Algebra and Geometry through Projective Spaces

SF2724 Topics in Mathematics IV
Spring 2015, 7.5 hp

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May 12, 2015

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Chapter 1

Topological spaces

1.1 The definition of a topological space

One of the first definitions of any course in calculus is that of a continuous function:

Definition 1.1.1. A map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at the point $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. It is simply called continuous if it is continuous at every point.

This ϵ-δ-definition is tailored to functions between the real numbers; the aim of this section is to find an abstract definition of continuity.

As a first reformulation of the definition of continuity, we can say that f is continuous at $x \in \mathbb{R}$ if for every open interval V containing $f(x)$ there is an open interval U containing x such that $f(U) \subseteq V$, or equivalently $U \subseteq f^{-1}(V)$. The original definition would require x and $f(x)$ to be at the centers of the respective open intervals, but as we can always shrink V and U, this does not change the notion of continuity.

We can even do without explicitly mentioning intervals. Let us call a subset $U \in \mathbb{R}$ *open* if for every $x \in U$ there, U contains an open interval containing x. We now see that a f is continuous everywhere if and only if $f^{-1}(V)$ is open whenever V is.

Thus by just referring to open subsets of $\mathbb{R}$, we can define continuity. We are now in a place to make this notion abstract.

Definition 1.1.2. Let X be a set. A *topology* on X is a collection $\mathcal{U} \subseteq \mathcal{P}(X)$ of subsets of X, called the *open sets*, such that:

(Top1) $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$;

(Top2) If $X,\ Y \in \mathcal{U}$ then $X \cap Y \in \mathcal{U}$;

(Top3) If $\mathcal{V} \subseteq \mathcal{U}$ is an arbitrary subset of open sets then $\bigcup \mathcal{V} = \bigcup_{V \in \mathcal{V}} V \in \mathcal{U}$.

A space X together with a topology $\mathcal{U}$ is called a topological space; we will usually abuse notation and say that X is a topological space without mentioning $\mathcal{U}$.
A function $f\colon X \to Y$ between topological spaces is called continuous if preimages of open sets under f are open.

Confusingly, a *closed* subset $A \subseteq X$ is not one that is not open but rather one such that the complement $X - A$ is open. Thus there are many subsets that are neither open nor closed (for instance, half-open intervals in $\mathbb{R}$ or the subset $\mathbb{Q} \subset \mathbb{R}$) and a few that are both (for instance, the empty set).
Since the preimage of the complement of a set under a map is the complement of the preimage, we might equally well define a continuous map to be one such that preimages of closed sets are closed.

Example 1.1.3. The standard topology on the space $\mathbb{R}^n$ has as open sets those sets $U \subseteq \mathbb{R}^n$ such that for every $x_0 \in U$ there is an $\epsilon > 0$ such that

$$B_\epsilon(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < \epsilon\} \subseteq U$$

Lemma 1.1.4. *A set $A \subseteq \mathbb{R}^n$ is closed if and only if for every convergent sequence $x_i \in A$, $x = \lim_{i\to\infty} x_i \in A$.*

Proof. We show the "if" direction first. Let $A \subseteq \mathbb{R}^n$ be a subset with the convergent sequence property, and assume A is not closed, i. e., $\mathbb{R}^n - A$ is not open. Then there exists a point $x \in \mathbb{R}^n - A$ such that for every $\epsilon = \frac{1}{n}$ there is a point $x_n \in B_\epsilon(x_0)$ which belongs to A. But this sequence converges to x, which therefore has to lie in A, contrary to our assumption.
Conversely, assume $\mathbb{R}^n - A$ is open and (x_i) is a convergent sequence in A with limit x. If it was true that $x \neq A$ then there would be some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \mathbb{R}^n - A$. Since x_i converges to x, almost all x_i would have to lie in $\mathbb{R}^n - A$, contrary to our assumption. $\square$

Example 1.1.5. Any set X can be given two extreme topologies: one where *all* subsets are open (this is called the *discrete topology*) and one where no subsets except $\emptyset$ and X itself are open (this is called the *indiscrete* or *trivial topology*).

Definition 1.1.6. A *homeomorphism* between two topological spaces X and Y is a bijective, continuous map $f\colon X \to Y$ whose inverse f^{-1} is also continuous.

1.2 Subspaces, quotient spaces, and product spaces

Let X be a topological space and $Y \subset X$ a subset. Then Y inherits a topology by defining the open sets of Y to be the intersections $U \cap Y$, where U is open in X. This topology is called the *subspace topology*. Note that in the case where Y is not itself open, open sets in Y are not necessarily open in X.

Example 1.2.1. Let $X = \mathbb{R}$ and $Y = [0,1] \subseteq \mathbb{R}$ be the closed unit interval. Then the intervals $I_1 = (\frac{1}{3}, \frac{2}{3})$ and $I_2 = (\frac{2}{3}, 1]$ are both open in Y because

$$I_1 = (\frac{1}{3}, \frac{2}{3}) \cap Y \quad \text{and} \quad I_2 = (\frac{2}{3}, \frac{4}{3}) \cap Y.$$

However, only I_1 is open in X because the point $1 \in I_2$ is not contained in an open interval which itself is contained in I_2.

Now let X be a topological space and let $\sim$ be an equivalence relation on X. There is a natural surjective map $p\colon X \to X/\sim$ to the set of equivalence classes under the relation $\sim$, sending a point x to its equivalence class $[x]$. Again, we can define an induced topology on $X/\sim$ by decreeing that a subset $V \subseteq X/\sim$ is open if and only if its preimage $p^{-1}(V)$ is open in X.

It may be worthwhile to stop to verify that this indeed defines a topology. For (Top1), not that $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(X/\sim) = X$, so those two extreme subsets are open. Next, if V_1 and $V_2 \subseteq X/\sim$ are open then $p^{-1}(V_1 \cap V_2) = p^{-1}(V_1) \cap p^{-1}(V_2)$, which is open in X by (Top2), thus so is $V_1 \cap V_2$ in $X/\sim$. Lastly, if $\mathcal{V}$ is a family of open sets in $X/\sim$ then $p^{-1}\left(\bigcup \mathcal{V}\right) = \bigcup_{V \in \mathcal{V}} p^{-1}(V)$ is open in X, proving (Top3).

A special case of the quotient space construction, which is of particular interest, is the quotient space with respect to a group action. Let G be a group acting on a topological space X. We can define an equivalence relation $\sim_G$ on X by $x \sim_G y$ if and only if there exists some $g \in G$ such that $g.x = y$. In this way, the quotient space X/G of equivalence classes, or G-orbits, becomes a topological space.

Quotient spaces can be quite ill-behaved for random equivalence relations. In practice, one wishes the projection map $p\colon X \to X/\sim$ to not only be continuous but also *open*, i. e. images of open subsets of X are open in $X/\sim$. Luckily, this is always the case for group actions:

Lemma 1.2.2. *Let G be a group acting continuously on a topological space X. Then the projection map $p\colon X \to X/G$ is open.*

Proof. Let $U \subseteq X$ be open. We need to show that $p(U)$ is open, i. e. by the definition of quotient topology, that $p^{-1}(p(U))$ is open in X. Note that the translates gU of U are all open because they are the inverse images of U under the continuous map $X \xrightarrow{g^{-1}} X$. Thus

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU,$$

as a union of open sets, is open. $\square$

Given two topological spaces X and Y, the product $X \times Y$ becomes a topological space with the *product topology* which is defined as follows: a subset $U \subseteq X \times Y$ is open iff it is an arbitrary union of sets of the form $U \times V$, where U is open in X and V is open in Y.

1.3 Separation

The indiscrete topology $\{\emptyset, X\}$, defined on any space X, is not a very useful one because geometrically, all points are clumped together – they cannot be *separated* from each other. More precisely, there are no nonconstant continuous maps $X \to Y$ for any "reasonable" space Y such as the real line.

Definition 1.3.1. A topological space X is *Hausdorff* if for any choice of two distinct points x, $y \in X$ there are disjoint open sets U, V in X such that $x \in U$ and $y \in V$.

The indiscrete topology is manifestly not Hausdorff unless X is a singleton. The standard topology on $\mathbb{R}^n$ is Hausdorff: for $x \neq y \in \mathbb{R}^n$, let d be half the Euclidean distance between x and y. Then $U = B_d(x)$ and $V = B_d(y)$, the open balls of radius d centered at x resp. y, fulfill the requirements.

A convenient alternative way to define Hausdorffness is as follows:

Lemma 1.3.2. *Let X be a topological space and $\Delta \subseteq X \times X$ the diagonal, i. e. the set $\Delta = \{(x,x) \in X \times X\}$. Then X is Hausdorff if and only if Δ is closed in $X \times X$.*

Proof. Let us first assume Δ is closed. Let x, y be two distinct points in X. Then (x, y) lies in the open set $X \times X - \Delta$. By the definition of the product topology, there are open sets U, V containing x and y, respectively, such that $U \times V \subseteq X \times X - \Delta$. Thus U and V are disjoint open sets separating x and y.

Conversely, assume X is Hausdorff. It suffices to produce an open set $U \times V$ for every point $(x, y) \in X \times X - \Delta$ such that $x \in U$, $y \in V$, and such that $U \times V \subseteq X \times X - \Delta$. Any two separating open sets U and V of x and y will work for this, and they exist because X is Hausdorff. $\square$

Hausdorffness is inherited by subspaces, but not necessarily by quotient spaces. However, we have:

Lemma 1.3.3. *Let $X/\sim$ be a quotient space of a Hausdorff space X by an equivalence relation $\sim$ such that the projection map $p\colon X \to X/\sim$ is open. Define*

$$D = \{(x, y) \in X \times X \mid x \sim y\}.$$

Then $X/\sim$ is Hausdorff if and only if D is closed in $X \times X$.

Proof. Let us first show that D is closed if $X/\sim$ is Hausdorff. We have that

$$D = (p \times p)^{-1}(\Delta),$$

where $\Delta = \{(x, x) \in (X/\sim) \times (X/\sim)\}$ is the diagonal. Since $p \times p$ is continuous and Δ is closed by the assumption that $X/\sim$ is Hausdorff, D is also closed. We did not need p to be open for this direction. Conversely, if $(X \times X) - D$ is open in $X \times X$ then $(p \times p)((X \times X) - D)$ is open in $(X/\sim \times X/\sim)$ because p is assumed to be an open map. But that image is $(X/\sim \times X/\sim) - \Delta$ because p is surjective. □

1.4 Compactness

A collection $\mathcal{U}$ of open subsets of a space X is called an *open cover* if their union is all of X. The following definition is of central importance in topology:

Definition 1.4.1. A space X is *compact* if any open cover of X contains a finite subcover, i. e. we can choose a finite subset $\mathcal{V} \subseteq \mathcal{U}$ which is still a cover.

To check compactness using the definition can be awkward. The following is a useful criterion for compactness:

Theorem 1.4.2 (Heine–Borel)**.** *Any closed and bounded subset of $\mathbb{R}^n$ is compact.*

We begin with proving a lemma:

Lemma 1.4.3. *Closed subsets of compact sets are compact.*

Proof. Let A be a closed subset of a compact set X, and let $\mathcal{U}$ be an open cover of A. Then $\mathcal{U}' = \mathcal{U} \cup \{X - A\}$ is an open cover of X since $X - A$ is open. By the compactness of X, it must contain a finite subcover $\mathcal{V}' = \mathcal{V} \cup \{X - A\}$. (The set $X - A$ must be part of it unless $X = A$, in which case there is nothing to prove.) Then $\mathcal{V}$ is a finite subcover of $\mathcal{U}$ of A. □

Proof of the Heine–Borel Theorem. By Lemma 1.4.3, it suffices to show that for any $a > 0$, the cube $Q_0 = [-\frac{a}{2}, \frac{a}{2}]^n \subseteq \mathbb{R}^n$ is compact because by definition, a bounded set must lie in such a cube.

We will prove this by contradiction. Let $\mathcal{U}$ be an open cover of Q_0 that does not have a finite subcover. Divide Q_0 into 2^n subcubes of half the side length; at least one of these cubes, let's say Q_1, cannot be covered by finitely many elements from $\mathcal{U}$. Continue in this way, producing nested cubes Q_i with side lengths $\frac{a}{2^i}$.

Now choose a point $x_i \in Q_i$ for each i. This sequence is Cauchy and thus, because of the completeness of $\mathbb{R}^n$, has a limit x. By Lemma 1.1.4, $x \in Q_i$ for all i. Now let $U \in \mathcal{U}$ be a set containing x. Then $B_\epsilon(x) \subseteq U$ for some $\epsilon > 0$ because U is open. But then for i large enough, $Q_i \subseteq B_\epsilon(x) \subseteq U$, showing that Q_i did not, after all, need infinitely many members of $\mathcal{U}$ to be covered, but only one. □

Example 1.4.4. The unit sphere $S^n \subseteq \mathbb{R}^{n+1}$, consisting of all vectors of length 1, is compact. Indeed, it is defined as the preimage of the closed set $\{1\} \subseteq \mathbb{R}$ under the continuous map

$$|-|\colon \mathbb{R}^{n+1} \to \mathbb{R}^{\geq 0}$$

and hence is closed; that it is bounded is part of the definition.

Lemma 1.4.5. *Let X be a compact topological space and $\sim$ an equivalence relation on X. Then $X/\sim$ is also compact.*

Proof. As before, let $p\colon X \to X/\sim$ denote the quotient map. Let $\mathcal{V} = \{V_i\}_{i \in I}$ be an open cover of $X/\sim$. Then $\{p^{-1}(V_i)\}_{i \in I}$ is an open cover of X by the definition of the quotient topology. Hence it contains a finite subcover $\{p^{-1}V_{i_1}, \cdots, p^{-1}V_{i_n}\}$. But then $\{V_{i_1}, \ldots, V_{i_n}\}$ is a cover of $X/\sim$ as well. □

Remark 1.4.6. In algebraic geometry, the term "compact" is often understood to mean "compact and Hausdorff"; what we call compact here would be called "quasicompact".

1.5 Countability

This section is somewhat technical but required for the correct definition of manifolds later on.

Definition 1.5.1. A space X with topology $\mathcal{U}$ is said to satisfy the *second countability axiom*, or shorter, to be *second-countable*, if there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ which is closed under finite intersections and such that $\mathcal{U}$ is the smallest topology containing $\mathcal{V}$.

Second-countability is thus a condition that restricts the size of the topology on a space. For instance, an uncountable set with the discrete topology is not second-countable. An equivalent way of phrasing the condition is that every open set $U \in \mathcal{U}$ can be written as the union of those open sets in $\mathcal{V}$ that lie in U:

$$U = \bigcup_{\substack{V \in \mathcal{V} \\ V \subseteq U}} V. \tag{1.1}$$

Since there are at most $2^{|\mathcal{V}|}$ different ways of taking arbitrary unions of elements of $\mathcal{V}$, a second-countable topology can not have more open sets than the cardinality of $\mathbb{R}$.

The reader may be curious as to what the first countability axiom says. We will not be needing it here, but for the sake of completeness, it is local second-countability: a space is first-countable if every point x is contained in some open set U that is second-countable.

Lemma 1.5.2. *For any n, the standard topology on $\mathbb{R}^n$ is second-countable.*

Proof. Let $\mathcal{V} = \{B_q(x) \mid q \in \mathbb{Q}, x \in \mathbb{Q}^n\}$. Then $\mathcal{V}$ is countable, and (1.1) holds basically by the completeness of $\mathbb{R}^n$. □

Lemma 1.5.3. *Any subspace of a second-countable space is second-countable. A quotient space $X/\sim$ of a second-countable space X is second-countable if the projection $p\colon X \to X/\sim$ is open.*

Proof. Let X be second countable with a countable subset $\mathcal{V}$ satisfying (1.1). If $A \subseteq X$ then $\mathcal{V}' = \{V \cap A \mid V \in \mathcal{V}\}$ is a countable subset of the topology of A satisfying (1.1) for open subsets U of A.

For the statement about quotients, the open sets of $X/\sim$ are exactly the images under the open projection map of opens in X, hence $\mathcal{V}' = \{p(V) \mid V \in \mathcal{V}\}$ fits the bill. □

Chapter 2

The definition of projective space

The n-dimensional real projective space is defined to be the set of all lines through the origin in $\mathbb{R}^{n+1}$; a similar definition works for the complex projective space or, in fact, projective spaces over any ring k. It is denoted by $\mathbb{R}P^n$ resp. $\mathbb{C}P^n$ by topologists and $P^n(\mathbb{R})$ resp. $P^n(\mathbb{C})$ (or $\mathbb{P}^n(\mathbb{R})$ or $\mathbb{P}^n_{\mathbb{R}}$ etc.) by algebraic geometers:

$$\mathbb{R}P^n = P^n(\mathbb{R}) = \{L \leq \mathbb{R}^{n+1} \mid L \text{ 1-dimensional linear subspace}\}.$$

Unfortunately, using this definition it is quite awkward to try to define a sensible topology on $\mathbb{R}P^n$; we will do this in the next section.

2.1 Projective spaces as quotients

A line through the origin in $\mathbb{R}^{n+1}$ is uniquely determined by any other point on that line, that is, any point $x \in \mathbb{R}^{n+1} - \{0\}$. That is, we have a surjective map

$$p\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$$

sending a point $(x_0, \dots, x_n)$ to the subspace spanned by that vector. This map is surjective, but clearly not injective since a line contains many points. More precisely, we define an equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{0\}$ by decreeing that two points x, y are equivalent if and only if they both lie on a line through the origin, that is, $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if there is a $\lambda \in \mathbb{R} - \{0\}$ such that $x_i = \lambda y_i$ for all $0 \leq i \leq n$.

It is now clear that p factors through a bijective map

$$\overline{p}\colon (\mathbb{R}^{n+1} - \{0\})/\sim \to \mathbb{R}P^n.$$

We define the standard topology on $\mathbb{R}P^n$ to be the quotient topology under this identification.

Another way of looking at this construction is to define a group action of the multiplicative group $\mathbb{R}^\times$ on $\mathbb{R}^{n+1} - \{0\}$ by scalar multiplication. Then $\mathbb{R}P^n \cong (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^\times$.

The equivalence class of a point $(x_0, \ldots, x_n)$ in $\mathbb{R}P^n$ is customarily denoted by $[x_0 : \cdots : x_n]$ or $(x_0 : \cdots : x_n)$.

We will give another couple of constructions of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ as quotients. These alternative construction will be useful later.

Lemma 2.1.1. *Let $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\} \subseteq \mathbb{C}^{n+1} - \{0\}$ denote the $2n+1$-dimensional standard sphere. The group S^1 of complex numbers of absolute value 1 acts on S^{2n+1} by scalar multiplication. Then*

$$\mathbb{C}P^n \cong S^{2n+1}/S^1.$$

Similarly, $\mathbb{R}P^n \cong S^n/\{\pm 1\}$.

Proof. We will only prove the complex case. The inclusion $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1} - \{0\}$ is equivariant with respect to the S^1-action, thus we get an induced continuous map

$$S^{2n+1}/S^1 \to (\mathbb{C}^{n+1} - \{0\})/S^1 \twoheadrightarrow (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times,$$

where the last map is the projection associated to the group inclusion $S^1 \subset \mathbb{C}^\times$. It is clear that this map is bijective. The inverse map can be described as

$$(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^\times \to S^{2n+1}/S^1; \quad [x_0 : \cdots : x_n] \mapsto [\frac{1}{|x|}(x_0, \cdots, x_n)],$$

which is continuous, thus the desired homeomorphism is established. $\square$

If we write S^n as a union of its upper and its lower hemisphere, $S^n = D^n_+ \cup D^n_-$ (both parts including the equator), we observe that for every point $x \in S^n$, either $x \in D^n_+$ or $-x \in D^n_+$. Thus we have proved:

Lemma 2.1.2. *There is a homeomorphism $\mathbb{R}P^n \cong D^n/\sim$, where $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ and the equivalence relation $\sim$ identifies antipodal points on the boundary S^{n-1}.*

A similar construction works for $\mathbb{C}P^n$ and is left to the reader.

2.2 $\mathbb{R}P^1$ and $\mathbb{C}P^1$

Lemma 2.2.1. *There are homeomorphisms $\mathbb{R}P^1 \cong S^1$ and $\mathbb{C}P^1 \cong S^2$.*

Proof. Consider the map $f\colon S^1 \to S^1$ given by $f(z) = z^2$, where we think of $z \in S^1 \subseteq \mathbb{C}$ as a unit complex number. Then f is surjective and $f(z) = f(-z)$, thus it factors through a bijection

$$\overline{f}\colon S^1/\{\pm 1\} \cong \mathbb{R}P^1 \to S^1.$$

Moreover, $\overline{f}$ is a homeomorphism because it is continuous and open, the latter because f is open.

For the complex case, we have to employ some other methods. We think of the 2-sphere S^2 as the "one-point compactification" of $\mathbb{C} = \mathbb{R}^2$. The stereographic projection gives a homeomorphism $\sigma\colon S^2 - \{N\} \to \mathbb{C}$, where N denotes the north pole of S^2. Now we define

$$f\colon \mathbb{C}P^1 \to S^2$$

by

$$f([z_1 : z_2]) = \begin{cases} \sigma^{-1}\left(\frac{z_1}{z_2}\right); & \text{if } z_2 \neq 0 \\ N; & \text{if } z_2 = 0. \end{cases}$$

This is well-defined and bijective; we have to check that it is continuous and open. That it is that away from the point $[1 : 0]$ resp. the north pole is obvious. □

2.3 $\mathbb{R}P^2$

The projective plane $\mathbb{R}P^2$ is an example of a *non-orientable surface*; it can be obtained by taking a Möbius strip and attaching a two-dimensional disk along its single boundary circle. This cannot be embedded in $\mathbb{R}^3$, but it can be embedded into $\mathbb{R}^4$:

Lemma 2.3.1. *The map $f\colon S^2 \subseteq \mathbb{R}^3 \to \mathbb{R}^4$ given by*

$$f(x, y, z) = (xy, xz, y^2 - z^2, 2yz)$$

induces an embedding of $\mathbb{R}P^2$.

Proof. First note that if we change signs on x, y, z simultaneously, the image of f does not change, thus f is well-defined on $\mathbb{R}P^2$. We leave it to the reader to check that this map is injective and closed. □

2.4 $\mathbb{R}P^3$

Lemma 2.4.1. *There is a homeomorphism $SO(3) \cong \mathbb{R}P^3$.*

Proof. Recall from Lemma 2.1.2 that we can describe $\mathbb{R}P^3$ as a quotient of D^3, where antipodal points on the boundary are identified. Define a map

$$\phi\colon D^3 \to SO(3)$$

as follows: a point $\alpha x \in D^3$, where $0 \leq \alpha \leq 1$ and $x \in \mathbb{R}^3$ is a unit vector, is mapped to the rotation around x by $\alpha\pi$ in the positive direction. This factors through $\mathbb{R}P^3$ because a rotation by π and the rotation by π in the other direction are the same. One can write down a more explicit formula for this map and verify that it is continuous; this is left to the reader. We will, however, show that the map is bijective. Injectivity is obvious since all the rotations thus produced are distinct. To see that every element in $SO(3)$ is a rotation, and hence in the image of ϕ, let $\lambda_1, \lambda_2, \lambda_3$ be the three complex eigenvalues of $A \in SO(3)$. Then one of the λ_i has to be real (a degree-3 real polynomial has a real root), while the other two are complex conjugated, and $\lambda_1\lambda_2\lambda_3 = \det A = 1$. This can only happen if 1 is one of the eigenvalues. If x is an associated eigenvector, it spans a stable axis, and A is some rotation around this axis. $\square$

Chapter 3

Topological properties of projective spaces

3.1 Point-set topological properties

Proposition 3.1.1. *The projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are compact.*

Proof. By Lemma 2.1.1, $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are quotients of spheres. Spheres are compact by Theorem 1.4.2, and quotients of compact spaces are compact by Lemma 1.4.5. $\square$

Proposition 3.1.2. *The projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are Hausdorff.*

Proof. We will give the proof for $\mathbb{C}P^n$ only. Define a map

$$f\colon \big(\mathbb{C}^{n+1} - \{0\}\big) \times \big(\mathbb{C}^{n+1} - \{0\}\big) \to \mathbb{R}$$

by

$$f(x, y) = f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i \neq j} |x_i y_j - x_j y_i|^2.$$

We observe that $f(x, \lambda x) = 0$ for all $\lambda \in \mathbb{C}$. Conversely, if $f(x, y) = 0$ then $x_i y_j = x_j y_i$ for all $i \neq j$ and thus x and y are linearly dependent.

We thus conclude that $f^{-1}(0) = \{(x, y) \mid x \sim y\}$ under the equivalence relation of linear dependence that defines $\mathbb{C}P^n$ as a quotient. Since $\{0\} \subseteq \mathbb{R}$ is closed and f is continuous, $\{(x, y) \mid x \sim y\}$ is closed in $(\mathbb{C}^{n+1} - \{0\}) \times (\mathbb{C}^{n+1} - \{0\})$. By Lemma 1.3.3, $\mathbb{C}P^n$ is Hausdorff. $\square$

Proposition 3.1.3. *The projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are second countable.*

Proof. They are open quotients of subspaces of $\mathbb{R}^n$ and hence second countable by Lemmas 1.5.2, 1.5.3, and 1.2.2. $\square$

3.2 Charts and manifold structures

An important property of projective spaces is that they are smooth manifolds.

Definition 3.2.1. Let M be a second countable Hausdorff space. A *chart* on M is a homeomorphism $\phi\colon U \to V$, where U is an open subset of M and V is an open subset of $\mathbb{R}^n$ for some n. An *atlas* on M is a collection of charts $\phi_\alpha\colon U_\alpha \to V_\alpha$ such that the U_α together cover M.
The space M is called a topological manifold if it has an atlas.

This is a useful definition, but not quite what we're after; we want to do analysis on manifolds, so there should be a notion of "differentiable function" on it.

Definition 3.2.2. An atlas $(\phi_\alpha\colon U_\alpha \to V_\alpha)_\alpha$ on a topological manifold M is called *smooth* if whenever U_α and U_β have nontrivial intersection $U_{\alpha\beta}$, the map

$$\phi_\alpha(U_{\alpha\beta}) \xrightarrow{\phi_\alpha^{-1}} U_{\alpha\beta} \xrightarrow{\phi_\beta} \phi_\beta(U_{\alpha\beta})$$

is not only a homeomorphism but a diffeomorphism of open subsets of $\mathbb{R}^n$.
A function $f\colon M \to \mathbb{R}$ is smooth with respect to a smooth atlas ϕ_α if $f \circ \phi_\alpha^{-1}\colon V_\alpha \to \mathbb{R}$ is smooth. A function $f\colon M \to N$ between manifolds with smooth atlases is called smooth if for every smooth function $g\colon N \to \mathbb{R}$, the function $g \circ f\colon M \to \mathbb{R}$ is also smooth.

We would now like to say that a "smooth manifold" is a topological manifold together with a smooth atlas. Although one can read such a statement in the literature, this is not correct. Two different smooth atlases can give rise to the same class of smooth functions and in that case, we do not want to consider those manifolds as different.

Definition 3.2.3. A *smooth manifolds M* is a topological manifold together with an equivalence class of smooth atlases. Here an atlas ϕ_α is equivalent to an atlas ψ_β if the identity map $\mathrm{id}\colon (M, \phi_\alpha) \to (M, \psi_\beta)$ is smooth.

We will sometimes omit the word "smooth" and just speak of a "manifold", it being understood that it is smooth.

Theorem 3.2.4. *The projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are smooth manifolds of dimensions n and $2n$, respectively.*

Proof. We have already seen than projective spaces are second countable and Hausdorff. Let $k = \mathbb{R}$ or $\mathbb{C}$. We define charts on kP^n as follows:

$$U_i = \{[x_0 : \cdots : x_n] \in kP^n \mid x_i \neq 0\} \quad (i = 0, \ldots, n).$$

Then we have homeomorphisms

$$\phi_i \colon U_i \to k^n; \quad [x_0 : \cdots : x_n] \mapsto \Big(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\Big),$$

where the ith entry (which would be $\frac{x_i}{x_i} = 1$) is omitted. Clearly the U_i cover kP^n as every point in kP^n has some nonzero coordinate. Moreover, the map

$$\psi_i \colon k^n \to U_i; \quad (x_1, \ldots, x_n) \mapsto [x_1 : \cdots : 1 : \cdots : x_n],$$

where the entry 1 is in the ith slot, is a continuous inverse for ϕ_i.
Now consider the change-of-coordinate functions $\phi_i \circ \phi_j^{-1}$ (for ease of notation, let's assume $i < j$):

$$\begin{aligned}(x_1, \ldots, x_n) &\mapsto [x_1, \cdots, 1, \cdots, x_n] \\ &\mapsto \Big(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{j-1}}{x_i}, \frac{1}{x_i}, \frac{x_{j+1}}{x_i}, \ldots \frac{x_n}{x_i}\Big).\end{aligned}$$

This is clearly a smooth functions, thus the ϕ_i exhibit a smooth atlas for kP^n. □

3.3 Cell structures

A manifold structure on a space of interest, like projective spaces, is crucial for its geometric study, but for its topological properties, it is often more useful to have a more combinatorial description. Topologists like to work with the category of so-called *CW complexes*. To get an intuition for this, consider what a graph is: it consists of vertices (0-dimensional "cells") and edges (1-dimensional "cells"), and the end points of the edges are identified (glued) to certain vertices. A CW-complex is a higher-dimensional generalization of this.
Denote by D^n the standard n-dimensional disk of vectors in $\mathbb{R}^n$ of norm ≤ 1; its boundary is the sphere S^{n-1}.

Definition 3.3.1. Let $\phi_\alpha \colon S^{n-1} \to X$ be a collection of maps. Then we define a new space

$$Y = X \cup_{\phi_\alpha} \coprod_\alpha D^n = (X \sqcup \coprod_\alpha D^n)/\sim,$$

where the equivalence relation is given by $\phi_\alpha(x) \sim x_\alpha$. Here $x \in S^{n-1}$ and x_α denotes the point x in the αth summand of $\coprod_\alpha D^n$.
We call ϕ_α the *attaching maps*, the various D^n *n-cells*, and say that Y is obtained from X by attaching (a number of) n-cells.

Definition 3.3.2. A *CW-complex* is a topological space X with a filtration

$$X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X$$

such that:

- $X^{(0)}$ is discrete,
- $X^{(n)}$ is obtained from $X^{(n-1)}$ be attaching n-cells for all $n > 0$.
- $\bigcup_{n \geq 0} X^{(n)} = X$, and

The subspace $X^{(n)}$ is called the *n-skeleton*. A CW-complex is said to be of dimension n if $X^{(n)} = X^{(n+1)} = \cdots = X$. A CW-complex is *finite* if it is of finite dimension and X^n is obtained from X^{n-1} by attaching only finitely many cells.

Theorem 3.3.3. *The projective space $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching a single n-cell. Also $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a single $2n$-cell.*

In particular, $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are finite, n- resp. $2n$-dimensional CW-complexes with exactly one cell in every resp. every even dimension.

Proof. By Lemma 2.1.2,

$$\mathbb{R}P^n \cong D^n / \sim,$$

where the equivalence relation $\sim$ identifies antipodal points on the boundary S^{n-1}. Define $\phi \colon S^{n-1} \to \mathbb{R}P^{n-1}$ to be the standard quotient map. Then $\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \cup_\phi D^n$.

A similar construction works for $\mathbb{C}P^n$ and is left to the reader. $\square$

3.4 Euler characteristic

A classical result from graph theory is that for every finite planar graph $\Gamma = (V, E)$, i. e. a graph one can embed onto a 2-dimensional sphere, the number $\chi(\Gamma) = \#V - \#E + \#F$, the difference between the number of vertices and the number of edges plus the number of 2-dimensional faces, is always 2. This says that this number 2, called the Euler number, is an invariant of the sphere itself and independent of the graph. The number will be different if we allow ourselves to embed the graph e.g. on a donut (what will it be then?). We can think of a graph embedded in a surface as giving rise to a 2-dimensional CW-complex with 1-skeleton the graph itself and 2-cells the faces of the graph.

The following theorem requires some knowledge of algebraic topology and is beyond our scope in this course:

Theorem 3.4.1. *Let X be a finite CW-complex, and let n_i denote the number of i-cells. Then the* Euler characteristic

$$\chi(X) = \sum_{i=0}^{\infty} (-1)^i n_i$$

is independent of the CW-structure.

Example 3.4.2. A point has Euler characteristic 1 because it consists of a single 0-cell. The n-dimensional sphere has Euler characteristic 2 for even n and 0 for odd n because it can be given a CW-structure with one 0-cell and one n-cell.

Example 3.4.3. The torus $S^1 \times S^1$ has Euler characteristic 0 because it can be given a CW structure with one 0-cell, two 1-cells (the longitudinal and latitudinal great circles), and one 2-cell.

Proposition 3.4.4. *We have that $\chi(\mathbb{C}P^n) = n+1$ and $\chi(\mathbb{R}P^n) = \begin{cases} 0; & n \text{ odd} \\ 1; & n \text{ even} \end{cases}$*

Proof. This follows directly from the CW structure of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ from Thm. 3.3.3. □

Chapter 4

Curves in the projective plane

We will in this chapter study different aspects of *plane curves* by which we mean curves in the projective plane defined by polynomial equations. Here we will start with the more classical setting and consider a plane curve as the set of solutions of one *homogeneous* equation in three variables.

We will start by choosing a field, k, which in most cases can be thought of as either $\mathbb{R}$ or $\mathbb{C}$, but sometimes, it is interesting also to look at $\mathbb{Q}$ or finite fields.

The first definition we might try is the following.

Definition 4.0.5. A *plane curve* C is the set of solutions in $\mathbb{P}^2_k$ of a non-zero homogeneous equation

$$f(x, y, z) = 0.$$

Example 4.0.6. The equation $x^2 + y^2 + z^2 = 0$ defines a degree two curve over $\mathbb{C}$ but over $\mathbb{R}$ it gives the empty set.

The equation $x^2 = 0$ has a solution set consisting of the line $(0 : s : t)$ while the degree of the equation is two.

The example above shows us that the definition does not give us a one-to-one correspondence between curves and equations.

4.1 Lines

We will start by the easiest curves in the plane, namely *lines*. These are defined by linear equations

$$ax + by + cz = 0 \tag{4.1}$$

where $(a, b, c) \neq (0, 0, 0)$. Observe that any non-zero scalar multiple of (a, b, c) has the same set of solutions, which shows us that we can *parametrize* all the lines in $\mathbb{P}^2_k$ by another projective plane with coordinates $[a : b : c]$.

Theorem 4.1.1. *Any two distinct lines in $\mathbb{P}^2$ intersect at a single point.*

Proof. The condition that the lines are distinct is the same thing as the equations defining them being linearly independent, which gives a unique solution to the system of equations. □

Theorem 4.1.2. *Any line in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.*

Proof. By a change of coordinates the equation of a line can be written as $x = 0$ and the solutions are given by $[0 : s : t]$ where $(s, t) \neq (0, 0)$, which as a set equals $\mathbb{P}^1$.
In fact, using this parametrization, we can define a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$, which has the given line as the image.
We will come back to what we mean by *isomorphism* later on in order to make this more precise. □

4.1.3 The dual projective plane $(\mathbb{P}^2)^*$

As mentioned above, the coefficients a, b, c of Equation 4.1, give us natural coordinates on the space of lines in $\mathbb{P}^2$ and we will call this the *dual projective plane*, denoted by $(\mathbb{P}^2)^*$.

Theorem 4.1.4. *The set of lines through a given point in $\mathbb{P}^2$ is parametrized by a line in $(\mathbb{P}^2)^*$.*

Proof. Equation 4.1 is symmetric in the two sets of variables, $\{x, y, z\}$ and $\{a, b, c\}$. Thus, fixing $[x : y : z]$ gives a line in $(\mathbb{P}^2)^*$. □

4.1.5 Automorphisms of $\mathbb{P}^2$

A linear change of coordinates on $\mathbb{P}^2_k$ is given by a non-singular 3×3-matrix with entries in k:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Because of the identification $[x : y : z] = [\lambda x : \lambda y : \lambda z]$, the scalar matrices correspond to the identity. The resulting group of automorphisms is called $\mathrm{PGL}(3, k)$.

4.2 Conic sections

We will now focus on quadratic plane curves, or *conics*. These are defined by a homogeneous quadratic equation

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0.$$

4.2.1 Conics as the intersection of a plane and a cone

The name *conic* is short for *conic section* and comes from the fact that each such curve can be realized as the intersection of a plane and a circular cone

$$x^2 + y^2 = z^2$$

in $\mathbb{P}^3$.

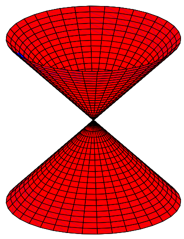


Figure 4.1: The circular cone

4.2.2 Parametrization of irreducible conics

The conic section is *irreducible* if the polynomial defining it is not a product of two non-trivial polynomials.

Theorem 4.2.3. *If C is a plane irreducible conic with at least two rational points, then C is isomorphic to $\mathbb{P}^1_k$.*

Proof. Let P be a rational point of C and let L denote the line in $(\mathbb{P}^2)^*$ parametrizing lines through P. In the coordinates of each line, the polynomial equation reduces to a homogeneous quadratic polynomial in two variables with at least one rational root. Without loss of generality, we may assume that P is $[0 : 0 : 1]$ and the equation of C has the form

$$ax^2 + bxy + cy^2 + dxz + eyz = 0.$$

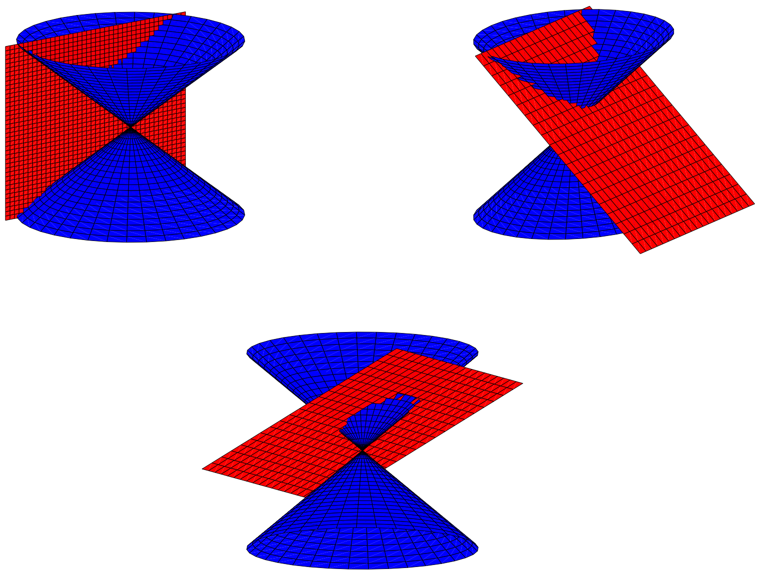


Figure 4.2: The hyperbola, parabola and ellipse as a plane sections of a cone

The lines through P are parametrized by a $\mathbb{P}^1$ with coordinates $[s:t]$ and we get the residual intersection between the curve and the line $sx+ty=0$ as

$$R = [est - dt^2 : dst - es^2 : cs^2 - bst + at^2].$$

Since C has another rational point, Q, we cannot have $d = e = 0$ since C is irreducible. Hence the residual point R is not equal to P except for one $[s,t]$. Moreover, by the next exercise, we get that the three coordinates are never zero at the same time (the first two are zero when $[s:t] = [d:e]$). Hence we have a non-trivial map from $\mathbb{P}^1$ to $\mathbb{P}^2$ whose image is in C. If the image was a line, C would be reducible and we conclude that C is the image of the map. $\square$

Exercise 4.2.4. Let C be a conic passing through the point $[0:0:1]$, i.e., having equation of the form

$$ax^2 + bxy + cy^2 + dxz + eyz = 0.$$

Show that C is reducible if and only if $cd^2 - bde + ae^2 = 0$ under the assumption that C has at least two rational points.

Example 4.2.5. The example $x^2 + y^2 = 0$ with k a field with no square root of -1 shows that we cannot drop the condition that C has at least two rational points.

4.2.6 The parameter space of conics

Exactly as for the lines, we have that the equation

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

defines the same curve when multiplied with a non-zero constant. Hence all the conics can be parametrized by a $\mathbb{P}^5$ with coordinates $[a : b : c : d : e : f]$.

In this *parameter space* we can look at loci where the conics have various properties. For example, we can look at the locus of degenerate conics that are double lines. These are parametrized by a $\mathbb{P}^2$ and the locus of such curves is the image of the *Veronese* embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$ defined by

$$[s : t : u] \mapsto [s^2 : 2st : t^2 : 2su : 2tu : u^2].$$

If we want to look at all the curves that are degenerate as a union of two lines, we look at the image of the map

$$\Phi \colon \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

given by

$$\begin{aligned}&([s_1 : t_1 : u_1], [s_2 : t_2 : u_2]) \\ &\qquad\qquad \mapsto [s_1s_2 : s_1t_2 + t_1s_2 : t_1t_2 : s_1u_2 + u_1s_2 : t_1u_2 + u_1t_2 : u_1u_2].\end{aligned}$$

The image of Φ is a *hypersurface* in $\mathbb{P}^5$ which means that it is defined by one single equation in the coordinates $[a : b : c : d : e : f]$.

Exercise 4.2.7. Find the equation of the hypersurface defined by the image of the map $\Phi \colon \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ defined above.

4.2.8 Classification of conics

When we want to classify the possible conics up to projective equivalence, we need to see how the group of linear automorphisms acts. One way is to go back to our knowledge of quadratic forms. If 2 is invertible in k, i.e., if k does not have characteristic 2, we may write the equation

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

as $Q(x, y, z) = 0$, where Q is the *quadratic form* associated to the matrix

$$A = \frac{1}{2}\begin{bmatrix} 2a & b & d \\ b & 2c & e \\ d & e & 2f \end{bmatrix}.$$

Now, a matrix P from $\mathrm{PGL}(3, k)$ acts on A by

$$Q \mapsto P^T AP.$$

Theorem 4.2.9. *Up to projective equivalence, the equation of a conic can be written in one of the three forms*

$$x^2 = 0, \quad x^2 + \lambda y^2 = 0 \quad \textit{and} \quad x^2 + \lambda y^2 + \mu z^2 = 0.$$

Proof. The first thing that we observe is that the rank of the matrix is invariant. If the rank is one, we can choose two of the columns of P to be in the kernel of A and hence after a change of coordinates, the equation is $\lambda x^2 = 0$, but this is equivalent to $x^2 = 0$.

If the rank is two, we choose one of the columns to be a generator of the kernel and we get that we can assume that $d = e = f = 0$. By completing the square, we can change it into $\kappa x^2 + \mu y^2$, which is equivalent to $x^2 + \lambda y^2$, where $\lambda = \mu/\kappa$.

If the rank is three, proceed by completing the squares in order to write the form as $x^2 + \lambda y^2 + \mu z^2$. □

Remark 4.2.10. In order to further characterize the conics, we need to know about the multiplicative group of our field. In particular, we need to know the quotient of k^* by the subgroup of squares.

Theorem 4.2.11. *Let $k = \mathbb{C}$. Then there are only three conics up to projective equivalence:*

$$x^2 = 0, \quad x^2 + y^2 = 0 \quad \textit{and} \quad x^2 + y^2 + z^2 = 0.$$

Proof. Since every complex number is a square, we can change coordinates so that $\lambda = \mu = 1$ in Theorem 4.2.9. □

Theorem 4.2.12. *Let $k = \mathbb{R}$. Then there are four conics up to projective equivalence:*

$$x^2 = 0, \quad x^2 + y^2 = 0, \quad x^2 - y^2 = 0 \quad \textit{and} \quad x^2 + y^2 - z^2 = 0.$$

Proof. Here, only the positive real numbers are squares and we have to distinguish between the various signs of λ and μ. If $\lambda = \mu = 1$ we get the empty curve, so there is only one non-degenerate curve $x^2 + y^2 = z^2$. □

4.2.13 The real case vs the complex case

4.2.14 Pascal's Theorem

We will look at a classical theorem by Pascal about conics.

Theorem 4.2.15 (Pascal's Theorem)**.** *Let C be a plane conic and H be a hexagon with its vertices on C. The three pairs of opposite sides of the hexagon meet in three collinear points.*

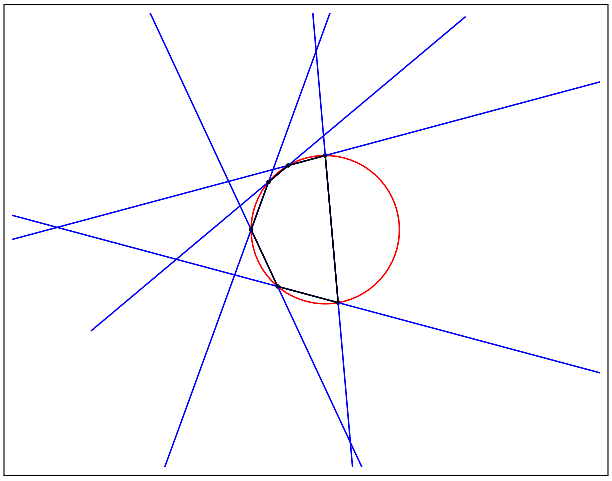


Figure 4.3: Pascal's Theorem

There are several ways to understand this theorem and we will now look at one way.

Proof. Start by dividing the lines into two groups of three lines so that no two lines in the same group intersect on the conic C.

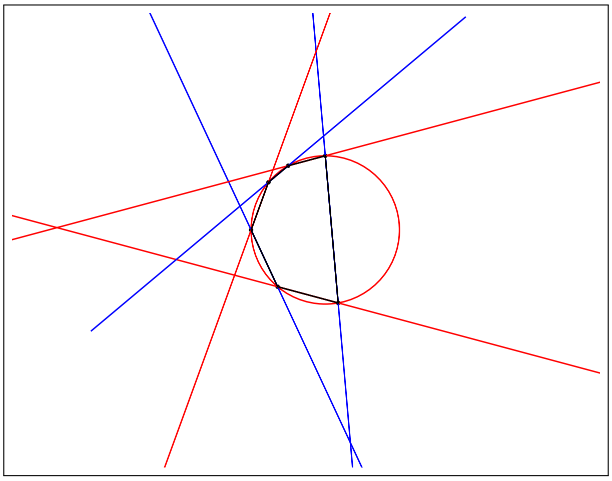


Figure 4.4: The two groups of lines

Each group of three lines defines a cubic plane curve, given by the product of the three linear equations defining the lines. Since each line in one group meets each of the lines from the other group, we have nine points of intersections of lines from the two groups. Six of these are on the conic and it remains for us to prove that the remaining three are collinear.

Choose two of the points and take the line L through them. Together with the conic, the line defines a cubic curve, i.e., there is a cubic polynomial vanishing on the line and the conic. In particular, this cubic curve passes through eight of our nine points. We already have two cubic curves passing through all nine points. If the last cubic didn't pass through all nine points, we would have three linearly independent cubic polynomials passing through our eight points.

Denote the three cubic polynomials by f_1, f_2 and f_3. They can generate seven or eight linearly independent polynomials of degree four. If they generate eight, we get that it will generate a space of codimension 7 in all higher degree, by multiplication by a linear form not passing through any of the points. If they generate only seven linearly independent forms of degree four, we must have two linearly independent syzygies, i.e., relations of the form

$$\begin{cases} \ell_1 f_1 & + & \ell_2 f_2 & + & \ell_3 f_3 & = & 0, \\ \ell_4 f_1 & + & \ell_5 f_2 & + & \ell_6 f_3 & = & 0. \end{cases}$$

Since there is a unique solution to this system up to multiplication by a polynomial, we get that

$$(f_1, f_2, f_3) = \ell(\ell_2\ell_6 - \ell_3\ell_5, \ell_3\ell_4 - \ell_1\ell_6, \ell_1\ell_5 - \ell_2\ell_4)$$

showing that the three cubics share a common linear factor. However, this cannot be the case, since the two original cubics did not have a common factor.

We conclude that the cubic passing through eight of the nine points also pass through the ninth, which shows that the three that were not on the conic have to be collinear. $\square$

The property that any cubic passing through eight of the nine points also has to pass through the ninth point is known as the Cayley–Bacharach property and similar consequences occur in much more general situations.

Chapter 5

Cubic curves

When we move to cubic curves, we have ten coefficients of the equation

$$a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3 = 0.$$

Thus, as in the case of lines and conics, we can use a projective space to parametrize all cubics and in this case we get $\mathbb{P}^9$. As the group of automorphisms of $\mathbb{P}^2$ has dimension 8, we expect that there should be at least a one-dimensional family of non-isomorphic cubics.

As in the case of conics, we have a number of degenerate cases where the cubic is reducible. We get several different ways the cubic polynomial could factor. If we have linear factors, they could all be equal, two distinct or three distinct. In the case when there are three distinct factors, they can share a common zero or not. This can be summarized as

$$x^3 = 0, \qquad x^2y = 0 \qquad , xy(x+y) = 0 \qquad \text{or} \qquad xyz = 0.$$

When the cubic polynomial has a linear and an irreducible quadratic factor, we get different cases depending on whether the line is tangent to the conic or not which gives the two possibilities

$$x(x^2 + y^2 - z^2) \qquad \text{and} \qquad (x - z)(x^2 + y^2 - z^2).$$

5.1 Normal forms for irreducible cubics

Definition 5.1.1. L is a *tangent line* to C at P if the restriction of the equation of C to L has a root of multiplicity at least two at P.

Definition 5.1.2. A point P on a curve C is *non-singular* if there is a unique tangent line of C at P.

Definition 5.1.3. A non-singular point P of a curve C is a *flex* point of C if the tangent of C at P intersects C with multiplicity at least three at P.

Theorem 5.1.4. *The equation of an irreducible cubic with at flex point can be written as*

$$y^2z = x^3 + ax^2z + bxz^2 + cz^3$$

after a change of coordinates.

Proof. Let C be the curve defined by the equation

$$a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3 = 0.$$

Assume that $[0 : 1 : 0]$ is a flex point with tangent line $z = 0$. Then, when restricting the equation to the line, we need to get $x^3 = 0$, forcing $a_1 = a_3 = a_6 = 0$.

If $a_7 = 0$ we get that the restriction of the equation of C to the line $x = 0$ is $a_8yz^2 + a_9z^3 = 0$. Thus $x = 0$ is a second tangent line to C at P. Since P is a flex point, it is non-singular and we deduce that $a_7 \neq 0$.

We can now change change variables with $y = y' + \alpha x + \beta z$ so that there will be no other terms involving y' than $(y')^2$. Thus we get to the desired normal form.

The irreducibility gives that $a_0 \neq 0$ since otherwise $z = 0$ would be a component. Thus we can get the leading term on the right hand side to be x^3. □

Exercise 5.1.5. Find the normal form for the Fermat cubic $x^3 + y^3 = z^3$.

5.2 Elliptic curves

Definition 5.2.1. A non-singular cubic curve is called en *elliptic curve.*

Theorem 5.2.2. *The cubic curve defined by the equation*

$$y^2z = f(x, z)$$

is non-singular if and only if $f(x, z)$ has no multiple factors.

Proof. Without loss of generality, we can assume that the point is $P = [0 : y_0 : 1]$. The lines though P are $sx + t(y - y_0z) = 0$, for $[s : t]$ in $\mathbb{P}^1$. When $t = 0$ we get the line $x = 0$ which is tangent to C if and only if $c = y_0 = 0$.

For $t \neq 0$ we substitute in $y^2z = x^3 + ax^2z + bxz^2 + z^3$ to get

$$x(t^2x^2 + (at^2 - s^2)xz + (bt + 2sy_0)tz^2) = 0$$

which has a double root at P if and only if $(bt + 2sy_0)t = 0$. Thus we get a unique tangent line, unless $c = y_0 = b = 0$, where we get $x = 0$ and $y = y_0$ as tangent lines. □

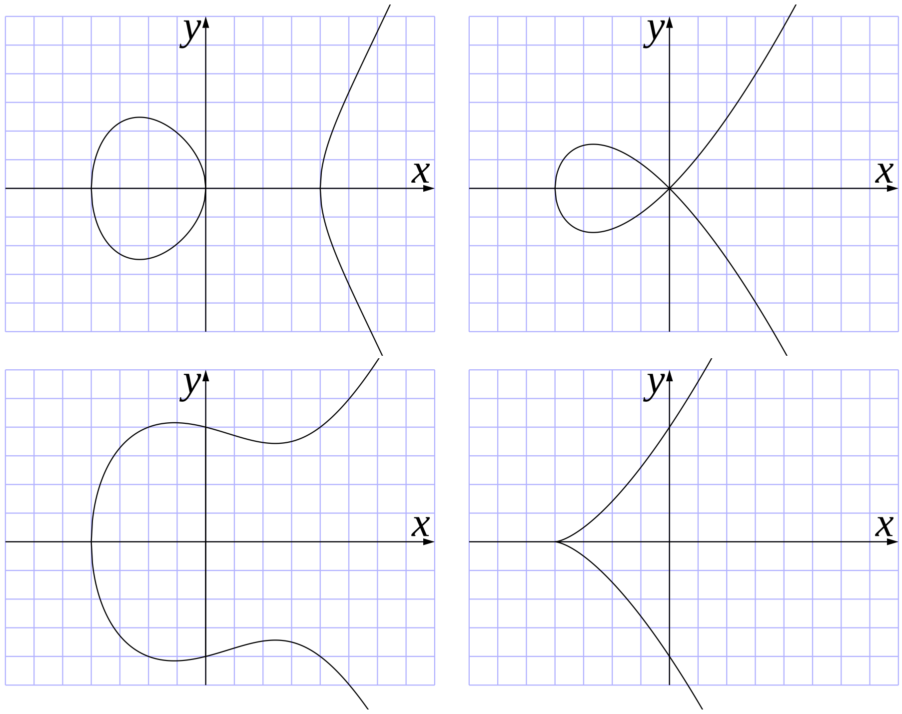


Figure 5.1: Different kinds of cubics in normal form

5.2.3 The group law on an elliptic curve

The elliptic curves are special in many ways. One of them is that there is a commutative group law on the set of rational points of an elliptic curve.

The restriction to any line of the equation of a cubic curve gives a homogeneous cubic equation in two variables. If this equation has two rational solutions, the third has to be rational as well.

Definition 5.2.4. Choose a flex point O of the elliptic curve C. If P and Q are points on C we define the sum $P+Q$ to be the third point on the line through O and the third point on the line through P and Q. Observe that if $P = Q$, we take the tangent line at P.

Theorem 5.2.5. *The addition defines a commutative group law on the set of points of C.*

Proof. The commutativity is clear from the definition. The identity element is given by O since the line through O and P meets the curve in a point Q and then the line through O and Q is the same as the line before, which shows that

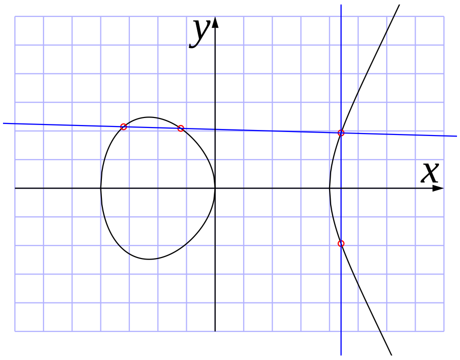


Figure 5.2: The addition on an elliptic curve

$O + P = P$. The inverse of P is given as the point Q on the line through O and P.

The associativity is more involved and we will refer to other sources for a proof of that. □

5.2.6 A one-dimensional family of elliptic curves

The normal form $y^2z = x^3 + ax^2z + bxz^2 + cz^3$ does not specify an elliptic curve up to isomorphism. As we have seen before, the right hand side has distinct factors. We can translate one of them to $x = 0$ and scale one of them to $x = z$. This leaves us with the normal form

$$y^2z = x(x - z)(x - zw)$$

where $w \neq 0$ and $w \neq 1$.

5.2.7 Flex points on an elliptic curve

Theorem 5.2.8. *The set of flex points on C form an elementary 3-group.*

Proof. The flex points can be shown to be zeroes of the Hessian form (cf. Exercise 5.2.14), which shows that there are at most finitely many flex points. If P is a flex point, we have that $3P = 0$ since the tangent through P meets C only at P. The sum of two flex points is again a flex point as $3P = 0$ and $3Q = 0$ implies that $3(P + Q) = 0$. Thus the set of flex points on an elliptic curve form a finite subgroup where all non-trivial elements have order 3, i.e, an elementary 3-group. □

Exercise 5.2.9. Show that an elliptic curve over $\mathbb{R}$ cannot have more than three flex points.

5.2.10 Singularities and the discriminant

Among the irreducible cubics, there are two kinds of singular curves; nodal cubics and cuspidal cubics. Both of these singular curves are rational curves and are images of a degree three map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$. In the normal form they can be written as

$$y^2z = x^3 \quad \text{and} \quad y^2z = x^3 - x^2z$$

We can localize the singularities of C by the Jacobian ideal since they correspond to zeroes of the gradient of the polynomial defining C.

Example 5.2.11. Let C be the nodal cubic defined by $F(x, y, z) = y^2z - x^3 + x^2z$. The gradient is given by

$$\nabla F = (-3x^2 + 2xz, 2yz, y^2 + x^2)$$

which is zero only at $[0 : 0 : 1]$.

Example 5.2.12. Let C be the cuspidal cubic defined by $F(x, y, z) = y^2z - x^3$. Then we get

$$\nabla F = (-3x^2, 2yz, y^2)$$

which again is zero only at $[0 : 0 : 1]$.

Exercise 5.2.13. Define the rational cubic curve C as the image of the map $\Phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ given by

$$\Phi([s : t]) = [s^3 : st^2 : t^3], \qquad [s : t] \in \mathbb{P}^1.$$

Find the singular point of C and determine whether C is nodal or cuspidal.

As we have seen, the general cubic curve is non-singular, but there is an eight-dimensional family of singular curves given by the nodal cubics. One way to see that the family of singular cubics is eight-dimensional is to look at the curves that are singular at a given point $[x_0 : y_0 : z_0]$. We have a two-dimensional choice of the point and for each point, we have three linear conditions on the coefficients of the cubic giving us a $\mathbb{P}^6$ of curves singular at the given point. We can describe this as a $\mathbb{P}^6$-bundle over $\mathbb{P}^2$.

The locus $X \subseteq \mathbb{P}^9$ parametrizing singular cubics is defined by a single polynomial called the *discriminant*. It is a difficult task to compute this polynomial which is of degree 12.

Exercise 5.2.14. Let C be a cubic plane curve over $\mathbb{C}$. Show that the Hessian, i.e., the determinant of

$$\begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} \\ \frac{\partial^2 F}{\partial z \partial x} & \frac{\partial^2 F}{\partial z \partial y} & \frac{\partial^2 F}{\partial z^2} \end{bmatrix}$$

vanishes exactly at the singular points of C and on the flex points of C.

Chapter 6

Bézout's Theorem

On $\mathbb{P}^1$ we have that any polynomial of degree d has exactly d roots counted with multiplicity, at least when we are working over $\mathbb{C}$ or any algebraically closed field. We will now look at a generalization of this called *Bézout's Theorem*, which states that two plane curves of degree d and e with no common component intersect in exactly $d \cdot e$ points counted with multiplicity.

There are a couple of difficulties that we have to overcome in order to prove this. The first is to properly define what *multiplicity* means in the statement of the theorem.

6.1 The degree of a projective curve

As we have seen before, when a homogeneous polynomial of degree d defining a plane curve is restricted to a line with coordinates $[s : t]$, we either get zero or a homogeneous polynomial of degree d in s and t. In the first case, the line was a component of C and in the second case, we get a polynomial which factors into a product of d linear factors if our field is algebraically closed. From now on, we will assume that this is the case. Moreover, we will assume that the polynomial defining our curve has the lowest possible degree, so that there are no multiple factors in the factorization into irreducible polynomials. We call such a polynomial *reduced*.

With these conventions, the following definition makes sense.

Definition 6.1.1. A plane curve C has degree d if a general line in $\mathbb{P}^2$ intersect C in d distinct points.

6.2 Intersection multiplicity

Let C_1 and C_2 be plane curves defined by reduced polynomials f_1 and f_2 with no common factors. In order to define the intersection multiplicity of C_1 and C_2 at their points of intersection, we will first change coordinates in order to move the common point P to $[0 : 0 : 1]$. When looking locally around this point, we can dehomogenize the polynomials by substituting $z = 1$. Let $F_1 = f_1(x, y, 1)$ and $F_2 = f_2(x, y, 1)$ be the polynomials we obtain in this way. We have $F_1, F_2 \in k[x, y]$, but we can also see them as formal power series in the ring $k[[x, y]]$, which has the advantage that any polynomial which is non-zero at the origin $(0, 0)$ is invertible. In this way, we can concentrate only at what happens at the origin. From F_1 and F_2 we get an ideal $I = (F_1, F_2) \subseteq k[[x, y]]$ and we can define the quotient ring $k[[x, y]]/I$.

Definition 6.2.1. The intersection multiplicity of C_1 and C_2 at $P = [0 : 0 : 1]$ is given by

$$I_P(f_1, f_2) = \dim_k k[[x, y]]/(F_1, F_2)$$

We will need a couple of properties of the intersection multiplicity.

Theorem 6.2.2. *If f, g and h are homogeneous polynomials in $k[x, y, z]$ with no common factors, we have*

(1) $I_P(f, gh) = I_P(f, g) + I_P(f, h)$

(2) $I_P(f, g + fh) = I_P(f, g)$ *if* $\deg g = \deg f + \deg h$.

Proof. At the moment, we will refer to other sources for the proof of the first statement, which requires more knowledge in power series rings.
The second statement follows from the definition since $(f, g) = (f, g + fh)$ as ideals in $k[x, y, z]$ and hence also their images in $k[[x, y]]$ under the map $k[x, y, z] \to k[[x, y]]$ sending z to 1. □

Exercise 6.2.3. Show that if P is a non-singular point of C_1 and C_2 such that the tangents of C_1 and C_2 at P are distinct, then $I_P(f, g) = 1$ where f and g are the homogeneous polynomials defining C_1 and C_2.

6.3 Proof of Bézout's Theorem

Theorem 6.3.1 (Bézout's Theorem)**.** *If C_1 and C_2 are plane curves defined by homogeneous polynomials f and g of degree d and e, they intersect in $d \cdot e$ points,* *counted with multiplicity, i.e.,*

$$\sum_P I_P(f,g) = d \cdot e.$$

Proof. If one of the polynomials splits into a product of linear factors, we can use Theorem 6.2.2 (1) to conclude the theorem.

We will use Theorem 6.2.2 to make reductions until we can assume that one of the polynomial splits into a product of linear factors. By Theorem 6.2.2 (1) we can assume that f and g are irreducible.

The basic step will be the following. Write the polynomials as

$$\begin{array}{rcl} f(x,y,z) & = & z^{d'}h_0(x,y) + z^{d'-1}h_1(x,y) + \cdots + h_{d'}(x,y) \\ g(x,y,z) & = & z^{e'}k_0(x,y) + z^{e'-1}k_1(x,y) + \cdots + k_{e'}(x,y) \end{array}$$

With no loss of generality, we may assume that $d' \geq e'$. Then we can define

$$f_1 = k_0 f - h_0 z^{d'-e'} g$$

which will have lower degree in z than f. We have that

$$I_P(f_1, g) = I_P(k_0 f, g) = I_P(k_0, g) + I_P(f, g), \quad \forall P.$$

Since k_0 is a polynomial in two variables, it splits into a product of linear forms. Since g was assumed to be irreducible we know that k_0 is not a factor of g. Since we know the theorem holds for k_0 and g we can deduce it for f and g if we know it holds for f_1 and g. For this we can use induction on the degree of z in the polynomials. $\square$

Exercise 6.3.2. Follow the proof of Bézout's theorem above starting with the curves $zy^2 = x^3 - xz^2$ and $x^2 + y^2 = z^2$. What are all the intersection points and their multiplicities at the end of the reduction?

6.4 The homogeneous coordinate ring of a projective plane curve

As we have seen, the polynomial ring $k[x,y,z]$ plays an important role in the study of $\mathbb{P}^2$. This is the *homogeneous coordinate ring* of $\mathbb{P}^2$.

If C is defined by the homogeneous polynomial $f \in k[x,y,z]$, we get the homogeneous coordinate ring of C as $R_C = k[x,y,z]/(f)$.

Theorem 6.4.1. *(1) C is irreducible if and only if R_C is a domain.*

(2) C is reduced if and only R_C has no nilpotent elements.

The homogeneous coordinate ring of C is graded, i.e., we can write it as

$$R_C = \bigoplus_{i \geq 0} [R_C]_i$$

so that $[R_C]_i [R_C]_j \subseteq [R_C]_{i+j}$.

Definition 6.4.2. The *Hilbert function* of C is given by $H_C(i) = \dim_k [R_C]_i$, for $i = 0, 1, 2, \ldots$.

Theorem 6.4.3. *If C has degree d, the Hilbert function of C is given by*

$$H_C(i) = \binom{i+2}{2} - \binom{i+2-d}{2} = di + \frac{d(3-d)}{2},$$

for $i \geq d-1$.

Proof. The Hilbert function of C is given by the vector space dimension of $k[x,y,z]/(f)$ in degree i. We have the following short exact sequence

$$0 \to k[x,y,z] \to k[x,y,z] \to R_C \to 0$$

where the first map is multiplication by f. Thus the dimension of R_C in degree i is the difference between the dimension of $k[x,y,z]$ in degree i and degree $i-d$. If $i \geq d-1$, these dimensions are given by the formula in the statement of the theorem. $\square$

Lemma 6.4.4. *The homogeneous polynomial g defines an injective map $R_C \longrightarrow R_C$ if and only if g doesn't vanish completely on any component of C.*

Proof. Suppose that $gh = 0$ in R_C for some homogeneous polynomial h. This means that $gh = qf$ for some homogeneous polynomial q and since $k[x,y,z]$ is a unique factorization domain, we conclude that each irreducible factor of f must be a factor of either g or h. If none of them divides g, we must have that $h \in (f)$ so $h = 0$ in R_C. Thus g defines an injective map if it doesn't vanish on any component of C.

If g does vanish on some component, we will be able to find $h \neq 0$ in R_C with $gh = 0$ showing that the map is not injective. $\square$

Chapter 7

Affine Varieties

7.1 The polynomial ring

Let $\mathbb{C}$ denote the field of complex numbers, and let $\mathbb{C}[x_1, \ldots, x_n]$ denote the polynomial ring in n variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$. Elements f in $\mathbb{C}[x_1, \ldots, x_n]$ are polynomials in $x_1, \ldots, x_n$, that is finite expressions of the form

$$f = \sum c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

with c_α in $\mathbb{C}$. Polynomials are added and multiplied in the obvious way, and $\mathbb{C}[x_1, \ldots, x_n]$ indeed forms a ring; a commutative unital ring.

7.2 Hypersurfaces

To any $f \in \mathbb{C}[x_1, \ldots, x_n]$ we let $Z(f)$ denote the zero set of the element f, that is

$$Z(f) = \{(a_1, \ldots, a_n) \in \mathbb{C}^n \mid f(a_1, \ldots, a_n) = 0\}.$$

For non-constant polynomials f the zero set $Z(f)$ is referred to as a hypersurface. Clearly we have that the union satisfies

$$Z(f) \cup Z(g) = Z(fg).$$

In order to describe intersections of hypersurfaces it is convenient to use ideals, a notion we recall next.

7.3 Ideals

A non-empty subset $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ that is closed under sum, and closed under multiplication by elements of $\mathbb{C}[x_1, \ldots, x_n]$, is called an ideal. The zero element is an ideal, and the whole ring is an ideal.

If $\{f_\alpha\}_{\alpha \in \mathscr{A}}$ is a collection of elements in $\mathbb{C}[x_1, \ldots, x_n]$ they generate the ideal $I(f_\alpha)_{\alpha \in \mathscr{A}}$ that consists of all finite expressions of the form

$$I(f_\alpha)_{\alpha \in \mathscr{A}} = \left\{ \sum_{\alpha \in \mathscr{A}} g_\alpha f_\alpha \mid g_\alpha \in \mathbb{C}[x_1, \ldots, x_n], g_\alpha \neq 0 \text{ finite indices } \alpha \right\}.$$

The zero ideal is generated by the element 0, and this is the only element that generates the zero ideal. We have that $(1) = \mathbb{C}[x_1, \ldots, x_n]$, so the element 1 generates the whole ring.

Noetherian ring The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is an example of a Noetherian ring which means that any ideal is in fact finitely generated. Thus, if I is an ideal generated by the collection $\{f_\alpha\}_{\alpha \in \mathscr{A}}$, then there exists a finite subset $f_1, \ldots, f_m$ of the collection that generates the ideal $I(f_\alpha)_{\alpha \in \mathscr{A}} = I(f_1, \ldots, f_m)$.

7.4 Algebraic sets

Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. We let $Z(I)$ denote the intersection of the zero sets of the elements in I, that is

$$Z(I) = \bigcap_{f \in I} Z(f).$$

A subset of $\mathbb{C}^n$ of the form $Z(I)$ for some ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is called an *algebraic set*. One verifies that if the ideal I is generated by $f_1, \ldots, f_m$ then we have that

$$Z(I) = Z(f_1) \cap Z(f_2) \cap \cdots \cap Z(f_m).$$

In particular we have that $Z(f) = Z(I(f))$, where $I(f)$ is the ideal generated by f.

Union and intersection Let $\{I_\alpha\}_{\alpha \in \mathscr{A}}$ be a collection of ideals in $\mathbb{C}[x_1, \ldots, x_n]$. Their set theoretic intersection is an ideal we denote by $\cap_{\alpha \in \mathscr{A}} I_\alpha$. Their union $I(\cup_\alpha I_\alpha)$ is the ideal generated by their set theoretic union. If the collection is finite, we have the product $I_1 \cdots I_m$, which denotes the ideal where the elements are finite sums of products $f_1 \cdots f_m$, with $f_i \in I_i$, for $i = 1, \ldots, m$.

7.5 Zariski topology

Lemma 7.5.1. *We have that the algebraic sets in $\mathbb{C}^n$ satisfy the following properties.*

(1) Finite unions $\cup_{i=1}^m Z(I_i) = Z(I_1 \cdots I_m) = Z(\cap_{i=1}^m I_i)$.

(2) Arbitrary intersections $\cap_\alpha Z(I_\alpha) = Z(I(\cup_\alpha I_\alpha))$.

We have furthermore that $Z(1) = \emptyset$ and that $Z(0) = \mathbb{C}^n$.

Proof. This is an excellent exercise. □

The lemma above shows that the collection of algebraic sets satisfy the axioms for the *closed* sets of a topology. This particular topology where the closed sets are the algebraic sets is called the Zariski topology.

Open sets When defining the topology on a set, it is customary to define what the open sets are. The open sets are the complements of the closed sets, so having defined what the closed sets are we also know what the open sets are. But, we could have started the other way around. A collection $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of subsets of a set X that contains X and $\emptyset$, and which is closed under finite intersections, and arbitrary unions, define the open sets of a topology on X.

7.6 Affine varieties

Definition 7.6.1. We let $\mathbb{A}^n$ denote the vector space $\mathbb{C}^n$ endowed with the Zariski topology. The space $\mathbb{A}^n$ is called the affine n-space.

Example 7.6.2. The affine plane is our favorite example. For any element $f \in \mathbb{C}[x, y]$ the zero set $Z(f)$ is by definition a closed set. The intersection of two hypersurfaces - or curves - is

$$Z(f) \cap Z(g) = Z(f, g),$$

the collection of points corresponding to their common intersections - which typically is a finite set of points. For instance, let $f = x - y + 1$ and $g = y^2 - x^3$.

Example 7.6.3. Show that the open sets $D(f) = \mathbb{A}^n \setminus Z(f)$, with $f \in \mathbb{C}[x_1, \ldots, x_n]$ form a basis for the topology. That is, any open can be written as a union of the basic opens $D(f)$. In the usual, strong, topology, the open balls form a basis for the topology.

Example 7.6.4. The open sets in $\mathbb{A}^n$ are big: Show that any two non-empty opens U and V in $\mathbb{A}^n$ has a non-empty intersection. In particular we get that $\mathbb{A}^n$ is not a Hausdorff space.

Example 7.6.5. Show that $\mathbb{A}^n$ is compact; for any open cover $\{U_\alpha\}$ of $\mathbb{A}^n$ a finite subcollection will be a covering.

Definition 7.6.6. A (non-empty) topological space X is called irreducible if X can not be written as the union $X = X_1 \cup X_2$ of two proper closed subsets X_1 and X_2 of X.

Example 7.6.7. The affine line $\mathbb{A}^1$ is irreducible. Because any non-zero polynomial f is such that $Z(f)$ is a collection of finite points. It follows that closed, proper, subsets of $\mathbb{A}^1$ are collections of finite points. And in particular we can not write $\mathbb{A}^1$ as a union of two finite sets, hence $\mathbb{A}^1$ is irreducible.

Example 7.6.8. If X is irreducible, then any non-empty open $U \subseteq X$ is also irreducible (Exercise 7.6.13). In particular if we let $U = \mathbb{A}^1 \setminus (0)$, then U is irreducible even if the picture you draw indicates that the space U is not even connected. A topological space X is not connected if it can be written as a union $X = X_1 \cup X_2$ of two proper closed subsets where $X_1 \cap X_2 = \emptyset$. In particular a space that is not connected is in particular not irreducible.

Definition 7.6.9. An irreducible algebraic set is an affine algebraic variety.

Example 7.6.10. Let $I = (xy) \subseteq \mathbb{C}[x, y]$ denote the ideal generated by the element $f = xy$. Then the algebraic set

$$Z(xy) = Z(x) \cup Z(y),$$

where the two sets $Z(x) \subset Z(xy)$ and $Z(y) \subset Z(xy)$ are proper subsets. Hence $Z(xy)$ is not an algebraic variety.

Example 7.6.11. Let $I = (x^2+y^3) \subseteq \mathbb{C}[x, y]$. The ideal I is generated by $f = x^2+y^3$, which is an irreducible element - which means that the element f can not be written as a product $f = f_1 \cdot f_2$ in a non-trivial way. It follows that $Z(f)$ is an algebraic variety.

Noetherian spaces A topological space X is called Noetherian space if any descending chain of closed subsets

$$X \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots$$

stabilizes, that is there exists an integer n such that $X_n = X_{n+i}$, for all integers $i \geq 0$.

Exercise 7.6.12. Show that $\mathbb{A}^n$ is a Noetherian space.

Exercise 7.6.13. Let $U \subseteq X$ be a non-empty open set, with X irreducible. Show that U is also irreducible.

Exercise 7.6.14. Show that any topological space X can be written as a union of irreducible subsets, called irreducible components of the space X. If X is an algebraic variety it has only a finite set of irreducible components.

7.7 Prime ideals

Definition 7.7.1. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a *proper* ideal. The ideal I is said to be a prime ideal if

$$gf \in I \quad \text{implies that } f \text{ or } g \text{ is in } I.$$

Example 7.7.2. Let $I = (xy) \subseteq \mathbb{C}[x, y]$ denote the ideal generated by the element $f = xy$. Then any element in I can be written as $F \cdot f$, with $F \in \mathbb{C}[x, y]$. In particular we get that neither x nor y is in I, but clearly xy is. Thus $I = (xy)$ is not a prime ideal.

Example 7.7.3. Let $I = (x^2+y^3) \subseteq \mathbb{C}[x, y]$. The ideal I is generated by $f = x^2+y^3$, which is an irreducible element. It follows that the ideal $I = (x^2 + y^3)$ is a prime ideal.

Exercise 7.7.4. A non-zero element $f \in \mathbb{C}[x_1, \ldots, x_n]$ is irreducible if $f = f_1 \cdot f_2$ implies that at least one of the factors f_1 or f_2 is a unit. Show that an ideal I generated by an irreducible element f implies that the algebraic set $Z(I)$ is irreducible. Give an example of an irreducible hypersurface $Z(f)$ where f is not an irreducible element.

Exercise 7.7.5. Show that an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a prime ideal if and only if the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I$ is an integral domain. Recall that a ring A is called an integral domain if A is not the zero ring, and if $f \cdot g = 0$ then either $f = 0$ or $g = 0$. Show furthermore that an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a maximal ideal if and only if the quotient ring $\mathbb{C}[x_1, \ldots, , x_n]/I$ is a field. Recall that an integral domain A is called a field if every non-zero element is invertible, and that a prime ideal not properly contained in any other prime ideal is maximal ideal.

Example 7.7.6. Ideals of the form $I = (x_1 - a_1, \ldots, x_n - a_n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$, with $(a_1, \ldots, a_n) \in \mathbb{C}^n$, are prime ideals. For instance, the quotient ring $\mathbb{C}[x_1, \ldots, x_n]/I = \mathbb{C}$, is a field (see Exercise 7.7.5). Note that

$$Z((x_1 - a_1, \ldots, x_n - a_n)) = \cap_{i=1}^n Z(x_i - a_i) = (a_1, \ldots, a_n).$$

Exercise 7.7.7. The ideal $I = (xy) \subseteq \mathbb{C}[x, y]$ is not prime, but the ideals (x) and (y) are prime ideals. Show that $I = (x) \cap (y)$, and use this to describe the irreducible components of $Z(I)$.

7.8 Radical ideals

Definition 7.8.1. The radical of an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is the set

$$\sqrt{I} = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f^m \in I \text{ for some } m \geq 0\}.$$

One shows that the radical $\sqrt{I}$ is also an ideal. Clearly we have an inclusion $I \subseteq \sqrt{I}$. We say that the ideal I is radical if $I = \sqrt{I}$.

Example 7.8.2. That any prime ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is a radical ideal, follows almost directly from the definitions. The converse is not the case. Consider for instance the ideal $I = (xy) \subseteq \mathbb{C}[x, y]$ discussed in Example 7.7.2. The ideal $I = (xy)$ is not prime, but radical.

Example 7.8.3. The ideal $I = (x^2) \subseteq \mathbb{C}[x, y]$ is not prime, nor radical. The element x is not in I, but x^2 is in I. Note that the set $Z(x^2)$ is an irreducible hypersurface (see Exercise 7.7.4).

Theorem 7.8.4 (Hilberts Nullstellenzats)**.** *Algebraic sets in $\mathbb{A}^n$ are in one to one correspondence with the set of radical ideal in $\mathbb{C}[x_1, \ldots, x_n]$. Furthermore, we have that an algebraic set is irreducible if and only if its corresponding radical ideal is a prime ideal.*

Corollary 7.8.5. *Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be any proper ideal. Then we have that its radical ideal*

$$\sqrt{I} = \bigcap_{\substack{(a_1,\ldots,a_n)\in\mathbb{C}^n \\ I\subseteq(x_1-a_1,\ldots,x_n-a_n)}} (x_1 - a_1, \ldots, x_n - a_n)$$

where the intersection is taken over the set of maximal ideals containing I.

Proof. The radical of an ideal I is the intersection of all prime ideals containing I, which is a standard fact. The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ is an example of a Jacobson ring, which means that any intersection of prime ideals is in fact given by the corresponding intersection of maximal ideals. Thus, the radical of I is the intersection of all maximal ideals containing I. By the Nullstellensatz, or the weak version of it, the maximal ideals are all of the form $(x_1 - a_1, \ldots, x_n - a_n)$, with $a_i \in \mathbb{C}$, $i = 1, \ldots, n$. $\square$

Exercise 7.8.6. Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be an ideal. Show that for any integer n we have that

$$Z(I) = Z(I^n).$$

Exercise 7.8.7. Show that for any ideal I we have $Z(I) = Z(\sqrt{I})$.

Exercise 7.8.8. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a proper, radical ideal. Assume that I is not prime. Show that there exists elements f and g such that

$$Z(I) = Z(I + f) \cup Z(I + g)$$

is a union of proper subsets. Here $I + f$ means the ideal generated by f and elements of I. Thus if $f_1, \ldots, f_m$ generate I, then $f, f_1, \ldots, f_m$ generate $I + f$.

Exercise 7.8.9. Let $Z \subseteq \mathbb{A}^3$ be the algebraic set defined by the ideal

$$I = (x^2 - yz, xz - x) \subset \mathbb{C}[x, y, z].$$

Show that Z has three components, and describe their corresponding prime ideals.

Exercise 7.8.10. We identify $\mathbb{C}^2$ with $\mathbb{C} \times \mathbb{C}$. Show that the Zariski topology on $\mathbb{C}^2$ is not given by the product topology of $\mathbb{A}^1$ with $\mathbb{A}^1$.

7.9 Polynomial maps

Lemma 7.9.1. *Let $F_i = F_i(x_1, \ldots, x_m)$ be an ordered sequence of n polynomials in m variables ($i = 1, \ldots, n$). The induced map $F\colon \mathbb{A}^m \longrightarrow \mathbb{A}^n$, sending $a = (a_1, \ldots, a_m)$ to $(F_1(a), \ldots, F_n(a))$ is a continuous map.*

Proof. We show that the map is continuous by showing that the inverse image of closed sets are closed. Let $f(y_1, \ldots, y_n)$ be a polynomial in n variables. Let $f \circ F = f(F_1(x), \ldots, F_n(x))$, which is a polynomial in the m variables $x = x_1, \ldots, x_m$. Note that

$$F^{-1}(Z(f)) = \{a \in \mathbb{A}^m \mid (F_1(a), \ldots, F_n(a)) \in Z(f)\},$$

which means that

$$F^{-1}(Z(f)) = \{a \in \mathbb{A}^m \mid f(F_1(a), \ldots, F_n(a)) = 0\} = Z(f \circ F).$$

It follows that if $I \subseteq \mathbb{C}[y_1, \ldots, y_n]$ is an ideal generated by $f_1, \ldots, f_N$, then $F^{-1}(Z(I))$ is the algebraic set given by the ideal $J \subseteq k[x_1, \ldots, x_m]$ generated by $f_1 \circ F, \ldots, f_N \circ F$. $\square$

A map $F\colon \mathbb{A}^m \longrightarrow \mathbb{A}^n$ given by polynomials as in the Lemma, is a polynomial map.

7.10 Maps of affine algebraic sets

If $F\colon \mathbb{A}^m \longrightarrow \mathbb{A}^n$ is a polynomial map that would factorize through an algebraic set $Y \subseteq \mathbb{A}^n$, then we have a map $F\colon \mathbb{A}^n \longrightarrow Y$. Restricting such a map to an algebraic set $X \subseteq \mathbb{A}^m$ gives a map $F\colon X \longrightarrow Y$, which is how we will define a map of algebraic sets.

Remark Note that our definition of a map between affine algebraic sets, requires an embedding into affine spaces.

Example 7.10.1. The two polynomials $F_1(t) = t^2$ and $F_2(t) = t^3$ determine the polynomial map $F\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^2$. The map F then sends a point $a \mapsto (a^2, a^3)$. Let $Y \subseteq \mathbb{A}^2$ be the curve given by the polynomial $f(x,y) = y^2 - x^3$ in $\mathbb{C}[x,y]$, that is $Y = Z(f)$. For any scalar a we have that the pair (a^2, a^3) is such that $f(a^2, a^3) = 0$. In order words we get an induced map

$$F\colon \mathbb{A}^1 \longrightarrow Y.$$

Definition 7.10.2. Two affine algebraic sets X and Y are isomorphic, or simply equal, if there exists polynomial maps $F\colon X \longrightarrow Y$ and $G\colon Y \longrightarrow X$ such that $F \circ G = \mathrm{id}$ and $G \circ F = \mathrm{id}$.

Example 7.10.3. The map $F\colon \mathbb{A}^1 \longrightarrow Y = Z(y^2 - x^3) \subseteq \mathbb{A}^2$ discussed in Example 7.10.1 is a homeomorphism (see Exercise 7.10.5). However, even if the map F gives a homeomorphism between $\mathbb{A}^1$ and the curve $Y \subseteq \mathbb{A}^2$, these two varieties are not considered as equal. The inverse $G\colon Y \longrightarrow \mathbb{A}^1$ to F is not a polynomial map.

Example 7.10.4. Let $p\colon \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ be the projection on the first factor, thus $p(a,b) = a$. Which is a polynomial map. The fiber over a point $a \in \mathbb{A}^1$ is the "vertical" line $Z(x-a)$, where the hypersurface $x - a \in \mathbb{C}[x,y]$. Consider now the polynomial

$$G(x,y) = y^3 + g_1(x)y^2 + g_2(x)y + g_3(x).$$

We get an induced projection map $p_1\colon Z(G) \longrightarrow \mathbb{A}^1$. The fiber over a point $a \in \mathbb{A}^1$ is $Z(x-a, G)$, which is given by

$$g(y) = y^3 + g_1(a)y^2 + g_2(a)y + g_3(a) \in \mathbb{C}[y].$$

The three roots of $g(y)$ are the three points lying above a.

Exercise 7.10.5. Verify that the map $F\colon \mathbb{A}^1 \longrightarrow Y$ in Example 7.10.1 is a homeomorphism, and that the inverse map is not a polynomial map: You first check that the map is bijective, and from Lemma 7.9.1 you have that it is continuous. Then to be able to conclude that the map is a homeomorphism it suffices to verify that the map F gives a bijection between the closed sets of $\mathbb{A}^1$ and Y. The closed sets on Y are the intersection of closed sets on $\mathbb{A}^2$ with Y. The non-trivial closed sets are given by a finite collection of points, and then you have verified that F is a homeomorphism. Then you need to convince yourself that the inverse map G is not a polynomial map.

Exercise 7.10.6. Show that the variety $Z(xy-1) \subseteq \mathbb{A}^2$ is isomorphic to $\mathbb{A}^1 \setminus 0$.

Exercise 7.10.7. Show that a basic open $\mathbb{A}^n \setminus Z(f)$ is an algebraic variety by identifying it with $Z(tf - 1) \subseteq \mathbb{A}^{n+1}$.

Chapter 8

Projective varieties

We will define the projective space as a certain quotient space where we identify lines in affine space.

8.1 Projective n-space

Definition 8.1.1. We let $\mathbb{P}^n$ denote the topological space we obtain by taking the quotient space of $\mathbb{A}^{n+1} \setminus (0, \ldots, 0)$ modulo the equivalence relation

$$(a_0, \ldots, a_n) \simeq (\lambda a_0, \ldots, \lambda a_n),$$

with non-zero scalars $\lambda \in \mathbb{C}$. The space $\mathbb{P}^n$ is called projective n-space. The equivalence class of a vector $(a_0, \ldots, a_n)$ we denote by

$$[a_0 : a_1 : \cdots : a_n].$$

Quotient topology Recall the notions of quotient topology discussed in Section 1.2.

Exercise 8.1.2. Show that $\mathbb{P}^n$ is a Noetherian space.

8.2 Homogeneous polynomials

Let $S = \mathbb{C}[X_0, \ldots, X_n]$ denote the polynomial ring in the variables $X_0, \ldots, X_n$. Let $S_d(X_0, \ldots, X_n) = S_d$ denote the vector space of degree $d \geq 0$ forms in $X_0, \ldots, X_n$. The monomials $\{X_0^{d_0} \cdots X_n^{d_n}\}$ where $d = d_0 + \cdots + d_n$, form a basis for the vector space S_d.

We have the decomposition of vector spaces

$$S = \mathbb{C}[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$$

into homogeneous parts. An element $F \in S$ is homogeneous, and of degree d, if $F \in S_d$. As the zero polynomial is in S_d for any $d \geq 0$, it has no well-defined degree.

Exercise 8.2.1. Compute the dimension of $S_d(X_0, \dots, X_n)$, that is, the Hilbert function $H_{\mathbb{P}^n}(d)$ (cf. Definition 6.4.2).

Exercise 8.2.2. The Hilbert series of the graded polynomial ring is the formal expression $H(t) = \sum_{d \geq 0} \dim S_d(X_0, \dots, X_n) t^d$. Show that

$$H(t) = \prod_{i=0}^{n} \frac{1}{1-t}.$$

8.3 Hypersurfaces in projective space

Any homogeneous polynomial $F(X_0, \dots, X_n)$ will satisfy

$$F(\lambda a_0, \dots, \lambda a_n) = \lambda^d F(a_0, \dots, a_n)$$

where d is the degree of $F(X_0, \dots, X_n)$, and $a_0, \dots, a_n$ is any vector in $\mathbb{C}^{n+1}$. Thus, if $a = (a_0, \dots, a_n)$ is an element of $Z(F) \subseteq \mathbb{A}^{n+1}$, then the whole line spanned by a is in $Z(F)$. It follows that the set $Z(F) \setminus (0, \dots, 0)$ in $\mathbb{A}^{n+1} \setminus (0, \dots, 0)$ is invariant with respect to the equivalence relation. In particular we get, by taking the quotient, a closed subset $\overline{Z}(F)$ in $\mathbb{P}^n$. Any homogeneous element F of degree $d \geq 1$ will vanish on $(0, \dots, 0)$ in $\mathbb{A}^{n+1}$, that is $(0, \dots, 0) \in Z(F)$. If $Z(F)$ contains other points as well, then we get a non-empty subset $\overline{Z}(F)$ in $\mathbb{P}^n$, and these we refer to as hypersurfaces in projective n-space.

Lectures 4-6 The projective curves discussed in lectures 4-6 are examples of hypersurfaces in the projective plane.

8.4 Homogeneous ideals

An ideal I in the polynomial ring $S = \mathbb{C}[X_0, \dots, X_n]$ is a *homogeneous ideal* if there exist homogeneous elements $F_1, \dots, F_m$ that generates the ideal, $I = (F_1, \dots, F_m)$. A homogeneous ideal I can be decomposed as $I = \bigoplus_{d \geq 0} I_d$.

Any homogeneous ideal $I \subseteq S$ defines a closed set

$$\overline{Z}(I) = \cap_{i=1}^{m} \overline{Z}(F_i) \subseteq \mathbb{P}^n,$$

where $F_1, \dots, F_m$ is a collection of homogeneous elements that generate I. Note that any closed subset $Z \subseteq \mathbb{P}^n$ is of the form $Z = \overline{Z}(I)$, for some homogeneous ideal $I \subseteq \mathbb{C}[X_0, \dots, X_n]$. We refer to the closed sets in $\mathbb{P}^n$ as algebraic sets.

Exercise 8.4.1. Let F be a homogeneous polynomial in $\mathbb{C}[X_0, \dots, X_n]$, and let I be the homogeneous ideal generated by $X_0F, X_1F, \dots, X_nF$. Show that

$$\overline{Z}(F) = \overline{Z}(I)$$

as subsets of $\mathbb{P}^n$.

Proposition 8.4.2. *The projective n-space $\mathbb{P}^n$ has a open cover by affine n-spaces. In particular we have identification of varieties*

$$\mathbb{P}^n \setminus \overline{Z}(X_i) = \mathbb{A}^n,$$

for every $i = 0, \dots, n$, and these open sets cover $\mathbb{P}^n$.

Proof. Let $Z(x_0 - 1)$ be the affine variety in $\mathbb{A}^{n+1}$ where the first coordinate is 1. We will prove that $Z(x_0 - 1)$ can be identified with $U_0 = \mathbb{P}^n \setminus \overline{Z}(X_0)$. The remaining cases are proved similarly. Restricting the projection map $\pi \colon \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$, gives a map

$$\pi_| \colon Z(x_0 - 1) \longrightarrow U_0 = \mathbb{P}^n \setminus \overline{Z}(X_0)$$

that sends $(1, a_1, \dots, a_n) \mapsto [1 : a_1 : \cdots : a_n]$. We define a map $s \colon U_0 \longrightarrow Z(x_0 - 1)$ by sending

$$s([X_0 : X_1 : \cdots : X_n]) = (1, X_1/X_0, \dots, X_n/X_0).$$

It is clear that $\pi_|$ is a bijection, with s being its inverse. To conclude that $\pi_|$ is a homeomorphism, we need to show that s is continuous. As closed sets in $\mathbb{A}^{n+1}$ are intersections of hypersurfaces, it suffices to show that the inverse image $s^{-1}(Z(f))$ is closed, for hypersurfaces. We have that $s^{-1}(Z(f)) = \pi_|(Z(f))$. The identification of $\mathbb{A}^n$ with $Z(x_0 - 1) \subseteq \mathbb{A}^{n+1}$, of varieties, is clear. Thus, the polynomial f can be considered as a polynomial in n variables $x_1, \dots, x_n$. The polynomial

$$F(X_0, \dots, X_n) = X_0^{\deg f} f(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0})$$

is homogeneous, and by definition $\overline{Z}(F)$ is closed in $\mathbb{P}^n$. We have furthermore that

$$\pi_|(Z(f)) = \overline{Z}(F) \cap U_0,$$

hence closed in U_0. $\square$

Example 8.4.3. Note that setting the first coordinate to 1, and not to say 2, is a choice that corresponds, locally, to a section of the projection map $\mathbb{A}^{n+1}\setminus 0 \longrightarrow \mathbb{P}^n$.

Example 8.4.4. Note that the construction in the proof is a bit ad hoc. We have a homeomorphism identifying $\mathbb{A}^n$ with $\mathbb{P}^n \setminus \overline{Z}(X_i)$, for each i. And we use this identification to give the variety structure on $\mathbb{P}^n$, locally. One could ask why this particular chosen structure was right. It is, and that can be proved using a completely different approach and a different definition of what projective space means.

Gluing Another way of construction the projective n-space is by gluing, which is related to the manifold structure of Theorem 3.2.4. We recall some of it here. Quite generally, if we have a collection of topological spaces $\{U_i\}_{i\in\mathscr{A}}$ we can glue these together along specified intersections: Assume that we have an inclusion of open subsets $U_{i,j} \subset U_i$, for all indices $i, j \in \mathscr{A}$, and homeomorphisms $\varphi_{i,j} \colon U_{i,j} \longrightarrow U_{j,i}$ that satisfies the co-cycle condition

$$\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$$

when restricted to $U_{i,j} \cap U_{i,k}$, for all $i, j, k \in \mathscr{A}$. Then one can check that the data equals the data of an equivalence relation on the disjoint union $\sqcup U_i$. We can then form the quotient space X by identifying, or gluing, the spaces U_i and U_j together along $U_{i,j} = U_{j,i}$ (identified with $\varphi_{i,j}$).

Exercise 8.4.5. Let $U_i = \mathbb{A}^n$, where $i = 0, \ldots, n$ is fixed. For any $j = 0, \ldots, n$ we let $U_{i,j} = \mathbb{A}^n \setminus Z(x_j)$, and we let

$$\varphi_{i,j} \colon U_{i,j} \longrightarrow U_{j,i}$$

be the map given by the following composition. For the fixed i, we have the map $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)$ putting the 1 on the i'th coordinate, for $i = 0, \ldots, n$. As subsets of $\mathbb{A}^{n+1}$ we have a natural identifications of $U_{i,j}$ with $U_{j,i}$. If $j \leq i$ then the j'th coordinate in $U_{i,j}$ will be invertible, and we consider the map

$$(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n) \mapsto a_j^{-1}(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_n)$$

identifying $U_{i,j}$ with $U_{j,i}$. Write up the situation with $j > i$, and show that the identifications satisfy the co-cycle condition. Show furthermore that the gluing of $\sqcup_{i=0}^n U_i$ along the identifications $U_{i,j}$ with $U_{j,i}$ gives $\mathbb{P}^n$. In fact the images of U_i in the quotient space are precisely the open sets $\mathbb{A}^n = \mathbb{P}^n \setminus \overline{Z}(X_i)$ given in Proposition 8.4.2.

Definition 8.4.6. An irreducible algebraic set in $\mathbb{P}^n$ is a projective variety.

Proposition 8.4.7. *Any projective variety $Z \subseteq \mathbb{P}^n$ is given by a homogeneous ideal I that is a prime ideal in $\mathbb{C}[X_0, \dots, X_n]$.*

Proof. Let $\pi \colon \mathbb{A}^{n+1} \setminus 0 \longrightarrow \mathbb{P}^n$ be the quotient map, where $0 = (0, \dots, 0)$. A set $X \subseteq \mathbb{A}^{n+1}$ is irreducible if and only if $X_0 = X \setminus 0$ is irreducible or empty in $\mathbb{A}^{n+1} \setminus 0$. If X_0 is irreducible, then $\pi(X_0)$ is also irreducible since the quotient map π is continuous. Thus we see that for any prime ideal $I \subseteq \mathbb{C}[X_0, \dots, X_n]$ which is also homogeneous, we have that $\overline{Z}(I)$ is irreducible (or empty). Conversely, let $Z \subseteq \mathbb{P}^n$ be an irreducible set. Then $Z = \overline{Z}(I)$ for some homogeneous radical ideal $I \subseteq \mathbb{C}[X_0, \dots, X_n]$. If I is not prime, then there exists elements F and G not in I but where $FG \in I$. One checks that we can assume the elements F and G to be homogeneous elements. But, then we have, as in Exercise 7.8.8,

$$Z(I + F) \cup Z(I + G) = Z(I),$$

with proper subsets $Z(I + F) \subset Z(I)$, and $Z(I + G) \subset Z(I)$. It follows that $\overline{Z}(I) = \overline{Z}(I + F) \cup \overline{Z}(I + G)$ is the union of two proper closed subsets. Hence $\overline{Z}(I)$ was not irreducible after all. We can therefore conclude that the ideal I was prime. $\square$

Exercise 8.4.8. The projective set C in $\mathbb{P}^3$ given by the homogeneous ideal generated by

$$XW - YZ, \quad XZ - Y^2 \quad \text{and} \quad YW - Z^2$$

in $\mathbb{C}[X, Y, Z, W]$ is called the twisted cubic. If we let $[t : u]$ be projective coordinates for the projective line, then the twisted cubic is the set of points in $\mathbb{P}^3$ of the form $[t^3 : t^2u : tu^2 : u^3]$.

Exercise 8.4.9. The intersection of varieties is not always a variety. Consider the two surfaces Q_1 and Q_2 in $\mathbb{P}^3$, given by the ideals generated by the quadratic polynomials

$$F = Z^2 - YW \quad \text{and} \quad G = XY - ZW$$

in $\mathbb{C}[X, Y, Z, W]$. Show that both $Q_1 = Z(F)$ and $Q_2 = Z(G)$ are varieties. Identify furthermore their intersection $Q_1 \cap Q_2$ as the union of a twisted cubic, and a line. In fact, we have that the intersection is given as the union of the line $Z(X, W)$ and the twisted cubic $C = Z(F, G, Y^2 - XZ)$ (see Exercise 8.4.8).

Exercise 8.4.10. The ideal generated by the union does not always describe the intersection. Let C be the curve given by the ideal $I(C) = (X^2 - YZ)$, and let L be the line $I(L) = (Y)$. Show that the intersection $C \cap L$ is a point $P \in \mathbb{P}^2$. Compute the homogeneous prime ideal $I(P)$ corresponding to the point P, and deduce that $I(C) + I(L) \neq I(P)$. Draw a picture to explain the situation.

Exercise 8.4.11. Let $P = [1 : 0 : 0 : 0]$ and $Q = [0 : 1 : 0 : 0]$ be points in $\mathbb{P}^3$. There is a unique line L passing through P and Q. A parametric description of the line is the set $tP + uQ$, with projective parameters $[t : u] \in \mathbb{P}^1$. Describe the homogeneous prime ideal $I \subseteq \mathbb{C}[x, y, z, w]$ defining L.

Exercise 8.4.12. Consider the quadratic surface $Q \subset \mathbb{P}^3$ given by the polynomial $XY - ZW$ in $\mathbb{C}[X, Y, Z, W]$. Show that Q contains two families of lines L_P and N_P parametrized by points $P = [t : u] \in \mathbb{P}^1$, with the following property.

$$L_P \cap L_{P'} = N_P \cap N_{P'} = \emptyset \quad \text{if} \quad P \neq P',$$

and

$$L_P \cap N_{P'} = \text{a point} \quad \text{for all } P, P'.$$

Hint, use that $XY - ZW$ is the determinant of the matrix

$$\begin{bmatrix} X & Z \\ W & Y \end{bmatrix}.$$

Chapter 9

Maps of projective varieties

9.1 Quasi-projective varieties

Before we continue with what should be a map between projective varieties, it turns out that it is convenient to define the notion of quasi-projective sets.

Definition 9.1.1. A set $X \subseteq \mathbb{P}^n$ is quasi-projective if it is locally closed; meaning that there exists a (algebraic) closed set $Z \subseteq \mathbb{P}^n$ containing $X \subseteq Z$ as an open subset.

Example 9.1.2. Any algebraic set $X \subseteq \mathbb{P}^n$ is quasi-projective since X is open in itself, and an algebraic set is by definition closed in $\mathbb{P}^n$. Any open subset $U \subseteq X$ of an algebraic set $X \subseteq \mathbb{P}^n$ is by definition quasi-projective. In particular the affine n-space $\mathbb{A}^n$ is quasi-projective since $\mathbb{A}^n = \mathbb{P}^n \setminus \overline{Z}(X_0)$ is open in $\mathbb{P}^n$. Any open subset $U \subseteq \mathbb{A}^n$ is quasi-projective since it will also be open in $\mathbb{P}^n$.

Exercise 9.1.3. Show that any affine algebraic set $X = Z(I) \subseteq \mathbb{A}^n$ is quasi-projective by showing that there is a homogeneous ideal $J \subseteq \mathbb{C}[X_0, \dots, X_n]$ such that

$$X = \overline{Z}(J) \cap \mathbb{P}^n \setminus \overline{Z}(X_0).$$

Then X will be open in $\overline{Z}(J)$. I think you will prove something as: For any $f \in \mathbb{C}[t_1, \dots, t_n]$, let $h(f) = x_0^{\deg f} f(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0})$. Then $h(f)$ is a homogeneous element of $\mathbb{C}[X_0, \dots, X_n]$. For any homogeneous polynomial $F \in \mathbb{C}[X_0, \dots, X_n]$ we let $d(F) = F(1, t_1, \dots, t_n)$. We have that

$$d(h(f)) = f \quad \text{and} \quad h(d(F)) = x_0^N F \quad \text{some} \quad N \geq 0.$$

Exercise 9.1.4. Let $Y \subseteq \mathbb{A}^n$ be an algebraic set, and let $I(Y)$ be an ideal defining Y. We view Y as a subset of $\mathbb{P}^n$, by identifying $\mathbb{A}^n = \mathbb{P}^n \setminus Z(X_0)$. The closure $\overline{Y}$

is the smallest closed subset containing Y, and is then by definition an algebraic set. Show that the $h(I(Y))$, obtained by applying the h function in Exercise 9.1.3 to all elements of $I(Y)$, generates the homogeneous ideal $I(\overline{Y})$ describing $\overline{Y}$.

Exercise 9.1.5. Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic, that is the image of $\mathbb{A}^1 \longrightarrow \mathbb{A}^3$ by the map $t \mapsto (t, t^2, t^3)$ (cf. Exercise 8.4.8). Use this example to show that if $f_1, \dots, f_r$ generates an ideal $I(Y)$, then $h(f_1), \dots, h(f_r)$ does not generate the ideal $I(\overline{Y})$ of its closure in projective space (see Exercise 9.1.4).

The affine varieties are the building blocks The notion of quasi-projective sets includes both projective and affine algebraic sets. Note that sets as $\mathbb{A}^2 \setminus (0,0)$ are also quasi-projective, these are however not affine algebraic sets. At first glance it now would appear as the notion of quasi-projective varieties is to generous, forcing us to also deal with sets which are not build up via affine charts. However, as we saw in Exercise 7.10.7 the basic open sets are affine algebraic sets. So indeed $\mathbb{A}^2 \setminus = \mathbb{A}^2 \setminus Z(x) \cup \mathbb{A}^2 \setminus Z(y)$ is the union of varieties.

9.2 Regular maps

Let $F_0, \dots, F_n$ be $n+1$ homogeneous polynomials each of the same degree, in $m+1$ variables. Having these polynomials we get a polynomial map

$$F = (F_0, \dots, F_n) \colon \mathbb{A}^{m+1} \longrightarrow \mathbb{A}^{n+1}. \tag{9.1}$$

The fact that the polynomials all have the same degree guarantees that the map of affine spaces respects the equivalence classes defining the projective space. In order to get an induced map of projective spaces we need furthermore that the common zero of the polynomials is the origin only.

Definition 9.2.1. Let $X \subseteq \mathbb{P}^m$ be a quasi-projective set. A regular map $F \colon X \longrightarrow \mathbb{P}^n$ is a sequence $F = (F_0, \dots, F_n)$ of homogeneous polynomials in $m+1$-variables, all of same degree, and where the polynomials $F_0, \dots, F_n$ do not simultaneously vanish on X. The last condition is that

$$X \cap \overline{Z}(F_0) \cap \cdots \cap \overline{Z}(F_n) = \emptyset.$$

Example 9.2.2. Consider the polynomial map $\mathbb{A}^3 \longrightarrow \mathbb{A}^2$ given by the two linear polynomials $F_0 = X$ and $F_1 = Y - Z$. So the polynomial map sends

$$(a, b, c) \mapsto (a, b - c).$$

This will respect the equivalence classes defining the projective space, but will not induce a map from the projective plane to the projective line. This is because the line $(0, t, t)$ is sent to $(0, 0)$. In order words we get an induced map

$$\mathbb{P}^2 \setminus [0 : 1 : 1] \longrightarrow \mathbb{P}^1$$

defined on the complement of a point in the projective plane. The regular map defined above can not be extended to a regular map defined on the whole of projective plane.

Example 9.2.3. Let $Q \subset \mathbb{P}^2$ be the quadratic curve given by the equation $X^2 + Y^2 + Z^2$. As the point $[0 : 1 : 1]$ is not on Q, the map described in the previous example gives a map $F \colon Q \longrightarrow \mathbb{P}^1$ that takes

$$[X : Y : Z] \mapsto [X : Y - Z].$$

So the map $F \colon Q \longrightarrow \mathbb{P}^1$ is given by the homogeneous polynomials $F_0 = X$ and $F_1 = Y - Z$ in the variables X, Y, Z.

9.3 Maps of projective varieties

We are now ready to define what we mean with a map of projective varieties.

Definition 9.3.1. Let $X \subseteq \mathbb{P}^m$ be a quasi-projective algebraic set. A map $F \colon X \longrightarrow \mathbb{P}^n$ is a finite collection of regular maps $F_i \colon X_i \longrightarrow \mathbb{P}^n$ $(i = 1, \ldots, N)$, where

(1) we have that $X_i \subseteq X$ is an open subset, for each $i = 1, \ldots, N$,

(2) the opens cover $X = \cup_{i=1}^N X_i$,

(3) restricted to intersections $X_i \cap X_j$ the regular maps F_i and F_j agree, for all $i, j \in 1, \ldots, N$.

Furthermore, if $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ are two projective sets, then a map $F \colon X \longrightarrow Y$ is a map $F \colon X \longrightarrow \mathbb{P}^n$ that factors via the inclusion $Y \subseteq \mathbb{P}^n$. Note that a regular map $F \colon X \longrightarrow \mathbb{P}^n$ is in particular a map of projective varieties.

Example 9.3.2. Let $C \subset \mathbb{P}^2$ be the quadratic curve given by the equation $X^2 - Y^2 + Z^2 = 0$. The map in Example 9.2.2 induces, by restriction, a map

$$F_1 \colon C \setminus [0 : 1 : 1] \longrightarrow \mathbb{P}^1$$

that sends $[X : Y : Z]$ to $[X : Y - Z]$. In a similar way we have a map $F_2\colon C \setminus [0 : -1 : 1] \longrightarrow \mathbb{P}^1$ that sends

$$[X : Y : Z] \mapsto [Y + Z : X].$$

There is nothing wrong with these two maps F_1 and F_2, both being given by polynomials, but none of these two can be extended to a polynomial map from the whole curve. However, together they describe a map. Note that when $X \neq 0$ we have

$$[X : Y - Z] = [1 : \frac{Y - Z}{X}] = [1 : \frac{Y^2 - Z^2}{X(Y + Z)}] = [Y + Z : \frac{X^2}{X}].$$

In other words, the two regular maps F_1 and F_2 are equal when restricted to

$$C \setminus [0 : 1 : 1] \cup [0 : -1 : 1].$$

Therefore, together the two maps F_1 and F_2 describe a map from the union

$$F\colon C \longrightarrow \mathbb{P}^1.$$

It should be clear from the example above that we need to accept that the maps of projective varieties can only be locally defined by polynomials. However, when one have defined the maps locally then one can wonder how small or local these defining charts need to be. The situation is not that bad, as the following statement shows.

Proposition 9.3.3. *A map $F\colon X \longrightarrow Y$ of affine algebraic sets is given by a polynomial map.*

Proof. We do not prove it here, but we want to point out the following. Any affine algebraic set X is a quasi-projective set (Exercise 9.1.3). Hence it makes sense to talk about maps of affine algebraic sets, considered as quasi-projective sets. If $X \subseteq \mathbb{A}^n$, then $X = Z(I)$ for some ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$. One shows that the maps $X \longrightarrow \mathbb{A}^m$ are simply m elements in the quotient ring $\mathbb{C}[x_0, \ldots, x_n]/I$. And it follows that maps of affine algebraic sets are the same as polynomial maps. $\square$

Exercise 9.3.4. What should two isomorphic projective varieties mean? You probably can imagine several reasonable definitions, and these are probably all correct.

Exercise 9.3.5. The degree of a map of projective curves is the number of points in a generic fiber. What is the degree of the map $F\colon Q \longrightarrow \mathbb{P}^1$ given in Example 9.3.2

9.4 The Veronese embedding

Let $d \geq 1$ be a given integer. For each integer $n \geq 0$ we will describe the Veronese map

$$v_d \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

where $N+1$ is the number of monomials in $n+1$ variables, in degree d, i.e. $N+1$ is the dimension of $S_d(X_0, \dots, X_n)$. Define the set

$$\mathscr{D}_d^n = \mathscr{D} = \{\underline{d} = (d_0, \dots, d_n) \in \mathbb{N}^{n+1} \mid d_0 + \cdots + d_n = d\}. \tag{9.2}$$

Then clearly the monomials in $X_0, \dots, X_n$ of degree d correspond, naturally, to the elements of $\mathscr{D}$. If $\underline{d} \in \mathscr{D}$ then the corresponding monomial $X^{\underline{d}} = X_0^{d_0} \cdots X_n^{d_n}$. We have the polynomial map $\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{N+1}$, that sends a vector $(a_0, \dots, a_n)$ to $(a^{\underline{d}})_{\underline{d} \in \mathscr{D}}$. The map respects the equivalence relation defining the projective spaces, since the monomials are all homogeneous of the same degree, and hence gives an induced map $v_d \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$ taking

$$[X_0 : \cdots : X_n] \mapsto [X^{\underline{d}}]_{\underline{d} \in \mathscr{D}}.$$

Proposition 9.4.1. *Let $d \geq 1$ and $n \geq 0$ be two given integers, and let $\mathbb{C}[X_{\mathscr{D}}]$ be the graded polynomial ring with variables indexed by the set $\mathscr{D}$ given in 9.2. The Veronese map $v_d \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$ identifies $\mathbb{P}^n$ with the Veronese variety $V_{n,d}$ given by the homogeneous ideal*

$$I = (X_{\underline{d}} X_{\underline{d}'} - X_{\underline{e}} X_{\underline{e}'}) \subset \mathbb{C}[X_{\mathscr{D}}],$$

where the quadratic equations $X_{\underline{d}} X_{\underline{d}'} - X_{\underline{e}} X_{\underline{e}'}$ is formed for every quadruple $\underline{d}, \underline{d}', \underline{e}, \underline{e}'$ in $\mathscr{D}$ such that $\underline{d} + \underline{d}' = \underline{e} + \underline{e}'$.

Proof. Let $\underline{i}$ denote the element in $\mathscr{D}$ that has the number d in coordinate i, and zero elsewhere (with $i = 0, \dots, n$). Thus $\underline{i}$ corresponds to the monomial X_i^d. For each $i = 1, 0, \dots, n$ we let

$$U_i = \overline{Z}(I) \cap \mathbb{P}^N \setminus \overline{Z}(X_{\underline{i}}).$$

Then one checks that the open sets $U_0, \dots, U_n$ cover the algebraic set $\overline{Z}(I)$. We will next indicate how maps $G_i \colon U_i \longrightarrow \mathbb{P}^n$ are defined by simply giving the definition of G_0. Let $d(i) \in \mathscr{D}$ be the element $d(i) = (d-1, 0, \dots, 1, \dots, 0)$, where the 1 appears in coordinate i (and $i = 1, \dots, n$). We define the map $G_0 \colon U_0 \longrightarrow \mathbb{P}^n$ by

$$G_0([X_{\underline{d}}]) = [X_{\underline{0}} : X_{d(1)} : \cdots : X_{d(n)}].$$

One checks that the similarly defined maps G_i and G_j coincide on $U_i \cap U_j$ (for all pairs i, j), and hence we have a map

$$G\colon \overline{Z}(I) \longrightarrow \mathbb{P}^n.$$

Clearly the maps G_i will factor via the open inclusion $\mathbb{P}^n \setminus \overline{Z}(X_i)$. Let F_i be the restriction of the Veronese map v_d to the open subset $\mathbb{P}^n \setminus \overline{Z}(X_i)$. Note that since $X_0 \neq 0$ on U_0, we have that

$$G_0([X_{\underline{d}}]) = X_0^d[1\colon \frac{X_1}{X_0}\colon \cdots \colon \frac{X_n}{X_0}].$$

One verifies that G_i is the inverse of F_i, so G is the inverse of F, and you have proven that the Veronese map is an isomorphism of projective varieties. $\square$

Example 9.4.2. The Veronese surface is given by the map $v_2\colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ that sends

$$[X : Y : Z] \mapsto [X^2 : XY : XZ : Y^2 : YZ : Z^2].$$

The defining equations for the Veronese surface is given by the quadratic polynomials in $\mathbb{C}[Z_0, \ldots, Z_5]$, given as the (2×2)-minors of

$$\begin{bmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{bmatrix}.$$

This is a neat way of describing all two pairs (d, d') and (e, e') of vectors, among $(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)$ that have the same sum $d + d' = e + e'$. There are nine (2×2)-minors, but since the matrix is symmetric, only six of them are relevant.

Exercise 9.4.3. Let $V \subseteq \mathbb{P}^5$ denote the Veronese surface, given in Example 9.4.2. Show that for any two points P and Q on V there exists a conic curve C on V passing through the points P and Q. The curve C will be an embedding of the projective line $\mathbb{P}^1$.

9.5 Veronese subvarieties

As the projective n-space $\mathbb{P}^n$ is identified with the Veronese variety $V_{d,n} \subseteq \mathbb{P}^N$, it means in particular that subvarieties $\overline{Z}(I)$ in $\mathbb{P}^n$ are identified with subvarieties in $V_{d,n}$. We will look a bit closer at that correspondence. Thus let $d \geq 1$ and $n \geq 0$ be fixed integers. Note that a homogeneous element $F \in \mathbb{C}[X_0, \ldots, X_n]$ of degree $d \cdot e$, naturally can be viewed as a polynomial in the monomials of degree d.

Furthermore, if $G \in \mathbb{C}[X_0, \ldots, X_n]$ is homogeneous of degree e, then $I = (X_0G, \ldots, X_nG)$ is homogeneous of degree one more, $e+1$, and we have that $\overline{Z}(G) = \overline{Z}(I)$ (see Exercise 8.4.1). Hence, if $I \subseteq \mathbb{C}[X_0, \ldots, X_n]$ is a homogeneous ideal generated by elements of degree $\leq e$, then we can form the ideal I' that is generated in degree equal to $d \cdot e$, and where $\overline{Z}(I) = \overline{Z}(I')$. However, the generators of I' correspond to degree e polynomials in the degree d monomials in $\mathbb{P}^N$, wherein the Veronese variety $V_{d,n}$ is.

Example 9.5.1. We have the Veronese map $v_2 \colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$, identifying the projective plane with the Veronese surfaces $V_{2,2}$. In the plane we have the cubic curve

$$C = \overline{Z}(X^3 + Y^3 + Z^3) \subset \mathbb{P}^2.$$

We write C as the intersection of the three quartics

$$X^4 + XY^3 + XZ^3, \quad X^3Y + Y^4 + YZ^3 \quad \text{and} \quad X^3Z + Y^3Z + Z^4.$$

These three polynomials of degree $2 \cdot 2$, correspond to the following degree 2 polynomials in $\mathbb{C}[Z_0, \ldots, Z_n]$,

$$Z_0^2 + Z_1Z_3 + Z_2Z_5, \quad Z_0Z_1 + Z_3^2 + Z_4Z_5 \quad \text{and} \quad Z_0Z_2 + Z_3Z_4 + Z_5^2.$$

Intersecting the zero set of these three polynomials with the Veronese surface $V_{2,2}$ gives $v_2(C) \subseteq \mathbb{P}^5$.

Exercise 9.5.2. Show that any projective variety is isomorphic to an intersection of a Veronese variety ($v_d(\mathbb{P}^n)$ for some n, d), with a linear space.

9.6 The Segre embeddings

We let $\mathbb{P}^m \times \mathbb{P}^n$ denote the set of ordered pairs of points $([X], [Y])$, with $[X]$ in $\mathbb{P}^m$ and $[Y]$ in $\mathbb{P}^n$. The Segre map

$$\sigma \colon \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

is defined by sending

$$([X_0 : \cdots : X_m], [Y_0 : \cdots : Y_n]) \mapsto [\cdots : X_iY_j : \cdots].$$

Proposition 9.6.1. *The image of the Segre map $\sigma \colon \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$ is the Segre variety $\Sigma_{m,n}$ defined by the homogeneous ideal*

$$I = (Z_{i,j}Z_{k,l} - Z_{i,l}Z_{k,j}) \subset \mathbb{C}[Z_{i,j}]_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}}.$$

Proof. □

Example 9.6.2. The image of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ by the Segre map is the quadratic surface given by

$$XW - YZ \in \mathbb{C}[X, Y, Z, W].$$

Product varieties Via the Segre map we identify $\Sigma_{m,n}$ with the product $\mathbb{P}^m \times \mathbb{P}^n$, and in particular we can give the product the structure of a variety. This is of course an ad hoc construction. We can define and talk about products of varieties, even if we do not define that concept here.

9.7 Bi-homogeneous forms

A polynomial $F(X,Y) \in \mathbb{C}[X_0,\ldots,X_n,Y_0,\ldots,Y_m]$ in two set of variables X and Y, is bihomogeneous of degree (d,e), if of the form

$$F(X,Y) = \sum_{|\alpha|=d,\,|\beta|=e} c_{\alpha,\beta} X^{\alpha} Y^{\beta},$$

where $c_{\alpha,\beta}$ are complex numbers, and $\alpha = (\alpha_0,\ldots,\alpha_n)$ is multi-index notation with $\mid \alpha \mid = \sum_{i=0}^{n} \alpha_i$. And similarly with $\beta = (\beta_0,\ldots,\beta_m)$. Note that any bigraded polynomial $F(X,Y)$ gives a well-defined pair of closed subsets $Z(F) \subseteq \mathbb{P}^n \times \mathbb{P}^m$.

Example 9.7.1. Let $C \subseteq \mathbb{P}^3$ be the twisted cubic, defined by the quadratic polynomials

$$F = XW - YZ, \quad G = XZ - Y^2 \quad \text{and} \quad H = YW - Z^2$$

in $\mathbb{C}[X,Y,Z,W]$. In $\mathbb{P}^3$ we have the Segre surface Σ being the image of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Example 9.6.2). The surface Σ is cut out by $F = XW - YZ$, and in particular we have that our curve $C \subset \Sigma$. As Σ is $\mathbb{P}^1 \times \mathbb{P}^1$ we can relate algebraic subvarieties of Σ with bihomogeneous ideals in $\mathbb{C}[X_0,X_1,Y_0,Y_1]$. The polynomial G restricted to $\mathbb{P}^1 \times \mathbb{P}^1$ is then the polynomial

$$X_0X_1Y^2 - X_0Y_1^2 = X_0 \cdot g \quad \text{where} \quad g = X_1Y_0^2 - X_0Y_1^2.$$

The polynomial g is bigraded of degree $(1,2)$. The restriction of $Z(G)$ to Σ is then the union of a line $Z(X_0)$ and the curve $Z(g)$. Similarly, the polynomial H becomes

$$X_0X_1Y_1^2 - X_1^2Y_0^2 = -X_1 g,$$

so $Z(H) \cap \Sigma$ is a line union the curve $Z(g)$. In other words the twisted curve C is given by a single polynomial g, of bidegree $(1,2)$ over the Veronese surface Σ.

Exercise 9.7.2. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties, given by homogeneous ideals

$$I_X \subseteq \mathbb{C}[X_0,\ldots,X_n] \quad \text{and} \quad I_Y \subseteq \mathbb{C}[Y_0,\ldots,Y_m].$$

We define the maps

$$i \colon X \longrightarrow \mathbb{P}^{m+n+1} \quad \text{and} \quad j \colon Y \longrightarrow \mathbb{P}^{m+n+1}$$

by $i(x) = [x : 0]$, and $j(y) = [0 : y]$, for any point $x \in X$ and $y \in Y$, and where 0 denotes the sequence of $m+1$, respectively $n+1$, zeros. Show that i identifies X with $i(X)$, and describe the ideal $I'_X \subseteq \mathbb{C}[X, Y]$ describing $i(X) \subseteq \mathbb{P}^{m+n+1}$. Show that

$$i(X) \cap j(Y) = \emptyset.$$

Let finally $I^e_X \subseteq \mathbb{C}[X, Y]$ denote the ideal generated by I_X, and similarly with I^e_Y. Show that the ideal $I^e_X + I^e_Y$ describes the join $J(X, Y) \subseteq \mathbb{P}^{m+n+1}$ consisting of all points on lines $L(x, y)$ between a point $x \in i(X)$ and a point $y \in j(Y)$.

Chapter 10

Projective toric varieties and polytopes: definitions

10.1 Introduction

Toric varieties are algebraic varieties related to the study of **sparse polynomials.** A polynomial is said to be sparse if it only contains prescribed monomials.

Let $A = \{m_0, \ldots, m_d\} \subset \mathbb{Z}^n$ be a finite subset of integer points. We will use the multi-exponential notation:

$$x^a = x_1^{a_1} \cdot x_2^{a_2} \cdots x_n^{a_n} \text{ where } x = (x_1, \ldots, x_n) \text{ and } a = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

Sparse polynomials of type A are polynomials in n variables of the form:

$$p(x) = \sum_{a \in A} c_a x^a.$$

For example if $A = \{(i, j) \in \mathbb{Z}_+^2 \text{ such that } i + j \leq k\}$ then the polynomials of type A are all possible polynomials of degree up to k.

Toric varieties admit equivalent definitions arising naturally in many mathematical areas such as: Algebraic Geometry, Symplectic Geometry, Combinatorics, Statistics, Theoretical Physics etc.

We will present here an approach coming from Convex geometry and will see that toric varieties represent a natural generalization of projective spaces.

There are two main features we will try to emphasize:

(1) toric varieties, X, are prescribed by **sparse polynomials**, in the sense that they are mapped in projective space via these pre-assigned monomials,

whose exponents span an integral polytope **polytope** P_X. You can think of a parabola parametrized locally by $t \mapsto (t, t^2)$. The monomials are prescribed by the points $1, 2 \in \mathbb{Z}$. The polytope spanned by these points is a segment of length 1, $[1, 2]$. Discrete data A (i.e., points in $\mathbb{Z}^n$) gives rise to a polytope P_A and in turn to a toric variety X_A allowing a geometric analysis of the original data. This turns out to be very useful in Statistics or Bio-analysis for example.

(2) Toric varieties are defined by **binomial ideals**, i.e., ideals generated by polynomials consisting of two monomials: $x^u - x^v$. In the example of the parabola all the points in the image are zeroes of the binomial: $y - x^2$. This feature is particularly useful in integer programming when one wants to find a vertex (of the associated polytope) that minimizes a certain (cost) function.

10.2 Recap example

Consider the ideal $(x^3 - y^2) \in \mathbb{C}[x, y]$.

(a) The generating polynomial is irreducible and thus the corresponding affine variety $X = Z(x^3 - y^2) \subset \mathbb{C}^2$ is an **irreducible affine variety**.

(b) Consider now the **algebraic torus** $\mathbb{C}^* = \mathbb{C} \setminus \{0\} \subset \mathbb{C}$. Notice that $\mathbb{C}^* = \mathbb{C} \setminus Z(x)$, a Zariski-open subset of $\mathbb{C}$. Consider now the map $\phi : \mathbb{C}^* \to \mathbb{C}^2$ defined as $\phi(t) = (t^2, t^3)$. Observe that $\mathrm{Im}(\phi) \subseteq X$ and that $\psi : \mathrm{Im}(\phi) \to \mathbb{C}^*$ defined as $\phi(x, y) = (y/x)$ is an inverse. It follows that $\mathbb{C}^* \cong \mathrm{Im}(\phi)$, i.e., $\mathbb{C}^* \subset X$.

(c) The open set $\mathbb{C}^*$ is also a group under multiplication. We can define a **group action** on X as follows:

$$\mathbb{C}^* \times X \to X, (t, (x, y)) \mapsto (t^2 x, t^3 y).$$

Notice that, by definition, the action restricted to $\mathbb{C}^* \subset X$ is the multiplication in the group.

We will call such a variety, i.e., a variety satisfying (a), (b) and (c), an **affine toric variety**.

10.3 Algebraic tori

Definition 10.3.1. A **linear algebraic group** is a Zariski-open set G having the structure of a group and such that the multiplication map and the inverse map:

$$m : G \times G \to G, i : G \to G$$

are morphisms of affine varieties.
Let G, G' be two linear algebraic groups, a **morphism** $G \to G'$ of linear algebraic groups is a map which is a morphism of affine varieties and a homomorphism of groups.

We will indicate the SET of such morphisms with $\mathrm{Hom}_{AG}(G, G')$.

Exercise 10.3.2. Show that when G, G' are abelian $\mathrm{Hom}_{AG}(G, G')$ is an abelian group.

Example 10.3.3. The classical examples of algebraic groups are:$(\mathbb{C}^*)^n, GL_n, SL_n$.

Definition 10.3.4. An n-dimensional *algebraic torus* is a Zariski-open set T, isomorphic to $(\mathbb{C}^*)^n$.

An algebraic torus is a group, with the group operation that makes the isomorphism (of affine varieties) a group-homomorphism. Hence an algebraic torus is a linear algebraic group.
From now on we will drop the adjective algebraic in algebraic torus.

Definition 10.3.5. Let T be a torus.

- An element of the abelian group $\mathrm{Hom}_{AG}(\mathbb{C}^*, T)$ is called a *one parameter subgroup* of T.

- An element of the abelian group $\mathrm{Hom}_{AG}(T, \mathbb{C}^*)$ is called a *character* of T.

Lemma 10.3.6. *Let $T \cong (\mathbb{C}^*)^n$ be a torus.*

$$\mathrm{Hom}_{AG}(T, \mathbb{C}^*) \cong \mathbb{Z}^n.$$

Proof. Because $\mathrm{Hom}_{AG}(T, \mathbb{C}^*) \cong (\mathrm{Hom}_{AG}(\mathbb{C}^*, \mathbb{C}^*))^n$ it suffices to prove that $\mathrm{Hom}_{AG}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$. Let $F : \mathbb{C}^* \to \mathbb{C}^*$ be an element of $\mathrm{Hom}_{AG}(\mathbb{C}^*, \mathbb{C}^*)$. Then $F(t)$ is a polynomial such that $F(0) = 0$. Moreover it is a multiplicative group homomorphism, e.g., $F(t^2) = F(t)^2$. It follows that $F(t) = t^k$ for some $k \in \mathbb{Z}$. $\square$

A **Laurent monomial** in n variables is defined by

$$t^a = t^{a_1} \cdot t^{a_2} \cdot \ldots \cdot t^{a_n}, \text{ where } a = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

Observe that t^a defines a function $(\mathbb{C}^*)^n \to \mathbb{C}^*$, i.e., t^a is a character of the torus $(\mathbb{C}^*)^n$. Such a character is usually denoted by $\chi^a : T \to \mathbb{C}^*$ where $\chi^a(t) = t^a$. Another important fact, whose proof can be found in [Hum75] is that:

Lemma 10.3.7. *Any irreducible closed subgroup of a torus (i.e., an irreducible affine sub-variety which is a subgroup) is a sub-torus.*

10.4 Toric varieties

Definition 10.4.1. An (affine or projective) **toric variety** of dimension n is an irreducible (affine or projective) variety X such that

(1) X contains an n-dimensional torus $T \cong (\mathbb{C}^*)^n$ as Zariski-open subset.

(2) the multiplicative action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

Example 10.4.2. $\mathbb{C}^n$ is an affine toric variety of dimension n.

Example 10.4.3. $\mathbb{P}^n$ is a projective toric variety of dimension n. The map $(\mathbb{C}^*)^n \to \mathbb{P}^n$ defined as $(t_1, \ldots, t_n) \mapsto (1, t_1, \ldots, t_n)$ identifies the torus $(\mathbb{C}^*)^n$ as a subset of the affine patch $\mathbb{C}^n \subset \mathbb{P}^n$. The action:

$$(t_1, \ldots, t_n) \cdot (x_0, x_1, \ldots, x_n) = (x_0, t_1 x_1, \ldots, t_n x_n)$$

is an extension of the multiplicative action on the torus.

Example 10.4.4. Consider the Segre embedding $\mathrm{seg} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$. Consider now the map $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ given by $\phi(t_1, t_2) = (1, t_1, t_2, t_1t_2)$. Observe that if one identifies $(\mathbb{C}^*)^2$ with the Zariski open $\mathbb{P}^1 \times \mathbb{P}^1 \setminus V(x_0x_1y_0y_1)$ then it is $\phi = \mathrm{seg}|_{(\mathbb{C}^*)^2}$. By Lemma 10.3.7 this image is a torus which shows that the torus $(\mathbb{C}^*)^2$ can be identified with a Zariski open of the Segre variety $\mathrm{Im}(\mathrm{seg}) \subset \mathbb{P}^3$. The torus action of $(\mathbb{C}^*)^2$ on $\mathrm{Im}(\mathrm{seg})$ defined by $(t_1, t_2) \cdot (x_0, x_1, x_3, x_a) = (x_0, t_1x_1, t_2x_3, t_1t_2x_4)$ is by definition an extension of the multiplicative self-action.

10.5 Discrete data: polytopes

Definition 10.5.1. A subset $M \subset \mathbb{R}^n$ is called a **lattice** if it satisfies one of the following equivalent statements.

(1) M is an additive subgroup which is discrete as a subset, i.e., there exists a positive real number ϵ such that for each $y \in M$ the only element x such that $d(x,y) < \epsilon$ is given by $y = x$.

(2) There are $\mathbb{R}$-linearly independent vectors $b_1, \ldots, b_n$ such that:

$$M = \sum_1^n \mathbb{Z} b_i = \{\sum_1^n c_i b_i, c_i \in \mathbb{Z}\}$$

A lattice of rank n is then isomorphic to $\mathbb{Z}^n$.

Definition 10.5.2. Let $A = \{m_1, \ldots, m_d\} \in \mathbb{Z}^n$ be a finite set of lattice points. A combination of the form

$$\sum a_i m_i, \text{ such that } \sum_a^d a_i = 1, a_i \in \mathbb{Q}_{\geq 0}$$

is called a **convex combination**. The set of all convex combinations of points in A is called the **convex hull** of A and is denoted by $\mathrm{Conv}(A)$.

Definition 10.5.3. A **convex lattice polytope** $P \subset \mathbb{R}^n$ is the convex hull of a finite subset $A \subset \mathbb{Z}^n$. The dimension of P is the dimension of the smallest affine space containing P.

In what follows the term polytope will always mean a convex lattice polytope.

Example 10.5.4. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. The polytope $\mathrm{Conv}(0, e_1, \ldots, e_n)$ is called the n-dimensional regular simplex and it is denoted by Δ_n.
Given a polytope $P = \mathrm{Conv}(m_0, m_1, \ldots, m_n)$, let $kP = \{m_1 + \ldots + m_k \in \mathbb{R}^n \text{ s.t. } m_i \in P\}$.

10.6 Faces of a polytope

Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. It can be described as the intersection of a finite number of upper half planes.

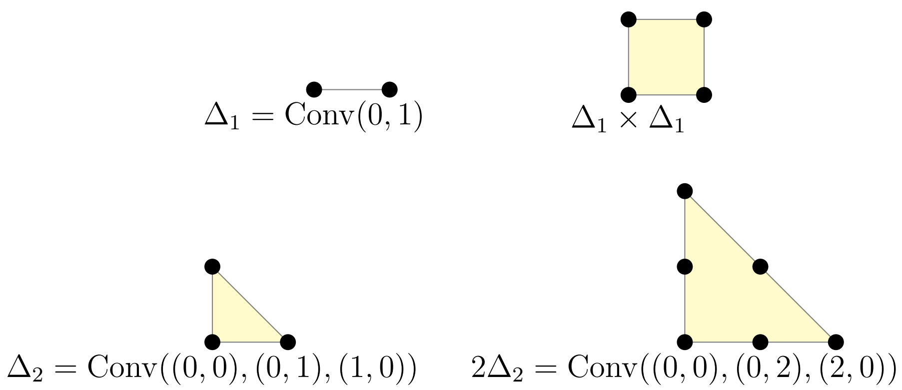


Figure 10.1: Examples of polytopes

Definition 10.6.1. Let $\xi \in \mathbb{Z}^n$ be a vector with integer coordinates and let $b \in \mathbb{Z}$. Define:

$$H^+_{\xi,b} = \{m \in \mathbb{R}^n \| \langle m, \xi \rangle \geq b\}, H_{\xi,b} = \{m \in \mathbb{R}^n \| \langle m, \xi \rangle = b\}$$

$H^+_{\xi,b}$ is called an **upper half plane** and $H_{\xi,b}$ is called an **hyperplane**.

Definition 10.6.2. Let $P \subset \mathbb{R}^n$ be a convex lattice polytope. We say that $H_{\xi,b}$ is a **supporting hyperplane** for P if $H_{\xi,b} \cap P \neq \emptyset$ and $P \subset H^+_{\xi,b}$.

It is immediate to see that a polytope has a finite number of supporting hyperplanes and that:

$$P = \bigcap_{i=1}^{s} H^+_{\xi_i,b_i}$$

Definition 10.6.3. A **face** of a polytope P is the intersection of P with a supporting hyperplane. P is considered an (improper) face of itself.
Faces are convex lattice polytopes as $\text{Conv}(S) \cap H_{\xi,b} = \text{Conv}(S \cap H_{\xi,b})$.
The dimension of the face is equal to the dimension of the corresponding polytope.
Let F be a face, then

- F is a **facet** if $\dim(F) = \dim(P) - 1$.
- F is a **edge** if $\dim(F) = 1$.
- F is a **vertex** if $\dim(F) = 0$.

Remark 10.6.4. Observe (and try to justify) that:

- All polytopes of dimension one are segments.

- All the edges of a polytope contain two vertices.
- $\mathrm{Conv}(S)$ contains all the segments between two points in S.
- Every convex lattice polytope P is the convex hull of its vertices.

Definition 10.6.5. Let $P, P' \subset \mathbb{R}^n$ be two n-dimensional polytopes. They are *affinely equivalent* if there is a lattice-preserving affine isomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ that maps P to P' and thus bijectively $P \cap \mathbb{Z}^n$ to $P' \cap \mathbb{Z}^n$.

Definition 10.6.6. Let P be a lattice polytope of dimension n.

- P is said to be *simple* if through every vertex there are exactly n edges.
- P is said to be *smooth* if it is simple and for every vertex m the set of vectors $(v_1 - m, \dots, v_n - m)$, where v_i is the first lattice point on the ith edge, forms a basis for the lattice $\mathbb{Z}^n$.

Remark 10.6.7. All polygons are simple (but not necessarily smooth).

Lemma 10.6.8. *A set of vectors $\{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ is a basis for the lattice $\mathbb{Z}^n$ if and only if the associated matrix B (having the v_i as columns) has determinant ± 1.*

Proof. If the determinant is ± 1, then the inverse matrix B^{-1} has integral elements which shows that $\{v_1, \dots, v_n\}$ is a basis. Conversely, let $\{v_1, \dots, v_n\} \subset \mathbb{Z}^n$ be a basis for the lattice $\mathbb{Z}^n$. Then there is an integral matrix U such that $I_n = BU$. Moreover one observes that the matrix U defines a lattice isomorphism and thus, because the determinant of U has to be an integer, $\det(U) = \pm 1$. $\square$

10.7 Assignment: exercises

(1) Consider a minimal hyperplane description of a lattice polytope P. In other words let $P = \bigcap_{i=1}^{s} H^+_{\xi_i, b_i}$ where s is the the minimum number of half-spaces necessary to cut out P. Show that P has s facets and that the vectors ξ_i are normal vectors to the associated facet. Moreover show that the pairs (ξ_i, b_i) are uniquely determined up to enumeration and multiplication by positive scalar factors.

(2) Classify, up to affine equivalence, all the smooth polygons containing at most 8 lattice points.

Chapter 11

Construction of toric varieties

11.1 Recap of example 10.4.4

Example 11.1.1. Consider the Segre embedding $\mathrm{seg} : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $((x_0, x_1), (y_0, y_1)) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$. Consider now the map $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^4$ given by $\phi(t_1, t_2) = (1, t_1, t_2, t_1t_2)$. Observe that if one identifies $(\mathbb{C})^2$ with the Zariski open $\mathbb{P}^1 \times \mathbb{P}^1 \backslash V(x_0x_1y_0y_1)$ then it is $\phi = \mathrm{seg}|_{(\mathbb{C}^*)^2}$. This image is a torus which shows that the torus $(\mathbb{C}^*)^2$ can be identified with a Zariski open of the Segre variety $\mathrm{Im(seg)} \subset \mathbb{P}^3$. The torus action of $(\mathbb{C}^*)^2$ on Im(seg) defined by $(t_1, t_2) \cdot (x_0, x_1, x_3, x_4) = (x_0, t_1x_1, t_2x_3, t_1t_2x_4)$ is by definition an extension of the multiplicative self-action.

Notice that in Example 11.1.1 the map defining the toric embedding and the torus action was given by characters associated to the vertices of the polytope $\Delta_1 \times \Delta_1$. Observe moreover that for this polytope the vertices coincide with all the lattice points in the polytope.
This is of course not always the case, the polytope $2\Delta_2$ for example is the convex hull of 3 vertices, but it contains $|2\Delta_2 \cap \mathbb{Z}^2| = 6$ lattice points.

Example 11.1.2. Let $A = 2\Delta_2 \cap \mathbb{Z}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)(0, 2), (2, 0)\}$. Consider the map defined by the associated characters and the following composition:

$$\phi_A : (\mathbb{C}^*)^2 \to \mathbb{C}^6 \to \mathbb{P}^5, (t_1, t_2) \mapsto (1, t_1, t_2, t_1t_2, t_1^2, t_2^2).$$

Observe that this map is the restriction of the 2-Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$. One sees as above that such a variety is a two dimensional projective toric variety.

The previous examples suggest a general construction:

11.2 Toric varieties from polytopes

Let T be an n-dimensional torus with character group $M \cong \mathbb{Z}^n$ and let $A = \{m_0, \ldots, m_d\} \subset M$. Consider the following action of T on $\mathbb{C}^{d+1}$

$$t \cdot (x_0, \ldots, x_d) = (\chi^{m_0}(t)x_0, \ldots, \chi^{m_d}(t)x_d).$$

This action yields an action on the projective space $\mathbb{P}^d$ as $t \cdot (\lambda x_0, \ldots, \lambda x_d) = \lambda(\chi^{m_0}(t)x_0, \ldots, \chi^{m_d}(t)x_d)$.

Let $x_0 \in \mathbb{P}^d$ be a general point, i.e., a point with non-zero homogeneous coordinates. The orbit $T \cdot x_0 = T_A \cong T$. The Zariski closure in $\mathbb{P}^d$ of the orbit $T \cdot x_0 \cong T$ is a projective algebraic variety containing a torus as a Zariski open set.

Let $X_A = \overline{T_A}$ be such a variety.

Alternatively:

Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope and let $A = P \cap \mathbb{Z}^n = \{m_0, \ldots, m_d\}$. Assume that $m_0 = 0$ and that P_A is contained in the positive orthant. Consider the monomial map defined by the associated characters:

$$\phi_A : (\mathbb{C}^*)^n \to \mathbb{C}^{d+1} \to \mathbb{P}^d, (t_1, \ldots, t_n) = t \mapsto (1 : t^{m_1} : \ldots : t^{m_d})$$

The image $\mathrm{Im}(\phi_A)$ is a torus T_A. Define X_A to be the Zariski closure of T_A. This means that X_A is the smallest subvariety of $\mathbb{P}^d$ containing T_A. Let $\mathscr{A}$ denote the $n \times (d+1)$ matrix whose columns are the vectors m_i.

Lemma 11.2.1. *Th variety X_A is a projective toric variety of dimension equal to $rank(\mathscr{A})$.*

Proof. Let $T_A = (\mathbb{C}^*)^r$ and consider the lattice of its characters: $\mathrm{Hom}_{AG}(T_A, \mathbb{C}^*) = \mathbb{Z}^r$. The map ϕ_A induces a map:

$$\mathrm{Hom}_{AG}((\mathbb{C}^*)^{d+1}, \mathbb{C}^*) \to \mathrm{Hom}_{AG}((\mathbb{C}^*)^n, \mathbb{C}^*);\ f \mapsto f \circ \phi_A$$

$$\psi_A : \mathbb{Z}^{d+1} \to \mathbb{Z}^n, e_i \mapsto m_i$$

where e_i are the elements of the standard lattice basis. We see that $\psi_A(\mathbb{Z}^{d+1}) = \mathbb{Z}^r$, and thus that $r = rank(\mathscr{A})$. $\square$

Exercise 11.2.2. Consider the $n-$dimensional standard simplex $\Delta_n = \mathrm{Conv}(e_0, e_1, \ldots, e_n)$, where $e_0 = 0$. Describe the projective toric variety associated to Δ_n and $2\Delta_n$.

Let $P \subset \mathbb{R}^n$ be an n-dimensional lattice polytope. The toric variety associated to P, denoted by X_P is the toric variety $X_{P \cap \mathbb{Z}^n}$.

11.3 Affine patching and subvarieties

11.4 Recap example

You have seen that $\mathbb{P}^n$ is the projective toric variety associated to the polytope Δ_n. By translating any vertex e_i to $e_0 = 0$ one can construct a map: $\phi^i : (\mathbb{C}^*)^n \to \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ defined by $t \mapsto (t^{e_0-e_i}, \ldots, 1, t^{e_n-e_i})$. The Zariski closure of $\mathrm{Im}(\phi_i)$ defines the affine patch of $\mathbb{P}^n$ where $x_i \neq 0$, i.e.

$$\overline{\mathrm{Im}(\phi_i)} = X_i$$

Notice that the map ϕ_i is the map defined by the lattice points:

$$A_i = \{e_0 - e_i, e_i - e_i, \ldots, e_n - e_i\}$$

We will see that projective toric varieties are in a sense a generalization of projective space as they are built by patching together affine toric varieties defined by the vertices of the polytope.

11.5 Affine patching

Let $P \subset \mathbb{R}^n$ be a polytope and let $A = P \cap \mathbb{Z}^n = \{m_0, \ldots, m_d\}$. For every $m_i \in A$ define $A_{m_i} = \{m - m_i | m \in A\}$. Consider $\phi_{A_m} : (\mathbb{C}^*)^n \to \mathbb{C}^d, t \mapsto (\ldots, t^{m_j - m_i}, \ldots)_{m_j \in A}$ and define:

$$X_m = \overline{\mathrm{Im}(\phi_{A_m})} \subset \mathbb{C}^d.$$

Note that X_m is an affine toric variety.

Proposition 11.5.1. *Notation as above. Let $V = \{v_1, \ldots, v_r\}$ be the set of vertices of P. Then*

$$X_A \cong \bigcup_{v_i \in V} X_{v_i}.$$

Proof. First notice that $X_{m_i} = X_A \cap X_i \subset \mathbb{P}^d$ and thus $X_A = \bigcup_{m \in A} X_m$. We prove the proposition if we show that for every $m \in A$ there is at least one vertex $v \in V$ such that $X_m \subseteq X_v$. As observed earlier, $P = \mathrm{Conv}(V)$. Let $m = \sum_{v_i \in V} k_i v_i$. After clearing denominators we can write $km = \sum_{v_i \in V} k_i v_i$, for $k_i \in \mathbb{Z}_{\geq 0}$. Notice that $t^m \neq 0$ iff $t^{km} = t^{\sum k_i v_i} = \Pi(t^{v_i})^{k_i} \neq 0$, which happens only if $t^{v_i} \neq 0$ for every $k_i \neq 0$. This shows that $X_m \subseteq X_{v_i}$ for every $k_i \neq 0$. $\square$

The vertices of the polytope defines the affine patches that build the associated toric variety. The following gives an intuition of how projective toric varieties are considered a generalization of the projective space.

Exercise 11.5.2. Let P be a polytope of dimension n and let $P \cap \mathbb{Z}^n = \{m_0, \ldots, m_d\}$. Show that

- $d \geq n$
- $d = n$ and $m_1, \ldots, m_n$ is a lattice basis (i.e., every vector in $\mathbb{Z}^n$ is an integral combination of $m_1, \ldots, m_n$) if and only if $P = \Delta_n$.

Let us now examine closer the category of smooth polytopes and the associated toric varieties. Let P be a smooth polytope and let m_0 be a vertex. After a lattice-preserving affine transformation we can assume that $m_0 = 0$ and that the primitive vectors on the n edges through m_0 are $e_1, \ldots, e_n$.

Lemma 11.5.3. *(Exercise) Let P be a smooth polytope. Then $X_v \cong \mathbb{C}^n$ for every vertex v.*

Observe that if P is an n-dimensional smooth lattice polytope, then a facet $F \subset P$ is a smooth polytope of dimension $(n-1)$. Denote by X_F the associated toric variety.

Lemma 11.5.4. *Let P be a smooth polytope. Then $X_P \setminus T_P = \bigcup_{F \text{ facet}} X_F$.*

Proof. Let $\dim(P) = n$, let V denote the set of vertices of P and $V(F)$ denote the set of vertices of F. First observe that:

$$X_P \setminus T_P = \bigcup_{v \in V} (X_v \setminus T_P) = \bigcup_{v \in V} \bigcup_{i} \{(x_1, \ldots, x_n) \in X_v \text{ s.t. } x_i = 0\}.$$

Let $v = (m_1, \ldots, m_n) \in V$, then are n facets passing through $v, F_1, \ldots, F_n$ such that $v_i = (m_i, \ldots, m_{i-1}, m_{i+1}, m_n) \in V(F_i)$. Clearly it is:

$$\{(x_1, \ldots, x_n) \in X_v \text{ s.t. } x_i = 0\} \cong X_{v_i} \subset X_{F_i}.$$

This proves that $X_P \setminus T_P \subseteq \cup_{F \text{ facet}} X_F$. But because for each facet it is $X_F = \cup_{w \in V(F)} X_w$ and $w = v_i$ for some $v \in V$, it is clearly

$$X_F \subset \bigcup_{\substack{v_i = w \\ w \in V(F)}} X_v \setminus T_N \text{ and thus } \bigcup_{F \text{ facet}} X_F \subseteq X_P \setminus T_P.$$

$\square$

11.6 Assignment: exercises

(1) Prove Lemma 11.5.3.

(2) Recall that $kP = \{m_1 + \ldots + m_k \text{ s.t. } m_i \in P\}$ and that if $P_1 \subset \mathbb{R}^{n_1}$, $P_2 \subset \mathbb{R}^{n_2}$ then $P_1 \times P_2 = \{(m_1, m_2) \text{ s.t. } m_1 \in P_1, m_2 \in P_2\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a polytope of dimension $\dim(P_1) + \dim(P_2)$ and whose faces are products of faces of the respective polytopes.

   (a) Describe the faces of the polytope $P = \Delta_1 \times 2\Delta_2$.

   (b) Is P smooth?

   (c) Describe the toric variety X_P as union of affine patches.

   (d) Describe the induced map $X_P \to \mathbb{P}^{11}$.

Chapter 12

More on toric varieties

12.1 Ideals defined by lattice points

Definition 12.1.1. A *semigroup* S is a set with an associative binary operation and an identity 0.
A semigroup is finitely generated if there is a finite subset $\mathscr{A} \subset S$ such that

$$S = \mathbb{N}\mathscr{A} = \{\sum_{m \in \mathscr{A}} a_m m \text{ s.t. } a_m \in \mathbb{N}\}.$$

Definition 12.1.2. A finitely generated semigroup $S = \mathbb{N}\mathscr{A}$ is called an *affine semigroup* if

- the binary operation is commutative; and
- it can be embedded in a lattice.

Let S be an affine semigroup, embedded in the lattice $\mathbb{Z}^n$. We associate to it the so called *semigroup algebra*:

$$\mathbb{C}[S] = \{\sum_{m \in S} c_m \chi^m \text{ s.t. } c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m\}$$

Lemma 12.1.3. *The semigroup algebra* $\mathbb{C}[S]$*is a subring of the ring of Laurent polynomials in d variables* $\mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$.

Proof. The proof is left as exercise. □

Consider an affine toric variety $X_{\mathscr{A}}$, associated to the finite subset $\mathscr{A} \subset \mathbb{Z}^n$. It clearly defines an affine semigroup $S_{\mathscr{A}}$ and a semigroup algebra

$$\mathbb{C}[S_{\mathscr{A}}] = \mathbb{C}[X_{\mathscr{A}}] = \mathbb{C}[\chi^{m_1}, ..., \chi^{m_d}]$$

(associated to the characters of the torus).

Remark 12.1.4. The semigroup algebra associated to the torus $T_{\mathscr{A}}$ is the algebra of all Laurent polynomials in n variables:

$$\mathbb{C}[T_{\mathscr{A}}] = \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$$

Note that $(\mathbb{C}^*)^n \cong V(x_1y_1 - 1, ..., x_ny_n - 1) \subset \mathbb{C}^{2n}$.

Let $A = \{m_0, \ldots, m_d\} \subset \mathbb{Z}^n$ as above. Consider the following two maps:

$$\psi_A^* : \mathbb{C}[y_0, \ldots, y_d] \to \mathbb{C}[x_1, \cdots, x_n], \psi_A : \mathbb{Z}^{d+1} \to \mathbb{Z}^n$$

defined as:

$$\psi_A^*(y_i) = x^{m_i} \text{ and } \psi_A(e_i) = m_i$$

Let $I_A = \text{Ker}(\psi_A^*)$ and $L = \text{Ker}(\psi_A)$. Let moreover $I = \{y^\alpha - y^\beta | \alpha, \beta \in \mathbb{N}^{d+1} \text{ and } \alpha - \beta \in L\}$.

Lemma 12.1.5. *I_A is a prime ideal of the ring $\mathbb{C}[y_0, \ldots, y_d]$.*

Proof. The kernel of a ring-morphism is always an ideal. Note that

$$\mathbb{C}[y_0, \ldots, y_d]/I_A \cong \mathbb{C}[x^{m_0}, \ldots, x^{m_d}]$$

and that $\mathbb{C}[x^{m_0}, \ldots, x^{m_d}]$ is an integral domain. $\square$

Proposition 12.1.6. *$I_A = I$.*

Proof. It is easily checked that $I \subseteq I_A$. Let $\alpha = \sum \alpha_i e_i, \beta = \sum \beta_i e_i \in \mathbb{N}^d$ such that $\alpha - \beta \in L$, i.e. $\sum_i \alpha_i m_i = \sum_i \beta_i m_i$. Then $t^{\sum \alpha_i m_i} = t^{\sum \beta_i m_i}$ and thus $\psi_A^*(y^\alpha - y^\beta) = 0$. Assume now that $I_A \setminus I \neq \emptyset$ and let $f \in I_A \setminus I$ be the element with minimal (after choosing a term order) leading coefficient y^α. After possibly rescaling we can write:

$$f = y^\alpha + f_1 \text{ ,where } f(x^{m_1}, \ldots, x^{m_d}) = 0.$$

It follows that f_1 has a monomial y^β such that $\phi_A^*(y^\alpha) = \phi_A^*(y^\beta)$ and thus $\alpha - \beta \in L$ which implies $y^\alpha - y^\beta \in I$ for $\alpha = \alpha_j e_j, \beta = \beta_j e_j$. It follows that $f_2 = f - (y^\alpha - y^\beta) \in I_A \setminus I$ is an element with lower leading term than f which is impossible. $\square$

12.2 Toric ideals

Definition 12.2.1. A prime ideal $I \subseteq \mathbb{C}[y_0, \ldots, y_d]$ is called a **toric ideal** if it is of the form I_A for some $A \subset \mathbb{Z}^d$.

Proposition 12.2.2. *(Homogeneous) toric ideals I define toric (projective) varieties and (projective) toric varieties are defined by (homogeneous) toric ideals.*

Proof. Consider a projective toric variety $X_A \subset \mathbb{P}^d$ defined by

$$A = \{m_0, \ldots, m_d\} \subset \mathbb{Z}^n.$$

Let $I \in \mathbb{C}[y_0, \ldots, y_d]$ be the homogeneous ideal defining X_A. By definition $f(x^{m_0}, \ldots, x^{m_d}) = 0$ for all $f \in I$ which implies $I \subseteq I_A$ and thus $V(I_A) \subseteq X_A$. On the other hand all the polynomials in I_A vanish on $\phi_A((\mathbb{C}^*)^n)$ which implies that $I_A \subseteq I(\phi_A((\mathbb{C}^*)^n))$ and thus $\phi_A((\mathbb{C}^*)^n) \subseteq V(I_A)$. But X_A is the smallest closed subvariety containing $\phi_A((\mathbb{C}^*)^n)$ which implies $X_A = V(I_A)$. $\square$

12.3 Toric maps

Definition 12.3.1. Let X, Y be toric varieties and let T_X, T_Y be the algebraic tori. A map $f : X \to Y$ is said to be a *toric map* if

(1) $f(T_X) \subseteq T_Y$;

(2) $f|_{T_X} : T_X \to T_Y$ is a group homomorphism.

Definition 12.3.2. A toric map $f : X \to Y$ is *equivariant* if

$$f(t \cdot x) = f(t) \cdot f(x).$$

Consider the map $\phi_A : X_A \hookrightarrow \mathbb{P}^d$. This is an equivariant toric map (we call it a toric embedding). In fact $\phi_A(T_X) \subset T_{\mathbb{P}^d}$ and they are related via the following:

$$T_{\mathbb{P}^d} = \mathbb{P}^d \setminus V(x_0 \cdot x_1 \cdot \cdots \cdot x_d).$$

$$1 \to \mathbb{C}^* \to (\mathbb{C}^*)^{d+1} \to T_{\mathbb{P}^d} \to 1$$

$$\phi_A : T_{X_A} \to (\mathbb{C}^*)^{d+1} \to T_{\mathbb{P}^d}.$$

Moreover

$$\phi_A(tx) = ((tx)^{m_0}, \ldots, (tx)^{m_d}) = \phi_A(t) \cdot \phi_A(x).$$

12.4 Fixed points

Let P be a smooth polytope of dimension n. and and let $V(P)$ denote the set of vertices. For every vertex $v \in V(P)$ there are n facets $F_1, \ldots, F_n$ passing through v. Notice that:

$$v = \bigcap_{i=1}^{n} F_i \quad \text{and} \quad \bigcap_{1}^{n} V(F_i) = (0, \ldots, 0) \in X_v \cong \mathbb{C}^n.$$

Every vertex $v \in V(P)$ corresponds to the point $0 \in X_v$ which is the unique point of X_v fixed by the torus action. This means that $|V(P)|$ corresponds to the number of fixed points in X_P.

Example 12.4.1. The torus action on $\mathbb{P}^n$ has $n+1$ fixed points: $(1:0:\ldots:0), (0:1:\ldots:0), \ldots, (0:\ldots:0:1)$.

12.5 Blow up at a fixed point

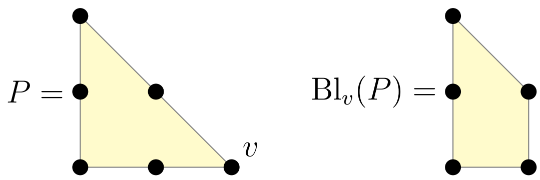
We will define a new polytope, obtained by truncating a vertex. This is not possible with every polytope and it is for this reason that in this chapter we make the following important assumption.

Definition 12.5.1. Let P be a smooth polytope of dimension n. A vertex v is called a **vertex of order** 2 if the length of all the n edges through v is at least 2.

Lat $P = \cap_1^r H^+_{\xi_i, b_i}$ and let v be a vertex of order 2. Let $F_1, \ldots, F_n$ be the facets containing v corresponding to $H_{\xi_1, b_1} \cap P, \ldots, H_{\xi_n, b_n} \cap P$. We will call the following polytope the **blow up of** P at v and will denote it by $\mathrm{Bl}_v(P)$:

$$\mathrm{Bl}_v(P) = (\cap_1^r H^+_{\xi_i, b_i}) \cap H^+_{\xi_v, -1}$$

where $\xi_v = \xi_1 + \ldots + \xi_n$.



The blow up polytope define a toric variety which will be denoted by $\mathrm{Bl}_{x(v)}(X)$ and called the *Blow up of* X *at the point* $x(v)$. Let $\dim(P) = n$, one can see immediately that:

(1) If $X \subset \mathbb{P}^d$ then $\mathrm{Bl}_{x(v)}(X) \subset \mathbb{P}^{d-1}$.

(2) Let $V(P) = \{m_0, \ldots, m_d\}$, with $v = m_d$ and let $e_1, \ldots, e_n$ be the first integer points on the edges through v. Then $V(\mathrm{Bl}_v(P)) = \{m_0, \ldots, m_{d-1}, e_1, \ldots, e_n\}$.

(3) $H_{\xi_v,-1} \cap \mathrm{Bl}_v(P) = \mathrm{Conv}(e_1, \ldots, e_n) \cong \Delta_{n-1}$

(4) If the facets of P are $H_{\xi_j,b_i} \cap P, i = 1, \ldots r$ then the facets of $\mathrm{Bl}_v(P)$ are $H_{\xi_j,b_i} \cap \mathrm{Bl}_v(P), i = 1, \ldots r$ together with $\delta_{n-1} = H_{\xi_v,-1} \cap \mathrm{Bl}_v(P)$.

(5) $\mathrm{Bl}_v(P)$ has the same dimension, n.

Geometrically what happened is that we introduced a $V(\Delta_{n-1}) = \mathbb{P}^{n-1}$ instead of the fixed point $x(v)$.

12.6 Assignment: exercises

(1) A rational normal curve of degree d is defined as the image of the degree d Veronese embedding of $\mathbb{P}^1$:

$$\mathbb{P}^1 \to \mathbb{P}^d \;\; (x_0 : x_1) \mapsto (x_0^d : x_0^{d-1}x_1 : x_0^{d-2}x_1^2 : \ldots : x_0 x_1^{d-1} : x_1^d)$$

Let P be a lattice polytope. Show that for every edge $L \subset P$, the toric variety $V(L)$ is smooth and isomorphic to a rational normal curve. What is the degree of such a rational curve?

(2) Let $a_0, \ldots, a_n$ be coprime positive integers. Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}$ given by:

$$t \cdot (x_1, \ldots, x_n) = (t^{a_0}x_0, \ldots, t^{a_n}x_n) = \mathbb{P}(a_0, \ldots, a_n).$$

The quotient $(\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ exists and it is called the **weighted projective space with weights** $a_0, \ldots, a_n$.

(a) In which sense is this a generalization of $\mathbb{P}^n$?

(b) We say that a polynomial $p(x) = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[x_0, x_1, x_2, \ldots, x_n]$ is $(a_0, a_1, \ldots, a_n)-$homogeneous of weighted degree s if every monomial x^α satisfies $\alpha \cdot (a_0, \ldots, a_n) = s$. Show that $f = 0$ is a well defined equation on $\mathbb{P}(a_0, \ldots, a_n)$ if and only if f is $(a_0, a_1, \ldots, a_n)-$homogeneous.

(c) Consider $\mathbb{P}(1, 1, d)$. Show that the map $\mathbb{P}(1, 1, d) \to \mathbb{P}^{d+1}$ defined by $(x_0, x_1, x_2) \to (x_0^d, x_0^{d-1}x_1, \ldots, x_0 x_1^{d-1}, x_1^d, x_2)$ is well defined.

(d) Show that $\mathbb{P}(1, 1, d)$ is a projective toric variety.

(e) Construct the polytope associated to $\mathbb{P}(1, 1, d)$.

(f) (*)[bonus point] Can you show (d) and (e) for any $\mathbb{P}(a_0, \ldots, a_n)$?

Chapter 13

Rational maps

13.1 Varieties

By a *variety* we will mean either an affine variety, a projective variety or a quasi-projective variety. Recall that both affine varieties and projective varieties are quasi-projective (Example 9.1.2 and Exercise 9.1.3). We will occasionally also mention abstract varieties which are defined analogously to smooth manifolds, cf. Definition 3.2.2.

Definition 13.1.1. An *abstract variety* X is an irreducible topological space that has an open covering $X = \bigcup_\alpha U_\alpha$ together with homeomorphisms $\phi_\alpha \colon U_\alpha \to V_\alpha$ where the V_α are affine varieties such that over the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the map

$$\phi_\alpha(U_{\alpha\beta}) \xrightarrow{\phi_\alpha^{-1}} U_{\alpha\beta} \xrightarrow{\phi_\beta} \phi_\beta(U_{\alpha\beta})$$

is an isomorphism of varieties.

An affine variety is trivially an abstract variety. Projective space has a canonical open covering by open affine pieces which gives it a structure of an abstract variety (Proposition 8.4.2). Quasi-projective varieties are also abstract varieties.

13.2 Rational maps

Let X and Y be varieties and let U and V be non-empty open subsets of X. Two maps $f \colon U \to Y$ and $g \colon V \to Y$ are *equivalent* if $f|_W = g|_W$ for some non-empty open subset $W \subseteq U \cap V$.

Definition 13.2.1. A *rational map* $f\colon X \dashrightarrow Y$ between varieties is an equivalence class of maps $U \to Y$ where $U \subseteq X$ is a non-empty open subset. We say that f is *defined over* an open subset $V \subseteq X$ if there exists a representative $V \to Y$ of f.

Remark 13.2.2 (Graphs). Let $f\colon X \to Y$ be a map between varieties. Then there is a map $(\mathrm{id}_X, f)\colon X \to X \times Y$. The *graph of* f is the image

$$\Gamma_f = \{(x, f(x)) \; : \; x \in X\} \subseteq X \times Y$$

of (id_X, f). The graph is closed and $X \to \Gamma_f$ is an isomorphism of varieties. Indeed, if f is a polynomial map

$$(x_1, x_2, \ldots, x_n) \longmapsto (p_1(x), p_2(x), \ldots, p_m(x)),$$

then Γ_f is defined by the polynomial equations $y_1 = p_1(x), \ldots, y_m = p_m(x)$. A general map f is locally given by polynomials so $\Gamma_f \cap (U_i \times Y)$ is closed in $U_i \times Y$ for some open covering $X = \bigcup_i U_i$. It follows that Γ_f is closed. The composition $\mathrm{pr}_1 \circ (\mathrm{id}_X, f)\colon X \to X \times Y \to X$ is the identity so $X \to \Gamma_f$ is an isomorphism.

Lemma 13.2.3. *Let X and Y be varieties and let $U \subseteq X$ be a non-empty open subset. If $f, g\colon X \to Y$ are two maps such that $f|_U = g|_U$, then $f = g$. In particular, a map $U \to Y$ has at most one extension $X \to Y$.*

Proof. Consider the product $X \times Y$ and the open subvariety $U \times Y$. We have seen that Γ_f and Γ_g are closed subvarieties of $X \times Y$. Let $\psi\colon X \to \Gamma_f$ denote the canonical isomorphism. The intersection $\Gamma_f \cap \Gamma_g$ can be identified with the closed subvariety $Z = \psi^{-1}(\Gamma_f \cap \Gamma_g)$ of X. But $f|_U = g|_U$ by assumption, so Z contains U. Since X is irreducible, it follows that $Z = X$ so $\Gamma_f = \Gamma_g$ and hence $f = g$. $\square$

Thus, if $f\colon X \dashrightarrow Y$ is a rational map that is defined over both U and V, then it is defined over $U \cup V$. The *domain of definition* of f is the maximal open subset $\mathrm{dom}(f) \subseteq X$ over which f is defined. Thus, f is represented by a unique map $\mathrm{dom}(f) \to Y$ that we also denoted by f. When the domain of definition of f is X, we have an ordinary map $X \to Y$. We can thus consider the set of maps $X \to Y$ as the subset of rational maps $f\colon X \dashrightarrow Y$ that are defined over X.

Exercise 13.2.4. Show that every rational map $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is represented by a partially defined function $f([x_0 : x_1 : \cdots : x_n]) = [f_0 : f_1 : \cdots : f_m]$ where the f_i are homogeneous polynomials in $x_0, \ldots, x_n$ of the same degree d and not all f_i are identically zero. Conversely, show that any set of homogeneous polynomials $f_0, \ldots, f_m$ that are not all identically zero and have the same degree gives rise to a rational map. When are two rational maps equal?

13.3 Birational varieties

Definition 13.3.1. A rational map $f\colon X \dashrightarrow Y$ is *birational* if there exists an open non-empty subset $U \subseteq X$ such that f is defined over U and induces an isomorphism onto an open subset $V \subseteq Y$. A map $f\colon X \to Y$ is birational if it is birational as a rational map.

Notation 13.3.2. By “birational map” we mean either a map or a rational map that is birational. To avoid this confusion, one sometimes uses “morphism” and “rational map” instead of morphism and map. Another choice is to use “regular map” do denote ordinary maps. A third choice is to write “birational rational map”. In our text, the difference will be made clear from the context or from the arrow being either solid or dashed.

Note that if $f\colon X \dashrightarrow Y$ is birational, then there exists a unique birational map $f^{-1}\colon Y \dashrightarrow X$ inverse to f.

Definition 13.3.3. We say that two varieties are *birationally equivalent*, or merely birational, if there exists a birational map between them (this is an equivalence relation). The *birational class* of a variety is the class of it under birational equivalence.

Remark 13.3.4. It can be shown that two affine varieties are birational if and only if the fraction fields of their coordinate rings are isomorphic. This also holds for projective varieties with a suitable definition of the fraction field of the homogeneous coordinate rings.

Example 13.3.5. Affine space $\mathbb{A}^n$ and projective space $\mathbb{P}^n$ are birational: there is a map $\mathbb{A}^n \to \mathbb{P}^n$ taking $(x_1, x_2, \dots, x_n)$ to $[1 : x_1 : x_2 : \cdots : x_n]$ which induces an isomorphism onto its image.

Example 13.3.6. The projective varieties $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birational but there is no map between them. There is a rational map

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\dashrightarrow \mathbb{P}^2 \\ ([x_0 : x_1], [y_0 : y_1]) &\mapsto [x_0/x_1 : y_0/y_1 : 1] \end{aligned}$$

that induces an isomorphism between the open subsets $\{x_0x_1y_0y_1 \neq 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ and $\{z_0z_1z_2 \neq 0\} \subset \mathbb{P}^2$. These open subsets are isomorphic to the algebraic torus $(\mathbb{C}^*)^2$. We will see in Section 14.6 that there exists a non-singular projective surface X and birational maps $X \to \mathbb{P}^1 \times \mathbb{P}^1$ and $X \to \mathbb{P}^2$.

Example 13.3.7. Two toric varieties of the same dimension n are birational: the algebraic torus $(\mathbb{C}^*)^n$ is an open subset of both. This generalizes the previous example.

Example 13.3.8 (Projection). The rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^m$, $[x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : x_1 : \cdots : x_m]$ is defined where $(x_0, x_1, \ldots, x_m) \neq (0, 0, \ldots, 0)$. This rational map will be treated in detail in Section 14.3.

Example 13.3.9 (Cremona transformation). The rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $[x : y : z] \mapsto [1/x : 1/y : 1/z] = [yz : zx : xy]$ is birational but not everywhere defined. It induces an isomorphism over the torus $(\mathbb{C}^*)^2 = \mathbb{P}^2 \setminus Z(xyz)$. This birational map is further discussed in Section 14.4.

Exercise 13.3.10. Show that a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by $f(x_0 : \cdots : x_n) = (f_0 : \cdots : f_n)$ is birational if and only if

(1) $f_0 \neq 0$, and

(2) the field extension $k(f_1/f_0, \ldots, f_n/f_0) \subseteq k(x_1/x_0, \ldots, x_n/x_0)$ is an isomorphism.

13.4 Birational geometry of curves

We list some facts on the birational structure of curves. Most of these facts can be found in [Har77, Ch. I.6]. Here by curve we mean a variety of dimension 1, so in particular irreducible.

(1) A curve C has a finite number of singular points. We denote this subset by C^{sing}.

(2) If $f \colon C' \to C$ is a birational map of curves, then f is an isomorphism over the open subset $f(C') \setminus C^{\text{sing}}$.

(3) There exists a unique desingularization $\widetilde{C} \to C$, that is, a projective[1] birational map $\widetilde{C} \to C$ such that $\widetilde{C}$ is non-singular. In commutative algebra terms, the desingularization is accomplished by taking the normalization/integral closure of the coordinate ring.

(4) If C is a curve, then there exists an embedding $C \subseteq \overline{C}$ where $\overline{C}$ is projective. More generally, every curve is either affine or projective [Har77, Exc. IV.1.3].

(5) In every birational class of curves, there exists a unique non-singular projective curve. Thus if C and C' are non-singular projective curves, then C and C' are birational if and only if C and C' are isomorphic.

[1]When C is projective, the map being projective means that $\widetilde{C}$ is projective. Over $\mathbb{C}$, a map is projective if $f^{-1}(W)$ is compact for every compact subset $W \subseteq C$ in the analytic topology.

Let C be a non-singular projective curve over $\mathbb{C}$. If we consider C with the analytic topology, inherited from $\mathbb{P}^n$, then C becomes a real compact orientable surface. These are classified by their *genus* g which is a non-negative integer counting the number of "holes" in the surface. If C is any curve over $\mathbb{C}$, then the *geometric genus* $p_g(C)$ of C is the genus of the unique non-singular projective curve in the birational class of C. By definition, the geometric genus of a curve is a *birational invariant*.

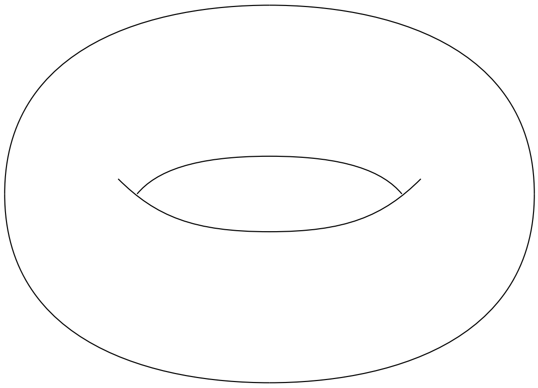


Figure 13.1: A compact orientable surface of genus 1.

Example 13.4.1 (Genus 0). There is exactly one projective non-singular curve of genus 0, the projective line $\mathbb{P}^1$. Thus, a curve C has geometric genus 0 if and only if it is *rational*, that is, admits a birational map $\mathbb{P}^1 \dashrightarrow C$ (if C is projective, this map is defined everywhere, see Corollary 13.6.2 below). Examples of singular rational curves are the nodal and cuspidal cubics, cf. §5.2.10. The twisted cubic (Exercises 8.4.8, 9.1.5 and 9.7.1) is a non-singular rational curve embedded into $\mathbb{P}^3$.

Example 13.4.2 (Genus 1). A real compact surface of genus 1 is a *torus*, that is isomorphic as a manifold to $S^1 \times S^1$ (not to be confused with the algebraic tori of toric varieties). The projective non-singular curves of genus 1 are the non-singular cubic curves (elliptic curves) with equation $y^2z = x(x-z)(x-\lambda z)$ where $\lambda \in \mathbb{C}\backslash\{0,1\}$, cf. §5.2.6. Some of these curves are actually isomorphic. The correct invariant is the so called j-invariant:

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

Two elliptic curves are isomorphic exactly when their respective j-invariants are equal.

Remark 13.4.3 (Arithmetic genus). Let $X \subseteq \mathbb{P}^n$ be a projective variety. Recall that the homogeneous coordinate ring of X is $k[X] := k[x_0, x_1, \ldots, x_n]/I(X)$. This is a graded ring. The *Hilbert function* of X is the function $h_X(d) = \dim_k k[X]_d$ (Definition 6.4.2). For all large d, $h_X(d)$ equals a polynomial, the *Hilbert polynomial* $HP_X(d)$. The *arithmetic genus* of X is the integer $p_a(X) := 1 - HP_X(0)$. The arithmetic genus of a plane curve C of degree d is

$$p_a(C) = \frac{(d-1)(d-2)}{2}$$

cf. Theorem 6.4.3. In particular, the arithmetic genus of a degree 2 curve is 0 and the arithmetic genus of a degree 3 curve is 1. It can be shown that if C is a non-singular projective curve over $\mathbb{C}$, then $p_g(C) = p_a(C)$. Note that a singular cubic curve has $p_g = 0$ and $p_a = 1$. In general, $p_g \leq p_a$ and the arithmetic genus is not a birational invariant.

13.5 A rational map from a surface

Consider the following function in two variables

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

This is a classic example of a function that has partial derivatives yet is not differentiable. Indeed, the partial derivatives (along any line through the origin) is zero but the function is not continuous.

From the perspective of algebraic geometry, this can be treated as a rational map

$$\begin{aligned} \mathbb{A}^2 &\overset{f}{\dashrightarrow} \mathbb{P}^1 \\ (x,y) &\longmapsto [xy : x^2+y^2] \end{aligned}$$

which is defined outside the origin. If we restrict f along the line $L\colon y = kx$, we obtain $f(x,kx) = k^2/(1+k^2)$ outside the origin, that is, the restriction of f to L extends to a *map*:

$$\begin{aligned} \mathbb{A}^1 &\overset{f|_L}{\dashrightarrow} \mathbb{P}^1 \\ x &\longmapsto [k^2 : 1+k^2] \end{aligned}$$

and similarly for $x = ky$. Note that the map is even constant (so the partial derivatives along any line is zero) but depends on the slope k. Thus, to extend f to a map everywhere, we have to take into account the direction we approach the origin. We do this by considering the set $\mathbb{A}^2 \times \mathbb{P}^1$ of pairs (P, L) where $P \in \mathbb{A}^2$ is a point and $L \subseteq \mathbb{A}^2$ is a line through the origin. Outside the origin we know what

$f(P)$ is and there is a unique L through P. At the origin, choosing a line L through the origin tells us what $f(P, L)$ should be. We thus end up with considering the following subset

$$\mathrm{Bl}_0\mathbb{A}^2 = \{(P, L) \ : \ P \in L\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$$

which is called the *blow-up* of $\mathbb{A}^2$ at the origin, cf. Section 14.1. We then define, as a map of sets:

$$\begin{aligned} g \colon \mathrm{Bl}_0\mathbb{A}^n &\longrightarrow \mathbb{P}^1 \\ (P, L) &\mapsto \begin{cases} f(P) & \text{if } P \neq 0 \\ [k^2 : 1 + k^2] & \text{if } P = 0, \text{ where } k \text{ is the slope of } L \end{cases} \end{aligned}$$

Exercise 13.5.1. Analyze the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ given by $f(x, y) = \frac{x^2 y}{x^4 + y^2}$.

13.6 Maps from curves into projective varieties

Theorem 13.6.1. *Let C be a curve, let $P \in C$ be a non-singular point and let X be a projective variety. Any map $f \colon C \setminus \{P\} \to X$ extends uniquely to a map $C \to X$.*

Proof. We will only treat the case where C is rational (the general case can be found in [Har77, Prop. I.6.8]. If $C' \to C$ denotes the desingularization, then it is enough to produce a map $C' \to X$ since $C' \to C$ is an isomorphism in a neighborhood of P. We can thus assume that C is non-singular. We may also compactify C and assume that C is projective. Then $C = \mathbb{P}^1$. Now use Exercise 13.2.4 and unique factorization... □

Corollary 13.6.2. *If C is a non-singular curve and X is a projective variety, then any rational map $C \dashrightarrow X$ extends to a map $C \to X$.*

Remark 13.6.3. In the theorem, the following conditions are crucial:

- P has to be non-singular (see Exercise 13.6.5),
- X has to be projective or, more generally, complete (see Exercise 13.6.5),
- C has to be a curve (see Section 13.5).

Remark 13.6.4. Complete varieties (see below) are characterized by the theorem.

Exercise 13.6.5. Show that Corollary 13.6.2 fails if:

(1) C is singular: consider a suitable rational map $C \dashrightarrow X = \mathbb{P}^1$ where C is a nodal cubic curve (Example 5.2.11).

(2) X is not projective: consider $\mathbb{P}^1 \dashrightarrow \mathbb{A}^1$, $[x : y] \mapsto (x/y)$.

One way to show that a rational map does not extend to a map of varieties, if we work over $\mathbb{C}$, is to prove that there is no continuous map in the analytic topology extending the rational map.

13.7 Appendix: complete varieties and theorems by Chow, Nagata and Hironaka

Definition 13.7.1. An abstract variety X, over $\mathbb{C}$, is *complete* if X equipped with the analytic topology is compact.

The following alternative definition works over any field.

Definition 13.7.2. An abstract variety X is *complete* if for every variety Y, the projection map $X \times Y \to Y$ takes closed sets to closed sets.

Projective varieties are complete and complete varieties are not too far away from being projective.

Theorem 13.7.3 (Chow's lemma, c. 1950, weak formulation)**.** *If X is a complete variety, then there exists a birational map $\widetilde{X} \to X$ where $\widetilde{X}$ is projective.*

A celebrated, and much deeper, theorem by Hironaka is that varieties are not too far away from being non-singular either.

Theorem 13.7.4 (Hironaka's desingularization theorem, 1964, weak formulation)**.** *If X is a variety, then there exists a birational map $\widetilde{X} \to X$ where $\widetilde{X}$ is non-singular.*

Theorem 13.7.5 (Nagata's compactification theorem, 1962)**.** *If X is an abstract variety, then there exists a complete variety $\overline{X}$ and an open embedding $i\colon X \to \overline{X}$, i.e., a map such that $i(X)$ is an open subvariety and $X \to i(X)$ is an isomorphism.*

When X is a quasi-projective variety, Nagata's theorem is almost trivial. Indeed, by definition (9.1.1), X is a locally closed subset of $\mathbb{P}^n$. This means that X is open inside a closed subset Z. Since Z is a projective variety, we have an open embedding $i\colon X \to Z$.

Example 13.7.6. Consider the parabola $C \subseteq \mathbb{A}^2$ defined by $f(x, y) = y - x^2 = 0$. This embeds into $\mathbb{P}^2$ by $(x, y) \mapsto (x : y : 1)$. The closure of C in $\mathbb{P}^2$ is the projective variety defined by the homogenized equation $F(x, y, z) = yz - x^2 = 0$ (a non-singular conic).

[picture]

We can also choose a different embedding into a projective variety. For example $C \to \mathbb{P}^2$ by $(x, y) \mapsto (x : xy : 1)$. The closure of C in $\mathbb{P}^2$ is the projective cubic curve C' defined by $s^3 - tu^2 = 0$. The curve C' is singular at $(s : t : u) = (0 : 1 : 0)$ (a cusp singularity). We have a partial inverse $C' \dashrightarrow C$ given by $(s : t : u) \mapsto (s/u, t/s) = (s/u, s^2/u^2)$ which is defined outside $(0 : 1 : 0)$.

Also see Exercises 9.1.3 and 9.1.4.

Remark 13.7.7. Every variety is birational to an affine variety, hence to a projective variety. Indeed, if X is an abstract variety, then there is an open non-empty subset $U \subseteq X$ such that U is an affine variety. This gives a birational map $X \dashrightarrow U$. Since U is affine, it is a closed subvariety of some $\mathbb{A}^n$ and $\mathbb{A}^n$ is an open subvariety of $\mathbb{P}^n$. This gives a rational map $X \dashrightarrow \mathbb{P}^n$. Admitting a *birational* map $X \dashrightarrow \mathbb{P}^n$ is a much stronger condition: one says that X is rational in this case, cf. a rational curve.

Chapter 14

Blow ups

14.1 Definition of blow-up

Recall that $\mathbb{P}^{n-1}$ is the set of lines in $\mathbb{A}^n$ through the origin 0. Thus $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is the set of pairs (P, L) where $P \in \mathbb{A}^n$ is a point and $L \subseteq \mathbb{A}^n$ is a line through the origin.

Definition 14.1.1. The blow-up of affine space $\mathbb{A}^n$ at the origin is the subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ consisting of pairs (P, L) where $P \in L$. The blow-up is denoted $\mathrm{Bl}_0\mathbb{A}^n$.

If $\mathbb{A}^n$ has coordinates $x_1, x_2, \ldots, x_n$ and $\mathbb{P}^{n-1}$ has coordinates $y_1, y_2, \ldots, y_n$, then the condition that $P \in L$ is the condition that $(x_1, x_2, \ldots, x_n) = \lambda(y_1, y_2, \ldots, y_n)$ for some $\lambda \in \mathbb{C}$. Equivalently, the conditions $x_i y_j = x_j y_i$ holds for all $1 \leq i < j \leq n$.

The projection map $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ induces a natural map $\pi\colon \mathrm{Bl}_0\mathbb{A}^n \to \mathbb{A}^n$ that takes (P, L) to P. If $P \neq 0$, then there is a unique line that passes through P. Thus, π is injective outside the origin. On the contrary, every line passes through the origin. Thus

(1) $\pi|_{\mathbb{A}^n \setminus 0}$ is an isomorphism, so π is birational.

(2) $\pi^{-1}(0) = \mathbb{P}^{n-1}$.

The fiber $\pi^{-1}(0)$ is called the *exceptional divisor*, where divisor signifies that it has codimension 1 in the blow-up.

Slightly more general, we may define:

Definition 14.1.2. Let X be a variety and let $P \in X$ be a point. Let $U \subseteq X$ be an open neighborhood of P and let $U \to \mathbb{A}^n$ be a map that identifies U with an open subset of $\mathbb{A}^n$ and maps P to the origin. The blow-up of X at P is the gluing of $X \setminus P$ with $(\mathrm{Bl}_0\mathbb{A}^n)|_U$ along $U \setminus P$.

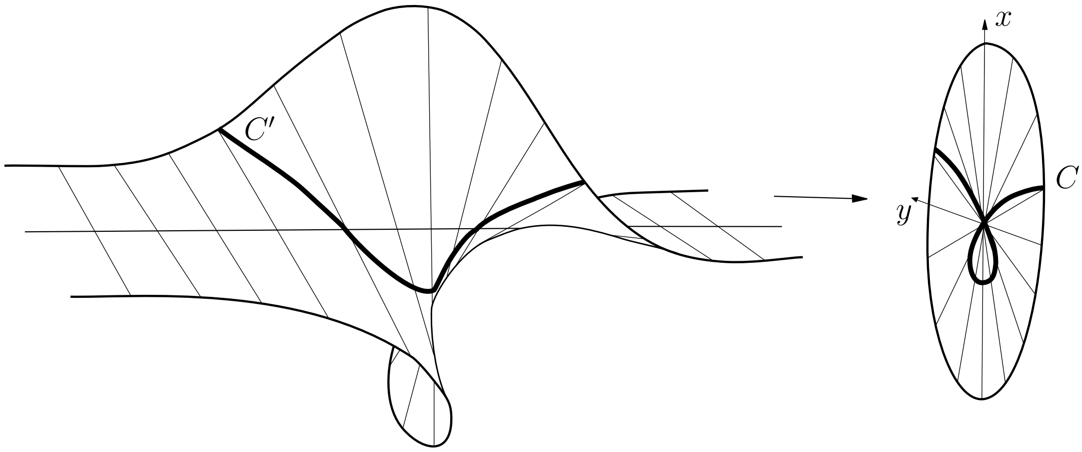


Figure 14.1: The blow-up of the affine plane at the origin depicted at the left. The exceptional divisor is the horizontal line. The curve C' in the blow-up maps to the curve C in the plane.

For the general definition of a blow-up of a variety at a non-singular point, see Section 14.7.

14.2 Charts of a blow-up

Consider the blow-up $Y = \mathrm{Bl}_0\mathbb{A}^2$ of the affine plane $\mathbb{A}^2$. We give the plane the coordinates x, y and the projective line the coordinates s, t. The blow-up Y is then the subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ given by the equations $xt = ys$ or in a more suggestive form "$x/y = s/t$".

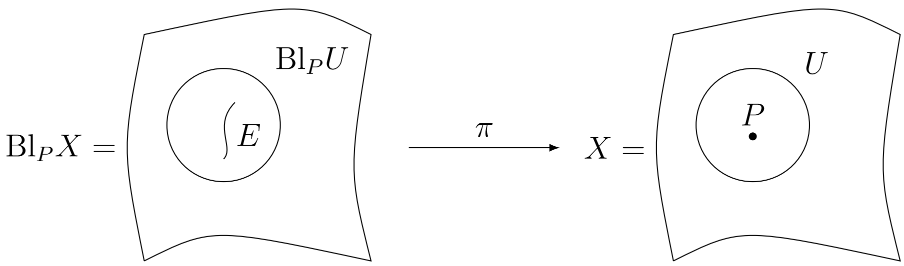


Figure 14.2: The blow-up of X at P is locally defined as the blow-up of U at P. Here $\mathrm{Bl}_P U = \pi^{-1}(U)$ which defines $\mathrm{Bl}_P U$ in terms of $\mathrm{Bl}_P\mathbb{A}^n$ if U is an open subspace of $\mathbb{A}^n$.

Recall that the projective line has a standard open covering by two affines: $U_1 = \{s \neq 0\}$ and $U_2 = \{t \neq 0\}$. On the first open we have $U_1 \cong \mathbb{A}^1$ with coordinate $k = t/s$ and on the second we have $U_2 \cong \mathbb{A}^1$ with coordinate $k^{-1} = s/t$.

The open covering of the projective line induces an open covering of the blow-up by affine varieties. The two open subsets $Y_1 = \{s \neq 0\} \subseteq Y$ and $Y_2 = \{t \neq 0\} \subseteq Y$ are isomorphic to closed subvarieties of $\mathbb{A}^2 \times \mathbb{A}^1$. On the first chart we use coordinates x, y, k and on the second chart coordinates x, y, k^{-1}. The equations become $y = kx$ and $x = k^{-1}y$ respectively.

In the first chart, the coordinate y is a polynomial in the other variables. This implies that the first chart is isomorphic to $\mathbb{A}^2$ with coordinates x, k. We write this as $\mathbb{A}^2_{x,k}$. Similarly, the second chart is isomorphic to $\mathbb{A}^2_{y,k^{-1}}$. Thus, the blow-up is locally isomorphic to the affine plane itself. In particular, it is non-singular.

The first chart contains everything of Y except the line $s = x = 0$ which is contained in the second chart as $x = k^{-1} = 0$ (or simply as $k^{-1} = 0$ in $\mathbb{A}^2_{y,k^{-1}}$). Similarly, the second chart contains everything of Y except the line $t = y = 0$ which is contained in the first chart as $y = k = 0$ (or simply as $k = 0$ in $\mathbb{A}^2_{x,k}$).

The exceptional divisor is given by the equations $x = y = 0$. In the first chart this equation becomes the line $x = 0$ in $\mathbb{A}^2_{x,k}$. In the second chart this equation becomes the line $y = 0$ in $\mathbb{A}^2_{y,k^{-1}}$. The parameter on the exceptional divisor is k and k^{-1} respectively and these two lines glue to a projective line $\mathbb{P}^1$ with coordinates $(s : t)$ where $k = t/s$ and $k^{-1} = s/t$.

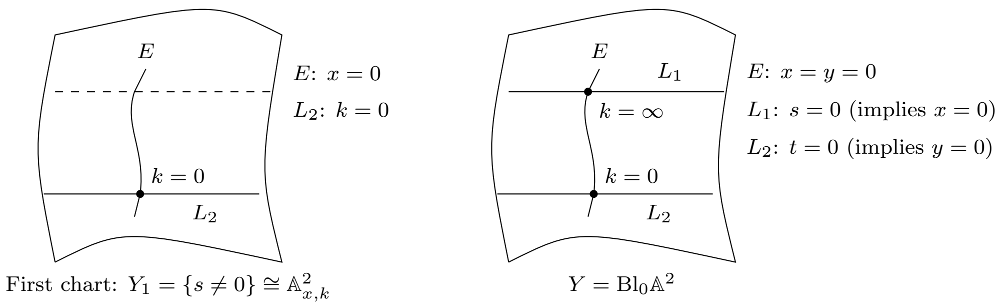


Figure 14.3: One of the charts of the blow-up of the affine plane. The slope is $k = y/x = t/s$ which gives the coordinate on E.

Exercise 14.2.1 (Blow-up of affine space in dimension 3). Describe affine charts for the blow-up of the origin in $\mathbb{A}^3$ and give equations for the exceptional divisor in these charts. Verify that locally the blow-up is isomorphic to $\mathbb{A}^3$ and that the exceptional divisor is a hyperplane.

14.3 Projections and blow-ups

Let us consider the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, $[x : y : z] \mapsto [x : y]$. This is defined outside the point $O = [0 : 0 : 1]$ which we can view as the origin in the affine chart $z \neq 0$. In this chart, a point $(x, y) = (x : y : 1)$ is mapped to $[x : y]$, that is, the line through the origin and (x, y). Any point outside this chart lies on the *line at infinity* $L_\infty = \{z = 0\}$. Any point $[x : y : 0]$ on L_∞ is the unique intersection point between L_∞ and a line L through the origin, namely the line $L = \{[x : y : \lambda] \ : \ \lambda \in k\}$ which consists of the points $[\lambda^{-1}x : \lambda^{-1}y : 1]$ $(\lambda \neq 0)$ and $[x : y : 0]$. We have thus arrived at our first description of the projection:

(I) The projection takes a point $P \in \mathbb{P}^2 \setminus O$ to the line $L = OP$.

But we saw that any such line L passes through a unique point at the line at infinity L_∞. The line at infinity is isomorphic to $\mathbb{P}^1$ and we can consider the projection as a map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 = L_\infty \subset \mathbb{P}^2$, $[x : y : z] \mapsto [x : y : 0]$. This gives the second description of the projection:

(II) The projection takes a point $P \in \mathbb{P}^2 \setminus O$ to the intersection of the line $L = OP$ and the line at infinity L_∞.

It is thus natural to speak of the projection from the point O onto the line L_∞. If we shift focus to a chart at the line of infinity, all lines through the origin are parallel. In the standard coordinates, these lines are orthogonal to L_∞ and we may talk about an orthogonal projection onto L_∞.

(III) The projection is the orthogonal projection onto the line at infinity L_∞.

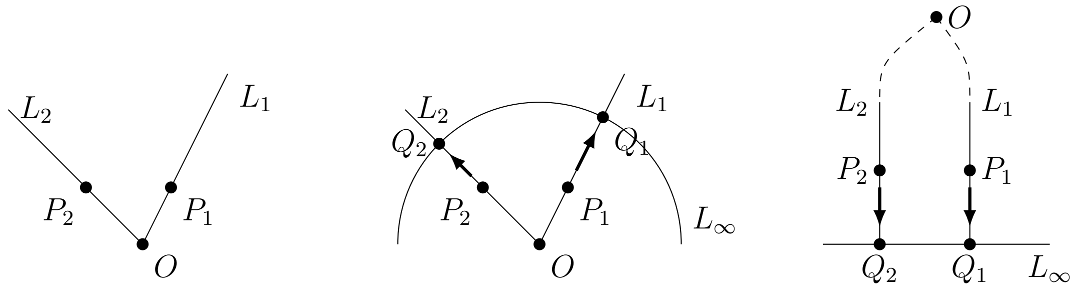


Figure 14.4: Pictures of the three different descriptions of the projection.

If we choose any line L_1 not passing through O, we may also describe the projection as the map taking a point P to the intersection point between $L = OP$ and L_1. The projection from O can thus be naturally identified with the projection onto any line not passing through O.

Blowing up Although the projection $p\colon \mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ is not defined at O, it becomes a well-defined map *after blowing up* O. Recall that the blow-up $\mathrm{Bl}_O\mathbb{A}^2$ sits inside $\mathbb{A}^2 \times \mathbb{P}^1$. The composition $p \circ \pi\colon Y \to \mathbb{A}^2 \dashrightarrow \mathbb{P}^1$ is given by $((x,y),[s:t]) \mapsto [x:y]$ but outside the exceptional divisor we have that $[x:y]=[s:t]$. Thus, $p\circ\pi$ is equivalent to the second projection $Y \to \mathbb{A}^2\times\mathbb{P}^1 \to \mathbb{P}^1$ that takes $((x,y),[s:t]) \mapsto [s:t]$ and this is defined on all of Y.

14.3.1 Conics and projections

In Theorem 4.2.3, it was shown that every irreducible conic C is isomorphic to $\mathbb{P}^1$ (we assume that the ground field k is algebraically closed so every point is rational). The proof is closely related with the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from a point on C. After a change of coordinates, we may assume that O lies on C. The projection then restricts to a map $C\setminus O \to \mathbb{P}^1$ that takes a point $P \in C\setminus O$ to the line through OP. Since C has degree two, any line through O meets C in at most one other point or is a component of C (but this would mean that C is reducible). This means that the projection $C\setminus O\to\mathbb{P}^1$ is injective. A point in $\mathbb{P}^1$, corresponding to a line L, is in the image exactly when L is not a tangent to C at O. Conversely, we may define $\mathbb{P}^1 \to C$ by sending a line L to the unique point of intersection with $C\setminus O$, or to O if L is tangent at O.

The composition $\mathbb{P}^1 \to C \dashrightarrow \mathbb{P}^1$ is the identity and it turns out that both maps are defined everywhere and isomorphisms. This is because any irreducible conic is non-singular and rational maps from non-singular curves are defined everywhere (Theorem 13.6.1).

Figure 14.5: Projection from a point on the conic and from a point not on the conic respectively.

If we choose a point O outside C, we obtain a map $C\to\mathbb{P}^1$ which is generically two-to-one since every line through O meets C at two points unless it is tangent to C.

Exercise 14.3.2. Given a non-singular conic C and a point $O \in \mathbb{A}^2\setminus C$, show that there are exactly two tangents of C through O. Thus the projection $C\to\mathbb{P}^1$ from O is two-to-one except in two points P_1 and P_2. We say that the map has *degree* 2 and that it is *ramified* in P_1 and P_2.

Let C be the conic $yz = x^2$. Consider the projection from $[1 : 0 : 0]$ giving rise to a map $C \to \mathbb{P}^1$ of degree 2. Where is this map ramified?

14.3.3 Cubics and projections

Let C be a plane cubic curve, let O be a point on C and consider the projection $C \setminus O \to \mathbb{P}^1$ from O. Since C has degree three, we expect that most lines through O should meet C in two other points. This means that the projection would be two-to-one and not one-to-one. An exception is if every line meets O in multiplicity at least two, that is, if O is singular. Thus, if C is a singular cubic curve we obtain a birational map $C \dashrightarrow \mathbb{P}^1$. By Theorem 13.6.1, the inverse is everywhere defined and gives a desingularization $\mathbb{P}^1 \to C$. This map takes a line L through O onto the unique point of $C \setminus O$ meeting L, or O itself if $(C \setminus O) \cap L = \emptyset$. Since C is singular, the map $C \dashrightarrow \mathbb{P}^1$ is not defined at the singular point.

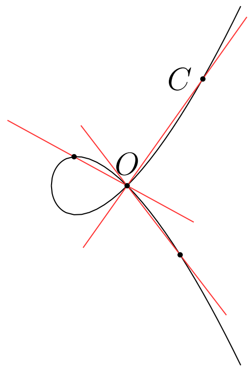


Figure 14.6: Projection from the singular point of a nodal cubic.

Exercise 14.3.4. If C is a cubic curve in normal form: $y^2z = x(x - z)(x - \lambda z)$, then C has a unique point $O = [0 : 1 : 0]$ on the line at infinity. The projection from this point gives a map $C \setminus O \to \mathbb{P}^1$. The point O is non-singular so this map extends to a map $C \to \mathbb{P}^1$ of degree 2. Make this map explicit and determine the ramification points of this map.

Exercise 14.3.5. Recall the twisted cubic curve $C \subset \mathbb{P}^3$ (Exercises 8.4.8, 9.1.5 and 9.7.1) which is a rational curve, that is, admits a birational map $\mathbb{P}^1 \to C$. In fact, C is even non-singular and hence isomorphic to $\mathbb{P}^1$. Find

(1) a point P on C such that the projection $C \dashrightarrow \mathbb{P}^2$ is defined everywhere, birational onto its image C' and such that $C' \subset \mathbb{P}^2$ is a conic.

(2) a point P outside C such that the projection $C \to \mathbb{P}^2$ is birational onto its image C' and C' is a cuspidal cubic.

(3) a point P outside C such that the projection $C \to \mathbb{P}^2$ is birational onto its image C' and C' is a nodal cubic.

Remark 14.3.6. It can be shown that any curve C is birational to a plane curve. More generally, any variety X of dimension r is birational to a hypersurface in $\mathbb{P}^{r+1}$ [Har77, Prop. I.4.9]. Also, if we start with $X \subset \mathbb{P}^n$, then there is a projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{r+1}$ that induces a birational map $X \to X'$ [Har77, Exc. I.4.9].

14.4 A Cremona transformation

Consider the *Cremona transformation*

$$\begin{aligned} \mathbb{P}^2 &\overset{f}{\dashrightarrow} \mathbb{P}^2 \\ [x : y : z] &\longmapsto [1/x : 1/y : 1/z] \end{aligned}$$

We give the right-hand side coordinates $[s : t : u]$. The map is rational since it is given by polynomials: $[s : t : u] = [yz : zx : xy]$. Note that f is its own inverse. The rational map is defined outside the three points $P_1 = [1 : 0 : 0]$, $P_2 = [0 : 1 : 0]$ and $P_3 = [0 : 0 : 1]$. We also note that the lines $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$ and $L_3 = \{z = 0\}$ are mapped to the points $Q_1 = [1 : 0 : 0]$, $Q_2 = [0 : 1 : 0]$ and $Q_3 = [0 : 0 : 1]$ in the image (although the map is undefined in the pairwise intersections of these lines). Conversely, the inverse map is defined outside Q_1, Q_2 and Q_3 and the lines $L_1' = \{s = 0\}$, $L_2' = \{t = 0\}$, $L_3' = \{u = 0\}$ are mapped to the points P_1, P_2 and P_3 respectively.

Let us study f locally around P_3. In the affine chart $z \neq 0$, the map is given by $(x, y) = [x : y : 1] \mapsto [y : x : xy]$ and is defined outside P_3. If we blow-up P_3, then we get a well-defined map $\mathrm{Bl}_{P_3}\mathbb{A}^2 \to \mathbb{P}^2$. Indeed, the map is given by $((x, y), [s : t]) \mapsto [t : s : sy] = [t : s : tx]$. Note that the restriction to the exceptional divisor $x = y = 0$ is an isomorphism onto the line L_3'.

Similarly, if we also blow-up P_1 and P_2, the map f becomes well-defined everywhere, that is, if π denotes the structure map of the blow-up, then we get a well-defined map $g \colon \mathrm{Bl}_{P_1,P_2,P_3}\mathbb{P}^2 \to \mathbb{P}^2$ sitting in the diagram

$$\begin{tikzcd} & \mathrm{Bl}_{P_1,P_2,P_3}\mathbb{P}^2 \arrow[dl, "\pi"'] \arrow[dr, "g"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "f"] & & \mathbb{P}^2 \end{tikzcd} \tag{14.1}$$

Let $E_1 = \pi^{-1}(P_1)$, $E_2 = \pi^{-1}(P_2)$ and $E_3 = \pi^{-1}(P_3)$ be the exceptional divisors and let $\widetilde{L}_1$, $\widetilde{L}_2$ and $\widetilde{L}_3$ be the strict transforms of L_1, L_2 and L_3 respectively, cf.

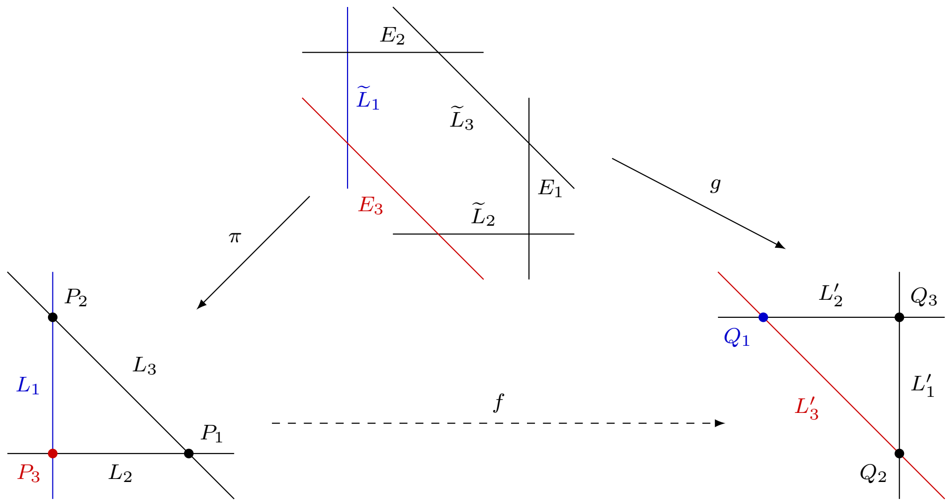


Figure 14.7: The Cremona transformation f and the blow-up in P_1, P_2 and P_3.

Figure 14.7. The map g takes the line $\widetilde{L}_i$ to the point Q_i for $i = 1, 2, 3$ since the rational map f takes the lines L_i to the points Q_i. As we noted above, the map g also induces an isomorphism between E_i and L'_i for $i = 1, 2, 3$.

Exercise 14.4.1. Show that g is the blow-up in Q_1, Q_2 and Q_3.

In the following section we will see that any birational map of non-singular surfaces factors as a "roof" of blow-ups at points as in (14.1).

14.5 Resolving the indeterminacy locus by blow-ups

A rational map $f\colon X \dashrightarrow Y$ between two non-singular surfaces is defined outside a finite number of points $P_1, P_2, \ldots, P_n \in X$. In fact, the following generalization of Theorem 13.6.1 holds: if $f\colon X \dashrightarrow Y$ is a rational map where X is non-singular and Y is projective, then f is defined outside a closed subset of codimension 2.

Given a birational map $f\colon X \dashrightarrow Y$, we thus have that both f and f^{-1} are defined outside a finite number of points on X and Y respectively. Nevertheless, it does not follow that X and Y are isomorphic after removing a finite number of points on X and Y.

Example 14.5.1. Let P_1 and P_2 be different points on $\mathbb{P}^2$ and consider $X_1 = \mathrm{Bl}_{P_1}\mathbb{P}^2$ and $X_2 = \mathrm{Bl}_{P_2}\mathbb{P}^2$. Since X_1 and X_2 are naturally birational to $\mathbb{P}^2$, there is a natural birational map $f\colon X_1 \dashrightarrow X_2$. If E_1 and E_2 denote the exceptional divisors and we identify P_2 with its inverse image in X_1 and P_1 with its inverse image in X_2, then f is defined over $X_1 \setminus P_2$ and f^{-1} is defined over $X_2 \setminus P_1$ but the largest open subsets over which X_1 and X_2 are isomorphic is $X_1 \setminus (E_1 \cup P_2) \cong X_2 \setminus (P_1 \cup E_2)$.

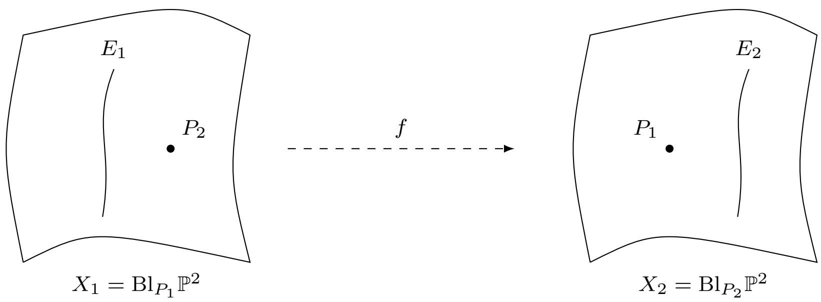


Figure 14.8: A simple birational map.

On the other hand, suppose $f\colon X \to Y$ is a birational map that is defined on all of X. We have seen that the inverse $f^{-1}\colon Y \dashrightarrow X$ is defined outside a finite set of points $Z \subset Y$ and it follows that $f|_{X\setminus f^{-1}(Z)}\colon X \setminus f^{-1}(Z) \to Y \setminus Z$ is an isomorphism.

Theorem 14.5.2. *Let $f\colon X \to Y$ be a birational map between non-singular surfaces and Z the finite subset of Y over which f is not an isomorphism. Then f is an iterated sequence of blow-ups at points in Z and at points in inverse images of Z. In particular, the fiber $f^{-1}(z)$ is a tree of $\mathbb{P}^1$'s for any $z \in Z$.*

Sketch of proof. Zariski's main theorem shows that the fiber $f^{-1}(z)$ is connected (but not necessarily irreducible) and hence of pure dimension 1 for every $z \in Z$. The irreducible components of the $f^{-1}(z)$'s are precisely the curves in X that are mapped to a point in Y. These curves are called *exceptional curves*. By the universal property of blow-ups (that we have not discussed), it follows that there is a map $X \to \mathrm{Bl}_z Y$ for any $z \in Z$. This map is surjective so at least one exceptional curve is not exceptional for $X \to \mathrm{Bl}_z Y$. The result now follows by induction on the number of exceptional curves. □

Theorem 14.5.3 (Resolving the indeterminacy locus)**.** *Given a birational map $f\colon X \dashrightarrow Y$ of non-singular projective surfaces, there exists a sequence of blow-ups $\widetilde{X} = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$ such that the composition $\widetilde{X} \to Y$ is defined everywhere.*

Sketch of proof. The closure of the graph of f gives a variety $Z \subset X \times Y$ and birational maps $Z \to X$ and $Z \to Y$. However, Z can be singular. By Hironaka (Theorem 13.7.4), there exists a birational map $\widetilde{Z} \to Z$ such that $\widetilde{Z}$ is projective and non-singular. Now apply Theorem 14.5.2 to the composition $\widetilde{Z} \to Z \to X$. □

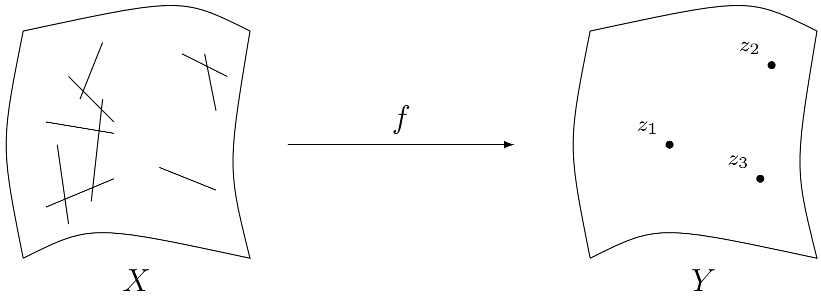


Figure 14.9: An illustration of a general birational map $f\colon X \to Y$.

By the previous theorem, $\widetilde{X} \to Y$ is a sequence of blow-ups so we have a "roof":

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

where both $\widetilde{X} \to X$ and $\widetilde{X} \to Y$ are sequences of blow-ups.

14.6 Birational geometry of rational surfaces

In the previous section we saw that every birational map between non-singular surfaces is given by a sequence of blow-ups and blow-downs, i.e., inverses of blow-ups. The set of non-singular surfaces form a graph where an edge is a blow-up. Two surfaces are then birational if and only if they belong to the same component of the graph. In this section we will study the component of *rational surfaces*, that is, those that are birational to $\mathbb{P}^2$.

Example 14.6.1. We have seen the following examples of rational surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, toric surfaces, and blow-ups of these.

Minimal surfaces Given a surface X we can blow up any point and obtain a new non-singular surface X' with a map $X' \to X$. Thus the graph has no maximal elements. The process of blow-down is more subtle. Given a line in X, it is not always possible to contract this line to a point. We say that a surface X is *minimal* if it does not admit any birational map $X \to Y$ with Y non-singular, or equivalently, if there is no blow-up $X \to Y$.

Hirzebruch surfaces Let $1 \leq a \leq b$ be integers and let $P_{a,b}$ be the polytope $\mathrm{Conv}(0, ae_1, e_2, be_1 + e_2)$. It can be shown that the toric surface $X_{P_{a,b}}$ only depends on $b - a$.

Definition 14.6.2. The *Hirzebruch surface* Σ_n, $n \geq 0$ is the toric surface associated to the polytope $P_{a,a+n}$ for any a.

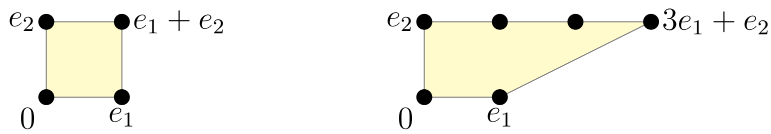


Figure 14.10: The polytopes $P_{1,1}$ and $P_{1,3}$ of the Hirzebruch surfaces Σ_0 and Σ_2.

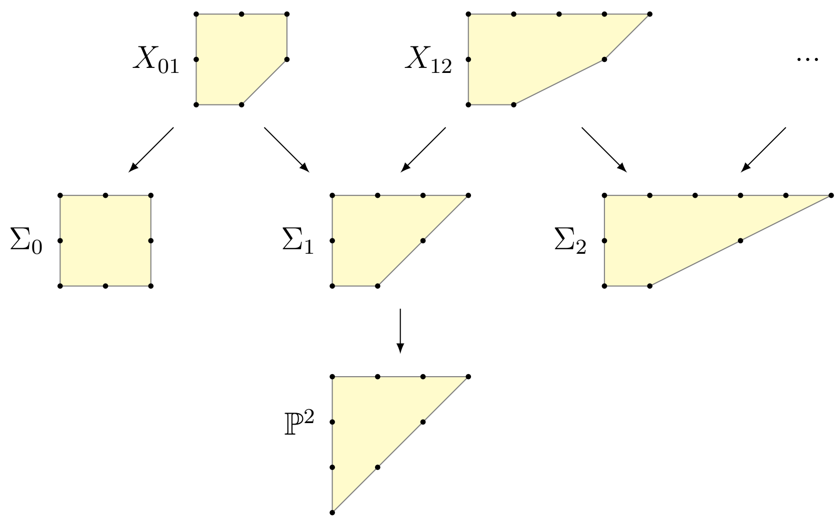
Exercise 14.6.3. Show that the Hirzebruch surfaces are non-singular (i.e., the polytopes are smooth) and that $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 14.6.4. *The minimal rational surfaces are $\mathbb{P}^2$, $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and Σ_n for $n \geq 2$. They are related via a diagram*

$$\begin{tikzcd}
& X_{01} \arrow[dl] \arrow[dr] & & X_{12} \arrow[dl] \arrow[dr] & & X_{23} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \\
\Sigma_0 & & \Sigma_1 \arrow[d] & & \Sigma_2 & & \Sigma_3 & \\
& & \mathbb{P}^2 & & & & &
\end{tikzcd}$$

where every arrow is a blow-up in a point.

If P is a smooth polytope and v is a vertex, then the blow-up at v as defined in §12.5 corresponds to the blow-up of the toric variety X_P in the point corresponding to v. The surfaces $X_{i,i+1}$ in the diagram above are also toric surfaces. In terms of polytopes we have the diagram:



14.7 Appendix: non-singular varieties and tangent spaces

For simplicity, we will only treat blow-ups of varieties that locally looks like affine space but in this section we will give a brief description of how the general case can be treated.

Just as a smooth manifold of dimension n locally looks like $\mathbb{R}^n$, a non-singular variety over $\mathbb{C}$ of dimension n locally looks like $\mathbb{C}^n$ in the analytic topology. However, in the Zariski topology a non-singular variety need not locally look like affine space.

Example 14.7.1. An elliptic curve is non-singular of dimension 1 but is not Zariski-locally isomorphic to the affine line. Indeed, if that was the case, then the elliptic curve would be birational to $\mathbb{P}^1$, but an elliptic curve has genus 1 whereas the projective line has genus 0 and the genus is a birational invariant.

A non-singular variety is rather close to $\mathbb{C}^n$ though. A non-singular point P in an n-dimensional variety X has, by definition, a *tangent space* of dimension n. In a neighborhood of P, there are then n functions $f_1, f_2, \ldots, f_n$ such that their differentials constitute a basis of the *cotangent space*. This gives a rational map $\varphi\colon X \dashrightarrow \mathbb{C}^n$, $x \mapsto (f_1(x), \ldots, f_n(x))$ defined in a neighborhood of P which is close to being an isomorphism. The technical term is that φ is étale which means that in the analytic topology φ is a local homeomorphism at P. The blow-up of X at P, denoted $\mathrm{Bl}_P X$ can now locally be defined as the closed subvariety of $X \times \mathbb{P}^{n-1}$ defined by the equations $y_i f_j(x) = y_j f_i(x)$ for all $i = 1, \ldots, n$ where $y_1, y_2, \ldots, y_n$ are the coordinates of $\mathbb{P}^{n-1}$.

Chapter 15

Singularities of curves

In this chapter, we will study some singular curves and describe how curves can be desingularized via blow-ups.

15.1 Strict transform of curves

Consider the *nodal cubic curve* $C = \{y^2 - x^2 - x^3 = 0\}$ in the plane as depicted in Figure 14.1. The inverse image of this curve in $Y = \mathrm{Bl}_0\mathbb{A}^2$ has the same equation. Let us study this equation in the two charts.

- Recall that the first chart is isomorphic to $\mathbb{A}^2_{x,k}$ where $y = kx$ (and $k = t/s$). In this chart the equation becomes $k^2x^2 - x^2 - x^3 = 0$ which factors as $x^2(k^2 - 1 - x) = 0$.
- The second chart is isomorphic to $\mathbb{A}^2_{y,k^{-1}}$ where $x = k^{-1}y$. In this chart, the equation becomes $y^2 - k^{-2}y^2 - k^{-3}y^3 = 0$ and factors as $y^2(1 - k^{-2} - k^{-3}y) = 0$.

We see that the inverse image of C has two components: (1) the exceptional divisor E (equation $x^2 = 0$ and $y^2 = 0$ respectively) and (2) a curve $\widetilde{C}$ with equations $k^2 - 1 - x = 0$ and $1 - k^{-2} - k^{-3}y = 0$ respectively. In global coordinates for Y, we can also express $\widetilde{C}$ as $t^2 - s^2 - s^2x = 0$. In the first chart, the curve is isomorphic to the affine line (with coordinate k). In the second chart, the curve is isomorphic to the affine line minus a point (coordinate $k^{-1} \neq 0$). We have thus found a *desingularization* $\widetilde{C}$ of C.

The curve $\widetilde{C}$ is called the *strict transform* (or birational transform, or proper transform) of C. It is obtained by removing the exceptional divisor from $\pi^{-1}(C)$ or, equivalently, taking the closure of $C \cap (\mathbb{A}^2 \setminus 0)$, which is isomorphic to $\pi^{-1}(C \cap (\mathbb{A}^2 \setminus 0))$, inside the blow-up $Y = \mathrm{Bl}_0\mathbb{A}^2$. As varieties, $\pi^{-1}(C) = E \cup \widetilde{C}$ but we see

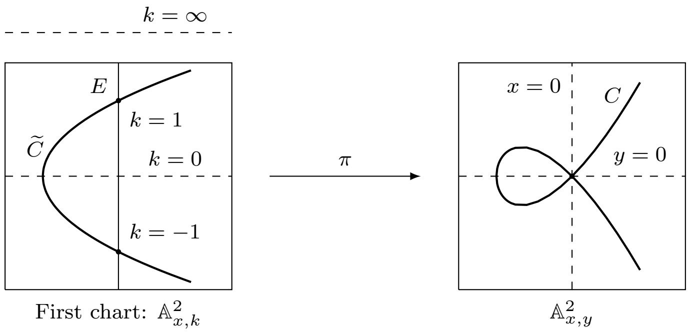


Figure 15.1: The blow-up of the affine plane and the strict transform $\widetilde{C}$ of the nodal cubic curve $C = \{y^2 - x^2 - x^3 = 0\}$.

from the equations that in this case the exceptional divisor also comes with the multiplicity 2.

Furthermore, we note that $\widetilde{C}$ intersects E in two points: $k = \pm 1$. These are contained in both charts and correspond to the two tangent directions of C at the origin: $y = x$ and $y = -x$. The whole curve $\widetilde{C}$ is contained in the first chart because there are no points on the line $k^{-1} = 0$. It is not contained in the second chart since there is a point on the line $k = 0$, namely $x = -1$.

Exercise 15.1.1. Describe the multiplicity of the exceptional divisor for an arbitrary polynomial $p(x, y) = 0$ (that is, when is the multiplicity 0, 1, 2, etc.). Can you phrase the multiplicity in terms of derivatives of p?

Exercise 15.1.2. Partial derivatives of polynomials makes sense in any characteristic but since $p = 0$ some unexpected behavior occurs (e.g., we cannot express the pth term in Taylor's theorem in terms of usual partial derivatives). Does your description in the previous exercise in terms of derivatives hold in positive characteristic?

Let us now consider the *cuspidal cubic curve* $C = \{y^2 = x^3\}$ in $\mathbb{A}^2$ and do the same analysis. In the two charts we obtain the equations

$$\begin{aligned} (kx)^2 - x^3 = 0 \quad &\Longleftrightarrow \quad x^2(k^2 - x) = 0 \\ y^2 - (k^{-1}y)^3 = 0 \quad &\Longleftrightarrow \quad y^2(1 - k^{-3}y) = 0 \end{aligned}$$

Again we note that $\pi^{-1}(C) = E \cup \widetilde{C}$ where the multiplicity along E is 2 and $\widetilde{C}$ has the equations $k^2 = x$ and $1 = k^{-3}y$ respectively. As before $\widetilde{C}$ is non-singular but we have a new phenomenon: E and $\widetilde{C}$ do not meet *transversally*, i.e., they are tangent to each other at their intersection point $k = 0$.

For various reasons, it is desirable to not only desingularize C but also arrange so that C becomes transversal to the exceptional divisor. This can be accomplished

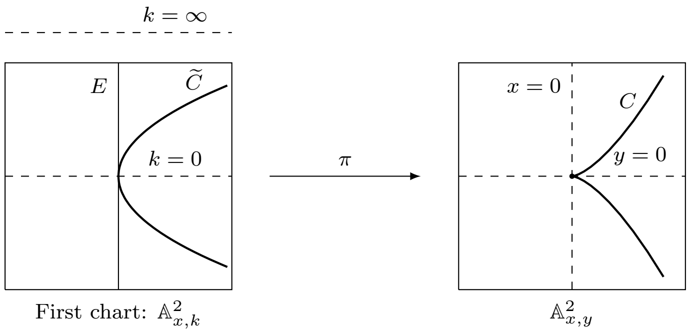


Figure 15.2: The blow-up of the affine plane and the strict transform $\widetilde{C}$ of the cuspidal cubic curve $C = \{y^2 - x^3 = 0\}$.

by one further blow-up. Then however, C meets both exceptional divisors in the same point and this is also undesirable. After a further blow-up the curve will meet all exceptional divisor transversally and not more than one at the same time. We say that C has *normal crossings* with the exceptional divisors.

After n blow-ups we denote the coordinates by x_n and y_n (in any chart), the exceptional divisor by E_n and the strict transform of the curve by C_n. We also denote the strict transforms of the ith exceptional divisors by E_i after the nth blow-up.

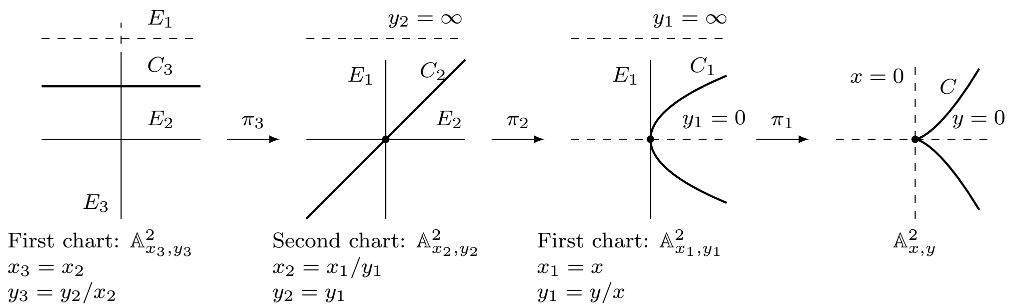


Figure 15.3: An iterated blow-up of the affine plane and the strict transforms of the cuspidal cubic curve $C = \{y^2 - x^3 = 0\}$.

Exercise 15.1.3 (Blow-up of affine space in dimension 3). Consider the blow-up $\pi\colon \mathrm{Bl}_0\mathbb{A}^3 \to \mathbb{A}^3$.

(1) Give equations for the strict transform of the hypersurface $x^2 + y^2 + z^2 = 0$ and determine the multiplicity of the exceptional divisor.

(2) Give equations for the strict transform of the hypersurface $x^2 - yz^2 = 0$ and

determine the multiplicity of the exceptional divisor. Is the strict transform less singular than the original surface?

Exercise: higher cusps

15.2 Order of vanishing and multiplicity

Consider as before a curve $C = \{f(x, y) = 0\}$ in $\mathbb{A}^2$. We can write the polynomial f as a sum of homogeneous polynomials:

$$f(x, y) = f_0 + f_1(x, y) + f_2(x, y) + \cdots + f_n(x, y)$$

where $f_d(x, y)$ is homogeneous of degree d, that is, every monomial is of degree d. Note that $f_d(0, 0) = 0$ for all $d > 0$. Thus $f_0 = 0$ if and only if $(0, 0) \in C$.

Next, consider a line $y = kx$ through the origin. The restriction of f to that line is $f(x, kx)$. Note that $f_d(x, kx)$ remains homogeneous of degree d, i.e., $f_d(x, kx) = c_d x^d$ for some constant c_d. If $f_0 = f_1(x, kx) = \cdots = f_{d-1}(x, kx) = 0$ and $f_d(x, kx) \neq 0$, then $f(x, kx)$ has a zero of multiplicity d at $x = 0$. In particular, $f_0 = f_1(x, kx) = 0$ if and only if the line is tangent at the origin. Recall that, by definition, C is singular at the origin if there is more than one tangent line. Since the set of tangent lines is a subspace, C is singular at the origin if and only if $f_0 = f_1(x, y) = 0$. This motivates the following definition.

Definition 15.2.1. The *order of vanishing* at the origin of $f(x, y)$ is

$$\operatorname{ord}_0 f = \min\{d \ : \ f_d \neq 0\}.$$

The order of vanishing measures how singular the origin is:

- $\operatorname{ord}_0 f > 0$ if the origin lies on C,
- $\operatorname{ord}_0 f = 1$ if C is smooth at the origin, and
- $\operatorname{ord}_0 f > 1$ if C is singular at the origin.

If $P = (x_0, y_0)$ is any point of $\mathbb{A}^2$, we define $\operatorname{ord}_P f(x, y) := \operatorname{ord}_0 f(x - x_0, y - y_0)$. If $f(x)$ is a polynomial in one variable, we define $\operatorname{ord}_{x_0} f$ as the multiplicity of the zero $x = x_0$. Note that $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ and $\operatorname{ord}_0 f(x)$ is the smallest d such that $c_d \neq 0$. If $f = 0$, then we let $\operatorname{ord}_P f = \infty$.

For curves, and varieties, embedded into higher-dimensional spaces, there is a more refined invariant called the *multiplicity* which has the three properties above. For curves in the affine plane, the multiplicity coincides with the order of vanishing. Also, the multiplicity does not depend on the choice of embedding.

Lemma 15.2.2. *Let $C = \{f(x,y) = 0\}$ be a curve in $\mathbb{A}^2$ and let $m = \operatorname{ord}_0 f$. Let $Y = \mathrm{Bl}_0\mathbb{A}^2$ with structure map $\pi\colon Y \to \mathbb{A}^2$. Then*

(1) $\pi^{-1}(C) = mE + \widetilde{C}$, that is, the exceptional divisor appears with multiplicity m in $\pi^{-1}(f)$.

(2) The restriction of $\widetilde{C}$ to $E = \mathbb{P}^1$ is given by the homogeneous equation $f_m = 0$. In particular, $\widetilde{C} \cap E$ has m points counted with multiplicity.

(3) $\sum_{P\in\widetilde{C}\cap E} \operatorname{ord}_P \widetilde{C} \leq \operatorname{ord}_0 C = m$.

Proof. Consider the first chart $x_1 = x$ and $y_1 = y/x$. In this chart f becomes

$$f(x,y) = f(x_1, x_1y_1) = f_0 + x_1f_1(1,y_1) + x_1^2f_2(1,y_1) + x_1^3f_3(1,y_1) + \dots.$$

By assumption f vanishes to order m which gives

$$f(x,y) = x_1^m\big(f_m(1,y_1) + x_1f_{m+1}(1,y_1) + \dots\big).$$

Here $x_1^m = 0$ is the equation for the exceptional divisor, with multiplicity m, and $\widetilde{C}$ is given by $g(x_1,y_1) = f_m(1,y_1) + x_1f_{m+1}(1,y_1) + \cdots = 0$. If we restrict $\widetilde{C}$ to E, then we obtain $g(0,y_1) = f_m(1,y_1)$.

In the second chart, we obtain

$$f(x,y) = y_1^m\big(f_m(x_1,1) + y_1f_{m+1}(x_1,1) + \dots\big).$$

and the restriction of $\widetilde{C}$ to E is given by $f_m(x_1,1)$. The first two statements follow. For the last statement, it is readily verified that $\operatorname{ord}_P \widetilde{C}\cap E \geq \operatorname{ord}_P \widetilde{C}$ (e.g., use local equations as above) and $\sum_{P\in\widetilde{C}\cap E} \operatorname{ord}_P \widetilde{C}\cap E = m$ since $\widetilde{C}\cap E$ is given by the homogeneous polynomial f_m of degree m. □

We conclude that $\widetilde{C}$ meets E in at most m points and that the sum of the multiplicity at these points is at most m. Thus, if $\widetilde{C}$ meets E at more than one point, then the multiplicity has decreased at all points.

We also see that if we blow-up at a non-singular point, then $\widetilde{C}$ meets E in a unique non-singular point.

Exercise 15.2.3. Let $P \in C$ and let $\widetilde{C}$ be the strict transform under the blow-up at P. Show that $\widetilde{C} \to C$ is an isomorphism if C is non-singular at P.

Example 15.2.4. Let C be the nodal cubic. We have seen that $m = 2$ and after one blow-up $\widetilde{C}\cap E$ consisted of two smooth points.

Example 15.2.5. Let C be the cuspidal cubic. We have seen that $m = 2$ and after one blow-up $\widetilde{C}\cap E$ consisted of one smooth point. In this case the intersection was not transversal which exactly means that the inequality $\operatorname{ord}_P \widetilde{C}\cap E \geq \operatorname{ord}_P \widetilde{C}$ is strict.

15.3 Resolution algorithm

We would like to resolve a singular curve using blow-ups. Since the blow-up at the point P is an isomorphism outside P, blow-ups at different points do not interact. This suggest the algorithm:

Algorithm 15.3.1. *Pick any singular point $P \in C$ and blow-up P. Replace C with $\widetilde{C}$ and repeat as long as there is a singular point.*

If the algorithm terminates, then we obtain a birational map $C' \to C$ from a non-singular curve. If C is projective, so is C'. The question is now whether the algorithm does terminate. Since we can measure the singularities using the multiplicity, it is enough to ensure that the multiplicity drops. Thus, let $m > 1$ be the maximal multiplicity of a point on C and apply the following algorithm:

Algorithm 15.3.2 (Multiplicity reduction)**.** *Pick any point $P \in C$ with multiplicity at m and blow-up P. Replace C with $\widetilde{C}$ and repeat as long as there is a point with multiplicity m.*

Lemma 15.2.2 shows that if $P' \in \pi^{-1}(P)$, then the multiplicity is at most m and it is smaller than m if $\pi^{-1}(P)$ has more than one point. It remains to show that eventually we either have more than one point or the multiplicity drops.

Exercise 15.3.3. Consider $f(x,y) = y^2 - x^n$. This curve is singular at the origin with multiplicity $m = 2$. Show that the multiplicity drops after $[n/2]$ blow-ups. Draw pictures for $n = 2, 3, 4, 5$.

15.4 Resolving curves in Weierstrass form

Even though the multiplicity need not drop after a blow-up, the equation becomes "less singular", as is easy to see in examples such as $y^2 = x^n$. It is however surprisingly difficult to measure this phenomenon. In this section, we will do this when $f(x,y)$ has a nice form.

In one variable, a polynomial or analytic function $f(x)$ can be written as $f(x) = x^d u(x)$ where $d = \operatorname{ord}_0 f$ and $u(0) \neq 0$. In higher dimension, there is a generalization for analytic functions:

Theorem 15.4.1 (Weierstrass preparation)**.** *Let $f(x_1, x_2, \ldots, x_n, y)$ be an analytic function such that $\operatorname{ord}_0 f = d$. Then (after possibly replacing y with $y = y + a_1x_1 + \cdots + a_nx_n$), we have that*

$$\begin{aligned} f(x_1, x_2, \ldots, x_n, y) = &\big(y^d + g_{d-1}(x_1, x_2, \ldots, x_n)y^{d-1} + \cdots + g_0(x_1, x_2, \ldots, x_n)\big) \\ &\cdot u(x_1, x_2, \ldots, x_n, y) \end{aligned}$$

where g_i and u are analytic functions such that $g_i(0,0,\ldots,0)=0$ and $u(0,0,\ldots,0)\neq 0$.

Note that since f has order d, every g_{d-i} has order at least i. Conversely, for any such g_is we have that $\operatorname{ord}_0 f = d$. The function u does not modify $C=\{f=0\}$ in a neighborhood of the origin. This motivates the following definition.

Definition 15.4.2. A polynomial $f(x,y)$ is a *Weierstrass polynomial* if $f(x,y) = y^d + g_{d-1}(x)y^{d-1} + \cdots + g_0(x)$ where $g_i(x)$ are polynomials such that $\operatorname{ord}_0 g_{d-i} \geq i$ for every i.

After the change of coordinates $y \mapsto y - g_{d-1}(x)/d$, we may also assume that $g_{d-1}(x)=0$.

Remark 15.4.3. There is an algebraic analogue of Theorem 15.4.1 where f is a polynomial, but the functions g_i and u will not be polynomials, only formal power series. For most purposes this is not a problem.

Lemma 15.4.4. *Let $f(x,y) = y^d + g_{d-2}(x)y^{d-2} + g_{d-3}(x)y^{d-3} + \cdots + g_0(x)$ be a Weierstrass polynomial defining the curve $C \subset \mathbb{A}^2$. Then*

(1) the order of vanishing at the origin is d;

(2) the strict transform $\widetilde{C}$ is contained in the first chart $x_1 = x$, $y_1 = y/x$.

(3) in the first chart, either (a) the multiplicity of $\widetilde{C}$ has dropped at every point of $\widetilde{C}\cap E$; or (b) $\widetilde{C}\cap E$ has a unique point $(x_1,y_1)=(0,0)$ and $\widetilde{C}$ is given by the Weierstrass polynomial

$$f_1(x_1,y_1) = y_1^d + h_{d-2}(x_1)y_1^{d-2} + h_{d-3}(x_1)y_1^{d-3} + \cdots + h_0(x_1)$$

where $\operatorname{ord}_0 h_i = \operatorname{ord}_0 g_i - i$.

Proof. (1) follows by definition. For (2), in the second chart we have $x_1 = x/y$ and $y_1 = y$ and obtain the equation

$$f(x,y) = f(x_1y_1, y_1) = y_1^d\big(1 + g_{d-2}(x_1y_1)y_1^{-2} + \cdots + g_0(x_1y_1)y_1^{-d}\big).$$

The exceptional divisor is given by $y_1 = 0$ and the strict transform is given by the second factor. The intersection $\widetilde{C}\cap\{x_1=0\}$ has the equation $1=0$ so $\widetilde{C}$ is contained in the first chart.

In the first chart, we have $x_1 = x$ and $y_1 = y/x$ and the equation:

$$f(x,y) = f(x_1, x_1y_1) = x_1^d\big(y_1^d + x_1^{-2}g_{d-2}(x_1)y_1^{d-2} + \cdots + x_1^{-d}g_0(x_1)\big)$$

so the strict transform is given by:

$$f_1(x_1, y_1) = y_1^d + h_{d-2}(x_1)y_1^{d-2} + h_{d-3}(x_1)y_1^{d-3} + \cdots + h_0(x_1)$$

where $h_{d-i}(x_1) = x_1^{-i} g_{d-i}(x_1)$ has order $\operatorname{ord}_0 g_i - i \geq 0$. If $\operatorname{ord}_0 g_i \geq 2i$ for all i, then f_1 is a Weierstrass polynomial. In particular $\operatorname{ord}_0 f_1 = d$ so $\widetilde{C} \cap E$ has a unique point, $(x_1, y_1) = (0, 0)$ of multiplicity d by Lemma 15.2.2.

If f_1 is not a Weierstrass polynomial, then we have to rule out that $\widetilde{C}$ has a point of multiplicity d. If $\operatorname{ord}_0 h_0 \geq 1$, then $(0, 0)$ is a point of multiplicity ≥ 1 but $< d$ and we are done by Lemma 15.2.2. In general, assume that $(x_1, y_1) = (0, \lambda)$ is a point on $\widetilde{C} \cap E$. Then

$$f_1(x_1, y_1 + \lambda) = y_1^d + d\lambda y_1^{d-1} + k_{d-2}(x_1)y_1^{d-2} + \cdots + k_0(x_1).$$

which has at most order of vanishing $d - 1$ (this is where the assumption that $g_{d-1} = 0$ is used). □

Exercise 15.4.5. Given an example of a Weierstrass polynomial of order d with $g_{d-1} \neq 0$ where $\widetilde{C}$ has a point $(x_1, y_1) = (0, \lambda) \neq (0, 0)$ with multiplicity d.

Corollary 15.4.6. *If C is a curve given by a Weierstrass polynomial $f(x, y) = y^d + g_{d-2}(x)y^{d-2} + g_{d-3}(x)y^{d-3} + \cdots + g_0(x)$, then the multiplicity of $\widetilde{C}$ at all points above 0 drops below m after*

$$\min\left\{\left\lfloor \frac{\operatorname{ord} g_i}{i} \right\rfloor\right\}$$

blow-ups.

15.5 Hypersurface of maximal contact

The Weierstrass form is quite remarkable. We can read of how many blow-ups it takes until the multiplicity drops and we can also immediately locate the only potential point of maximal multiplicity as the origin of the first chart. One says that the line $L = \{y = 0\}$ is a *hypersurface of maximal contact.* The strict transform of this line is $\widetilde{L} = \{y_1 = 0\}$ in the first chart (it is not contained in the second chart) and the intersection $E \cap \widetilde{L}$ is the only point that can have multiplicity d.

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