

# Duality Theory I: Basic Theory

*Notes from the Functional Analysis Course (Fall 07 - Spring 08)*

This section contains an important conceptual discussion on *duality*, which in a special case encodes the interplay between a locally convex space and its (topological) dual.

## A. Dual Pairings

**Convention.** Throughout this note  $\mathbb{K}$  will be one of the fields  $\mathbb{R}$  or  $\mathbb{C}$ , and all vector spaces are over  $\mathbb{K}$ .

**Definitions.** Suppose one has two vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$ . A *dual pairing of  $\mathcal{X}$  with  $\mathcal{Y}$*  is a bilinear<sup>1</sup> map  $\Phi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{K}$ , which is *non-degenerate*, in the following sense:

- If  $x \in \mathcal{X}$  is such that  $\Phi(x, y) = 0, \forall y \in \mathcal{Y}$ , then  $x = 0$ .
- If  $y \in \mathcal{Y}$  is such that  $\Phi(x, y) = 0, \forall x \in \mathcal{X}$ , then  $y = 0$ .

Equivalently,  $\Phi$  can be represented by two<sup>2</sup> linear injective maps

$$\# : \mathcal{X} \ni x \mapsto x^\# \in \mathcal{Y}', \tag{1}$$

$$\# : \mathcal{Y} \ni y \mapsto y^\# \in \mathcal{X}', \tag{2}$$

satisfying the equality

$$x^\#(y) = y^\#(x), \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}. \tag{3}$$

The bilinear map  $\Phi$  is simply given as  $\Phi(x, y) = x^\#(y) = y^\#(x)$ .

**Example 1.** Given an arbitrary vector space  $\mathcal{X}$ , and a linear subspace  $\mathcal{F} \subset \mathcal{X}'$ , which *separates the points of  $\mathcal{X}$*  (i.e. if  $x_1 \neq x_2$ , there exists  $f \in \mathcal{F}$  with  $f(x_1) \neq f(x_2)$ ), then the map  $\Phi : \mathcal{X} \times \mathcal{F} \ni (x, f) \mapsto f(x) \in \mathbb{K}$  is a dual pairing.

**Example 2.** We can specialize the above example to the following situation. Assume  $\mathcal{X}$  is a locally convex topological vector space. In particular the topology  $\mathfrak{T}$ , which  $\mathcal{X}$  is equipped with, is Hausdorff, and the dual space  $\mathcal{X}^* = (\mathcal{X}, \mathfrak{T})^*$  separates the points in  $\mathcal{X}$ . Then the map  $\Phi : \mathcal{X} \times \mathcal{X}^* \ni (x, \phi) \mapsto \phi(x) \in \mathbb{K}$  is a dual pairing.

## B. The Weak Dual Topologies

**Definitions.** Assume now a dual pairing is given, defined by the maps (1) and (2) satisfying (3). If we endow  $\mathcal{Y}'$  with the weak\* topology, we can apply the pull-back construction

<sup>1</sup> This means that  $\Phi$  is linear in each variable.

<sup>2</sup> Although the notation is abusive, it is very convenient!

(see LCVS II) using the map (1) to obtain a topology, denoted by  $w^\#$  on  $\mathcal{X}$ . Specifically, this is the *weakest topology which makes the map  $\# : \mathcal{X} \rightarrow (\mathcal{Y}', w^*)$  continuous*. Since the weak\* topology on  $\mathcal{Y}'$  is Hausdorff locally convex, and the map  $\#$  is linear and injective, the topology  $w^\#$  on  $\mathcal{X}$  is Hausdorff locally convex.

Similarly, one also constructs a Hausdorff locally convex topology  $w^\#$  on  $\mathcal{Y}$ , as the *weakest topology that makes the map  $\# : \mathcal{Y} \rightarrow (\mathcal{X}', w^*)$  continuous*.

The two topologies constructed above are called the *weak topologies associated with the dual pairing*.

**Remarks 1-2.** Use the notations as above.

1. Convergence in  $(\mathcal{X}, w^\#)$  and in  $(\mathcal{Y}, w^\#)$  can be characterized as follows:

(i)  $x_\lambda \xrightarrow{w^\#} x$  (in  $\mathcal{X}$ ), if and only if one of the following equivalent conditions holds:

$$\begin{aligned} x_\lambda^\#(y) &\rightarrow x^\#(y), \quad \forall y \in \mathcal{Y}, \\ y^\#(x_\lambda) &\rightarrow y^\#(x), \quad \forall y \in \mathcal{Y}. \end{aligned}$$

(ii)  $y_\lambda \xrightarrow{w^\#} y$  (in  $\mathcal{Y}$ ), if and only if one of the following equivalent conditions holds:

$$\begin{aligned} y_\lambda^\#(x) &\rightarrow y^\#(x), \quad \forall x \in \mathcal{X}, \\ x^\#(y_\lambda) &\rightarrow x^\#(y), \quad \forall x \in \mathcal{X}. \end{aligned}$$

2. In terms of seminorms:

(i) the topology  $w^\#$  on  $\mathcal{X}$  is defined by the family  $\mathfrak{P}_\mathcal{Y}^\mathcal{X} = \{p_y\}_{y \in \mathcal{Y}}$ , given by

$$p_y(x) = |y^\#(x)| = |x^\#(y)|, \quad x \in \mathcal{X}, y \in \mathcal{Y};$$

(ii) the topology  $w^\#$  on  $\mathcal{Y}$  is defined by the family  $\mathfrak{P}_\mathcal{X}^\mathcal{Y} = \{p_x\}_{x \in \mathcal{X}}$ , given by

$$p_x(y) = |x^\#(y)| = |y^\#(x)|, \quad x \in \mathcal{X}, y \in \mathcal{Y};$$

**Theorem 1.** Use the notations above.

(i) For any  $x \in \mathcal{X}$ , the linear functional  $x^\# : \mathcal{Y} \rightarrow \mathbb{K}$  is continuous in the  $w^\#$  topology.

(i') If we equip the space  $\mathcal{Y}^* = (\mathcal{Y}, w^\#)^*$  with the weak\* topology, the map

$$\# : (\mathcal{X}, w^\#) \ni x \mapsto x^\# \in (\mathcal{Y}^*, w^*) \tag{4}$$

is a topological linear isomorphism.

(ii) For any  $y \in \mathcal{Y}$ , the linear functional  $y^\# : \mathcal{X} \rightarrow \mathbb{K}$  is continuous in the  $w^\#$  topology.

(ii') If we equip the space  $\mathcal{X}^* = (\mathcal{X}, w^\#)^*$  with the weak\* topology, the map

$$\# : (\mathcal{Y}, w^\#) \ni y \mapsto y^\# \in (\mathcal{X}^*, w^*) \tag{5}$$

is a topological linear isomorphism.

*Proof.* By symmetry, it suffices to prove only (i) and (i'). Property (i) is quite trivial, by Remark 1.

To prove (ii) we first remark that (4) is injective, by definition.

Next we show that (4) is surjective. Suppose  $\phi : \mathcal{Y} \rightarrow \mathbb{K}$  is linear and  $w^\#$ -continuous, and let us prove that there exists (a unique)  $x \in \mathcal{X}$ , such that  $\phi = x^\#$ . Using Remark 2, the discussion from LCVS IV (Remark 3), there exist  $x_1, \dots, x_n \in \mathcal{X}$  and  $t_1, \dots, t_n > 0$ , such that

$$|\phi(y)| \leq t_1|x_1^\#(y)| + \dots + t_n|x_n^\#(y)|, \quad \forall y \in \mathcal{Y}.$$

Using Exercise ?? from HB, (applied to the seminorms  $p_j(y) = t_j|x_j^\#(y)|$ ,  $j = 1, \dots, n$ ), there exist linear functionals  $\phi_1, \dots, \phi_n : \mathcal{Y} \rightarrow \mathbb{K}$ , such that  $\phi = \phi_1 + \dots + \phi_n$ , and

$$|\phi_j(y)| \leq t_j|x_j^\#(y)|, \quad \forall y \in \mathcal{Y}, j = 1, \dots, n.$$

Fix for the moment  $j$ , and consider the linear functional  $\psi_j = t_j x_j^\# : \mathcal{Y} \rightarrow \mathbb{K}$ . Since  $|\phi_j(y)| \leq |\psi_j(y)|$ ,  $\forall y \in \mathcal{Y}$ , we must have the inclusion  $\text{Ker } \psi_j \subset \text{Ker } \phi_j$ , so  $\mathcal{X}/\text{Ker } \phi_j$  is a quotient of  $\mathcal{X}/\text{Ker } \psi_j$ . Since  $\psi_j$  is linear, we must have

$$1 \geq \dim(\text{Ker } \psi_j) \geq \dim(\text{Ker } \phi_j) \geq 0,$$

so either

(I)  $\dim(\text{Ker } \psi_j) = \dim(\text{Ker } \phi_j) (\leq 1)$ , in which case  $\text{Ker } \phi_j = \text{Ker } \psi_j$ , or

(II)  $1 = \dim(\text{Ker } \psi_j) > \dim(\text{Ker } \phi_j) = 0$ , in which case  $\text{Ker } \phi_j = \mathcal{X}$ .

In either case we get  $\phi_j = \alpha_j \psi_j$ , for some  $\alpha_j \in \mathbb{K}$  (in case (II),  $\alpha_j = 0$ ). So now we have  $\phi_j = \alpha_j t_j x_j^\#$ ,  $j = 1, \dots, n$ , so

$$\phi(y) = \alpha_1 t_1 x_1^\#(y) + \dots + \alpha_n t_n x_n^\#(y) = (\alpha_1 t_1 x_1 + \dots + \alpha_n t_n x_n)^\#(y), \quad \forall y \in \mathcal{Y},$$

i.e.  $\phi = (\alpha_1 t_1 x_1 + \dots + \alpha_n t_n x_n)^\#$ .

Having proven that (4) is a linear isomorphism, its continuity, as well as the continuity of its inverse, are tautologic by Remark 1, since the condition  $x_\lambda^\# \xrightarrow{w^*} x^\#$  (in  $\mathcal{Y}^*$ ) is equivalent to

$$x_\lambda^\#(y) \rightarrow x^\#(y), \quad \forall y \in \mathcal{Y},$$

which in turn is equivalent to  $x_\lambda \xrightarrow{w^\#} x$  (in  $\mathcal{X}$ ). □

We now re-visit the two examples from sub-section A.

**Example 1.** Given an arbitrary vector space  $\mathcal{X}$ , and a linear subspace  $\mathcal{F} \subset \mathcal{X}'$ , which separates the points of  $\mathcal{X}$ , then, as pointed out in sub-section A, we have a dual pairing between  $\mathcal{X}$  and  $\mathcal{X}$ . The weak topology  $w^\#$  on  $\mathcal{X}$  is referred to as the *weak  $\mathcal{F}$ -topology*, and is denoted by  $w^\mathcal{F}$ . The other weak topology  $w^\#$  on  $\mathcal{F}$  is simply the restriction of the weak\* topology (from  $\mathcal{X}'$ ) to  $\mathcal{F}$ .

**Example 2.** If  $(\mathcal{X}, \mathfrak{T})$  is a locally convex topological vector space (thus  $\mathfrak{T}$  is Hausdorff), then, as discussed in Example 2 from sub-section A, we have a dual pairing between  $\mathcal{X}$  and

$\mathcal{X}^*$ . The topology  $w^\#$  on  $\mathcal{X}$  is simply the *weak topology*  $w_{\mathfrak{T}}$ , implemented by  $\mathfrak{T}$ , which was introduced in LCVS IV. In particular (see LCVS IV), for every set  $\mathcal{S} \subset \mathcal{X}$ , one has the equalities

$$\overline{\text{conv } \mathcal{S}^{\mathfrak{T}}} = \overline{\text{conv } \mathcal{S}^{w_{\mathfrak{T}}}} = \overline{\text{conv } \mathcal{S}^{w^\#}}. \quad (6)$$

As before, the other topology  $w^\#$  on  $\mathcal{X}^*$  is simply the weak\* topology. As a consequence of Theorem 1, we also have a natural topological linear isomorphism

$$\# : (\mathcal{X}, w_{\mathfrak{T}}) \ni x \longmapsto x^\# \in (\mathcal{X}^*, w^*)^*,$$

defined by  $x^\#(\phi) = \phi(x)$ ,  $\forall x \in \mathcal{X}$ ,  $\phi \in \mathcal{X}^*$ .

**Exercise 1\*.** Equip the direct sum  $\mathcal{X} = \bigoplus_{\mathbb{N}} \mathbb{K}$  with the locally convex sum topology  $\mathfrak{T}$ . Show that the weak topology  $w_{\mathfrak{T}}$  is strictly weaker than  $\mathfrak{T}$ . Conclude that there does not exist locally convex spaces  $\mathcal{Y}$ , such that  $(\mathcal{X}, \mathfrak{T})$  is topologically linearly isomorphic to  $(\mathcal{Y}^*, w^*)$ .

**Exercise 2.** Let  $I$  be some non-empty set. Consider the spaces  $\mathcal{X}_{\text{prod}} = \prod_I \mathbb{K}$  and  $\mathcal{X}_{\text{sum}} = \bigoplus_I \mathbb{K}$ .

(i) Prove that the map  $\Phi : \mathcal{X}_{\text{prod}} \times \mathcal{X}_{\text{sum}} \rightarrow \mathbb{K}$ , defined by<sup>3</sup>

$$\Phi(x, y) = \sum_{i \in I} x_i y_i, \quad \forall x = (x_i)_{i \in I} \in \mathcal{X}_{\text{prod}}, y = (y_i)_{i \in I} \in \mathcal{X}_{\text{sum}},$$

establishes a dual pairing.

(ii) Prove that the topology  $w^\#$  on  $\mathcal{X}_{\text{prod}}$  coincides with the product topology  $\mathfrak{T}_{\text{prod}}$ .

(iii) Prove that the topology  $w^\#$  on  $\mathcal{X}_{\text{sum}}$  is strictly weaker than the locally convex sum topology. Therefore we only have an algebraic linear isomorphism  $\mathcal{X}_{\text{sum}} \simeq (\mathcal{X}_{\text{prod}}, \mathfrak{T}_{\text{prod}})^*$ .

## C. Polars and the Bipolar Theorem

As we have already seen in Example 2, the closure of convex hulls depends only on the interaction between the ambient space and its (topological) dual. Therefore, it is expected that the operation of taking closed convex hulls to admit an “abstract” characterization, within the framework of dual pairs.

**Definitions.** Suppose a dual pairing between two vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$  is given, defined by the maps (1) and (2) satisfying (3). For a non-empty set  $\mathcal{A} \subset \mathcal{X}$ , we define its *polar* (in  $\mathcal{Y}$ ) to be the set:

$$\mathcal{A}^\circ = \{y \in \mathcal{Y} : \text{Re } y^\#(a) \leq 1, \forall a \in \mathcal{A}\}.$$

Similarly, for a non-empty set  $\mathcal{B} \subset \mathcal{Y}$ , we define its polar (in  $\mathcal{X}$ ) to be the set:

$$\mathcal{B}^\circ = \{x \in \mathcal{X} : \text{Re } x^\#(b) \leq 1, \forall b \in \mathcal{B}\}.$$

**Exercise 3.** Prove that, using the notations as above, for two non-empty sets  $\mathcal{A} \subset \mathcal{B} \subset \mathcal{X}$ , one has the reverse inclusion  $\mathcal{A}^\circ \supset \mathcal{B}^\circ$ .

<sup>3</sup> The sum is clearly finite, since  $y_i \neq 0$  only for finitely many  $i \in I$ .

**Proposition 2.** *With the notations as above, if  $\mathcal{A} \subset \mathcal{X}$  is non-empty, then  $\mathcal{A}^\circ$  is convex,  $w^\#$ -closed in  $\mathcal{Y}$ , and contains 0. Likewise, if  $\mathcal{B} \subset \mathcal{Y}$  is non-empty, then  $\mathcal{B}^\circ$  is convex,  $w^\#$ -closed in  $\mathcal{X}$ , and contains 0.*

*Proof.* By symmetry, we only need to prove the first statement. The fact that  $\mathcal{A}^\circ$  contains 0 is trivial. To prove convexity, start with  $y, z \in \mathcal{A}^\circ$  and some  $t \in [0, 1]$ , and let us show that  $ty + (1 - t)z \in \mathcal{A}^\circ$ . This is, however, trivial, since by the linearity of the map (5) we have

$$\begin{aligned} \operatorname{Re}[(ty + (1 - t)z)^\#(a)] &= \operatorname{Re}[ty^\#(a) + (1 - t)z^\#(a)] = \\ &= t[\operatorname{Re}y^\#(a)] + (1 - t)[\operatorname{Re}z^\#(a)] \leq 1, \quad \forall a \in \mathcal{A}. \end{aligned}$$

Finally, to prove that  $\mathcal{A}^\circ$  is  $w^\#$ -closed, we start with some net  $(y_\lambda)$  in  $\mathcal{A}^\circ$ , which is  $w^\#$ -convergent to some  $y \in \mathcal{Y}$ , and we show that  $y \in \mathcal{A}^\circ$ . This is again trivial, since the condition  $y_\lambda \xrightarrow{w^\#} y$  implies  $y_\lambda^\#(a) \rightarrow y^\#(a)$ ,  $\forall a \in \mathcal{A}$ , so the inequalities  $\operatorname{Re}y_\lambda^\#(a) \leq 1$ ,  $\forall \lambda$ , will force  $\operatorname{Re}y^\#(a) \leq 1$ .  $\square$

**Theorem 2** (Bipolar Theorem). *Use the notations and hypotheses from the above definition. Let  $w^\#$  be the weak topology on  $\mathcal{X}$  associated with the dual pairing. For any subset  $\mathcal{A} \subset \mathcal{X}$ , the  $w^\#$ -closure of the convex hull of  $\mathcal{A} \cup \{0\}$  is given by the equality*

$$\overline{\operatorname{CONV}(\mathcal{A} \cup \{0\})}^{w^\#} = (\mathcal{A}^\circ)^\circ. \quad (7)$$

*Proof.* Using Proposition 2, we already know that the bi-polar  $(\mathcal{A}^\circ)^\circ$  is  $w^\#$ -closed (in  $\mathcal{X}$ ), convex, and contains 0. Note also that  $(\mathcal{A}^\circ)^\circ$  contains  $\mathcal{A}$ . Indeed, if  $a \in \mathcal{A}$ , then for every  $y \in \mathcal{A}^\circ$  one has  $a^\#(y) = y^\#(a)$ , so we now have

$$\operatorname{Re}a^\#(y) = \operatorname{Re}y^\#(a) \leq 1, \quad \forall y \in \mathcal{A}^\circ.$$

(The second inequality comes from the very definition of  $\mathcal{A}^\circ$ .)

Since  $(\mathcal{A}^\circ)^\circ$  is  $w^\#$ -closed, convex, and it contains  $\mathcal{A} \cup \{0\}$ , we have the inclusion

$$\overline{\operatorname{CONV}(\mathcal{A} \cup \{0\})}^{w^\#} \subset (\mathcal{A}^\circ)^\circ.$$

To prove the other inclusion, we start with some element  $x \in \mathcal{X}$  that does not belong to  $\overline{\operatorname{CONV}(\mathcal{A} \cup \{0\})}^{w^\#}$ , and we show that  $x \notin (\mathcal{A}^\circ)^\circ$ . First of all, by Theorem 1 from LCVS IV, we know there exist  $\phi \in (\mathcal{X}, w^\#)^*$  and  $s \in \mathbb{R}$ , such that

$$\operatorname{Re}\phi(a) \leq s < \operatorname{Re}\phi(x), \quad \forall a \in \mathcal{A} \cup \{0\}.$$

In particular (using  $a = 0$ ), it follows that  $\operatorname{Re}\phi(x) > s \geq 0$ , so if we take  $t = (\operatorname{Re}\phi(x) + s)/2$ , and we define  $\psi = t^{-1}\phi$ , then  $\psi : \mathcal{X} \rightarrow \mathbb{K}$  is still linear and  $w^\#$ -continuous, but now satisfies

$$\operatorname{Re}\psi(a) < 1 < \operatorname{Re}\psi(x), \quad \forall a \in \mathcal{A} \cup \{0\}.$$

By Theorem 1, we know that there exists (a unique)  $y \in \mathcal{Y}$ , such that  $\psi = y^\#$ , so the above inequalities now read:

$$\operatorname{Re}y^\#(a) < 1 < \operatorname{Re}y^\#(x), \quad \forall a \in \mathcal{A} \cup \{0\}.$$

In particular, from the first inequality we can conclude that  $y \in \mathcal{A}^\circ$ , and then the second inequality shows that  $x \notin (\mathcal{A}^\circ)^\circ$ .  $\square$

**Comment.** Many textbooks use a notion that is slightly different from ours, by defining (use the notations as above), for non-empty  $\mathcal{A} \subset \mathcal{X}$  and  $\mathcal{B} \subset \mathcal{Y}$ , the sets

$$\begin{aligned}\mathcal{A}^\square &= \{y \in \mathcal{Y} : |y^\#(a)| \leq 1, \forall a \in \mathcal{A}\}, \\ \mathcal{B}^\square &= \{x \in \mathcal{X} : |x^\#(b)| \leq 1, \forall b \in \mathcal{B}\}.\end{aligned}$$

**Definition.** With the above notations, the set  $\mathcal{A}^\square$  is called the *absolute polar* of  $\mathcal{A}$  (in  $\mathcal{Y}$ ), and the set  $\mathcal{B}^\square$  is called the *absolute polar* of  $\mathcal{B}$  (in  $\mathcal{X}$ ).

The result below deal with the “absolute” version of Proposition 2 and the relationship between the absolute and the “honest” polars.

**Proposition 3** *Use the notations as above. If  $\mathcal{A} \subset \mathcal{X}$  is a non-empty subset, then:*

- (i) *The absolute polar  $\mathcal{A}^\square$  is  $w^\#$ -closed, convex, and balanced (hence  $\mathcal{A}^\square \ni 0$ );*
- (ii)  *$\mathcal{A}^\square = [\text{BAL } \mathcal{A}]^\circ$ .*

(Recall that  $\text{BAL } \mathcal{A}$  denotes the *balanced hull* of  $\mathcal{A}$ , defined as  $\bigcup_{\substack{\alpha \in \mathbb{K} \\ |\alpha| \leq 1}} \alpha \mathcal{A}$ .)

*Proof.* (ii). Suppose first  $y \in \mathcal{A}^\square$ , and let us prove that  $y \in [\text{BAL } \mathcal{A}]^\circ$ , i.e.  $\text{Re } y^\#(\alpha a)$ , for all  $a \in \mathcal{A}$ , and every  $\alpha \in \mathbb{K}$  with  $|\alpha| \leq 1$ . This is, however, obvious, since

$$\text{Re } y^\#(\alpha a) \leq |y^\#(\alpha a)| = |\alpha| \cdot |y^\#(a)| \leq |\alpha| \leq 1.$$

Conversely, if  $y \in [\text{BAL } \mathcal{A}]^\circ$ , then we can choose, for every  $a \in \mathcal{A}$ , a scalar  $\alpha_a \in \mathbb{K}$  such that  $|\alpha_a| = 1$  and  $\alpha_a y^\#(a) = |y^\#(a)| (\in \mathbb{R})$ . In particular, for every  $a \in \mathcal{A}$ , we have

$$|y^\#(a)| = \alpha_a y^\#(a) = \text{Re}[\alpha_a y^\#(a)] = \text{Re } y^\#(\alpha_a a) \leq 1$$

(the last inequality follows from the assumption that  $y$  is in  $[\text{BAL } \mathcal{A}]^\circ$ ), so  $y$  indeed belongs to  $\mathcal{A}^\square$ .

(i). By part (ii) and Proposition 2, the absolute polar  $\mathcal{A}^\square$  is  $w^\#$ -closed, convex (and contains 0). If  $\alpha \in \mathbb{K}$  is such that  $|\alpha| \leq 1$ , then for every  $y \in \mathcal{A}^\square$  we have:

$$|(\alpha y)^\#(a)| = |\alpha| \cdot |y^\#(a)| \leq |\alpha| \leq 1,$$

which means that  $\alpha y$  is also in  $\mathcal{A}^\square$ . □

The “absolute” version of Theorem 2 is the following.

**Theorem 3** (Absolute Bipolar Theorem) *Use the notations and hypotheses from the preceding definition. Let  $w^\#$  be the weak topology on  $\mathcal{X}$  associated with the dual pairing. For any non-empty subset  $\mathcal{A} \subset \mathcal{X}$ , one has the equality*

$$\overline{\text{CONV}(\text{BAL } \mathcal{A})}^{w^\#} = (\mathcal{A}^\square)^\square. \quad (8)$$

*Proof.* By Proposition 3, we know that

$$\mathcal{A}^\square = [\text{BAL } \mathcal{A}]^\circ. \quad (9)$$

Since  $\mathcal{A}^\square$  is already balanced, again by Proposition 3, we get the equality  $(\mathcal{A}^\square)^\square = (\mathcal{A}^\square)^\circ$ , so going back to (9) we now have

$$(\mathcal{A}^\square)^\square = ([\text{BAL } \mathcal{A}]^\circ)^\circ.$$

The desired equality (8) now follows from the one above, and Theorem 2 (applied to  $\text{BAL } \mathcal{A}$ ).  $\square$

**Exercise 4.** Use the notations as above. Prove that, for a non-empty set  $\mathcal{A} \subset \mathcal{X}$ , one has:

- (i)  $(t\mathcal{A})^\circ = t^{-1}(\mathcal{A}^\circ), \forall t > 0;$
- (ii)  $(\alpha\mathcal{A})^\square = |\alpha|^{-1}(\mathcal{A}^\square), \forall \alpha \in \mathbb{K}, \alpha \neq 0.$

Besides (absolute) polars, there is a third construction associated with dual pairs, described in the following group of Exercises.

**Exercises 5-7.** Use notations as above. For non-empty  $\mathcal{A} \subset \mathcal{X}$  and  $\mathcal{B} \subset \mathcal{Y}$ , let:

$$\begin{aligned} \mathcal{A}^{\text{ann}} &= \{y \in \mathcal{Y} : y^\#(a) = 0, \forall a \in \mathcal{A}\}, \\ \mathcal{B}^{\text{ann}} &= \{x \in \mathcal{X} : x^\#(b)y^\#(a) = 0, \forall b \in \mathcal{B}\}. \end{aligned}$$

5. Prove the equalities<sup>4</sup>

$$\mathcal{A}^{\text{ann}} = (\text{SPAN } \mathcal{A})^\circ = (\text{SPAN } \mathcal{A})^{\text{ann}} = (\text{SPAN } \mathcal{A})^\square.$$

6. Prove that  $\mathcal{A}^{\text{ann}}$  is a  $w^\#$ -closed linear subspace in  $\mathcal{Y}$ .

7. Prove that  $(\mathcal{A}^{\text{ann}})^{\text{ann}} = \overline{\text{SPAN } \mathcal{A}}^{w^\#}$ .

**Definition.** The set  $\mathcal{A}^{\text{ann}}$  is called<sup>5</sup> the *annihilator of  $\mathcal{A}$*  (in  $\mathcal{Y}$ ), and the set  $\mathcal{B}^{\text{ann}}$  is called the *annihilator of  $\mathcal{B}$*  (in  $\mathcal{X}$ ).

**Comment.** The (Absolute) Bipolar Theorem has numerous application in Functional Analysis, especially in the setting from Example 2. Specifically, if  $(\mathcal{X}, \mathfrak{T})$  is a locally convex topological vector space, we can from a dual pairing with its topological dual  $\mathcal{X}^* = (\mathcal{X}, \mathfrak{T})^*$ , and then by Theorems 3 and 4, it follows that, for every  $\mathcal{A} \subset \mathcal{X}$ , one has the equalities

$$\overline{\text{CONV}(\mathcal{A} \cup \{0\})}^{\mathfrak{T}} = (\mathcal{A}^\circ)^\circ; \quad (10)$$

$$\overline{\text{CONV}(\text{BAL } \mathcal{A})}^{\mathfrak{T}} = (\mathcal{A}^\square)^\square. \quad (11)$$

Some subtle applications are illustrated in DT II.

**Exercise 8.** Let **op** be any one of the three operations: “ $\circ$ ” (polar), “ $\square$ ” (absolute polar), or “ann” (annihilator). Prove that, given a dual pairing between  $\mathcal{X}$  and  $\mathcal{Y}$ , for a non-empty set  $\mathcal{A} \subset \mathcal{X}$  one has the equality  $[(\mathcal{A}^{\text{op}})^{\text{op}}]^{\text{op}} = \mathcal{A}^{\text{op}}$ .

<sup>4</sup> Here  $\text{SPAN } \mathcal{A}$  denotes the *linear span* of  $\mathcal{A}$ .

<sup>5</sup> Some texts refer to this as the *orthogonal of  $\mathcal{A}$  in  $\mathcal{Y}$* , and denote it by  $\mathcal{A}^\perp$ .