Topology and the Four Color Theorem

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1 Introduction

This project will use a combination of graph theory and topology to investigate graph coloring theorems. To color a graph means to assign a color to each vertex in the graph so that two adjacent vertices are not the same color. A very famous coloring theorem is the Four Color Theorem. This answers the question "Can a planar graph be colored with four colors such that adjacent vertices are not colored with the same color?" Before we can begin to understand the answer to this question, we must know what a planar graph is. A planar graph is one that can be drawn on a plane in such a way that there are no edge crossings, i.e. edges intersect only at their common vertices. This question was first asked in 1852 by Francis Guthrie but at that time no one was able to give a definite answer. The problem was ignored until 1878 when Arthur Cayley presented it to the London Mathematical Society. Mathematicians presented proofs but each time gaps were found. However, in 1890 mathematician P.J. Haewood was able to give the first proof of the Five Color Theorem which stated that every planar graph can be colored with five colors [2]. In this project, I will provide a proof of the Five Color Theorem. It was not until quite recently that the Four Color Theorem was proven. This accomplishment was made by Kenneth Appel and Wolfgang Haken at the University of Illinois. However, not all mathematicians approve of their work because it relies mostly on work done by computer. There is still not a proof of the Four Color Theorem that can be completed by hand.

For this reason, we are going to put more focus on the coloring of graphs on a torus and other non-planar surfaces. P.J. Haewood was the first mathematician to describe a map of the torus. He was also the first to prove that each map on the torus is colorable with 7 colors. He did this by proving an inequality that provided an upper bound on the number of colors required to a color a graph on any surface. In the case of the torus, the inequality proved that the chromatic number for the torus had to be less than or equal to 7. Haewood was able to provide an example of a coloring on the torus which required 7 colors [2]. This example provided a lower bound for the chromatic number. Therefore, the chromatic number of the torus has to equal 7. I will provide a proof of this Seven Color Theorem.

The coloring theorems that will be discussed for planar graphs can actually be thought of as graphs on the sphere since a graph on the sphere can be made a graph on the plane by removing one point from the sphere.

I first learned of graph coloring and the Four Color Theorem when I took Math 423 and I found it to be very interesting. That is the main reason why I chose to work on this project. I knew that the Four Color Theorem was not able to be proven by hand, but when I heard that topology could be used to prove related theorems I thought that would be an intriguing project. Most people believe me when I tell them that the map of the world can be colored using only four colors but when I add the restrictions of the theorem they become a little more skeptical. In fact, I have presented the problem to my high school students who are in disbelief until we begin to discuss it. That is the other reason why I am glad that I chose this topic. There may not be a lot of practical applications to coloring a torus but I have been able to describe a collegiate level math problem to high school students and explain it in such a way that they can at least grasp the basic concepts. This has given some of them an interest in mathematics and curiosity in the subject which they hope to fill in their own educational experiences.

2 Definitions

The following definitions use a lot of notation. To make things a little easier, I have set up some common notation that will be used throughout the entire project. The number of vertices in a graph will be denoted by v, the number of edges will be denoted by e, and similarly the number of faces will be denoted by f.

I have already given the definition for a planar graph, but we should also have a clear definition for a graph.

Definition 1 *A* ***graph*** *is an ordered pair* (V, E) *where* V *is a set of vertices and* E *is a set of two element subsets of* V *called edges. Informally, a graph is a collection of vertices and edges that join pairs of the vertices.*

There is a mathematical term for the number of colors required to color a graph. This term is the chromatic number. Therefore, if we say that a graph is four colorable, then that graph has a chromatic number of 4. We are also going to be considering the coloring of surfaces, specifically the torus. Therefore, we also need to know the definition for the chromatic number of a surface.

Definition 2 *The* ***chromatic number*** $\chi(G)$ ***of a graph*** G *is the minimum number of colors required to color* G *such that two adjecent vertices are not the same color.*

Before we can define the chromatic number of a surface, we need to know the definition for a graph being embedded in a surface.

Definition 3 *The graph G can be* ***embedded*** *into the surface S if there exists a polyhedron P on S such that the 1-skeleton of P has a subgraph homeomorphic to G* [2].

Definition 4 *The* ***chromatic number*** $\chi(S)$ ***of the surface*** *S is the maximum chromatic number of the graphs embedded in S.*

Some of the theorems that we will be using only apply to critical graphs. However, the definition of a critical graph requires us to study each proper subgraph of the graph G. Therefore, let's first look at the definition for a proper subgraph.

Definition 5 *If $H = (V', E')$ and $G = (V, E)$ are two graphs and $V' \subseteq V$ and $E' \subseteq E$ and if $\{v, w\} \in E'$ then $v, w \in V'$, then H is called a subgraph of G. If $H \neq G$, then H is a* ***proper subgraph*** *of G* [2].

Definition 6 *A graph G is* ***critical*** *if each proper subgraph has a chromatic number smaller than that of G* [3].

Since we are going to be studying graphs on the torus, we will need to understand properties of polyhedra. The definition for a polyhedron relies on having a closed pseudograph, so let's first define a pseudograph.

Definition 7 *A* ***pseudograph*** *consists of a set of vertices V and a set of two element subsets or one element subsets of V called edges.*

Definition 8 *Given a finite number of polygons, let the total number of all of the sides be even. Suppose these sides are given in pairs. Label the sides with letters so that both sides of a pair get the same letter. Also, assume that each side is given an orientation indicated by an arrow. Now identify each pair of sides such that the heads of the two arrows coincide. Two superimposed sides are called an edge. The figure obtained is called a* ***polyhedron*** [2].

In order to relate topology and coloring theorems to each other, we are going to need to use the Euler Characteristic of a polyhedron.

Definition 9 *Let F be any polyhedron. Then after identification, the number $v - e + f$ is called the* ***Euler Characteristic*** *of the polyhedron and is denoted by $E(F)$* [1].

The polygons that form the polyhedron are called the faces of the polyhedron. Sometimes it is difficult to draw a polyhedron and therefore fully understand it. So, in most cases we will look at the plane representation of the polyhedron.

Definition 10 *The collection of the polygons, the pairing of the sides, and the orientation of the sides before identification is defined to be the* ***plane representation of the polyhedron*** [2].

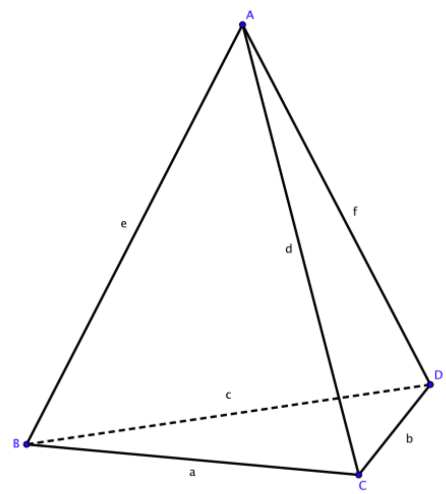

Figure 1: Tetrahedron

Example 1 Consider an octagon. We will want to label all of the edges with letters and we also want to make sure that each letter gets assigned to two different edges. Therefore, we will use the first four letters of the alphabet two times each for our labeling. Now, we can label the edges of the polygon using those letters in any order that we want. Once the labeling is finished, we will assign an orientation to each edge by drawing an arrow on the edge. The direction of the arrow is the direction of the edge. Consider the two edges that are labeled with an a. Give them opposite orientations. This is true for each pair of edges that are labeled the same letter. Now that we have labeled the edges of the polygon and assigned orientations to the edges, we are left with the plane representation of the polyhedron. To obtain the polyhedron, we will begin to match up the edges that are labeled with the same letter so that their orientations coincide. Therefore, we want the heads of the arrows to be pointing in the same direction after we match up the sides. After we have done that, we should end up with a closed figure which is the polyhedron. I should note that different polyhedra can result from the same polygon. The resulting polyhedron is completely dependent on the plane representation that is chosen.

Example 2 This example shows how to determine the plane representation of the tetrahedron. You start by labeling the sides of the tetrahedron. This is shown in Figure 1. Then, draw the tetrahedron net which is to draw an unfolding of the tetrahedron. You should label the edges of the net

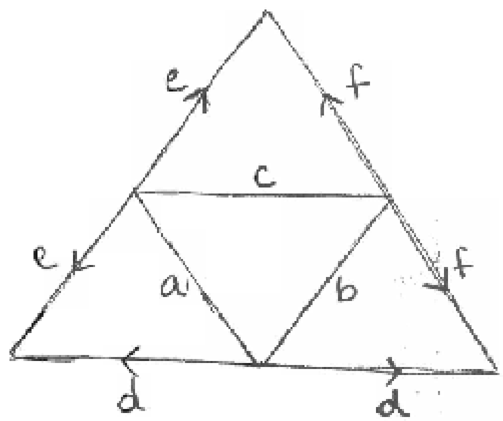

Figure 2: Tetrahedron net

the same way you labeled the tetrahedron. You will also want to draw in any arrows necessary so you know the orientation in which the sides must match up. The tetrahedron net is shown in Figure 2. Now, you can use the tetrahedron net to develop the plane representation. This is when the direction of the arrows is most important. Figure 3 is the plane representation of the tetrahedron in Figure 1.

Definition 11 *If P is a polyhedron we define the pseudograph consisting of all the edges and vertices of P to be the* ***1-skeleton*** *of P* [2].

In some cases, we may not want to consider a polyhedron in its entirety. If that is the case, we can study what is known as the partial polyhedron. When we study a partial polyhedron, there are edges of the polyhedron that are not being identified. These edges are called boundary edges. Once we identify the boundary edges and any vertices that are incident to at least one boundary edge, we can form a pseudograph.

Definition 12 *Consider a proper subset D of polygon sides. Let the number of sides in D be even. Label these sides with letters such that each letter appears exactly twice and assign arrows to the sides so that they are oriented in some manner. This gives us the plane representation of a partial polyhedron. When we match up the sides that are labeled with the same letter such that the direction of their arrows coincide, we have the* ***partial polyhedron****. There must be at least one polygon side that is not included in D. The polygon side(s) that are not included in the proper subset D are not labeled or oriented. These side(s) are called the* ***boundary edges of the partial polyhedron****. In a partial polyhedron the boundary edges and the vertices which are incident with at least*

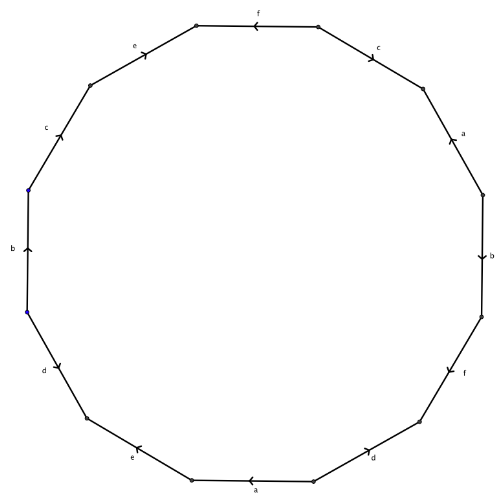

Figure 3: Plane representation of tetrahedron

one boundary edge form a pseudograph. This pseudograph is defined to be the ***boundary pseudograph of the partial polyhedron*** [2]*.*

We are also going to discuss two operations that can be applied to a polyhedron. These are a subdivision of dimension one or a composition of dimension one.

Definition 13 *A* ***subdivision of dimension one*** *is the process by which an edge a of the polyhedron P is divided into two new edges b and c by taking an inner point of a as an additional vertex* [2]*.*

Definition 14 *A* ***composition of dimension one*** *is the process by which two edges b and c that are incident to a common point are composed into one edge a by removing the common point. Essentially, this is the reverse of a subdivision of dimension one* [2]*.*

Some theorems that we are going to use require us to compare graphs in order to reach our result, however, we cannot just choose any two graphs that we want. The two graphs must be homeomorphic. The definition for homeomorphic also requires the definition for isomorphic graphs.

Definition 15 *Two graphs G, G' are called* ***isomorphic*** *if there exists a one-to-one correspondence between the set of vertices of G and the set of vertices of G' such that two vertices are adjacent in G if and only if the corresponding vertices in G' are adjacent* [2]*.*

Definition 16 *Two graphs G and G' are* ***homeomorphic*** *if G can be transformed into a graph isomorphic to G' solely by applying compositions of dimension one and subdivisions of dimension one a finite number of times* [2].

Also, there are a few proofs that require us to consider paths and circuits.

Definition 17 *A* ***path*** *is a sequence of distinct vertices with an edge connecting each pair of successive vertices.*

Definition 18 *A path is a* ***circuit*** *if there is an edge connecting the first and last vertices.*

3 Results

The first theorem we are going to look at is Euler's Formula. This theorem shows us the relationship between the number of vertices, edges, and faces in a polyhedron. We will be able to use the following results for graphs on any surface.

Theorem 19 *(Euler's Formula)Let S be a polyhedron with v vertices, e edges, and f faces. Let G be the 1-skeleton of S. Then $v - e + f = 2 - g$ for some $g \in \mathbb{N}$* [2].

This proof is beyond the scope of this project.

Now that we know Euler's Formula, we can prove a corollary to the theorem. This corollary provides us with an algorithm that we can use to find the number of edges a planar graph has as long as the graph has three or more vertices.

Corollary 20 *A graph with $v \geq 3$ vertices has at most $3v - 6 + 3g$ edges* [3].

Proof. Assume a graph has $v \geq 3$ vertices. Let each edge be made of two directed edges that go in opposite directions of each other. Then, to calculate the degree of a face, start at one vertex and walk your way around the face counting all of the directed edges you must take to return to the vertex you started at. This way of calculating the degree of a face is unique to this proof. Note then that the sum of the degrees of the faces is exactly twice the number of edges in the graph because each edge is counted exactly twice, either two times in the same face or once in 2 adjoining faces. Since each face must have degree greater than or equal to 3, it follows that

$$2e = \sum \deg(F) \geq 3f$$

where F is a face and the sum is taken over all of the faces. Therefore, $\frac{2}{3}e \geq f$. Using Euler's Formula, $f = e - v + 2 - g$, we get

$$e - v + 2 - g \leq \frac{2}{3}e.$$

Then, it follows that $\frac{e}{3} \leq v - 2 + g$ which shows us that $e \leq 3v - 6 + 3g$. ∎

The next theorem is commonly known as the Handshake Theorem. Basically, this explains how the number of edges is overcounted by a factor of 2. Consider a group of people who are meeting each other for the first time. They all shake hands with one another. Each handshake is counted by two different people. Therefore, if you would ask each person in the room the number of people they shook hands with and then took the sum of their responses, the answer would actually be double the number of handshakes that actually took place. The same is true when counting the number of edges in a graph. This is because each edge is incident to two vertices. Therefore, when we take the sum of the degrees of all of the vertices in a graph, each edge is being considered twice.

Theorem 21 *(Handshake Theorem)If $P_1, P_2, ..., P_v$ are the vertices in a graph, then $\sum_{i=1}^{v} \deg(P_i) = 2e$ where e is the number of edges in G* [2].

Proof. Let $P_1, P_2, ..., P_v$ be the vertices in a graph. The degree of a vertex is defined to be the number of edges incident to that vertex. Therefore, each edge is being counted twice since every edge has two endpoints. So, the sum of the degrees of the vertices will be twice the number of edges in the graph G. ∎

I have already mentioned that the Four Color Theorem cannot be proven by hand. However, the Five Color Theorem can be. Even though this theorem is less restrictive than the Four Color Theorem, it is still beneficial to see the proof.

Theorem 22 *Every planar graph is five colorable* [3].

Proof. Assume towards a contradiction that G is a planar graph with the fewest number of vertices that cannot be 5-colored. Let w be a vertex in G that has the minimum degree. Using the Handshake Theorem, $2e = \sum \deg(u)$ where u is the set of all the vertices in G. Suppose that there are v vertices and that each vertex has degree greater than or equal to 6. Then, the $\sum \deg(u)$ $\geq 6v$ which means that $2e \geq 6v$. Dividing both sides of the inequality tells us that $e \geq 3v$ which is greater than $3v - 6$. Thus, we have $e \geq 3v - 6$ which contradicts Corollary 20. So, we must negate our statement that each vertex has degree greater than or equal to 6. This gives us the statement that there exists at least one vertex with degree less than 6. We will let that vertex be w. So, we know that $\deg(w) < 6$.

Case 1 $\deg(w) \leq 4$. We will define the graph $G - w$ to be graph G with the vertex w removed and therefore all of the edges incident to w removed as well. Thus, the graph $G - w$ is 5-colorable by definition. At most 4 colors have been used for the neighbors of w. Therefore, there is one color left for w. So, G is 5-colorable which is a contradiction.

Case 2 $\deg(w) = 5$. The graph $G - w$ is 5-colorable. If at least 2 vertices neighboring w are the same color, then w can be the fifth color. If not, then we can assume the 5 neighbors of w are colors 1, 2, 3, 4, and 5. Construct a circle around w such that it intersects all of the edges between w and its 5 neighboring vertices and contains no other vertices or edges. Pick a point on the circle to be north. Then, we can label the vertices in clockwise order from north as w_1, w_2, w_3, w_4, w_5. Assume w_i is color i. Consider the subgraph of G that contains all of the vertices that are colors 1 and 3 along with the edges that connect only those vertices. If this subgraph does not contain a path between w_1 and w_3, then w_1 and w_3 are disconnected. If w_1 and w_3 are disconnected, then we can color w_3 color 1 instead. Take the connected component of the subgraph containing w_3, and change all of the color 3 vertices to color 1 and all of the color 1 vertices to color 3. This process will not have any effect on the rest of the graph since there are no vertices of color 1 or color 3 that are not included in the subgraph. Therefore, this allows color 3 for w. However, this is then a contradiction to our original statement that G is the smallest planar graph that isn't 5-colorable. Therefore, there must be a path between w_1 and w_3 in the subgraph. Now, let's consider w_2 and w_4. Once again, we must look at a subgraph of G, but this time we will look at the subgraph that contains the vertices that are colors 2 and 4 along with the edges that connect only those vertices. If there is no path in this subgraph that connects w_2 and w_4, then we can color them both color 2. Now, take the connected component of this subgraph that contains w_4, and change all of the color 4 vertices to color 2 and all of the color 2 vertices to color 4. Then, w can be color 4. However, this leads to the same contradiction as before. Thus, there must be a path between these two vertices as well. However, there is already a circuit between w_1 and w_3 which establishes two regions. I will call these the inside and outside regions. The vertex w_2 is in the inside region and w_4 is in the outside region. Therefore, the only way for a path to connect w_2 and w_4 is by crossing the circuit. So, there is no possible way to insert a path between w_2 and w_4 and maintain planarity. Thus, we must be able to color G with five colors.

$\blacksquare$

Theorem 23 K_5 *is not planar.*

I am first going to show a proof of this theorem using an algorithm I learned in Math 423. I believe that this proof allows you to use more intuition than actual rigor.

Proof. Start by constructing C_5 which is the cycle graph with 5 vertices. Then, we can add two edges that are in the center of our graph and still maintain planarity. Without loss of generality, choose the edge between v_1 and v_3 and the one between v_1 and v_4. Now, no other edges can be placed in the center without crossing over these two edges. We must connect v_5 to v_2 and v_3. We can make an edge from v_5 to v_2 that blocks v_1. Now, we have two choices for the edge between v_5 and v_3. We can either block v_1 and v_2 or block v_4. Keeping in mind that we still need an edge connecting v_2 and v_4, we see that either choice will result in a non-planar graph. Thus, we will choose to draw the edge between v_5 and v_3 by blocking v_1 and v_2. Of course, now that v_2 is blocked we cannot construct the edge from v_2 to v_4 so K_5 cannot be a planar graph. ■

I am now going to provide a more rigorous proof which uses the corollary of Euler's Formula.

Proof. The graph K_5 has 5 vertices and $\frac{5(4)}{2} = 10$ edges. From Corollary 20, we know that $e \leq 3v - 6$ in every planar graph. Therefore, we can see that this corollary is not satisfied since $3(5) - 6 = 9 < 10$. So K_5 is not planar. ■

The following theorem provides us with a relationship between the chromatic number of a graph and the degree of each vertex in the graph.

Theorem 24 *If G is critical with chromatic number χ, then the degree of each vertex of G is greater than or equal to $\chi - 1$* [2].

Proof. Let G be a critical graph with chromatic number χ. We will assume by contradiction that G has a vertex P of degree $d < \chi - 1$. We will label the d vertices adjacent to P by $P_1, P_2, ...P_d$. Next, remove vertex P and all of the edges connected to P. We now have the graph $G - P$ which is a proper subgraph of G. Thus, $\chi(G - P) < \chi(G)$ and there is a coloring of $G - P$ using less than or equal to $\chi - 1$ colors. Consider such a coloring of $G - P$. For the d vertices, $P_1, P_2, ...P_d$, no more than $d \leq \chi - 2$ different colors are used. Then, there is at least one of the $\chi - 1$ colors not being used for the d neighbors of P. Assigning this color to P gives a coloring of G with $\chi - 1$ colors or less. Therefore, χ is not the chromatic number of G which is a contradiction to the original assumption. Thus, the degree of each vertex of G is greater than or equal to $\chi - 1$. ■

There is also a relationship between the number of vertices, edges, and the chromatic number of a graph G. The proof of this theorem uses both the Handshake Theorem and the previous theorem.

Theorem 25 *If G is a critical graph with v vertices and e edges, and G has chromatic number χ, then the relation $(\chi - 1)v \leq 2e$ holds* [2].

Proof. Assume G is a critical graph with v vertices, e edges and chromatic number χ. By Theorem 24 each vertex of G has degree greater than or equal to $\chi - 1$. Then, using Theorem 21, we get $(\chi - 1)v \leq 2e$. ■

We know that two graphs can be homeomorphic. If that is the case, then the difference between the number of edges and vertices is the same for both graphs.

Theorem 26 *If the graphs G and G' are homeomorphic, then $v - e = v' - e'$* [2].

Proof. In a subdivision of dimension one, one edge is replaced by two edges and one additional vertex. Therefore, we are adding a vertex and adding an edge so the value of $v - e$ is unchanged. We must also consider if this holds for a composition of dimension one. In a composition of dimension one, two edges are composed into one edge by removing the common vertex. Therefore, we are subtracting a vertex and an edge so it stays balanced. Therefore, the value of $v - e$ is unchanged. ■

Usually, a vertex can have any degree greater than or equal to 1. This is not the case for vertices in a boundary pseudograph. The next theorem shows us that each vertex in a boundary pseudograph has degree 2. This unique quality will be useful later.

Theorem 27 *Each vertex of the boundary pseudograph of a partial polyhedron is of degree 2* [2].

Proof. Consider the plane representation of a polyhedron with a vertex v_0 incident with a boundary edge. If this vertex v_0 is incident with no labeled edge of the polygon then the degree of v_0 is 2 even after identifying all of the labeled edges. If not, the only other option for v_0 is for it to be incident to one boundary edge and one labeled edge. Then, it could get identified to another vertex that is either incident to a boundary edge and a labeled edge or one that is incident to two labeled edges. In regard to the first case, the identification process would be over and the vertex would have degree 2. In regard to the second case, the identification process would continue until the vertex was incident to one boundary edge. At that point, the identification process would have to stop and the vertex would have degree 2. ■

We have already established a bound for the Euler characteristic of a polyhedron. Now, we will prove that the Euler characteristic for a partial polyhedron also has an upper bound.

Theorem 28 *If T is a partial polyhedron, then $E(T) \leq 1$* [2].

Proof. Let T be a partial polyhedron. The boundary pseudograph B of T has only vertices of degree 2. Therefore, B consists of k circuits where $k \geq 1$. Let $s_1, s_2, ..., s_k$ be the lengths of these circuits. Then take k additional polygons, namely, an s_1- gon, an s_2- gon,..., and an s_k- gon. Identify the boundary of the s_i- gon with the corresponding circuit of boundary sides of T where

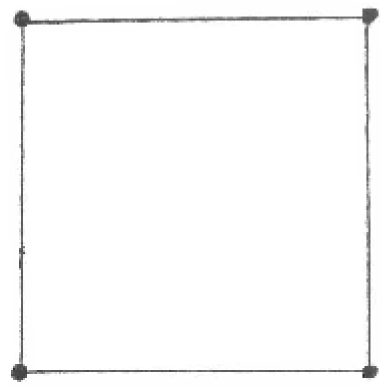

Figure 4: Graph G

$i = 1, 2, ..., k$. The polyhedron P which is obtained has $f + k$ polygons where f is the number of faces of T. So, we have

$$E(P) = v - e + (f + k) = E(T) + k \leq 2.$$

Since $k \geq 1$, we have shown that $E(T) \leq 1$. ■

The next theorem requires us to form partial polyhedra by cutting a polyhedron along a graph. This can be difficult to imagine, so we will look at two examples that show how partial polyhedra are formed. Let Figure 4 be Graph G. We are going to use composition of dimension one to make this a graph with only three vertices and three edges. Then we can embed the graph onto the tetrahedron. This is shown in Figure 5 and is now known as the graph G'. To create the partial polyhedra, we will cut along the edges of G'. This will form two partial polyhedra which are shown in Figure 6 and Figure 7. The first partial polyhedron, T_1 is the tetrahedron with one face removed. The second partial polyhedron, T_2 is the face of the tetrahedron. Now, we will change our initial graph slightly. Let Figure 8 be Graph H. Again, we will use composition of dimension one but this time we will get a graph, H', that has three vertices and only two edges. We will embed this graph onto the tetrahedron as shown in Figure 9. When we cut along the edges of H' this time it will not completely split the tetrahedron into seperate pieces. Instead, we are left with the polyhedron but one of the faces is now a flap, meaning it is only connected by one edge. Hopefully, this example helps explain the process of making the partial polyhedra. It also shows that in some instances, there is only one partial polyhedra formed.

Theorem 29 *Let G be a graph where each vertex has degree greater than or*

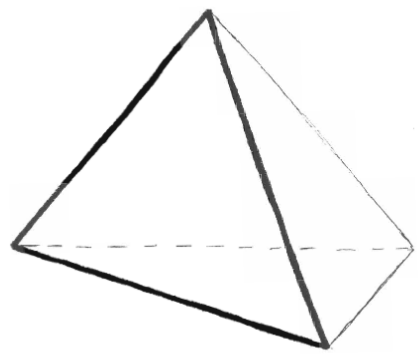

Figure 5: Graph G' on the Tetrahedron

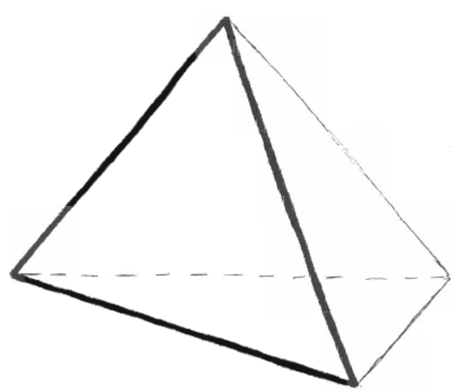

Figure 6: Partial Polyhedron T_1

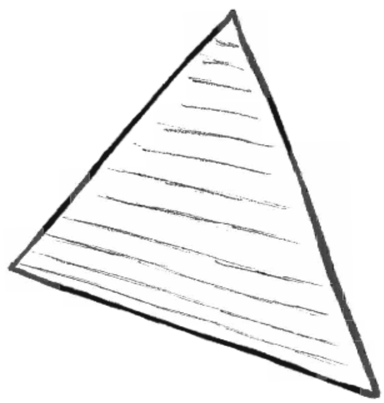

Figure 7: Partial Polyhedron T_2

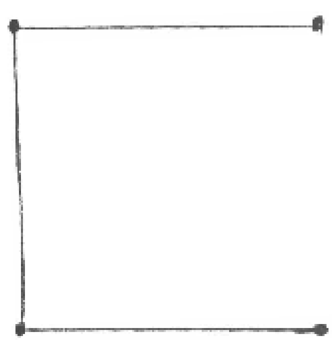

Figure 8: Graph H

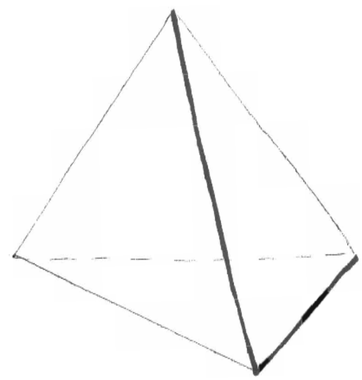

Figure 9: Graph H' on the Tetrahedron

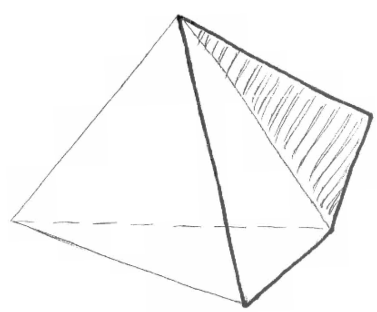

Figure 10: Partial Polyhedron

equal to 2. If G can be embedded into a closed surface S, then $e \leq 3v - 3E(S)$ [2].

Proof. Let G be a graph in which each vertex has degree greater than or equal to 2. Also, let v be the number of vertices and e be the number of edges in G. Assume G can be embedded into a closed surface S. Then, by definition, there exists a polyhedron P on S and a subgraph G' of the 1-skeleton of P which is homeomorphic to G. Each edge of G' corresponds to two polygon sides in the plane representation of P. Therefore, there are $2e'$ polygon sides which correspond to the edges of G'. Now, omit the labeling of these $2e'$ polygon sides. By doing this, the polyhedron is broken down into partial polyhedra $T_1, T_2, ...T_t$. You may think of this process as cutting the polyhedron along G'. Let $\beta_0^{(i)}$ be the number of vertices, $\beta_1^{(i)}$ be the number of edges, and $\beta_2^{(i)}$ be the number of faces in T_i where $i = 1, 2, ..., t$. Let $\beta_0, \beta_1, \beta_2$ be these values for the polyhedron P. We want to find these values. When we cut the polyhedron along G', one vertex could get split into several different partial polyhedron. Consider a disc around a specific vertex. If that disc gets split into two regions after the cutting process, then that specific vertex becomes two vertices. Similary, if the disc gets split into three regions, then the specific vertex will become a vertex in three different partial polyhedra. Therefore, we are adding $\deg(v) - 1$ vertices for every vertex in G'. So, the number of vertices in the polyhedron is

$$\begin{aligned} \beta_0 &= \sum_{i=1}^{t} \beta_0^{(i)} - \sum_{v \in G'} \deg v + \sum_{v \in G'} 1 \\ &= \sum_{i=1}^{t} \beta_0^{(i)} - \sum_{v \in G'} \deg v + v'. \end{aligned}$$

To find β_1, we will find the sum of the edges in all of the partial polyhedra. Here, we must remember that the edges we cut along became an edge in two different partial polyhedra. Therefore, we must subtract the number of edges in G'. So, the number of edges in the polyhedron is

$$\beta_1 = \sum_{i=1}^{t} \beta_1^{(i)} - e'.$$

The last value we need to find is the number of faces in the polyhedron. This is simply the sum of the faces in all of the partial polyhedra.

$$\beta_2 = \sum_{i=1}^{t} \beta_2^{(i)}$$

By the Handshake Theorem, the sum of the degrees of all vertices in the graph G' is equal to $2e'$. Let $E(S)$ be the Euler characteristic. Then, using Euler's formula we get

$$
\begin{aligned}
E(S) &= \beta_0 - \beta_1 + \beta_2 \\
&= \sum_{i=1}^{t} \left(\beta_0^{(i)} - \beta_1^{(i)} + \beta_2^{(i)}\right) - 2e' + v' + e' \\
&= \sum_{i=1}^{t} \left(\beta_0^{(i)} - \beta_1^{(i)} + \beta_2^{(i)}\right) + v' - e' \\
&= \sum_{i=1}^{t} E(T_i) + v^{'} - e^{'}
\end{aligned}
$$

Using Theorem 26 we obtain

$$E(S) = \sum_{i=1}^{t} E(T_i) + v - e.$$

Then, we can use Theorem 28 to get

$$E(S) \leq t + v - e.$$

Now, let's compare t to e. Since G has no vertex with degree less than or equal to 1, each circuit must have at least length 3. Each edge in G is represented by a path in G'. Therefore, the boundary pseudograph of each T_i represents at least 3 edges of G. Each edge in G corresponds to 2 or 1 of the partial polyhedra T_i. Thus, $3t \leq 2e$ so we obtain that

$$
\begin{aligned}
3E(S) &\leq 3t + 3v - 3e \\
3E(S) &\leq 3t + 3v - 2e - e \\
3E(S) &\leq 3v - e.
\end{aligned}
$$

Therefore, we have shown that if G can be embedded into a closed surface S, then $e \leq 3v - 3E(S)$. ∎

4 Main Result

The main goal of this project is to prove that every graph on the torus is seven colorable. I have included Figure 11 along with a description of the figure that can be used as some intuition into the 7 Color Theorem. I also have a rigorous proof of Haewood's Map Coloring Problem.

Figure 11 shows a torus that has been broken down into rectangles. Label one of the rectangles color 1. That rectangle is adjacent to six others. All six of these rectangles are adjacent to six others as well because of the way they wrap

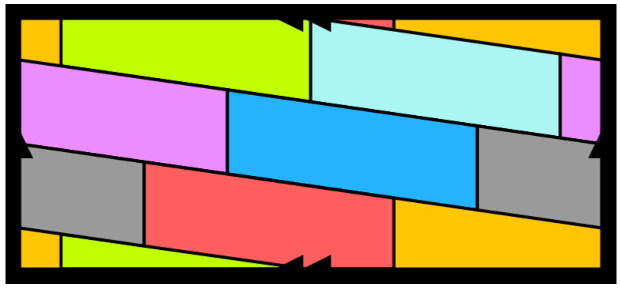

Figure 11: A map on the torus that requires 7 colors. http://en.wikipedia.org/wiki/Four_color_theorem

around the edges to form the torus. So, every vertex has degree 6. Therefore, in order to color this graph we must use at least 7 colors.

The following theorem requires us to know that there is an upper bound on the number of colors required to color any graph on a fixed surface. This is not immediately clear. The degree of a vertex can be as large as we want it to be. Intuitively, then, the chromatic numbers of the graphs on a surface could keep getting larger and larger. Therefore, it seems as if the chromatic number of a surface could be infinite. This is not the case, however. We will show that the 1-skeleton of a polyhedron is colorable. Euler's Formula states that $v-e+f=2-g$ where $g \in \mathbb{N}$. Earlier, we also determined that $e \leq 3v-6+3g$. Now, let's consider the critical graph G that cannot be seven colored. Let w be the vertex in G with the minimum degree. Suppose $\deg(w) \geq k \forall w \in G$. Then $kv \leq \sum \deg(w) = 2e$, so $\frac{kv}{2} \leq e$. We can now substitute this in for e in the above equation. Then, we have

$$\begin{aligned} \frac{kv}{2} &\leq 3v-6+3\dot{g} \\ kv &\leq 6v-12+6g \\ k &\leq \frac{6v-12+6g}{v} \\ k &\leq 6-12+6g \end{aligned}$$

This relationship now lets us set an upper bound to the chromatic number of the 1-skeleton of a polyhedron. Since every embedded graph on the torus is homeomorphic to a subgraph of the 1-skeleton, they can be colored as well. Therefore, every graph on the torus is colorable. For instance, if we set $g=2$ we find that

$$\begin{aligned} k &\leq 6-12+6(2) \\ k &\leq 6. \end{aligned}$$

Therefore, there must be at least one vertex in the graph that has degree less than or equal to 6 so the $\deg w \leq 6$. If we remove w from G, then G is seven colorable. Since w has degree less than or equal to 6, it can have at most 6 neighbors. Color the neighbors. If each of the neighbors needs a different color, then only 6 colors have been used. Therefore, there is a seventh color still available for w.

This seems to be a nice way to prove that the Seven Color Theorem for graphs on a torus is true, however, it is not as sharp as Theorem 30. This is because Theorem 30 can be used for graphs on any surface and it turns out that this method only works nicely for $g = 0$ or $g = 2$. Beyond that, this method extremely overestimates the number of colors required. It is a useful tactic though since it does show us that for a fixed surface there is an upper bound on $\chi(S)$.

Theorem 30 *If S is a closed surface with Euler characteristic $E(S) \neq 2$ then $\chi(S) \leq \left\lfloor \frac{7+\sqrt{49-24E(S)}}{2} \right\rfloor$ [2].*

Proof. There exists a graph G which can be embedded into S such that $\chi(G) = \chi(S)$. We can assume that G is critical, therefore, each proper subgraph of G has a chromatic number smaller than that of G. If G were not critical, we could choose a critical subgraph of G that has the same chromatic number. Since G is critical, we know that every vertex in G has degree at least $\chi - 1$ from Theorem 24. Also, K_4 can be embedded on every surface since it is planar and the degree of every vertex in K_4 is 3. Therefore, χ must be at least 3 so every vertex in G must have degree greater than or equal to 2. By Theorems 25 and 29 it follows that if $\chi(G) = \chi$ and $E(S) = E$ then $(\chi - 1)v \leq 2e$ and $e \leq 3v - 3E$ where v is the number of vertices and e is the number of edges in G. By substitution of those two inequalities, we find that $(\chi - 1)v \leq 6v - 6E$. Now, dividing both sides of this inequality by v gives

$$\chi - 1 \leq 6 - \frac{6E}{v}.$$

We will now assume that $E(S) \leq 0$ since we want to prove this true for graphs on a surface with $g = 0$. Since χ is defined to be the number of colors required to color the graph, we know that χ will not exceed the number of vertices in the graph and $\chi \geq 1$. Thus, $v \geq \chi$ so we can replace v with χ. This gives $\chi - 1 \leq 6 - \frac{6E}{\chi}$. We will now multiply by χ to get rid of the fraction which gives

$$\chi^2 - \chi \leq 6\chi - 6E.$$

To solve this inequality, we want to move everything to the left hand side which makes the right hand side 0. Thus, we have

$$\chi^2 - 7\chi + 6E \leq 0.$$

Using the quadratic formula, we can rewrite this inequality in the form

$$\left(\chi - \frac{7+\sqrt{49-24E}}{2}\right)\left(\chi - \frac{7-\sqrt{49-24E}}{2}\right) \leq 0.$$

Since $E \leq 0$, it follows that $\sqrt{49 - 24E} \geq 7$. Also, since $\chi \geq 1$, the second factor of our inequality will always be positive. It follows then that the first factor must be less than or equal to 0. Thus, we have proven that $\chi(S) \leq \left\lfloor \frac{7+\sqrt{49-24E(S)}}{2} \right\rfloor$.
■

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