A PROOF OF ZORN'S LEMMA

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Zorn's Lemma. *Let X be a partially ordered set. Suppose that every totally ordered subset of X has an upper bound. Then X has a maximal element.*

The following proof appears, for example, in [1, 2].

Proof. Suppose the contrary. Then it follows from the Axiom of Choice that for every totally ordered subset $A \subset X$, we can choose and fix an upper bound $g(A)$ in $X \setminus A$. For a subset $A \subset X$ and $a \in A$, denote

$$A_{<a} = \{x \in A : x < a\}.$$

We call a subset $A \subset X$ a *g-set* if

- A is totally ordered;
- A contains no infinite descending sequences, that is, there is no infinite sequences $a_1 > a_2 > \cdots$ in A;
- For every $a \in A$, $g(A_{<a}) = a$.

Note that $\varnothing$ is a g-set, and if A is a g-set then so is $A \cup \{g(A)\}$.

We prove that if $A, B \subset X$ are distinct g-sets, then either $A = B_{<b}$ for some $b \in B$, or $B = A_{<a}$ for some $a \in A$. To see this, let

$$C = \{c \in A \cap B : A_{<c} = B_{<c}\}.$$

We show that either $C = A$ or $C = A_{<a}$ for some $a \in A$. Suppose $C \neq A$. Then $A \setminus C$ has a smallest element a, for otherwise there would be an infinite descending sequence in A. It follows that $A_{<a} \subset C$. On the other hand, if there is $c \in C \setminus A_{<a}$, then $a \in A_{<c} \subset C$, a contradiction. So $C = A_{<a}$. Similarly, either $C = B$ or $C = B_{<b}$ for some $b \in B$. Suppose $C \neq A$ and $C \neq B$. Then $C = A_{<a} = B_{<b}$ for some $a \in A$ and $b \in B$. But

$$a = g(A_{<a}) = g(B_{<b}) = b.$$

So $a \in C$, a contradiction. It follows that either $C = A$ or $C = B$. If $C = A$, then $C \neq B$, and hence $A = C = B_{<b}$ for some $b \in B$. Similarly, if $C = B$, then $B = A_{<a}$ for some $a \in A$.

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Let E be the union of all g-sets. We prove that for $a \in E$, if A is a g-set containing a, then $A_{<a} = E_{<a}$. It is clear that $A_{<a} \subset E_{<a}$. To prove the converse, let $x \in E_{<a}$ and suppose B is a g-set containing x. If $B \subset A$, then $x \in A$ and hence $x \in A_{<a}$. If $B \not\subset A$, then $A = B_{<b}$ for some $b \in B$. Since $x < a$ and $a < b$, we have $x \in B_{<b} = A$ and hence $x \in A_{<a}$. This proves $E_{<a} \subset A_{<a}$.

We now verify that E is a g-sets:

- E is totally ordered: For any $a, b \in E$, there are g-sets A and B containing a and b, respectively. Since either $A \subset B$ or $B \subset A$, a and b are contained in a single g-set and hence are comparable.
- E contains no infinite descending sequences: Suppose E contains an infinite descending sequence $a_1 > a_2 > \cdots$. Let A be a g-set containing a_1. Then for $i \geq 2$ we have $a_i \in E_{<a_1} = A_{<a_1} \subset A$. It follows that A contains an infinite descending sequence, a contradiction.
- $g(E_{<a}) = a$ for every $a \in E$: Let A be a g-set containing a. Then $a = g(A_{<a}) = g(E_{<a})$.

This implies that E is the largest g-set. But $E \cup \{g(E)\}$ is also a g-set, a contradiction. $\square$

References

[1] H. Kneser, *Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom*, Math. Z. 53 (1950), 110–113.

[2] T. Szele, *On Zorn's lemma*, Publ. Math. Debrecen 1 (1950), 254–256, erratum 257.