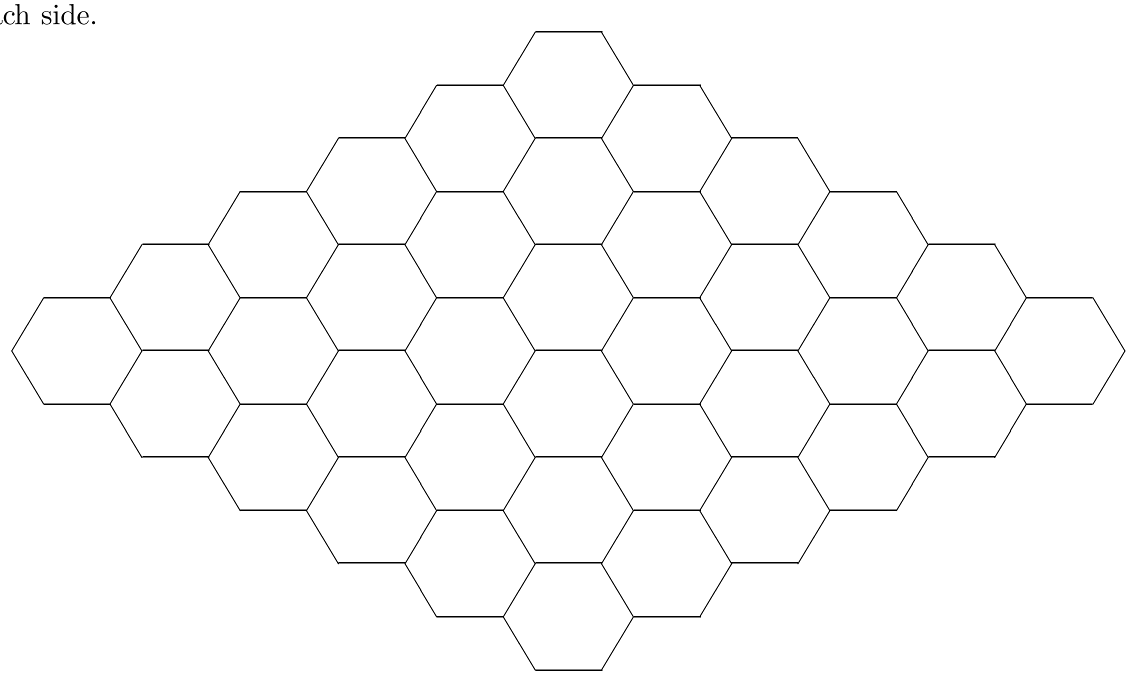
5 Brouwer's Fixed Point Theorem and the Jordan Curve Theorem

In this section we use the norm $|x| = \max\{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$ and take as the unit circle the set $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \ / \ |x| = 1\}$ with the usual topology inherited from $\mathbb{R}^2$. Thus $\mathbb{S}^1$ is the boundary of $\mathbb{I}^2$, where $\mathbb{I} = [0, 1]$ with the usual topology inherited from $\mathbb{R}$, so it is actually in the shape of a square which is homeomorphic to our usual concept of a circle. For a function f whose range is in $\mathbb{R}^2$ we denote by f_i the ith coordinate function of f.

Definition 5.1 *A* hex board *is an array of regular hexagons arranged into a diamond shape in such a way that there is the same number of hexagons along each side of the board.*

The game of hex *is played on a hex board in the following manner. Two players alternate and the aim of the game is to connect pairs of opposite sides. More precisely, if the players are labelled $\times$ and $\circ$ then one pair of opposite sides is labelled $\times$ and the other $\circ$. In turn each player labels a previously unlabelled hexagon with her or his symbol. The winner is the player who first obtains a connected path of adjacent hexagons stretching between the sides of that player's label.*

The picture below gives an example of a hex board in which there are six hexagons along each side.

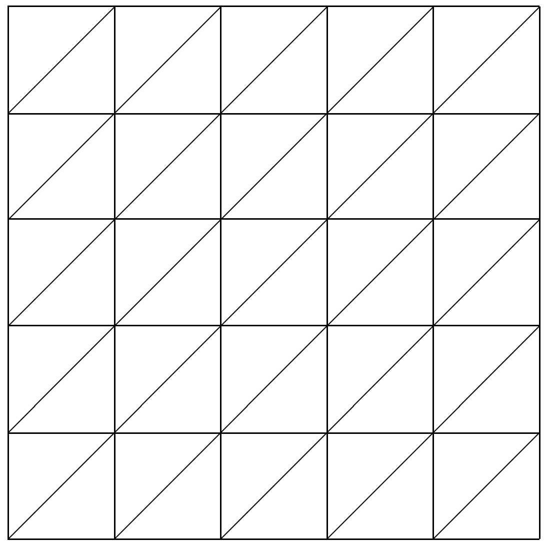


Theorem 5.2 *Every game of hex results in a winner.*

Proof. Suppose that the hex board has been filled by the symbols $\times$ and $\circ$ as described above. At each of the four corner hexagons of the board add a line segment from a free corner of the hexagon away from the board. Note that on either side of each of these line segments, as well as each of the edges of each hexagon, there is a region labelled $\times$ or $\circ$. Beginning at any one of the added segments, construct a path following edges according to the following rule: Whenever we come to a vertex, leave that vertex along an edge chosen to have $\times$ on one side and $\circ$ on the other. Clearly such a path must end. It cannot pass through the same vertex twice. Thus it must end at one of the other added segments. Then one side or other of the path will form a winning set of symbols. ∎

It can be shown that there will be only one winner.

We can modify the hex board so that the game is played on a square (which is the usual picture of $\mathbb{I}^2$). Subdivide the square as in the picture below. Now each player claims a vertex and whenever a player has claimed two adjacent vertices the edge joining them belongs to that player; thus the number of vertices in the subdivision of the square is the number of hexagons in the original hex board. Again the aim is to connect opposite sides.



Our proof of the next theorem makes use of Theorem 5.2. Conversely it may be shown that Theorem 5.2 follows from Theorem 5.3, but this will not be considered here.

Theorem 5.3 (Brouwer's Fixed Point Theorem) *If $f : \mathbb{I}^2 \to \mathbb{I}^2$ is any continuous function then there is $x \in \mathbb{I}^2$ with $f(x) = x$.*

Proof. Suppose $f : \mathbb{I}^2 \to \mathbb{I}^2$ is given. It suffices to show that f has approximate fixed points, i.e.

$$\text{for each } \varepsilon > 0 \text{ there is } x \in \mathbb{I}^2 \text{ so that } |f(x) - x| < \varepsilon. \qquad (*)$$

To show that (*) leads to a fixed point, note that for each natural number n there is $x_n \in \mathbb{I}^2$ so that $|f(x_n) - x_n| < \frac{1}{n}$. By compactness of $\mathbb{I}^2$ the sequence $\langle x_n \rangle$ has a convergent subsequence, which must converge to a fixed point of f.

We now verify (*). Suppose $\varepsilon > 0$. By uniform continuity there is $\delta > 0$ so that if $x, y \in \mathbb{I}^2$ satisfy $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$; we assume $\delta \leq \varepsilon$. Choose an integer $k > \frac{2}{\delta}$, and subdivide $\mathbb{I}^2$ into a hex board so that each side of $\mathbb{I}^2$ is divided into k equal pieces. Denote by V the vertices of this hex board. Let

$$H^+ = \{x \in V \ / \ f_1(x) - x_1 \geq \varepsilon\}, \qquad H^- = \{x \in V \ / \ x_1 - f_1(x) \geq \varepsilon\},$$

$$V^+ = \{x \in V \ / \ f_2(x) - x_2 \geq \varepsilon\}, \qquad V^- = \{x \in V \ / \ x_2 - f_2(x) \geq \varepsilon\},$$

where $x = (x_1, x_2)$.

No vertex of H^+ is adjacent to a vertex of H^-, for if $x \in H^+$ and $y \in H^-$ were vertices of a common edge in the hex board then $|x - y| < \delta$ so $|f(x) - f(y)| < \varepsilon$. However,

$$\begin{array}{rc}
x_1 - y_1 \geq -\varepsilon & \text{(because } \delta \leq \varepsilon) \\
f_1(x) - x_1 \geq \varepsilon & \text{(because } x \in H^+) \\
y_1 - f_1(y) \geq \varepsilon & \text{(because } y \in H^-) \\
\hline
f_1(x) - f_1(y) \geq \varepsilon & \text{(on adding),}
\end{array}$$

so $|f(x) - f(y)| \geq \varepsilon$, a contradiction. Furthermore, no vertex of H^+ has first coordinate equal to 1, nor does any vertex of H^- have first coordinate equal to -1. Thus $H^+ \cup H^-$ cannot form a winning set for a hex player wishing to join the sides $x_1 = -1$ and $x_1 = 1$. Similarly, $V^+ \cup V^-$ cannot form a winning set for a hex player wishing to join the sides $x_2 = -1$ and $x_2 = 1$. Thus by the hex theorem $H^+ \cup H^- \cup V^+ \cup V^-$ cannot cover all of V.

Now choose any $x \in V - (H^+ \cup H^- \cup V^+ \cup V^-)$. Then $|f(x) - x| < \varepsilon$. ∎

It is possible to prove the hex theorem from Brouwer's Fixed Point Theorem. The equivalence of these two theorems together with the modification of the hex board to fit the unit square suggests a modification of the game of hex based on the n-dimensional Brouwer Fixed Point Theorem. A hex board of size k will have vertices whose coordinates are all integer multiples of $\frac{2}{k}$ (in $\mathbb{I}^n$) and two vertices x and y will be adjacent provided that $|x - y| = \frac{2}{k}$ and either $x_i \leq y_i$ for each i or else $x_i \geq y_i$ for each i. This will be an n-player game, but there is a practical problem in trying to construct such a hex "board" when $n > 2$. When $n = 1$ the game is rather boring!

This proof is based on the paper "The game of hex and the Brouwer fixed-point theorem" by David Gale, appearing in the American Mathematical Monthly, December 1979, pp 818-827.

Definition 5.4 *A* Jordan curve *in $\mathbb{R}^2$ is the image of $\mathbb{S}^1$ under an embedding $e : \mathbb{S}^1 \to \mathbb{R}^2$.*

Sometimes a Jordan curve is called a *simple closed curve*.

Lemma 5.5 *Suppose that $h : \mathbb{I} \to \mathbb{I}^2$ and $v : \mathbb{I} \to \mathbb{I}^2$ are two paths with $h_1(-1) = -1$, $h_1(1) = 1$, $v_2(-1) = -1$ and $v_2(1) = 1$. Then for some $s, t \in \mathbb{I}$, $h(s) = v(t)$.*

Proof. Suppose, instead, that $h(s) \neq v(t)$ for all $s, t \in \mathbb{I}$. Then for all s, t we have $|h(s) - v(t)|) > 0$. Define $f : \mathbb{I}^2 \to \mathbb{I}^2$ by

$$f(s,t) = \left(\frac{v_1(t) - h_1(s)}{|h(s) - v(t)|}, \frac{h_2(s) - v_2(t)}{|h(s) - v(t)|} \right).$$

By Theorem 5.3 f has a fixed point, say $f(s_0, t_0) = (s_0, t_0)$. Furthermore, $f(\mathbb{I}^2) \subset \mathbb{S}^1$ so either $s_0 = \pm 1$ or $t_0 = \pm 1$. However, if $s_0 = 1$ then $f_1(s_0, t_0) = \dfrac{v_1(t_0) - h_1(1)}{|h(1) - v(t_0)|}$ is non-positive so cannot equal $s_0 = 1$. Similar problems arise if $s_0 = -1$ or if $t_0 = \pm 1$. This contradiction implies that $h(s) = v(t)$ for some s, t. ∎

Note that in this lemma $\mathbb{I}^2$ could be replaced by any rectangle.

Lemma 5.6 *Let J be a Jordan curve in $\mathbb{R}^2$. If $\mathbb{R}^2 - J$ is not connected, then each component of $\mathbb{R}^2 - J$ has J as its boundary.*

Proof. Let U be any component of $\mathbb{R}^2 - J$. Then no other component meets $\bar{U}$, so $\partial U = \bar{U} \cap (\mathbb{R}^2 - U) \subset J$. If $\partial U \neq J$ then there is an arc A in J containing ∂U. As there is at least one bounded component of $\mathbb{R}^2 - J$, we may choose a point p in such a component; if U itself is bounded then choose $p \in U$. Let D be a large disc centred at p containing J in its interior. Since A is homeomorphic to $\mathbb{I}$, Tietze's extension theorem guarantees a map $r : D \to A$ with $r|A = 1$. Define $f : D \to D - \{p\}$ by

$$f(x) = \begin{cases} r(x) & \text{if } x \in \bar{U} \\ x & \text{if } x \in \mathbb{R}^2 - U \end{cases} \quad \text{or} \quad f(x) = \begin{cases} x & \text{if } x \in \bar{U} \\ r(x) & \text{if } x \in \mathbb{R}^2 - U \end{cases},$$

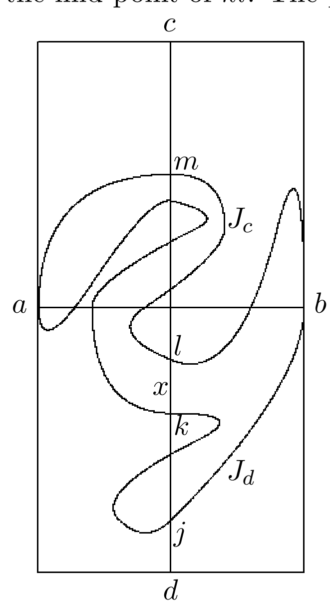
depending on whether or not U is bounded. As $\bar{U} \cap (\mathbb{R}^2 - U) \subset A$ and in A the map r is the identity, it follows that f is well-defined, and hence continuous. Let $g : D - \{p\} \to S(= \partial D)$ be obtained by outward projection from p and let $h : S \to S$ turn S through 180°. Then $hgf : D \to D$ has no fixed point, contradicting Brouwer's theorem. Thus $\partial U = J$. ∎

Theorem 5.7 (Jordan Curve Theorem) *The complement in* $\mathbb{R}^2$ *of a Jordan curve* J *consists of two components, each of which has* J *as its boundary.* $\mathbb{R}^2 - J$ *has exactly one unbounded component and each component of* $\mathbb{R}^2 - J$ *is path connected and open.*

Proof. We begin by showing that each component U of $\mathbb{R}^2 - J$ is open and path connected. Let $x \in U$. As J is compact it is also closed in $\mathbb{R}^2$ so $\mathbb{R}^2 - J$ is open and hence there is a small disc D centred at x which is disjoint from J. Then $U \cup D$ must be connected and hence, since U is a component of $\mathbb{R}^2 - J$, we must have $D \subset U$. Thus U is open. As $\mathbb{R}^2$ is locally path connected it follows that every open connected subset, such as U, is path connected.

By compactness of J there are points $a, b \in J$ for which the Pythagorean distance is greatest. Choose coordinates so that $a = (-1, 0)$ and $b = (1, 0)$. Then J lies in the rectangle $[-1, 1] \times [-2, 2] = R$, and the only points of J on ∂R are a and b. Let $c = (0, 2)$ and $d = (0, -2)$. The segment cd meets J by Lemma 5.5; let m be the point of $cd \cap J$ having maximum second coordinate. As the points a and b divide J into two arcs, we can denote the arc of J from a to b through m by J_c and the other arc by J_d. Let l be the point of $cd \cap J_c$ having least second coordinate.

The segment ld meets J_d; otherwise the two arcs J_d and $cm + \widetilde{ml} + ld$ violate Lemma 5.5. Let k be the point of $ld \cap J_d$ with greatest second coordinate and j the point of $ld \cap J_d$ with least second coordinate. Let x be the mid-point of kl. The picture below illustrates this.



Firstly the component U of $\mathbb{R}^2 - J$ containing x is bounded, for if it is not then there is a path in U from x extending outside R; let α denote the portion of this path from x to the first point, n, where it meets ∂R. If n has negative second coordinate then the paths $cm + \widetilde{ml} + lx + \alpha + \widehat{nd}$ and J_d violate Lemma 5.5, where $\widehat{nd}$ is the shortest path in ∂R from n to d. If n has positive second coordinate then the paths $dx + \alpha + \widehat{nc}$ and J_c violate Lemma 5.5. Thus no such path as α exists so U is bounded.

There can be no other bounded component, for if there were such a component V then the path $\beta = cm + \widetilde{ml} + lk + \widetilde{kj} + jd$ does not lie in this component. On the other hand, as β contains neither a nor b there are circular neighbourhoods A and B of a and b disjoint from β. By Lemma 5.6, $a, b \in \bar{V}$ so A and B contain points $u, v \in V$. As V is path connected there is a path $\widetilde{uv}$ in V from u to v. The paths $au + \widetilde{uv} + vb$ and β violate Lemma 5.5. ∎

This proof is based on the paper "The Jordan curve theorem via the Brouwer fixed point theorem" by Ryuji Maehara, appearing in the American Mathematical Monthly, December 1984, pp 641-643.

Exercises

1. Prove that each of the following two statements implies the other.

   - Every continuous function $f : \mathbb{I}^n \to \mathbb{I}^n$ has a fixed point.
   - There is no continuous function $r : \mathbb{I}^n \to \mathbb{S}^{n-1}$ such that $r(x) = x$ for each $x \in \mathbb{S}^{n-1}$.

   As the first statement is true when $n = 1, 2$ it follows that the second, the no retract theorem, also holds for $n = 1, 2$. Using algebraic topology one can quickly verify the second statement for all n from which Brouwer's Fixed Point Theorem for all n follows.

2. Describe a continuous function $r : \mathbb{I}^n \to \mathbb{S}^{n-1}$.

3. Is there a continuous function $r : \mathbb{I}^n \to \mathbb{S}^{n-1}$ such that $r(x) = -x$ for each $x \in \mathbb{S}^{n-1}$?

4. Extend Brouwer's Fixed Point Theorem to the following: every continuous function $f : \mathbb{I}^n \to \mathbb{R}^n$ such that $f(\mathbb{S}^{n-1}) \subset \mathbb{I}^n$ has a fixed point.

5. Let $\mathbb{P}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ / \ x_i \geq 0 \text{ for each } i\}$ and suppose that $f : \mathbb{P}^n \to \mathbb{P}^n$ is continuous. Prove that there are $x \in \mathbb{P}^n$ and $\lambda > 0$ such that $f(x) = \lambda x$.

6. Let A be an $n \times n$ matrix all of whose entries are non-negative. Prove that A has at least one non-negative eigenvalue.