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EXISTENCE OF THE SOLUTIONS OF FORCED PENDULUM EQUATION BY VARIATIONAL METHODS

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In această lucrare este reluat un rezultat de calcul variaţional şi prezentat într-o variantă îmbunătăţită a enunţului şi a demonstraţiei. Impreună cu alte două rezultate clasice ale calcului variaţional, teorema menţionată este implicată în demonstrarea unui rezultat ce caracterizează existenţa a cel puţin două soluţii de perioada T care nu diferă între ele printr-un multiplu de 2π *în anumite condiţii pentru ecuaţia pendulului forţat.*

A result of variational calculus is taken and presented in an improved form both of the statement and proof. Together with other two classical results, the announced theorem is used to demonstrate a statement which characterizes the existence of at least two T-periodic solutions which not differ by a multiple of 2π *in some prescribed conditions for the equation of the forced pendulum.*

Key words: variational calculus, forced pendulum, *T*-periodic solution, minimax theorem, critical point, $(PS)_{c,\,F}$ condition,

1. Introduction

In this paper two variant of the minimax theorem in Finsler manifold and a corollary of the second of them are discussed in order to obtain as an application of these three results a characterization of the solutions of forced pendulum equation.

A minimax theorem (here 2.3, [1], [2]) is retaken and improved, giving also an improved proof (by using Ekeland variational principle, [3], [4], [2]) together with other variant of minimax theorem in Finsler manifold with a corollary of it ([1], [2]).

This approach of the problem of the forced pendulum was used in papers [1-2]. A characterization of two *T*-periodic solutions for the forced pendulum equation which not differ by a multiple of 2π obtained under some conditions is here presented using the above three minimax results.

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For the classical forced pendulum equation,

$$\ddot{x} + A\sin x = f(t)$$

where $A=g/l$ is a constant with g being the gravity constant and l being the length of the pendulum, and f is a T-periodic function which is regarded as an external force, the existence or nonexistence of periodic solutions with prescribed minimal periods is involved in many modeling problems of similar complex fenomena. Other approaches with new results are obtained by applying also the critical point theory and by using a decomposition technique to better estimate the critical values associated with a variational functional [5].

2. Tools: two minimax results and some definitions

In order to prove one important variational theorem of minimax, one gives two classical statements which are used in the demonstrations of our statements.

2.1 Ekeland principle. *Let* (X, d) *be a complete metric space and* $\varphi : X \to \to (-\infty, +\infty]$ *bounded from below, lower semicontinuous and proper. For any* $\varepsilon > 0$ *and* u *of* X *with*

$$\varphi(u) \leq \inf \varphi(X) + \varepsilon$$

and for any $\lambda > 0$, *there exists* v_ε *in* X *such that*

$$\varphi(v_\varepsilon) < \varphi(w) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, w) \ \forall\ w \in X \setminus \{v_\varepsilon\}$$

and

$$\varphi(v_\varepsilon) \leq \varphi(u), \qquad d(v_\varepsilon, u) \leq \lambda$$

([3], [2]).

2.2 Ghoussoub deformation lemma. *Let* X *be a Finsler* C^1*-manifold connected and complete*, $\varphi : X \to \mathbf{R}$ *of* C^1*class*, *and* A, B *nonempty disjoint subsets*, A *closed and* B *compact*.

If $\|d\varphi(x)\| > 2\varepsilon$, $\varepsilon > 0$, $\forall\ x \in B$, *then* $\forall\ \lambda > 1$ *there exist* $g : X \to \mathbf{R}_+$ *continuous*, α *in* $C([0, 1] \times X; X)$ *and* t_0 *in* (0, 1] *such that*, $\forall$ t *from* $[0, t_0)$, *we have*

1° $\alpha(t,x) = x\ \forall\ x \in A$, $\alpha(0, x) = x\ \forall\ x \in X$;

2° $\rho(\alpha(t, x), x) \leq \lambda t\ \forall\ x \in X$;

3° $\varphi(\alpha(t, x)) - \varphi(x) \leq -\varepsilon g(x)t\ \forall\ x \in X$;

4° $g(x) = 1\ \forall\ x \in B$

([1], [2]).

Explanation. ρ is the Finsler metric on X. The elements of $C([0, 1] \times X; X)$ are called *deformations*.

Remark. Ghoussoub lemma remains true by removal only of the property 1° $\alpha(t, x) = x\ \forall\ x \in A$, in the case $A = \varnothing$ (obviously $B \neq \varnothing$) – the only change in the proof being undergone by the function ν (see (4)) : $\nu(x) = 1$ for $x \in B$, $\nu(x) = 0$ for $x \in X \setminus \bigcup_{k=1}^{n} V_k$.

Definition. Let X be a topological space, M a compact subset and $\mathscr{K}$ a nonempty set of compact nonempty subsets of X.

$\mathscr{K}$ is *homotopy-stable* with *boundary M* (*homotopy-stable*, in the case $M = \varnothing$) if

1° $M \subset A \ \forall A \in \mathscr{K}$;

2° For every α in $C([0, 1] \times X; X)$, with the property

$$\alpha(t, x) = x \ \forall (t, x) \in (\{0\} \times X) \cup ([0, 1] \times M),$$

and for every A in $\mathscr{K}$ we have

$$\alpha(\{1\} \times A) \in \mathscr{K}.$$

One presents the two announced minimax results.

2.3 Minimax theorem. *Let* X *be a Finsler* C^1*-manifold connected complete*, φ: X → **R** *of* C^1 *class*, $\mathscr{K}$ *a set of nonempty compact subsets homotopy-stable with boundary* M *and*

$$c = c(\varphi, \mathscr{K}) := \inf_{A \in \mathscr{K}} \sup \varphi(A).$$

If

(F0) $$\sup \varphi(M) < c,$$

then for every sequence $(A_n)_{n\geq 1}$ *from* $\mathscr{K}$ *with* $\lim_{n\to\infty} \sup \varphi(A_n) = c$ *there is a sequence* $(x_n)_{n\geq 1}$ *in* X *with the properties*

1° $\lim_{n\to\infty} \varphi(x_n) = c$,

2° $\lim_{n\to\infty} \|d\varphi(x_n)\| = 0$,

3° $\lim_{n\to\infty} \text{dist}(x_n, A_n) = 0$.

The statement of the theorem remains true in the case $M = \varnothing$ *considering* $\sup \varphi(\varnothing) = -\infty$.

Proof. For every A_n $\exists \rho > 0$ such that $c \leq \sup \varphi(A_n) \leq c + \rho$, let γ_n be the infimum of this numbers ρ, thus $c \leq \sup \varphi(A_n) \leq c + \gamma_n$. We have $\lim_{n\to\infty} \gamma_n = 0$: supposing *par absurdum* the contrary, one finds $u > 0$ such that $\gamma_{k_n} > u \ \forall n$, (γ_{k_n}) a subsequence of (γ_n); but for $n \geq N$ $\sup \varphi(A_n) \leq c + u$, hence for a $k_n > N$ we have $\sup \varphi(A_{k_n}) \leq c + u < c + \gamma_{k_n}$, a contradiction . Then denoting $\varepsilon_n := \gamma_n + \frac{1}{n}$, we have

$$\forall n \ \ c \leq \sup \varphi(A_n) \leq c + \varepsilon_n , \ \varepsilon_n > 0 \text{ and } \varepsilon_n \to 0.$$

This relation shows that if, by means of (1) $c \le \sup \varphi(A) < c + \varepsilon^2$, where $A \in \mathscr{K}$ and $\varepsilon > 0$, one can find x_ε in X such that

$$c \le \varphi(x_\varepsilon) \le c + \varepsilon^2,\ \|d\varphi(x_\varepsilon)\| \le 4\varepsilon,\ \text{dist}\,(x_\varepsilon, A) \le \varepsilon,$$

then the proof is finished.

Consider the subset $\mathscr{D}$ of $C([0, 1] \times X; X)$ of the deformations η with

$$\eta(t, x) = x \text{ for } (t, x) \in (\{0\} \times X) \cup ([0, 1] \times M) \qquad (2)$$

and

$$\sup\{\rho(\eta(t, x), x) : (t, x) \in [0, 1] \times X\} < +\infty \qquad (3)$$

(ρ the Finsler metric on X).

Endow $\mathscr{D}$ with the distance (use (3) and the triangle inequality)

$$d(\eta_1, \eta_2) = \sup\{\rho(\eta_1(t, x), \eta_2(t, x)) : (t, x) \in [0, 1] \times X\} \qquad (4)$$

and one obtains a complete metric space (attention, X is complete!).

Consider the function $\Phi : \mathscr{D} \to \mathbf{R}$,

$$\Phi(\eta) = \sup\{\eta(1, x) : x \in A\}. \qquad (5)$$

This definition is correct since $x \to \varphi(\eta(1, x))$ is continuous and A compact. Φ is lower bounded by c ($\eta(\{1\} \times A) \in \mathscr{K}$!) and lower semicontinuous.

Designate by η_1 the deformation from $\mathscr{D}$

$$\eta_1(t, x) = x \text{ on } [0, 1] \times X \text{ (it is correct, } \rho(x, x) = 0).$$

We have

$$\Phi(\eta_1) = \sup \varphi(A) \overset{(1)}{<} c + \varepsilon^2 \le \inf\{\Phi(\eta) : \eta \in \mathscr{D}\} + \varepsilon^2. \qquad (6)$$

Apply Ekeland principle (2.1) with ε^2 and $\lambda = \varepsilon$, $\exists\ \eta_0$ in $\mathscr{D}$ such that

$$\Phi(\eta_0) \le \Phi(\eta_1), \qquad (7)$$

$$d(\eta_0, \eta_1) \le \varepsilon, \qquad (8)$$

$$\Phi(\eta_0) - \varepsilon d(\eta, \eta_0) \le \Phi(\eta)\ \forall\ \eta \in \mathscr{D}. \qquad (9)$$

Let be $B_0 := \{x \in \eta_0(\{1\} \times A) : \varphi(x) = \Phi(\eta_0)\}$, a nonempty compact set ($\eta_0(\{1\} \times A)$ is compact). By the hypothesis (F0) we have

$$B_0 \cap M = \varnothing \qquad (10)$$

(*par absurdum*, $\eta_0(\{1\} \times A) \in \mathscr{K}$ (the homotopy-stability), the definition of c). It remains to prove

$$\exists\ x_\varepsilon \text{ in } B_0 \text{ such that } \|d\varphi(x_\varepsilon)\| \le 4\varepsilon, \qquad (11)$$

since $x_\varepsilon \in B_0 \Rightarrow c \le \varphi(x_\varepsilon) < c + \varepsilon^2$ ($\varphi(x_\varepsilon) = \Phi(\eta_0) \overset{(7)}{\le} \Phi(\eta_1) \overset{(6)}{<} c + \varepsilon^2$) and dist $(x_\varepsilon, A) \le \varepsilon$ ($d(\eta_0, \eta_1) \overset{(8)}{\le} \varepsilon \Rightarrow \sup_{x\in A} \rho(\eta_0(1, x), x) \le \varepsilon$, but $x_\varepsilon = \eta_0(1, a)$ with $a \in A$, hence $\rho(x_\varepsilon, a) \le \varepsilon$, dist $(x_\varepsilon, A) \le \varepsilon$).

Let us prove (11). Suppose *par absurdum* the contrary, (12) $\|d\varphi(x)\| > 4\varepsilon$ $\forall\ x \in B_0$. Let λ be from the interval (1,2). One applies Lemma 2.2 to the compact nonempty disjoint sets M and B_0 ((10),(12)), there exists g, α and t_0 with the agreed properties. For every τ from $(0, t_0)$ consider the function $\eta_\tau : [0,1] \times X \to \to X$,

$$\eta_\tau(t, x) = \alpha(t\tau, \eta_0(t, x)). \tag{13}$$

$\eta_\tau \in \mathscr{D}$: η_τ is continuous; $(t, x) \in (\{0\} \times X) \cup ([0,1] \times M) \Rightarrow \eta_\tau(t, x) = \alpha(t\tau,$ (14) $\eta_0(t, x)) = \alpha(t\tau, x) = x$ (Lemma 2.2, 1°); $\sup\{\rho(\eta_\tau(t, x), x): t \in [0, 1], x \in X\} < +\infty$ (use (13) and 2° of Lemma 2.2 with a triangle inequality).

$$d(\eta_\tau, \eta_0) = \sup\{\rho(\eta_\tau(t, x), \eta_0(t, x)): (t, x) \in [0, 1] \times X\} = \sup\{\rho(\alpha(t\tau, \eta_0(t, x)), \eta_0(t, x)):$$

$(t, x) \in [0, 1] \times X\} \overset{2.2, 2^0}{\leq} \lambda\tau$ and hence, taking into account (9),

$$\Phi(\eta_\tau) \geq \Phi(\eta_0) - \varepsilon\lambda\tau. \tag{15}$$

Let x_τ be in A such that (16) $\Phi(\eta_\tau) = \varphi(\eta_\tau(1, x_\tau))$ ((5), A is compact). Then taking into account (15), (16) and (57),

$$\varphi(\eta_\tau(1, x_\tau)) - \varphi(\eta_0(1, x)) \geq -\varepsilon\lambda\tau \ \forall\, x \in A. \tag{17}$$

On the other hand, according to 3° of Lemma 2.2,

$$\varphi(\eta_\tau(1, x_\tau)) - \varphi(\eta_0(1, x_\tau)) = \varphi(\alpha(\tau, \eta_0(1, x_\tau)) - \varphi(\eta_0(1, x_\tau)) \leq - 2\varepsilon\tau g(\eta_0(1, x_\tau)).$$

Combining (17) and (18), one finds

$$-\varepsilon\lambda \leq -2\varepsilon g(\eta_0(1, x_\tau)). \tag{19}$$

Take $\tau = \frac{1}{m}$, $m \in \mathbf{N}$ and let $(x_{\frac{1}{k_m}})_{m\geq 1}$ be a convergent subsequence of $(x_{\frac{1}{k_m}})_{m\geq 1}$, $x_{\frac{1}{k_m}} \to x_0$. Obviously $x_0 \in A$. Replacing into (13) and passing to the limit for $m \to \infty$, we get

(20) $\lim\limits_{n\to\infty} \eta_{\frac{1}{k_m}}(t, x_{\frac{1}{k_m}}) = \lim\limits_{n\to\infty} \alpha(\frac{t}{k_m}, \eta_0(t, x_{\frac{1}{k_m}})) = \alpha(0, \eta_0(t, x_0)) = \eta_0(t, x_0)$ (take $t = 0$ in $2.2, 1^\circ$). But $\varphi(\eta_\tau(1, x_\tau)) \overset{(15),(16)}{\geq} \Phi(\eta_0) - \varepsilon\lambda\tau$. Take $\tau = \frac{1}{k_m}$, then the limit for $m \to \infty$ and combining with (20) one finds $\varphi(\eta_0(1, x_0)) \geq \Phi(\eta_0)$, whence $\varphi(\eta_0(1, x_0)) \overset{(5)}{=} \Phi(\eta_0)$, i.e. $\varphi(\eta_0(1, x_0)) \in B_0$. Consequently, $g(\eta_0(1, x_0)) = 1$ (Lemma 2.2, 4°), which, confronted with (19) in which $\tau = \frac{1}{k_m}$ and passing to the limit, gives $\lambda \geq 2$, contradiction, and hence (11).

Finally, in the case $M = \varnothing$, it only remains to point out the remark to Lemma 2.2 in order to finish the proof (for instance (14) remains true since $M = \varnothing$). ■

Definition. The sequence $(A_n)_{n\geq 1}$ from $\mathscr{K}$ is *min-maxing* for φ if

$$\lim_{n\to\infty} \sup \varphi(A_n) = c.$$

2.4 Minimax theorem. *Let* X *be a complete connected Finsler* C^1*-manifold*, φ: X $\to$ **R** *of* C^1 *class*, $\mathscr{K}$ *a set of nonempty compact subsets homotopy-stable with boundary* M,

$$c = c(\varphi, \mathscr{K}) := \inf_{A\in\mathscr{K}} \sup \varphi(A) \in \mathbf{R}$$

and F *a closed nonempty subset of* X. *If*

(F1) $$F \cap M = \emptyset \text{ and } F \cap A \neq \emptyset \ \forall\, A \in \mathscr{K}$$

and

(F2) $$\inf \varphi(F) \geq c,$$

then for every sequence $(A_n)_{n\geq 1}$ *from* $\mathscr{K}$ *which is min-maxing for* φ *there is a sequence* $(x_n)_{n\geq 1}$ *in* X *with the properties*

1° $\lim_{n\to\infty} \varphi(x_n) = c$,

2° $\lim_{n\to\infty} \|d\varphi(x_n)\| = 0$,

3° $\lim_{n\to\infty} \text{dist}\,(x_n, F) = 0$,

4° $\lim_{n\to\infty} \text{dist}\,(x_n, A_n) = 0$.

The assertion of the theorem remains also true in the case M = $\emptyset$.

Remark. Theorem 2.4 justifies the name of „strong form" with respect to the Theorem 2.3, since if (F0) is verified, then $F := \{x \in X : \varphi(x) \geq c\}$ satisfies (F1) and (F2), and consequently there is a sequence $(x_n)_{n\geq 1}$ in X with the properties 1°, 2°, 3° and 4° , and 1°, 2° and 4° are stated in Lemma 2.3.

Definition. Let $(A_n)_{n\geq 1}$ be a sequence from $\mathscr{K}$. φ verifies

$(PS)_{c,\,F}$ *condition along of* $(A_n)_{n\geq 1}$

if every sequence $(x_n)_{n\geq 1}$ from X with the properties

$$\lim_{n\to\infty} \varphi(x_n) = c,\ \lim_{n\to\infty} \|d\varphi(x_n)\| = 0,\ \lim_{n\to\infty} \text{dist}\,(x_n, F) = 0 \text{ and } \lim_{n\to\infty} \text{dist}\,(x_n, A_n) = 0$$

has a convergent subsequence.

2.5 *Corollary*. *In the assumptions of Theorem* 2.4, *if in addition* φ *verifies* $(PS)_{c,\,F}$ *along a min-maxing sequence* $(A_n)_{n\geq 1}$ *from* $\mathscr{K}$, *then there is a critical point* x_0 *of* φ *at the level* c *situated in* F *and* $\lim_{n\to\infty} \text{dist}\,(x_0, A_{k_n}) = 0$.

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in X given by Theorem 2.4 and a convergent subsequence $(x_{k_n})_{n\geq 1}$, $x_{k_n} \to x_0$. Then, since $\varphi(x_{k_n}) \to \varphi(x_0)$ and $d\varphi(x_{k_n}) \to d\varphi(x_0)$, we have $\varphi(x_0) = c$ and $d\varphi(x_0) = 0$. Moreover, since $\text{dist}\,(x_0, F) \leq \rho(x_0, x_{k_n}) + \text{dist}\,(x_{k_n}, F)$, we have $x_0 \in F$. It is also obvious that $\text{dist}\,(x_0, A_{k_n}) \to 0$. ■

Remark. Using the notations $K_c := \{x \in X : \varphi(x) = c, \nabla_\beta (x) = 0\}$ and $A_\infty :=$ $\{x \in X : \lim_{n\to\infty} \text{dist}(x, A_n) = 0\}$, the assertion of Corollary 2.5 can be expressed by

$$F \cap A_\infty \cap K_c(\varphi) \neq \emptyset.$$

3. Formulation of the problem for the forced pendulum

Let $h : \mathbf{R} \to \mathbf{R}$ be a 2π-periodic function of C^1 class and $f : \mathbf{R} \to \mathbf{R}$ a T-periodic continuous function. Consider the differential equation

$$\ddot{u} + h'(u) = f(t) \text{ a.e.} \tag{21}$$

(the forced pendulum equation for $h(u) = -\cos u$; the point indicates the derivative with respect to t, the time).

We use Theorem 2.3 and Corollary 2.5 to obtain a proposition about the existence of the solutions of (21).

Let H be the Hilbert space of T-periodic absolutely continuous functions u on $\mathbf{R}$ for which $\int_0^T |\dot{u}(t)|^2 \, dt < +\infty$ with the scalar product

$$u \cdot v = \int_0^T (\dot{u}\dot{v} + uv) dt . \tag{22}$$

H is a connected complete Finsler manifold.

The solutions of (21) are the critical points of the action – functional $\varphi : H \to \mathbf{R}$,

$$\varphi(u) = \int_0^T [\frac{1}{2}\dot{u}^2 - h(u(t)) + f(t)u(t)] \, dt \tag{23}$$

(the coherence of the assertion is provided by the corresponding theorems of Lebesgue, Jordan and Riesz).

4. Solutions of the problem and their characterization

2.6 *Suppose* $\int_0^T f(t)\,dt = 0$. *Then the equation* (21) *has at least two* T-*periodic solutions which not differ by a multiple of* 2π.

Proof. Put for every u in H

$$k_{\mathrm{u}} := \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} u(t)\, dt,\ v_{\mathrm{u}} := u - k_{\mathrm{u}}. \tag{24}$$

Thus

$$\varphi(u) = \int_0^{\mathrm{T}} (\frac{1}{2}\dot{u}^2 - h(v_{\mathrm{u}} + k_{\mathrm{u}}) + f\, v_{\mathrm{u}})\, dt. \tag{25}$$

Obviously $\varphi(u + 2\pi) = \varphi(u)$.

Prove

$$\varphi \textit{ is lower bounded}. \tag{26}$$

One finds, using Schwarz and Wirtinger inequalities ($a := \sup h(\mathbf{R})$, $\rho > 0$),

$$\varphi(u) \geq \int_0^{\mathrm{T}} \frac{1}{2}\dot{v}_{\mathrm{u}}^2\, dt - aT - \left(\int_0^{\mathrm{T}} f^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\mathrm{T}} v_{\mathrm{u}}^2 dt\right)^{\frac{1}{2}} \geq \int_0^{\mathrm{T}} \frac{1}{2}\dot{v}_{\mathrm{u}}^2 dt - aT - \tag{27}$$

$$-\rho\left(\int_0^{\mathrm{T}} f^2 dt\right)^{\frac{1}{2}} \left(\int_0^{\mathrm{T}} \dot{v}_{\mathrm{u}}^2 dt\right)^{\frac{1}{2}} = \left(\int_0^{\mathrm{T}} \dot{v}_{\mathrm{u}}^2 dt\right)^{\frac{1}{2}} [\, \frac{1}{2}\left(\int_0^{\mathrm{T}} \dot{v}_{\mathrm{u}}^2 dt\right)^{\frac{1}{2}} - \rho\left(\int_0^{\mathrm{T}} f^2 dt\right)^{\frac{1}{2}}\,] - aT$$

and consequently (26).

Consider, for every N in $\mathbf{N}$,

$$F_{\mathrm{N}} := \{u \in H : \left|\frac{1}{T}\int_0^{\mathrm{T}} u(t)dt\right| \leq 2N\pi\}. \tag{28}$$

Prove

$$\varphi \textit{ verifies } (\mathrm{PS})_{\mathrm{c},F_{\mathrm{N}}} \textit{ for every } \mathrm{c} \in \mathbf{R} \textit{ and } \mathrm{N} \in \mathbf{N}. \tag{29}$$

Let $(u_{\mathrm{n}})_{\mathrm{n}\geq 1}$ be a sequence in H with (30) $\lim_{\mathrm{n}\to\infty} \varphi(u_{\mathrm{n}}) = c$, $\lim_{\mathrm{n}\to\infty} \varphi'(u_{\mathrm{n}}) = 0$ and $\lim_{\mathrm{n}\to\infty} \mathrm{dist}\,(u_{\mathrm{n}}, F_{\mathrm{N}}) = 0$. Using again Wirtinger inequality to (27) we find that $\left(v_{\mathrm{u_n}}\right)_{\mathrm{n}\geq 1}$ is bounded in H. Moreover, (30) allows us to admit that $\left(k_{\mathrm{u_n}}\right)_{\mathrm{n}\geq 1}$, and consequently also $(u_{\mathrm{n}})_{\mathrm{n}\geq 1}$, is bounded in H. Since H has a compact embedding in $\mathbf{L}^2(\mathbf{R})$, there is a subsequence $(u_{\mathrm{n}})_{\mathrm{n}\geq 1}$, denote it identically, with the properties

(31) $u_{\mathrm{n}} \to u$ uniformly on $[0, T]$, $\dot{u}_{\mathrm{n}} \to \dot{u}$ weakly in $\mathbf{L}^2(\mathbf{R})$.

Then, for m and n arbitrary,

$\varphi'(u_{\mathrm{n}} - u_{\mathrm{m}})(u_{\mathrm{n}} - u_{\mathrm{m}}) = \int_0^{\mathrm{T}} (\dot{u}_{\mathrm{n}}^2 - \dot{u}_{\mathrm{m}}^2)\mathrm{dt} - \int_0^{\mathrm{T}} h'(u_{\mathrm{n}} - u_{\mathrm{m}})(u_{\mathrm{n}} - u_{\mathrm{m}})dt \geq \|\dot{u}_{\mathrm{n}} - \dot{u}_{\mathrm{m}}\|_2^2 - \alpha\|u_{\mathrm{n}} - u_{\mathrm{m}}\|_\infty$, $\alpha > 0$. Taking into account (30) and (31) one gets $\|\dot{u}_{\mathrm{n}} - \dot{u}_{\mathrm{m}}\|_2 \to 0$ for

m, $n \to \infty$, this imposes $(u_n)_{n\geq 1}$ Cauchy sequence and consequently convergent in H.

Now combining (26) with (29) we get that φ has a global minimum u_0 in H and $\varphi(u_0 + 2\pi) = \varphi(u_0)$.

Let $\mathscr{K}$ be the set of continuous paths in H that joint u_0 with $v := u_0 + 2\pi$ ($v \in H$!). $\mathscr{K}$ is homotopy-stable with boundary $M := \{u_0, v\}$. Let $F := S_\rho$ be the sphere centred at u_0 with radius ρ, $\rho < \|u_0 - v\|$. F is dual of $\mathscr{K}$ and (F3) is verified:

(32) $$\sup \varphi(M) = \sup \{\varphi(u_0), \varphi(u_0 + 2\pi)\} = \varphi(u_0) \leq \inf \varphi(F),$$

u_0 being a global minimum point.

Let be $c := c(\varphi, \mathscr{K})$. Only the following two situations are possible and in each of these we find two T-periodic solutions of (21) which do not differ by a multiple of 2π.

The case $c = \varphi(u_0)$. Let S_ρ be as above, $\rho < \|2\pi\|$. Since for N high enough we have $S_\rho \subset F_N$, (29) validates the assertion φ verifies $(PS)_{c, S_\rho}$. As (F1) and (F2) are also verified with respect to S_ρ, Corollary 2.5 gives a critical point u_ρ for φ in S_ρ at the level c, i.e. u_ρ is a solution of (21). There is ρ_0 such that $u_{\rho_0} \neq u_0 + 2\pi$, in the opposite case, contracting the spheres in u_0, one finds a sequence $(u_n)_{n\geq 1}$, $u_n = u_0 + 2\pi$, with $u_n \to u_0$, contradiction. Thus u_0 and u_{ρ_0} verify the statement.

The case $c > \varphi(u_0)$. Since $\sup \varphi(M) \overset{(2.2)}{=} \varphi(u_0)$, we have in fact $\sup \varphi(M) < c$, apply Theorem 2.3 and one finds $(u_n)_{n\geq 1}$ a sequence in H with $\lim_{n\to\infty} \varphi(u_n) = c$, $\lim_{n\to\infty} \varphi'(u_n) = 0$. But the periodicity of φ allows to admit $u_n \in F_1$ $\forall\, n \geq 1$, then enter in action (29) and one gets v_0 a critical point at the level c. Since $c \neq \varphi(u_0)$, *a fortiori* $v_0 \neq u_0 + 2\pi$. ■

5. Applications

For the classical forced pendulum equation, h from (21) has the particular form:

$$h(u) = -\cos u. \tag{33}$$

The functional $\varphi : H \to \mathbf{R}$ takes its particular form:

$$\varphi(u) = \int_0^T [\frac{1}{2}\dot{u}^2 + \cos(u(t)) + f(t)u(t)]dt\,. \tag{34}$$

When we have in the classical forced pendulum equation a function f which verifies: $\int_0^T f(t)\,dt = 0$, the equation (21) has at least two T-periodic solutions which not differ by a multiple of 2π (see Proposition 2.6). These solutions are obtained as critical points of the action - functional φ.

If the problem of classical pendulum equation is under the conditions asked in Proposition 2.6, we can construct u_0 as a global minimum for φ. Another solution, different from u_0 not by a multiple of 2π, can be obtained using Corollary 2.5 when $c = \varphi(u_0)$ and by using Theorem 2.3 if $c > \varphi(u_0)$.

6. Conclusions

One variational result was improved. It was applied, together with other two classical variational statements, to state a proposition characterizing the existence of T-periodic solutions.

One can construct at least two T-periodic solutions which differ by a multiple of 2π, first like the global minimum of the action - functional φ and the second as another critical point for the same function φ.

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