

## ***Comparing Proportional Representation Electoral Systems: Quotas, Thresholds, Paradoxes and Majorities***

MICHAEL GALLAGHER\*

The relationship between electoral systems can be examined on a number of dimensions. Seat allocation methods are conveniently divided into two groups: those based on largest remainders and those based on highest averages. The single transferable vote has its own distinct characteristics. Focusing on certain elements – the quota, thresholds, paradoxes and the conditions under which a majority of seats can be won – enables comparisons to be drawn between seat allocation methods. Certain seat allocation methods conventionally seen as variants of proportional representation (PR) cannot be regarded as such. PR methods can be rank ordered according to whether, when complete proportionality is not attainable, they display electoral bias towards larger or smaller parties. However, a definitive ordering is elusive, since some methods that are generally more favourable to larger parties can in some circumstances set lower thresholds of representation than methods generally favourable to smaller parties.

Electoral systems can be compared on the basis of many criteria, such as their proportionality or the characteristics of political systems (two-party or multi-party systems, stable government and so on) with which they seem to be associated. This article takes a slightly more abstract approach and seeks instead to establish an ordering of seat allocation methods. It will rank the formulae in order of the degree to which, when some disproportionality is unavoidable, they tend to be favourable or unfavourable to large parties, comparing the methods on the basis of the quotas they use and the thresholds they entail. It will also compare the methods in terms of their vulnerability to paradox and the circumstances under which parties can win or be deprived of a majority of parliamentary seats. It will look at eleven seat allocation methods, including all of those employed in Western Europe. These consist of three largest remainders methods (using the Hare, Droop and Imperiali quotas), seven highest averages methods (d'Hondt, Sainte-Laguë, modified Sainte-Laguë, Imperiali, the Danish method, equal proportions and smallest divisors/Adams), and the single transferable vote. An overview of the eleven methods is presented in Figure 1. We shall start by illustrating the methods, thereby establishing the basis for comparison.

\* Department of Political Science, Trinity College, University of Dublin. I am grateful to Arend Lijphart, Ivor Crewe and the anonymous referees for comments on an earlier draft. The usual disclaimer applies.

<i>Largest remainders method</i>	<i>Variants</i>	<i>Quota</i>	<i>Countries employing</i>
Award a seat to each party for every quota in its total of seats; to fill any unfilled seats, reward the largest remainders (see Table 1)	LR-Hare  LR-Droop LR-Imperiali	$v/s$  $v/(s+1)$ $v/(s+2)$	Austria (lower), Belgium (lower), Denmark (higher), Germany (higher), Italy (higher)  Greece (lower) Italy (lower)
<i>Highest averages method</i>	<i>Variants</i>	<i>nth divisor</i>	<i>Sequence (first five divisors)</i>
Award seats sequentially to the parties according to the 'average' each presents for the next seat. Each party's vote total is divided by the <i>n</i> th divisor from a prescribed sequence, where ( <i>n</i> - 1) is the number of seats it has already won (see Table 2)	Imperiali d'Hondt  Modified Sainte-Laguë Sainte-Laguë Equal proportions Danish Adams	$(n+1)/2$ <i>n</i>  $(10n-5)/7^*$ $2n-1$ $\sqrt{n(n-1)}$ $3n-2$ <i>n</i> - 1	1, 1.5, 2, 2.5, 3 1, 2, 3, 4, 5  1, 2, 14, 3, 57, 5, 6.43 1, 3, 5, 7, 9 0, 1.41, 2.45, 3.46, 4.47 1, 4, 7, 10, 13 0, 1, 2, 3, 4
<i>Single transferable vote method</i>	<i>Variants</i>	<i>Sequence (first five divisors)</i>	<i>Countries employing</i>
Candidates with Droop quota are elected. Unfilled seats are filled by transferring surplus votes from elected candidates and by transferring votes from low-placed candidates			Belgium (municipal elections) Austria (higher), Belgium (higher), Finland, France (1986), Iceland (higher), Israel, Luxembourg, Netherlands, Portugal, Spain, Switzerland Denmark (lower), Norway, Sweden Denmark (higher from 1945 to 1953) USA (for allocating Representatives to states) Denmark (for awarding seats within parties) —
<i>Single transferable vote method</i>	<i>Variants</i>	<i>Sequence (first five divisors)</i>	<i>Countries employing</i>
Candidates with Droop quota are elected. Unfilled seats are filled by transferring surplus votes from elected candidates and by transferring votes from low-placed candidates			Ireland, Malta

Fig. 1. Eleven seat allocation methods

Note: In many countries, different formulae are used for awarding seats at constituency level and at national/regional level, designated here by lower and higher tier respectively. \* If  $n = 1$ , then  $n$ th divisor is 1.  $v =$  votes.  $s =$  seats.

Source for last column: Michael Gallagher, Michael Laver and Peter Mair, *Representative Government in Western Europe* (New York and London: McGraw-Hill, 1992), p. 156; Thomas T. Mackie and Richard Rose, *The International Almanac of Electoral History*, 3rd edn (London: Macmillan, 1991).

## THREE TYPES OF SEAT ALLOCATION METHOD

*Largest Remainders Methods*

The operation of the largest remainders (LR) method entails the calculation of a quota based on the number of seats at stake and the number of votes cast. Each party is awarded as many seats as it has full quotas. If this leaves some seats unallocated, each party's 'remainder' is calculated by deducting from its vote total the number of votes it has already used up by winning seats. The unallocated seats are then awarded to the parties that present the largest remainders. The different methods vary in the quotas they use. One variant (henceforth LR-Hare) employs the Hare (or 'natural') quota, which equals the number of votes divided by the number of seats. A second (LR-Droop) uses the Droop or Hagenbach-Bischoff quota, which involves dividing the number of votes not by the number of seats but by the number of seats plus one (the resulting value is sometimes rounded up to the next integer). The third (LR-Imperiali) divides the number of votes by the number of seats plus two. The general method is illustrated in Table 1. Once party *A* has used up three quotas of votes and party *B* one quota, it is party *C* that has the largest remainder and therefore wins the last seat. If LR-Hare was used in this case, the quota would be 20,000 votes (100,000 divided by 5), but the outcome would be the same, since *C*'s remainder of 12,000 would exceed the only other remainder, *B*'s 8,000 votes.

TABLE 1 *Allocation of Seats by Largest Remainder Method with the Droop Quota*

Party	Votes	Full quotas (first stage seats)	Remainder	Further seats	Total seats
<i>A</i>	60,000	3	10,000	0	3
<i>B</i>	28,000	1	11,333	0	1
<i>C</i>	12,000	0	12,000	1	1
Total	100,000	4	33,333	1	5

Note: There are 100,000 votes and 5 seats, so the Droop quota is  $100,000/6$ , that is, 16,667.

If the LR-Imperiali method was used, the quota would be 14,286 votes (100,000 divided by 7), and all five seats would be allocated without the need to consider remainders, since *A* has over four full quotas and *B* over one quota – this time *C* would lose out. The LR-Imperiali method is not always this straightforward, since it can happen, especially if there are few parties, that the number of full quotas exceeds the number of seats. For example, if in the above example there were only three seats to be awarded, the Imperiali quota would be 20,000 (100,000 divided by 5), and so party *A* would have exactly three quotas and *B* 1.4 quotas. In Italy, where the Imperiali quota

is used to allocate the constituency seats, this problem occasionally occurs, and in such cases the operation is replaced by the Droop quota.<sup>1</sup>

There is an infinite range of possible LR methods, varying in terms of the quota they employ – for example, one could use any of the quantities  $v/(s + 3)$ ,  $v/(s + 0.5)$ ,  $v/(s - 1)$ ,  $v/2s$ , or whatever. Outside a certain range (which is determined for each case by the distribution of votes among the parties and the number of seats at stake), however, quotas will not work as the basis for seat allocation methods, because they award either too many or too few seats. In the case illustrated in Table 1, a quota of less than 14,000 would ‘award’ six seats initially, while a quota of more than 30,000 would award only one initially, so that even if each of the three parties’ remainders is rewarded, only four seats are allocated. The Droop quota is the lowest value that guarantees that the number of full quotas possessed by the various parties cannot be larger than the number of seats. The highest useful value cannot be so simply defined; it is, in fact, equal to the smallest divisors (Adams) quota, which we discuss later.

We have seen that LR-Imperiali employs the lowest quota and yet, in the case illustrated in Table 1, it is the most generous to the largest party, awarding four seats to party *A* while LR-Hare and LR-Droop give it only three seats. At first thought it might seem counter-intuitive that a high quota helps smaller parties and a low quota larger parties.<sup>2</sup> Surely a low quota is easier to attain and is thus preferable from the viewpoint of smaller parties?<sup>3</sup> In fact a low quota helps larger parties, because it makes it more likely that all the seats will be awarded at the first stage and no remainders will be rewarded. Smaller parties will thus be disadvantaged. If the quota is increased, the remainder of a party whose vote total is less than the quota is not reduced at all, whereas the remainders of larger parties are reduced. Moreover, the larger the party, the greater the decrease in its remainder. Larger parties have to use up more of their votes for every seat they win.

This can be illustrated by taking an example where four seats are to be awarded and 40,000 votes are cast, with party *A* winning 33,000 votes and party *B* 7,000. Using the Droop quota (8,001) or the d’Hondt quota (8,250), party *A* receives all four seats, as it has four quotas. But if the Hare quota (10,000) were employed, party *A* would have to use up 30,000 of its votes just to earn three seats, and its remainder of 3,000 would be well below *B*’s remainder of 7,000. When the quota is raised by 2,000 votes, it is true that party *B* falls 2,000 votes further adrift of its target of one quota, but in relative terms it still benefits, because party *A* falls 8,000 votes further adrift of its

<sup>1</sup> Mario Amoroso, ‘Italy’, in Geoffrey Hand, Jacques Georgel and Christoph Sasse, eds, *European Electoral Systems Handbook* (London: Butterworths, 1979), pp. 140–69, at p. 158.

<sup>2</sup> The terms ‘smaller’ and ‘larger’ are entirely relative. For a more formal demonstration that larger parties benefit from a low quota, see Appendix.

<sup>3</sup> For example, Douglas Rae, *The Political Consequences of Electoral Laws*, revised edn (New Haven and London: Yale University Press, 1971), p. 34, says that, compared with LR-Hare, LR-Imperiali lowers the price of the first seat, thereby ‘helping weak parties’.

target of four quotas. The larger the party, the more it suffers from an increase in the quota.

### *Highest Average Methods*

Highest average methods operate, at least at first sight, on a different principle from that of largest remainders methods. Under highest average methods, each party competes for each seat in sequence as if at an auction. The bid it makes for the seat is determined by its original number of votes and by how many seats it has already won – its bid is reduced each time it receives a seat, as its original vote total is divided by progressively larger numbers. The variation between the methods lies in the sequence of number employed as divisors. The basic idea is illustrated in Table 2, for the highest average method devised in 1910 by the French mathematician Sainte-Laguë.<sup>4</sup> This employs the sequence 1, 3, 5, 7, etc. Party *A*'s bid of 60,000 votes for the first seat is obviously the highest, so it receives this seat. Its 'average' is then reduced by dividing its vote total by the second divisor in the Sainte-Laguë sequence, 3. Consequently, it can bid only 20,000 votes for the second seat, and since *B* can make a better offer, *B* receives it. Party *A* receives the third seat, and with *A* and *C* each bidding 12,000 votes for the fourth seat, some tie-breaking mechanism would need to be invoked if there were only four seats at stake.

TABLE 2 *Allocation of Seats by Sainte-Laguë Highest Average Method*

Party	Votes	Votes divided by first divisor (1)	Votes divided by second divisor (3)	Votes divided by third divisor (5)	Total seats
<i>A</i>	60,000	60,000(1)	20,000(3)	12,000(4 = )	3
<i>B</i>	28,000	28,000(2)	9,333		1
<i>C</i>	12,000	12,000(4 =)			1
Total	100,000				5

*Note:* The numbers in brackets after the parties' vote totals indicate the award of a seat; thus party *A* is awarded the first seat, party *B* the second, and so on, with a tie for the fourth seat.

The seven methods we are considering use different sequences of divisors, which are set out in Figure 1. They are ranged in ascending order of the rapidity with which the value of successive divisors increases. Some sequences are often expressed in a different form, with a first divisor other than 1. For example, the Imperiali sequence may be expressed as 2, 3, 4, 5, etc., and the modified Sainte-Laguë sequence as 1.4, 3, 5, 7, etc. Dividing each term by the first number in such a sequence, so that it is converted to a sequence beginning with 1,

<sup>4</sup> For an English translation of his article, see Arend Lijphart and Robert W. Gibberd, 'Thresholds and Payoffs in List Systems of Proportional Representation', *European Journal of Political Research*, 5 (1977), 219–44, pp. 241–2.

does not alter the effect of the sequence, and it facilitates comparisons between different methods. This is not possible, of course, when the first divisor is 0, as is the case with two of the highest average methods we discuss. One of these is the equal proportions method of Hill and Huntington, which was endorsed by committees of the American National Academy of Sciences in 1929 and 1948 as the fairest way of allocating seats to states and is used as a baseline of fairness by Carstairs: it uses the sequence  $\sqrt{0*1}$ ,  $\sqrt{1*2}$ ,  $\sqrt{2*3}$ , etc.<sup>5</sup> The other is the method of smallest divisors devised by John Quincy Adams in 1832 and often named after him, whose divisor sequence is 0, 1, 2, 3, etc.<sup>6</sup> Although the Adams method has never been used for allocating seats, it is important as a delimiting case, as we shall see.

An alternative way of presenting these sequences in a common format would be to express them as  $d$ ,  $d + 1$ ,  $d + 2$ , etc., which can be done for all except equal proportions, where the increase in divisors is not uniform. Where the difference between consecutive divisors is uniform, the method can be a proportional representation method only if  $d$  has a value from 0 to 1 inclusive. When  $d = 0$ , we have the Adams method;  $d = 1/3$  gives the Danish method;  $d = 1/2$  gives Sainte-Laguë;  $d = 1$  gives d'Hondt. In the Imperiali highest averages method,  $d = 2$ , and so this is not a PR method, as we show later.

Of these seven methods, each except for Adams is or has been used as an allocation method, but only two methods, d'Hondt and modified Sainte-Laguë, are used in Europe at general elections to award seats to parties in accordance with their votes (see Figure 1).<sup>7</sup> Modified Sainte-Laguë differs from pure Sainte-Laguë only from the perspective of parties that have yet to win a seat, which find it harder to achieve this than would be the case under the pure version; once each party has won a seat, the pure and modified Sainte-Laguë methods are identical. Needless to say, the list of potential highest average methods is infinite, since we could pick any rule we like to generate the sequence of divisors. For example, we could devise a method that was midway between d'Hondt and Sainte-Laguë by taking the divisors 1, 2.5, 4, 5.5, etc.

Clearly, the more rapidly the divisors increase, the more rapidly the averages presented by a large party decrease to the level of the first average of a small party, and this factor determines how small parties fare in comparison with

<sup>5</sup> Andrew McLaren Carstairs, *A Short History of the Electoral Systems in Western Europe* (London: George Allen and Unwin, 1980), pp. 26–9, 42.

<sup>6</sup> Michel L. Balinski and H. Peyton Young, *Fair Representation: Meeting the Ideal of One Man, One Vote* (New Haven and London: Yale University Press, 1982), pp. 26–8.

<sup>7</sup> For equal proportions, see Edward V. Huntington, 'A New Method of Apportionment of Representatives', *Quarterly Publication of the American Statistical Association*, 17 (1921), 859–70. For the circumstances in which the Danish method is used, see Lars Nørby Johansen, 'Denmark', in Hand, Georgel and Sasse, *European Electoral Systems Handbook*, pp. 29–57, p. 46. Pure Sainte-Laguë was used at the first four post-war Danish elections – I owe this information to Arend Lijphart. The only reference of which I am aware to the employment of Imperiali highest averages is to be found in G. Van Den Bergh, *Unity in Diversity: A Systematic Critical Analysis of all Electoral Systems* (London: B. T. Batsford, 1956), pp. 25, 96, where it is stated that the method is used for municipal elections in Belgium.

large ones. Once a party has won three seats, it presents an average under Imperiali that is still 40 per cent of its vote total, while under the Danish method its bid is only a tenth of its original vote. However, as between two parties that have each already won three seats, the two methods are not so different, even though the respective divisors (2.5 and 10) might seem far apart. As Carstairs points out, what is important is not the numerical value of each divisor but the ratio between pairs of divisors.<sup>8</sup> If we compare the ratios between adjacent divisors for all methods, there is a steady convergence. The ratio between the fourth and the third divisors in the case of Imperiali is 1.25, while for the Danish method it is 1.43. The difference between the ratios of the tenth to the ninth divisors shrinks almost to the point of invisibility: it is 1.10 for Imperiali and 1.12 for the Danish method. The ratio of the 1,000th to the 999th divisor is exactly 1.001 in the case of Imperiali and 1.001002 for the Danish method. One practical consequence of this is that as district magnitude (the number of seats per constituency) approaches infinity, the seat allocations produced by all highest average methods become identical (and highly proportional).

The equal proportions and Adams methods appear to be deviant cases, because they begin with a divisor of 0. Clearly, this is extremely favourable to small parties; it means that every party presents an initial average that is infinitely large, and so each party that has won at least one vote will qualify for a seat before even the largest party qualifies for two seats. These two methods were devised to meet the American problem of deciding how Representatives should be allocated to states rather than the question of how seats should be shared among parties, and since each state is automatically entitled to one Representative, an initial divisor of 0 creates no difficulties. Neither method could be used to award seats to parties, in the absence of a threshold, unless the first divisor were replaced by a positive number, whose value would determine how it compared with the other methods. For example, if the first divisor in the Adams sequence were changed to 0.5, then Adams would become identical to d'Hondt as between a party that had won one seat and a party that had won none. These methods are not irrelevant to solving the European problem of awarding seats to parties, for they could well be employed in situations where thresholds debar small parties. For example, at Swedish elections the thirty-nine supplementary seats are used to bring about a proportional allocation of the total number of 349 seats among those parties that win at least 4 per cent of the national votes. Hence, even the smallest party qualifying for a share of these seats must have won at least 4 per cent of 349 (that is, 13.96) Hare quotas. The same would apply in other countries where the seats are shared only among those parties that have attained a threshold so high that qualifying parties must almost by definition be entitled to at least one seat: examples include Austria, Denmark, Germany, Iceland, the Netherlands

<sup>8</sup> Carstairs, *Short History of Electoral Systems*, p. 29.

and Norway. In these countries, equal proportions and Adams would be perfectly feasible allocation methods.

Although equal proportions and Adams stand out by having a first divisor of 0, in all other respects they can be placed in the ranking order straightforwardly enough. The equal proportions method, once each party has won about four seats, is virtually identical to (though very slightly more generous to small parties than) pure Sainte-Laguë. The ratio of fourth to third divisors is 1.414 for equal proportions and 1.40 for Sainte-Laguë; the ratio of tenth to ninth divisors is 1.1180 for equal proportions and 1.1176 for Sainte-Laguë; and for the ratio of the 1,000th to 999th divisors, the two methods are identical to eight decimal places. As for the Adams method, it clearly belongs on the bottom line of Figure 1. Its rationale is that it minimizes the under-representation – that is, it maximizes the seats:votes ratio – of the most under-represented party,<sup>9</sup> and as such it is the mirror image of d’Hondt, which minimizes the over-representation of the most over-represented party. It represents the extreme case; no formula that is more generous to small parties than Adams can plausibly claim to be a PR formula. It is easy to see that the ratio between successive divisors in the Adams sequence is as great as it possibly can be in any sequence in which one divisor is formed by adding a fixed sum to the previous divisor, which is the case for all the methods discussed here except equal proportions. For sequences of this type, we can call the first divisor  $d$  and the constant added to the previous divisor  $c$ , so the second divisor is  $d + c$ , the third is  $d + 2c$ , and so on. The ratio of the  $(n + 1)$ th divisor to the  $n$ th is thus

$$\frac{d + nc}{d + (n - 1)c}.$$

This takes a maximum value when  $d$  equals 0, as is the case with the Adams sequence; in all other cases, it will always be less than  $n/(n - 1)$ .

At the other end of the scale, d’Hondt displays the smallest ratios between successive divisors of any PR formula. Imperiali highest averages is a method of seat allocation, but this does not make it a PR formula. Any true PR method will always produce perfect proportionality (a complete correspondence between vote and seat shares) when this can be achieved. In the case illustrated in Tables 1 and 2, the smallest district magnitude for which perfect proportionality can be achieved is twenty-five seats. At this district magnitude, six of the seven highest average methods we are discussing duly produce a 15–7–3 outcome.<sup>10</sup> Imperiali, however, awards the seats on a 16–7–2 basis, because party  $A$ ’s 16th average (7,059) is larger than  $C$ ’s third average (6,000). Whereas the other methods all strive to minimize disproportionality, disagreeing only on

<sup>9</sup> Balinski and Young, *Fair Representation*, p. 104.

<sup>10</sup> So do LR-Hare and LR-Droop, but LR-Imperiali does not function effectively as its low quota of 3,704 votes would ‘allocate’ twenty-six seats initially. It too, therefore, cannot be classified as a PR method. We are excluding STV from this part of the discussion as it embodies a more sophisticated conception of ‘perfect proportionality’ by considering voters’ lower preferences.



exactly how disproportionality is best conceptualized,<sup>11</sup> Imperiali highest averages does not have the achievement of perfect proportionality as its main aim. In the words of Van Den Bergh, its aim is to obtain a disproportionate result by giving an advantage to the large parties.<sup>12</sup>

D'Hondt is at one end of the range of PR methods, and any method that errs more on the side of the larger parties is not genuinely proportional, in that it does not guarantee to produce perfect proportionality even when this is achievable. To see this, suppose there are just two parties, *A* and *B*, with votes  $v_A$  and  $v_B$ , and the ratio between their votes, after the highest common factor has been removed, is  $a:b$ . Then perfect proportionality can be achieved when the total number of seats is divisible by  $(a + b)$ . In these situations, the parties' last bids are the same: when there are  $(a + b)$  seats to be awarded  $v_A/a$  equals  $v_B/b$ , while in the general case, when there are  $n(a + b)$  seats to be awarded,  $v_A/na$  equals  $v_B/nb$ . Consequently, the two parties present identical averages or bids for the penultimate seat, and giving them each a further seat achieves perfect proportionality. However, party *A* would be over-represented at *B*'s expense if its  $(a + 1)$ th divisor were larger than *B*'s  $b$ th divisor, in which case, with  $(a + b)$  seats to be awarded, *A* would win at least  $(a + 1)$  seats and *B* at most  $(b - 1)$ . Under d'Hondt this cannot happen, but if we replace the d'Hondt sequence 1, 2, 3, etc. with the sequence 1,  $2 - f$ ,  $3 - 2f$ ,  $4 - 3f$ , etc., where  $f$  is a fraction lying between 0 and 1, then it becomes possible. In this case, *A* will qualify for an extra seat at *B*'s expense, in a situation where  $(a + b)$  seats are available and perfect proportionality is possible, if its  $(a + 1)$ th average is greater than *B*'s  $b$ th average – that is, if  $v_A/(a + 1 - af)$  is greater than  $v_B/(b - (b - 1)f)$ . This is the case if  $f$  is greater than

$$\frac{a + 1 - b\left(\frac{v_A}{v_B}\right)}{\frac{v_A}{v_B} - b\left(\frac{v_A}{v_B}\right) + a}$$

Since  $v_A/v_B$  equals  $a/b$ , this reduces to a requirement that  $f$  be greater than  $b/a$ .

For example, if party *A* wins 40,000 votes and party *B* 10,000 votes, then  $a = 4$  and  $b = 1$ , and perfect proportionality is possible if the number of seats is divisible by 5. With five seats at stake, *A* will win all five if its fifth average, of  $40,000/(5 - 4f)$ , exceeds *B*'s first average of 10,000. If the value  $f$  defined in the last paragraph has a value greater than  $1/4$ , then *A*'s fifth divisor is less than 4 and its fifth average is more than 10,000. Among sequences where the first divisor equals 1, only when  $f$  equals 0, as it does in the d'Hondt sequence, or has a negative value, as in all other sequences, can a highest average method be regarded as a PR system. All PR highest average methods

<sup>11</sup> Michael Gallagher, 'Proportionality, Disproportionality and Electoral Systems', *Electoral Studies*, 10 (1991), 33–51.

<sup>12</sup> Van Den Bergh, *Unity in Diversity*, p. 96.

lie within the range bounded at one end by d'Hondt and at the other end by Adams.

Comparing the ratios between divisors enables the seven highest average methods to be ranked as in Figure 1, the only contentious ranking being the placing of equal proportions, which must be ranked alongside Adams as far as the awarding of the first seat is concerned but which is thereafter very close to Sainte-Laguë. In order to compare these methods with the largest remainders methods we have already discussed, we have to look at highest average methods in terms of quotas.

### *Quotas and Highest Average Methods*

At first sight it might appear that largest remainders and highest average methods operate in completely different ways – the former employ a quota and the latter do not – and that comparisons are therefore impossible. However, it has already been observed<sup>13</sup> that highest average methods can also be viewed in terms of quotas, with the quota now depending to some extent on the distribution of votes between parties rather than being determined solely by the total numbers of votes and seats as is the case with largest remainders methods.

In practice, when seats are to be awarded to parties under a highest average formula, the calculations are usually done by using a quota rather than by laboriously calculating all the quotients as in Table 2. The method is simply to pick a plausible number and see how many seats it allocates. Under the d'Hondt formula, the aim is to find a number that will allocate exactly the right number of seats at the first stage. If the first number tried (usually the Droop quota) allocates too few, the d'Hondt formula is applied to the remainders.<sup>14</sup> In the case illustrated in Tables 1 and 2, any number between 14,001 and 15,999 inclusive would suffice.<sup>15</sup> Under pure Sainte-Laguë, the aim would be to find a value for the quota such that the number of full quotas, plus the number of remainders that equal or exceed half a quota, comes to exactly the number of seats to be awarded.

For our purposes, the quota used by the various formulae can be calculated from allocation tables of the sort illustrated in Table 2 above. For d'Hondt, the quota equals the last value to which a seat was awarded, which in the case illustrated in Table 2 would be 15,000, the value (its fourth quotient) that *A* presents to win the fifth seat. If a largest remainders allocation was to be made to the parties on the basis of Table 1, with a quota of 15,000,

<sup>13</sup> Arend Lijphart, 'Degrees of Proportionality of Proportional Representation Formulas', in Bernard Grofman and Arend Lijphart, eds, *Electoral Laws and Their Political Consequences* (New York: Agathon, 1986), pp. 170–9, at pp. 172–5; Balinski and Young, *Fair Representation*.

<sup>14</sup> Carstairs, *Short History of Electoral Systems*, p. 19.

<sup>15</sup> The method is illustrated by Van Den Bergh, *Unity in Diversity*, pp. 68–72.

then it can be seen that at the first stage party *A* would be awarded four seats and party *B* one seat, giving the d'Hondt outcome of 4–1–0. There would be no unallocated seats, so no 'remainders' would have to be taken into account.

For Sainte-Laguë, the quota equals twice the last value to which a seat was awarded. In the case shown in Table 2, this value was 12,000, so the quota equals 24,000. Applying the largest remainders method to the voting distribution in Table 1 with a quota of 24,000 would award party *A* two seats and party *B* one seat at the first stage. Parties *A* and *C*, each with a remainder of 12,000 (exactly half the quota), would tie for the fourth seat, as they do in Table 1.

It is possible to calculate a quota only when the sequence of divisors increases in a regular fashion. In modified Sainte-Laguë, however, the gap between the first and second divisors is smaller than the gap between other adjacent divisors. This irregularity was introduced, in the three Scandinavian countries where modified Sainte-Laguë is employed, to avoid making life too easy for small parties. When modified Sainte-Laguë is used in a five-seat constituency where the votes are distributed as in Tables 1 and 2, the seats are awarded on a 3–2–0 basis, since *B*'s third average of 9,333 is larger than *C*'s first average of 8,571 (12,000 divided by 1.4). However, there is no quota that, when applied to the figures in Tables 1 and 2, would produce the 3–2–0 outcome that modified Sainte-Laguë gives. If we wished to look at modified Sainte-Laguë in quota terms, we would have to use two different quotas; one that applies to parties that are in contention for their first seat, and a second (and lower) quota that applies to larger parties. Thus, in this case, the quota for parties *A* and *B* is 18,667 (twice the value with which *B* wins the fifth seat), but a party needs 13,067 (9,333 times 1.4) votes to receive a first seat, so a party with fewer than 13,067 votes is treated as if the quota was twice this value, i.e. 26,133. This might seem to conflict with the point made earlier to the effect that higher quotas help smaller parties. However, there is no conflict, for this rule applies only if all parties face the same quota – any party contending with a high quota will obviously be at a disadvantage if its competitors face a lower quota.

Quotas for the other highest average methods are not difficult to compute. The Imperiali highest average quota is half the value of the last average rewarded, a number that is usually so low that it would 'award' more seats than are at stake. The equal proportions quota is equal to the last value to which a seat is awarded. Under the Danish method, the quota is three times the value of the last average rewarded. For the Adams formula, the quota is equivalent to the last average rewarded – this quota also has the property of being the highest value that can be used as the basis of a largest remainders allocation, since any larger value would award too few seats even if every remainder were rewarded.

Before going on to compare the allocation methods in terms of quotas, we are now in a position to turn to the eleventh method under discussion, the single transferable vote (STV).

*Single Transferable Vote*

Under STV, the voter is faced with a ballot paper containing the names of all candidates and ranks them in order of preference. Candidates whose first preference votes amount to or exceed the quota (in practice, the Droop quota is always used) are elected at once. If this leaves some seats unfilled, the count proceeds with the distribution of the 'surplus' votes of elected candidates, i.e. the number such candidates have in excess of the quota, which are distributed to the other candidates in proportion to the second preferences marked on them. If there are still vacancies, the lowest placed candidate is eliminated, and his or her votes are transferred to the other candidates, again according to the second preferences marked on them. If the candidate awarded the second preference on a transferred ballot paper is unable to receive it, by having already been either elected or eliminated, the paper is transferred according to the third preference, or the fourth if the third-ranked candidate cannot benefit from it, and so on. If a transferred ballot paper contains no further preferences for any candidate who can receive it, it becomes 'non-transferable' and plays no further part in the count. The count continues until all the seats are filled.<sup>16</sup>

The distinctive nature of STV makes direct comparison with the previous methods difficult. In all the other methods, votes are regarded as being cast for parties, and the task of the formula is then merely to allocate seats to parties in accordance with the size of their vote. But under STV, the electoral system itself does not aggregate the votes cast for the various candidates of the one party. That would be illogical given that STV gives voters the power to rank all candidates, regardless of party, in order of their preferences; STV is candidate-centred rather than party-centred. Consequently, votes cast for a poorly-supported candidate may transfer, on his or her elimination, to a candidate of a different party. A party, or a candidate, with relatively little first preference support but with wide acceptability to many voters may fare better in terms of seats than one with more hard-core support but with little ability to attract lower preferences from supporters of other parties or candidates. If the former type of party is 'over-represented' in relation to its first preference votes and the latter type (Sinn Féin in Northern Ireland provides a good example) is under-represented, this should be regarded as a consequence of the logic of STV rather than necessarily as disproportionality. When discussing quotas we shall make the simplifying assumption that all transfers remain within the party fold (as, in fact, tends to happen in Malta, though not in Ireland), but it should be made clear that this does not give a full picture of the way STV can operate in practice.

The rationale for STV's use of the smaller Droop quota rather than the Hare quota is rarely considered. The reason usually advanced is that the Droop quota is all that a candidate actually requires for election. For example, in the case illustrated in Tables 1 and 2, the Droop quota of 16,667 would be

<sup>16</sup> For a full account of how STV operates in Ireland, see Geoffrey Hand, 'Ireland', in Hand, Georgel and Sasse, *European Electoral Systems Handbook*, pp. 121–39.

sufficient to guarantee a candidate's election no matter what the quota was, since it is impossible for five other candidates each to exceed this number. Consequently, it is argued, any votes a candidate has over and above this number are surplus to requirements, and should be freed to be transferred to the voters' second preferences. To set the quota at 20,000 would be to tie up, and thus cause to be wasted, an extra 3,333 votes, and unnecessarily prevent that number of voters from contributing effectively to the outcome. However, it is clear that roughly a Droop quota must be 'wasted' anyway; in the above example, if five candidates are ultimately elected with 16,667 votes each, the remaining 16,665 are unused and ineffective. Why is this preferable to their lying unused in the possession of elected candidates, as would happen if the Hare quota were employed? According to a strong proponent of STV, the answer is that with the Hare quota these votes are withdrawn from active participation in the election at an earlier stage,<sup>17</sup> but the significance of this is not explained. More persuasively, Lakeman also points out that using the Hare quota would make it possible for a party with a majority of the votes to end up with only a minority of the seats, a point that is discussed in the last section of this article. Newland maintains that using the Hare quota would cause smaller parties to be over-represented at the expense of larger ones,<sup>18</sup> but it would be at least as plausible to argue that the Droop quota, like the d'Hondt formula, enables larger parties to be over-represented at the expense of smaller ones.

The second argument put forward in favour of using the Droop quota is that it allows for 'sincere' voting, by removing any possible benefit from tactical voting. In 1881, Henry Droop argued that his quota, unlike Hare's, ensured that parties had no incentive to organize the distribution of their votes among their candidates: 'nothing can be gained by dividing the votes equally, or lost by dividing them unequally'.<sup>19</sup> But although using the Droop rather than the Hare quota reduces the problem, it cannot eliminate it. In practice, the distribution of votes among a party's candidates can determine the number of seats it wins, and the main political parties in the Republic of Ireland engage in extensive 'management' of their votes in order to maximize their seats. For example, if a party is aiming for two seats, then if one of its two candidates trails behind the other, that candidate may be eliminated from the count, whereas if they run level with each other they may both remain in the count long

<sup>17</sup> Enid Lakeman, *How Democracies Vote*, 4th edn (London: Faber and Faber, 1974), p. 147.

<sup>18</sup> Robert A. Newland, *Comparative Electoral Systems* (London: The Arthur McDougall Fund, 1982), p. 64. A similar point is made by I. D. Hill, 'Some Aspects of Elections: To Fill One Seat or Many', *Journal of the Royal Statistical Society A*, 151 (1988), 243–75, at pp. 253–4, 273; and by Stein Rokkan, 'Electoral Systems', in David L. Sills, ed., *International Encyclopedia of the Social Sciences*, Vol. 5 (New York: Crowell Collier and Macmillan, 1968), pp. 6–21, at p. 15.

<sup>19</sup> Quoted in John H. Humphreys, *Proportional Representation: A Study in Methods of Election* (London: Methuen, 1911), p. 138. See also Lakeman, *How Democracies Vote*, p. 148; Newland, *Comparative Electoral Systems*, p. 64.

enough to attract transfers from other parties' candidates and secure election. When votes become non-transferable during the count, due to some voters expressing preferences for only a subset of the candidates on the ballot paper, this becomes especially important. An example comes from the five-member Cork North-Central constituency in the Irish general election of 1989, where the allied parties of Fine Gael and the Progressive Democrats had enough votes to win a total of two seats. The only uncertainty was whether Fine Gael would win two seats and the PDs none, or whether each party would take one seat. The lack of an even distribution of votes among the two Fine Gael candidates proved crucial to the outcome. On the final count, the Progressive Democrat candidate had 5,748 votes (0.84 quotas), while the two Fine Gael candidates had 12,532 votes (1.82 Droop quotas) between them – one had 6,875 votes and the other 5,657. If the two Fine Gael candidates had each had 0.91 quotas, they would both have been elected, but because one had a full quota and the other 0.82 of a quota, the Progressive Democrat took the last seat ninety-one votes ahead of the second Fine Gael candidate.<sup>20</sup>

In the case illustrated in Tables 1 and 2, party *A* has 3.60 Droop quotas, party *B* 1.68 and party *C* 0.72. If no vote cast for a candidate of one party ever transfers to a candidate of another party, and transfers of votes thus remain wholly within party lines, then the allocation of seats will depend on how votes are distributed among candidates within parties. If party *A* has three candidates with a quota each and a fourth with 0.6 of a quota, then it will win only three seats. But if its four candidates each have 0.9 of a quota, it is certain of four seats.

This enables us to compare STV with previous methods. If, in the case illustrated in Tables 1 and 2, the split of votes among *A*'s candidates is uneven, with three having a full quota each and the fourth 0.6 of a quota, STV operates like LR-Droop. If *A*'s votes are evenly split among its candidates, STV closely resembles d'Hondt. Under the assumption that no votes transfer across party lines, a party's ideal strategy under STV is to ensure that all of its candidates have exactly the same number of votes at every state of the count, and that when any one is eliminated, his or her votes transfer evenly among all the party's other candidates. If all parties were able to achieve this, and if no transfer crossed party lines, STV would operate very much like d'Hondt.

#### QUOTAS

We are now in a position to start making direct comparisons between the eleven allocation methods we are discussing. Looking at methods in terms of the quotas they employ highlights an essential similarity between them. Each can be seen as a largest remainder method awarding each party a seat for

<sup>20</sup> Vote management at this election may have affected the outcome in eleven of the forty-one constituencies. It is discussed in Michael Gallagher, 'The Election Results and the New Dáil', in Michael Gallagher and Richard Sinnott, eds, *How Ireland Voted 1989* (Galway: Centre for the Study of Irish Elections, 1990), pp. 68–93, at pp. 81–3.

every full quota it has and giving the unallocated seats to the parties with the highest remainders. The difference between them lies in the different quotas they use and the consequences of the various quotas, which render each method a 'special case' of largest remainders. D'Hondt employs such a low quota that all the seats are allocated at once, so in practice no remainder is ever rewarded. The Sainte-Laguë quota is fixed so that only parties whose unused votes amount to half a quota or more see their remainders rewarded. Under the Danish method, all remainders (and only those remainders) equal to or more than a third of the quota are rewarded. The Adams method pitches the quota so high that every remainder, even a remainder of zero, brings a party an extra seat (provided, of course, that the number of seats equals or exceeds the number of parties). Even STV can be conceived in these terms, with the unique feature that candidates with remainders (or, more precisely, their voters) can transfer those remainders among themselves so that eventually, if no votes were to become non-transferable at any stage and if all surpluses were distributed, only one candidate, the runner up, would have a remainder (which is not rewarded), whose size would be marginally less than a quota.

The equal proportions method uses a quota that ensures that the remainders rewarded are those (and only those) belonging to parties whose total number of votes when divided by the quota is greater than or equal to the geometric mean (i.e. the square root of the product) of the two numbers  $n$  and  $(n + 1)$ , where  $n$  is the number of full quotas the party has. Like all the other methods, equal proportions rewards the largest remainders, but unlike them it measures the 'largeness' of remainders in relative terms (based on the size of each party) rather than in absolute terms. Thus one party will win a tenth seat if its vote total comes to 9.487 quotas (the square root of 9 times 10), while a smaller one will win a third seat provided its vote total equals 2.449 quotas (the square root of 2 times 3). All the other methods we are discussing would regard the first party's remainder in this example as being larger than the second party's remainder, but the philosophy underlying the equal proportions method is that, in relation to the respective sizes of the parties, the two remainders are identical.<sup>21</sup>

The quotas for the various methods as applied to the vote distribution illustrated in Tables 1 and 2 are set out in Table 3, under a varying number of seats at stake. The size of the quota increases as we move from those formulae that are most generous to the larger parties to those that give the benefit of the doubt to the smaller parties. When there are five seats at stake, the only possible seat allocations that can be given by a regular formula are 4–1–0 and 3–1–1, and the quota could vary widely without affecting the outcome. To be precise, a quota anywhere between 14,001 and 15,999 inclusive will give a 4–1–0 result, while any value from 16,001 to 30,000 will produce 3–1–1. A value of 16,000 would award three initial seats to party *A* and one to party *B*, and there would be a three-way tie for the fifth seat, as each party would

<sup>21</sup> This philosophy is outlined in Huntington, 'A New Method of Apportionment'.

have a remainder of 12,000. A value of 14,000 or fewer awards too many seats (at least six) initially, and a value of more than 30,000 awards too few.

TABLE 3 *Electoral Quotas under Different Formulae*

Formula	2 seats	5 seats	10 seats	25 seats	100 seats
Imperiali highest averages	20,000	12,000	7,000	3,500	966
LR-Imperiali	25,000	14,286	8,333	3,704	980
D'Hondt	30,000	15,000	9,333	4,000	1,000
STV/LR-Droop	33,333	16,667	9,091	3,846	990
Modified Sainte-Laguë	40,000*	18,667*	10,909	4,138	1,008
LR-Hare	50,000	20,000	10,000	4,000	1,000
Sainte-Laguë	56,000	24,000	10,909	4,138	1,008
Equal proportions	$\infty$	24,495	10,954	4,140	1,008
Danish	84,000	25,714	11,250	4,186	1,011
Adams	$\infty$	30,000	12,000	4,285	1,017

*Note:* These are the quotas in contests involving three parties, *A*, *B* and *C*, whose respective votes are 60,000, 28,000 and 12,000, as in Tables 1 and 2.

\*The quota for party *C* is 56,000 when there are two seats and 26,133 when there are five seats; the quota for party *B* is 56,000 when there are two seats.

It is noticeable that when district magnitude is small, none of the quotas is very close to the Hare or natural quota. As district magnitude increases, the various quotas start to converge around the Hare quota. The table shows also that the d'Hondt quota is liable to be lower than the Droop quota under some circumstances but higher under others; Lijphart is mistaken in asserting that the Droop quota is invariably the higher of the two.<sup>22</sup> The Droop and Hare quotas are fixed solely by the total number of seats and votes, whereas the other quotas fluctuate according to the distribution of votes among the parties. The Sainte-Laguë quota oscillates around the Hare quota – even though in Table 3 it is always higher than the Hare quota, it is simple to devise cases where it is lower. For example, if 100 votes are divided 30–25–20–9–8–8 among six parties, the Sainte-Laguë quota will be below the Hare quota if the number of seats equals one (Sainte-Laguë quota 60, Hare quota 100), four (Sainte-Laguë 20, Hare 25), five (Sainte-Laguë 18, Hare 20), and so on for other values. The equal proportions quota is slightly higher than the Sainte-Laguë quota, while the d'Hondt quota is usually close to the Droop quota. The Imperiali highest averages quota is considerably lower than the LR-Imperiali quota, and the similarity of names is highly misleading. Of course, we should not over-generalize from this purely illustrative example – in practice, the relationship between the highest average quotas on the one hand and the largest remainders quotas on the other varies depending on the precise arrangement of votes among the parties in any particular case. As we shall see in the next section,

<sup>22</sup> Lijphart, 'Degrees of Proportionality', p. 176.



it is possible for even the lowest of the largest remainders quotas, namely the LR-Imperiali quota, to exceed the Danish quota or, indeed, the quota of any highest average method except Adams.

#### THRESHOLDS

Of the various works on thresholds, the standard analysis is that by Lijphart and Gibberd, who concentrate on two thresholds.<sup>23</sup> *A vote share in excess of the threshold of exclusion* is always sufficient to secure a seat – this threshold is the maximum share that a party can possibly win while still failing to obtain a seat. The *threshold of representation* is the vote share that is necessary to win a seat – it is the minimum share with which a party can possibly obtain a seat. Their analysis covers only four of the methods we discuss here, but, the hard work having been done in their paper, it is straightforward to arrive at the thresholds presented in Figure 2.

Looking at the thresholds of exclusion, we see that four methods (d'Hondt, STV, LR-Imperiali and LR-Droop) have an identical threshold of one Droop quota. When there are only two parties, LR-Hare and Sainte-Laguë have a common threshold, of  $1/(2s)$ , but when there are more than two parties the respective thresholds are identical only if the number of seats is one less than the number of parties. The Imperiali highest average formula is a deviant case, since even the possession of a Droop quota does not guarantee a seat. For example, where there are five seats at stake, not even 25 per cent of the votes (1.5 Droop quotas) will guarantee a seat if another party wins the other 75 per cent, as then this party's fifth average will also be 25 per cent.

Both thresholds are illustrated in Tables 4 and 5 for eight of the allocation methods, under conditions when three, seven and ten parties compete for seats. The district magnitudes used are 3, 7 and 14 respectively – a district magnitude of 7 corresponds approximately to average district magnitude for the constituency seats in Belgium, Denmark, Greece, Iceland, Norway, Spain and Switzerland, while one of 14 is close to that used in Finland, Italy, Luxembourg, Portugal and Sweden.<sup>24</sup> No PR system uses district magnitude as low as 3 (although Ireland, Malta and the 1986 French system are not much higher), but taking this case illustrates well the important role of district magnitude in determining thresholds. The eight methods are arranged in order of their exclusion thresholds. Table 4 shows that district magnitude tends to have more impact on this threshold than does the number of parties. Equal proportions

<sup>23</sup> Lijphart and Gibberd, 'Thresholds and Payoffs'. See also Rokkan, 'Electoral Systems', pp. 13–15; Rae, *Political Consequences of Electoral Laws*, pp. 149–70; Douglas W. Rae, Victor Hanby and John Loosemore, 'Thresholds of Representation and Thresholds of Exclusion', *Comparative Political Studies*, 3 (1971), 479–88; Markku Laakso, 'The Maximum Distortion and the Problem of the First Divisor of Different PR Systems', *Scandinavian Political Studies*, 2 (1979), 161–9; Rein Taagepera and Matthew S. Shugart, *Seats and Votes: The Effects and Determinants of Electoral Systems* (New Haven and London: Yale University Press, 1989), especially pp. 274–7.

<sup>24</sup> Mackie and Rose, *International Almanac*, p. 510.

	Threshold of exclusion – maximum vote share with which it is possible not to win a seat	Threshold of representation – minimum vote share with which it is possible to win a seat
	If $s > (p - 1)$	
Imperiali highest averages	$2/(s + 3)$	$2/(s + 2p - 1)$
D'Hondt	$1/(s + 1)$	$1/(s + p - 1)$
STV	$1/(s + 1)$	0
Modified Sainte-Laguë	$1.4/(2s - p + 2.4)$	$1.4/(2s + 1.4p - 2.4)$
Pure Sainte-Laguë	$1/(2s - p + 2)$	$1/(2s + p - 2)$
Danish	$1/(3s - 2p + 3)$	$1/(3s + p - 3)$
LR-Imperiali	$1/(s + 1)$	$3/(p(s + 2))$
LR-Droop	$1/(s + 1)$	$2/(p(s + 1))$
LR-Hare	$(p - 1)/ps$	$1/ps$

Fig. 2. Thresholds of exclusion and representation under seat allocation methods

Source for d'Hondt, pure and modified Sainte-Laguë and LR-Hare thresholds: Lijphart and Gibberd, 'Thresholds and Payoffs', p. 225.

Notes:  $p$  denotes number of parties,  $s$  number of seats.

When  $(p - 1) \geq s$ , the threshold of exclusion is  $1/(s + 1)$  for all methods except Imperiali highest averages, where it remains at  $2/(s + 3)$ .

When  $s \geq (p - 1) \geq s/2$ , the modified Sainte-Laguë threshold of exclusion is  $1.4/(1.6s - 0.2p + 1.6)$ .

Under equal proportions and Adams, winning just one vote guarantees representation, provided there are as many seats as parties. Under these conditions each has a threshold of exclusion of zero and a threshold of representation of one vote.

For STV, 'vote' refers to first preferences only. We are no longer making the assumption that all transfers remain within the party fold.

and Adams are outliers, since they award a seat to every party that has won a vote; we can compare them directly with the other methods only when we consider the threshold of exclusion for a later seat. They then take their appointed places in the ranking given in Table 3. For example, if we consider the thresholds of exclusion for a second seat in a situation where there are two parties and five seats at stake, we find that the threshold for Imperiali highest averages is 37.5 per cent, for d'Hondt, STV, LR-Imperiali and LR-Droop it is 33.3 per cent, for LR-Hare and both types of Sainte-Laguë it is 30.0 per cent, for equal proportions it is 29.0 per cent, for the Danish method it is 28.6 per cent and for Adams it is 25.0 per cent.

Turning to the thresholds of representation, we again find identities when there are only two parties, between d'Hondt and LR-Droop on the one hand, and between Sainte-Laguë and LR-Hare on the other. This time STV differs from d'Hondt and LR-Droop, since it is possible for a candidate or (if transfers cross party lines) a party to win no first preferences and yet secure a seat due to vote transfers from other candidates. Table 5 shows more variation than did Table 4, and now a change in the number of parties can have as

TABLE 4 *Thresholds of Exclusion for Given District Magnitudes and Numbers of Competing Parties*

		Imperiali highest average	D'Hondt	LR- Imperiali	LR- Droop	Mod SL	LR-Hare	Pure SL	Danish
<i>s</i> =3	<i>p</i> =3	33.3	25.0	25.0	25.0	24.1	22.2	20.0	16.7
	<i>p</i> =7	33.3	25.0	25.0	25.0	25.0	25.0	25.0	25.0
	<i>p</i> =10	33.3	25.0	25.0	25.0	25.0	25.0	25.0	25.0
<i>s</i> =7	<i>p</i> =3	20.0	12.5	12.5	12.5	10.4	9.5	7.7	5.6
	<i>p</i> =7	20.0	12.5	12.5	12.5	12.3	12.2	11.1	10.0
	<i>p</i> =10	20.0	12.5	12.5	12.5	12.5	12.5	12.5	12.5
<i>s</i> =14	<i>p</i> =3	11.8	6.7	6.7	6.7	5.1	4.8	3.7	2.6
	<i>p</i> =7	11.8	6.7	6.7	6.7	6.0	6.1	4.3	3.2
	<i>p</i> =10	11.8	6.7	6.7	6.7	6.4	6.4	5.0	4.0

Note: *s* refers to number of seats, *p* to number of parties.

much effect as a change in district magnitude. Moreover, the ranking of the methods differs from that in Table 4. The largest remainders thresholds of representation maintain the same order among themselves when there is more than one seat at stake – the LR-Hare threshold is lower than the LR-Droop threshold, and this in turn is lower than the LR-Imperiali threshold. Similarly, among the highest average thresholds, when there is more than one seat at stake the Danish threshold is lower than the Sainte-Laguë threshold, which is lower than the d'Hondt threshold, which is lower than the Imperiali threshold.

TABLE 5 *Thresholds of Representation for Given District Magnitudes and Numbers of Competing Parties*

		Imperiali highest average	D'Hondt	LR- Imperiali	LR- Droop	Mod SL	LR-Hare	Pure SL	Danish
<i>s</i> =3	<i>p</i> =3	25.0	20.0	20.0	16.7	17.9	11.1	14.3	11.1
	<i>p</i> =7	12.5	11.1	8.6	7.1	10.4	4.8	9.1	7.7
	<i>p</i> =10	9.1	8.3	6.0	5.0	8.0	3.3	7.1	6.2
<i>s</i> =7	<i>p</i> =3	16.7	11.1	11.1	8.3	8.9	4.8	6.7	4.8
	<i>p</i> =7	10.0	7.7	4.8	3.6	6.5	2.0	5.3	4.0
	<i>p</i> =10	7.7	6.2	3.3	2.5	5.5	1.4	4.5	3.6
<i>s</i> =14	<i>p</i> =3	10.5	6.2	6.2	4.4	4.7	2.4	3.4	2.4
	<i>p</i> =7	7.4	5.0	2.7	1.9	4.0	1.0	3.0	2.2
	<i>p</i> =10	6.1	4.3	1.9	1.3	3.5	0.7	2.8	2.0

Note: *s* refers to number of seats, *p* to number of parties.

But if we compare the largest remainder thresholds with the highest average thresholds, some surprising results emerge. Lijphart and Gibberd have already shown that the LR-Hare thresholds of representation are consistently lower

than those of the main highest average methods.<sup>25</sup> It transpires that a more striking conclusion can be reached: under any largest remainder method, the threshold of representation may be lower than under any highest average method whose first divisor is greater than zero. This can be seen by comparing the thresholds in Figure 2. If we compare the threshold of LR-Imperiali (which has the lowest quota, and is therefore the least sympathetic towards small parties, of any of the largest remainder methods) with the pure Sainte-Laguë threshold, we can see that the LR-Imperiali threshold is lower than the Sainte-Laguë threshold if

$$\frac{3}{p(s+2)} \text{ is less than } \frac{1}{(2s+p-2)}.$$

This will be the case when  $s$  is greater than 1 and  $p$  is greater than 6.

For example, if there are two seats and seven parties, the LR-Imperiali threshold of representation is  $3/28$ , or 10.71 per cent, while the Sainte-Laguë threshold is  $1/9$ , or 11.11 per cent. Therefore, if one party has 34 per cent of the votes and the other six parties take equal shares of the remaining votes, each having 11 per cent, the largest party will win both seats under Sainte-Laguë, since its second average of 11.33 per cent is greater than the first average of each of the small parties. But under LR-Imperiali, one of the small parties will win a seat, since its remainder of 11 per cent exceeds that of the large party, which, with the LR-Imperiali quota standing at 25 per cent, is only 9 per cent. In an analogous fashion, it can be shown that the threshold of representation under LR-Imperiali is lower than under the Danish method when the number of seats exceeds one and the number of parties exceeds nine, as would occur in a two-seat constituency with ten parties if the votes were distributed among one party with 31.6 per cent of the votes and nine parties each with 7.6 per cent. If we devise a highest average method so generous to small parties that each divisor is 1,000 greater than the previous one (1, 1,001, 2,001 etc. then the LR-Imperiali threshold of representation will be lower than under this method if the number of seats is greater than one and the number of parties is greater than 3,000.

This relates to our earlier discussion of quotas, as it means that the quota produced by any highest average method can potentially be lower than the quota produced by any largest remainder method. For example, in the case of the two-seat constituency described above, where one party wins 31.6 per cent and nine other parties each win 7.6 per cent, so that a small party obtains a seat under LR-Imperiali but not under the Danish method, the LR-Imperiali quota is 25 per cent ( $100/4$ ) while the Danish quota is 23.7 per cent ( $31.6$  divided by 4 and multiplied by 3). There is, in other words, overlap between highest average and largest remainder methods. While in their general tendency LR-Droop is close to d'Hondt and LR-Hare to Sainte-Laguë, under certain circumstances any largest remainder formula can be more accommodating than

<sup>25</sup> Lijphart and Gibberd, 'Thresholds and Payoffs', p. 229.

a small party than any of the highest average methods we are considering here apart from Adams and equal proportions.

Comparing the thresholds of exclusion, we see that the values for LR-Droop and d'Hondt are identical: the vote share that guarantees a seat under one method guarantees it under the other. However, the two methods will not always produce the same outcome; in particular, a smaller party might win a seat under LR-Droop when a larger one would win it under d'Hondt. As we have already mentioned, the Droop quota is often used in practice as the first stage of a d'Hondt allocation, since it is the lowest quota that cannot possibly award too many seats. If all the seats can be awarded on the basis of full quotas, then LR-Droop and d'Hondt produce identical outcomes and the d'Hondt quota is equal to or larger than the Droop quota. If not, then LR-Droop rewards some of the remainders, while d'Hondt proceeds, in effect, to reduce the size of the quota until all the seats are awarded on the basis of full quotas. If the use of a lower quota makes any difference to the outcome, it will be a larger party that benefits at the expense of a smaller one, since a lower quota cannot benefit a smaller party against a larger one (see Appendix). In other words, where the two allocations differ, d'Hondt will be more generous to a larger party and LR-Droop to a smaller one.

The relationship between Sainte-Laguë and LR-Hare is slightly different. By comparing the expressions for the exclusion thresholds, it can be seen that the Sainte-Laguë threshold is lower than the LR-Hare threshold if the number of parties ( $p$ ) is greater than two and the number of seats is greater than  $(p - 1)$ . Thus, it may well happen (and does in several of the examples in Table 4) that the threshold of exclusion is lower under Sainte-Laguë than under LR-Hare, while, as we have seen (and as Table 5 illustrates), the threshold of representation can be lower under LR-Hare than under Sainte-Laguë.<sup>26</sup> The practical implication is that Sainte-Laguë and LR-Hare may give different allocations, and in these instances the type of party favoured varies from case to case. Compared with Sainte-Laguë, LR-Hare sometimes favours a larger party and sometimes it favours a smaller one. As we have seen, when there are only two parties, the two methods give identical results. If, with only two parties, party  $A$  has more than  $(2n - 1)$  times as many votes as party  $B$ , then party  $A$  wins  $n$  seats before party  $B$  wins one under both methods. Under Sainte-Laguë, the addition of further parties, attracting previously uncast votes, cannot possibly result in  $A$  losing a seat to  $B$ , since  $A$  still has more than  $(2n - 1)$  times as many votes as  $B$ . But under LR-Hare the addition of further parties might result in  $B$  gaining a seat from  $A$ , for as the number of votes increases, so the Hare quota increases, and  $A$ 's remainder over and above its  $(n - 1)$  quotas steadily diminishes, while  $B$ 's remainder, consisting of all its votes, obviously stays the same (for the general argument, see the Appendix). Thus it is possible for a larger party to fare better under Sainte-Laguë. An example would be a two-seat constituency where  $A$  wins sixty-eight votes and

<sup>26</sup> Both are demonstrated by Lijphart and Gibberd, 'Thresholds and Payoffs', pp. 228–9.

*B* wins twenty-two. Both methods produce a 2–0 allocation, but if a new party appears and attracts ten previously uncast votes, LR-Hare now results in a 1–1 allocation, since *A*'s remainder declines from twenty-three to eighteen, while the Sainte-Laguë allocation remains unaltered.

The opposite effect, of Sainte-Laguë being more generous than LR-Hare to a smaller party, can also arise. If party *A* has slightly less than  $(2n - 1)$  times as many votes as party *B*, then *B* always wins one seat before *A* wins *n* seats under Sainte-Laguë. This does not necessarily apply under LR-Hare when more than two parties compete. Assume there are more than two parties, and now add a seat, thereby reducing the Hare quota. We saw in the previous paragraph that an increase in the quota could cause a seat to change from a larger party to a smaller one and, in analogous fashion, the effect here is the opposite. The addition of an extra seat causes the quota to decrease and, as this happens, *A*'s remainder increases by more than *B*'s does and *B* might lose a seat to *A*. This surprising outcome, that an increase in the total number of seats can cause a small party to lose a seat, is widely known as the Alabama paradox, since it was first observed in the 1880 apportionment of seats in the American House of Representatives. It transpired that Alabama would receive eight seats in a 299-member House but only seven seats in a 300-member House. The reason was that the additional member increased every state's number of quotas by a factor of  $300/299$ , or 1.0033, and multiplying every state's old quotas by this number led to the remainders of two larger states, Illinois and Texas, overtaking that of Alabama and thus depriving it of the extra seat.<sup>27</sup>

Taking both quotas and thresholds into account, we are now in a position to rank our eleven allocation methods. The following list ranks them in order according to whether, when bias is inevitable, this tends to favour larger parties over smaller ones. From the most favourable to larger parties to the least favourable, the order is:

Imperiali highest averages  
 LR-Imperiali  
 D'Hondt  
 STV  
 LR-Droop  
 Modified Sainte-Laguë  
 LR-Hare/Sainte-Laguë  
 Equal proportions  
 Danish  
 Adams

Of these, the first, Imperiali highest averages, can be regarded as exceptional,

<sup>27</sup> Balinski and Young, *Fair Representation*, p. 39. Another example of the Alabama paradox is given by Carstairs, *Short History of Electoral Systems*, p. 25. In the case he illustrates, a hundred seats would be awarded on a 45–44–11 basis under both LR-Hare and Sainte-Laguë, while if the number of seats is increased to 101, Sainte-Laguë produces a 45–45–11 outcome whereas under LR-Hare the smallest party loses a seat and the outcome is 46–45–10.

since there is no guarantee that it will produce a perfectly proportional result even when this is possible; it favours the large parties even when disproportionality is not inevitable. LR-Imperiali is also outside the range of genuinely proportional methods, given its propensity to generate more full quotas than there are seats available. Modified Sainte-Laguë deserves to be placed between d'Hondt and Sainte-Laguë, since it makes it harder for small parties to win a first seat than the pure Sainte-Laguë version, but in situations where every party has won a seat, modified Sainte-Laguë ranks alongside pure Sainte-Laguë and LR-Hare.

This ordering is not immutable in all circumstances – it is qualified by the fact that, as we have just seen, the thresholds of representation for each of the largest remainders methods can be lower than those for any highest average method except Adams and equal proportions. Even though the circumstances under which, say, the LR-Imperiali threshold of representation is lower than the Danish threshold may seem artificial and improbable, the fact that they are possible means that we have to acknowledge a degree of overlap among the methods. In any specific case, LR-Imperiali, LR-Droop and LR-Hare, in that order, might all be below the Danish method, while, depending on how tightly or otherwise transfers follow party lines, STV could rank alongside d'Hondt or alongside Adams or anywhere in between.

#### PARADOXES AND MAJORITIES

When the various formulae are compared in terms of their practical utility, other criteria come into consideration. Largest remainders methods are often criticized because of their vulnerability to paradoxes. Besides the Alabama paradox, which we have already mentioned, they permit such anomalies as a party growing larger relative to another party and yet losing a seat to it, or the arrival of new parties causing seats to switch between other parties for no apparent reason. Paradoxes can occur whether the Hare quota or the Droop quota is used.<sup>28</sup> Balinski and Young, who outline these paradoxes, argue that they 'occur because at bottom the use of remainders to determine priority is simply not a proportional device. It does not adequately reflect the *relative* sizes' of the parties.<sup>29</sup> If this implies that LR-Hare is not a genuine PR method, then it is surely too harsh a judgement. It would also imply that only equal proportions among the methods we are discussing can be considered truly proportional, since it alone takes into account the sizes of the parties when deciding which remainders to reward.

Highest average methods are free from these paradoxes. Given a particular distribution of votes between parties, increasing the total number of seats at stake cannot (since the seats are awarded sequentially) possibly decrease the number any one party is awarded, so the Alabama paradox cannot arise. The

<sup>28</sup> Newland, *Comparative Electoral Systems*, pp. 40–2.

<sup>29</sup> Balinski and Young, *Fair Representation*, p. 69.

other two paradoxes mentioned above are also avoided; if party *A* received *n* seats before party *B* receives one, then it will always do so regardless of how many other parties, votes and seats are involved, and it will clearly also do so if the ratio between its and *B*'s votes widens.

Whether highest average methods are entirely free from paradoxes depends on what is held to constitute a paradox. Brams argues that the equal proportions (EP) method's credentials are questionable.<sup>30</sup> He observes that it has the 'bizarre property' that 'given that the total population remains constant, *increasing* the population of one state (and changing the proportions in one or more other states as well) may actually *decrease* the apportionment to the (now larger) state'. This, he says, 'gives EP solutions the overtones of a paradox'. If this is a paradox, every method stands indicted. For example, if there are three parties, 100 votes cast and five seats to fill, the largest party will win four seats under Sainte-Laguë if the votes fall 70–22–8 but only three if they fall 76–12–12. Under d'Hondt, it wins four seats with a vote distribution of 58–28–14 but only three with a distribution of 64–18–18. The same can happen under LR-Hare (compare the vote distributions 70–22–8 with 72–14–14) and LR-Droop (compare 62–28–10 with 64–18–18).

A more sustainable criticism of highest average methods is that they may produce seat allocations that deviate significantly from the number of Hare quotas in the parties' votes. A party with *n* full Hare quotas plus a remainder may win more than (*n* + 1) seats (especially under d'Hondt) or less than (*n* – 1) (especially under Adams). This is very unlikely to occur under Sainte-Laguë, which as we have seen can be regarded as the highest averages equivalent of the LR-Hare method, but it is still possible.

The ideal method might seem to be one that incorporates the best of both worlds, by avoiding the paradoxes of largest remainders while ensuring that no party receives more seats than its Hare quotas rounded up or fewer than its Hare quotas rounded down. For a while, in the mid-1970s, it was believed that such a method had been discovered, but it subsequently transpired that this method was vulnerable to paradox and, indeed, that no such 'ideal' method exists.<sup>31</sup> This being the case, pure Sainte-Laguë has a clear advantage at the theoretical level over its largest remainders equivalent, LR-Hare.

The various formulae also differ in their implications for tactical mergers and splitting. Mergers are encouraged by the d'Hondt method, since a party cannot possibly benefit from splitting into smaller components and running each as a separate list. At the other end of the scale, the Adams method encourages splitting into several equal-sized lists – a party cannot lose by doing this. Under the Danish method, a party is more likely to gain than to lose by running a number of lists. Pure Sainte-Laguë and LR-Hare do not

<sup>30</sup> Steven J. Brams, *Paradoxes in Politics* (New York: Free Press, 1976), p. 157.

<sup>31</sup> Michel L. Balinski and H. Peyton Young, 'The Quota Method of Apportionment', *American Mathematical Monthly*, 84 (1975), 450–5; Balinski and Young, *Fair Representation*, pp. 79–81 and 129–34.



in principle favour either splitting or mergers, although in any particular case one or other might prove advantageous. For example, in the case illustrated in Table 1, party *A* would have been better advised to split itself into four lists each with 15,000 votes, as it would then have won four of the five seats under both Sainte-Laguë and LR-Hare compared with the three it wins when it runs as one list with 60,000 votes. If party *A* does not perform this tactical splitting, *B* could use the device to gain a second seat by presenting two lists each with 14,000 votes. On the other hand, it may be that list *C* in the same case is in fact a merger between the lists of two smaller parties, which would not have won a seat had they not merged. Generally speaking, methods more favourable to large parties than Sainte-Laguë and LR-Hare are likely to lead to mergers paying dividends and methods more favourable to small parties are less likely to lead to payoffs from mergers and to encourage list-splitting.

The only method that is designed to produce the same outcome whether a party or an alliance splits its votes among many lists or concentrates them on one list is STV. In effect, under the electoral system parties aiming to win more than one seat have no option but to split their votes (among their candidates), in that STV does not aggregate the votes for a party's several candidates. The counting process then implements 'mergers' between the candidates of each party, and between the groups of voters supporting candidates of different parties, to the extent that the voters express this through their use of lower preferences. However, as we saw earlier, not even STV can entirely eliminate the possibility that 'vote management' will benefit a party.

Another criterion on which the methods can be compared is that of how many votes a party needs to win a majority when there is an odd number of seats at stake. Only the methods that use a low quota – the two Imperialis, d'Hondt, LR-Droop and STV (if transfers remain within the party fold) – guarantee that a party that wins a majority of the votes will win a majority of the seats in such cases. This is obvious in the case of LR-Droop and STV. If there are  $(n - 1)$  seats at stake in a 100-vote constituency (where  $n$  is an even number), then the Droop quota equals  $100/n$ . A party with more than fifty votes will thus have at least  $n/2$  full quotas, ensuring that it wins this many seats, i.e. a majority. Under d'Hondt, parties can never lose by merging, as mentioned above.<sup>32</sup> Thus, if one party has a majority of votes, its opponents cannot win more seats separately than if they merged into one party. If they took this step the merged party would still have a minority of the votes, and in a two-party contest d'Hondt would obviously give the larger party more seats than the smaller one. Therefore, a party with a majority of votes will always receive a majority of the seats (provided there is an odd number at stake) regardless of the spread of votes among its opponents.

Under other methods (highest average methods where, if the first divisor is one, the interval between successive divisors is greater than one, or largest remainders methods where the quota is larger than the Droop quota), a party

<sup>32</sup> This is demonstrated by Balinski and Young, *Fair Representation*, p. 90.

might win most of the votes and yet wind up with a minority of the seats. In a three-seat constituency, for example, a party with anything less than 60 per cent of the votes will win only one seat under Sainte-Laguë if two other parties divide the remaining votes equally between them; the corresponding figure for LR-Hare is 55.6 per cent. Under the Danish system, a party that wins 66 per cent of the votes in a three-seat constituency will take only one seat if two other parties each win 17 per cent. Under Adams, a party will never win a majority, no matter how many votes it has, if the remaining votes are spread among  $(n + 1)/2$  or more parties, where  $n$  is the number of seats in the constituency. Thus, although Sainte-Laguë might seem to be more fair than d'Hondt, in that it is as likely to reward mergers as splits, proponents of d'Hondt might argue that *it* is more fair in crucial situations, in that, unlike Sainte-Laguë, it prevents any risk of a majority being defeated by a more tactically aware minority that splits its votes to maximum advantage. The same case could be made for LR-Droop against LR-Hare and for STV's use of the Droop rather than the Hare quota.<sup>33</sup> In countries where the distribution of parliamentary seats is decided by an allocation at the national level (Israel, the Netherlands, and the Federal Republic of Germany except in 1990), this could be quoted in support of d'Hondt, since it is the only proportional highest average method that guarantees that there is no prospect of a party that wins a majority of votes failing to be rewarded with a majority of seats.

#### CONCLUSION

To rank seat allocation methods in order of proportionality would be a subjective exercise, entailing the selection of one criterion as the acid test of proportionality. If true proportionality consists of minimizing the over-representation of the most over-represented party, then d'Hondt is the most proportional method. This method also ensures that a party with a majority of votes will win a majority of seats when an odd number of seats is at stake. If, on the other hand, true proportionality entails ensuring that voters' shares of a parliamentary representative are as equal as possible, then either Sainte-Laguë or equal proportions is the most proportional, depending on how we measure 'as equal as possible'. Although the outcomes of some elections held under PR methods are clearly less proportional than others, by any criterion, this is often a function of district magnitude as much as of the PR formula employed. PR methods themselves are not innately more proportional or less proportional than each other. Each PR method works by minimizing its own particular conception of disproportionality, and so is by definition the most proportional method according to its own lights – it alone embodies the conception of pro-

<sup>33</sup> For example, Van Den Bergh, *Unity in Diversity*, p. 25. The other side of the argument is that a large party can often win a majority of seats with a minority of votes under d'Hondt, LR-Droop and STV if the other votes are scattered among a number of opponents. This may happen under Sainte-Laguë and LR-Hare, though it would require a greater degree of fragmentation of the other votes, and cannot happen under Adams.

portionality that it was devised to maximize.<sup>34</sup> Not all seat allocation methods are examples of proportional representation: both Imperiali highest averages and LR-Imperiali fall outside the range of genuinely PR methods.

This article has concentrated not on ‘proportionality’ but on other dimensions of seat allocation, which has entailed focusing on the quotas used by all PR methods. This makes it clear why larger parties fare better under methods that use a low quota, such as LR-Droop, STV and d’Hondt, than under methods with a higher quota, like LR-Hare, equal proportions and (usually) Sainte-Laguë. It permits the construction of a provisional ranking order embracing a variety of seat allocation methods that at first sight appear to be very different. When thresholds are taken into account it becomes evident that this ranking order is not immutable, because when many parties compete the number of votes with which a party might win a seat under largest remainder methods tends to be lower than the number needed under highest average methods. Under these circumstances, even those largest remainders methods that tilt the odds most heavily in favour of larger parties can actually be more accommodating to smaller parties than are most highest average methods.

The selection of one of these methods in preference to another will have consequences for party representation, but the consequences will vary depending on district magnitude and the number of competing parties. Consideration of paradoxes highlights the advantages of highest averages methods over largest remainders methods. When the share of votes that a party needs in order to win a majority of seats is taken into account, the fact that only certain methods guarantee that a party with a majority of votes within a constituency will not end up with a minority of the seats is one reason why such methods, particularly d’Hondt, tend to be the most widely employed. A systematic analysis of seat allocation methods can increase our knowledge, but the choice of the most suitable electoral system in any specific context remains a matter of subjective decision.

#### APPENDIX: PROOF THAT LOW QUOTAS ASSIST LARGER PARTIES

Among a number of parties competing for seats in a given constituency, there are two parties  $X$  and  $Y$ .  $X$  wins  $v_X$  votes (comprising  $x$  full quotas and a remainder  $r_X$ ) while  $Y$  wins  $v_Y$  votes (comprising  $y$  full quotas and a remainder  $r_Y$ ).  $v_X$  is greater than  $v_Y$ . Thus, calling the quota  $q$ , the votes of the two parties can be expressed as

$$\begin{aligned}v_X &= xq + r_X \\v_Y &= yq + r_Y\end{aligned}$$

It can be shown that a reduction in the value of the quota  $q$  may result in the switching of a seat from  $Y$  to the larger  $X$ , but cannot cause a seat to switch from  $X$  to the smaller  $Y$ .

To take the first case, suppose that initially party  $Y$ ’s remainder is rewarded but  $X$ ’s remainder is not (so  $X$  wins  $x$  seats and  $Y$  wins  $(y + 1)$  seats). As the quota is

<sup>34</sup> Gallagher, ‘Proportionality, Disproportionality and Electoral Systems’.

reduced, each party's remainder increases. For every vote by which the quota decreases,  $r_X$  increases by  $x$  votes and  $r_Y$  increases by  $y$  votes, since each party is able to 'free' one vote previously locked up in each of its full quotas. Since  $x > y$ ,  $X$ 's remainder is increasing more rapidly than  $Y$ 's initially larger remainder, and eventually the point will come when  $r_X$  either overtakes  $r_Y$  (in which case either both remainders are rewarded, and  $X$  has gained a seat at the expense of another party, or neither remainder is rewarded and  $Y$  has lost a seat to another party, or  $r_X$  is rewarded but  $r_Y$  is not, and  $X$  has gained a seat from  $Y$ ) or becomes greater than the quota (in which case  $X$  gains a seat, possibly at the expense of  $Y$ ). Thus, a reduction in the size of the quota can cause a seat to switch from a smaller to a larger party.

The reasons why a reduction in the size of the quota cannot possibly cause a seat to switch from a larger to a smaller party are essentially similar. This time assume that  $r_X$  is rewarded (clearly, if  $r_X$  is not rewarded, a reduction in the size of the quota cannot possibly lead to  $X$  losing a seat, since its vote total of more than  $x$  quotas guarantees it  $x$  seats even at the initial value of the quota). If  $r_X \geq r_Y$  then, as the quota decreases,  $v_X$  will be worth  $(x + 1)$  quotas before  $v_Y$  is worth  $(y + 1)$  quotas, since  $X$ 's remainder increases more rapidly than  $Y$ 's, as outlined in the previous paragraph. Therefore, even if  $Y$  picks up an extra seat this cannot happen at the expense of  $X$ . If  $r_Y > r_X$  then, since  $r_X$  is rewarded, clearly  $r_Y$  must be as well, so with the quota at the initial value  $q$   $X$  wins  $(x + 1)$  seats and  $Y$  wins  $(y + 1)$  seats. In this case, as the quota is gradually reduced it is possible that  $v_Y$  will be worth  $(y + 1)$  full quotas before  $v_X$  is worth  $(x + 1)$  full quotas. However, when this happens  $Y$ 's new remainder, over and above its  $(y + 1)$  full quotas, must be below  $X$ 's remainder over and above its  $x$  full quotas. Hence it can never happen that  $Y$  wins  $(y + 2)$  seats while  $X$  is reduced to  $x$  seats. Thus, when the quota is reduced, a smaller party cannot gain a seat at the expense of a larger one.

In an entirely analogous way, it is straightforward to show that an increase in the size of the quota might lead to a seat moving from a larger to a smaller party but cannot lead to one moving in the other direction.