \leftarrow i :: l;$
return l

Privacy

If queries are 1-sensitive, then $(\sqrt{M}\epsilon, \delta')$-diff. private

Tools

- advanced composition
- accuracy-dependent privacy
- optimal subset coupling

Perspectives and further directions

Language-based techniques

- for provable security and differential privacy
- based on probabilistic couplings

Open questions

- semantical foundations of approximate couplings
- applications to security (complexity of attacks)