Complex analysis, the $\overline{\partial}$-Neumann problem, and Schrödinger operators

Friedrich Haslinger

Fakultät für Mathematik, Universität Wien
Erwin Schrödinger Institute of Mathematical Physics, Wien

Preface

The subject of this book is complex analysis in several variables and its connections to partial differential equations and to functional analysis. We concentrate on the Cauchy-Riemann equation ($\overline{\partial}$-equation) and investigate the properties of the canonical solution operator to $\overline{\partial}$, the solution with minimal L^2-norm. The first chapters contain a discussion of Bergman spaces and of the solution operator to $\overline{\partial}$ restricted to holomorphic L^2-functions in one complex variable, pointing out that the Bergman kernel of the associated Hilbert space of holomorphic functions plays an important role. We investigate operator properties like compactness and Schatten-class membership, also for the solution operator on weighted spaces of entire functions (Fock-spaces). In the third chapter we generalize the results to several complex variables and explain some new phenomena which do not appear in one variable.

In the following we consider the general $\overline{\partial}$-complex and derive properties of the complex Laplacian on L^2-spaces of bounded pseudoconvex domains and on weighted L^2-spaces. The key result is the Kohn-Morrey formula, which is presented in different versions. Using this formula the basic properties of the $\overline{\partial}$-Neumann operator - the bounded inverse of the complex Laplacian - are proved. In the last years it turned out to be useful to investigate an even more general situation, namely the twisted $\overline{\partial}$-complex, where $\overline{\partial}$ is composed with a positive twist factor. In this way one obtains a rather general basic estimate, from which one gets Hörmander's L^2-estimates for the solution of the Cauchy-Riemann equation together with results on related weighted spaces of entire functions, such as that these spaces are infinite-dimensional if the eigenvalues of the Levi-matrix of the weight function show a certain behavior at infinity. In addition, it is pointed out that some L^2-estimates for $\overline{\partial}$ can be interpreted in the sense of a general Brascamp-Lieb inequality.

The next chapter contains a detailed account of the application of the $\overline{\partial}$-methods to Schrödinger operators, Pauli and Dirac operators and to Witten-Laplacians. Returning to the $\overline{\partial}$-Neumann problem we characterize compactness of the $\overline{\partial}$- Neumann operator using a description of precompact subsets in L^2-spaces. Compactness of the $\overline{\partial}$-Neumann operator is also related to properties of commutators of the Bergman projection and multiplication operators.

In the last part we use the $\overline{\partial}$-methods and some spectral theory to settle the question whether certain Schrödinger operators with magnetic field have compact resolvent. It is also shown that a large class of Dirac operators fail to have compact resolvent. Finally we exhibit some situations where the $\overline{\partial}$-Neumann operator is not compact.

In the appendices we collect results from spectral theory of unbounded, self-adjoint operators, a description of precompact subsets in L^2-spaces and prove Gårding's inequality, results which are used to handle compactness of the $\overline{\partial}$-Neumann operator. Additionally, we prove Ruelle's lemma and indicate that a certain form of the Kohn-Morrey formula can be explained by the concept of curvature on certain Kähler manifolds.

The prerequisites for reading the book are a knowledge of some spectral theory of unbounded, self-adjoint operators on Hilbert spaces and elements of complex analysis and partial differential equations.

Most of the material of the book stems from various lectures of the author given at the Erwin Schrödinger Institute of Mathematical Physics (ESI) in Vienna and at CIRM, Luminy , during programs on the $\overline{\partial}$-Neumann operator in the last years. The author is indebted to both institutions, ESI and CIRM, for their help and hospitality.

University of Vienna,

Friedrich Haslinger

Contents

1. Bergman spaces

Let $\Omega \subseteq \mathbb{C}^n$ be a domain and the Bergman space

$$A^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \text{ holomorphic } : \ \|f\|^2 = \int_\Omega |f(z)|^2 \, d\lambda(z) < \infty\},$$

where λ is the Lebesgue measure of $\mathbb{C}^n$. The inner product is given by

$$(f, g) = \int_\Omega f(z) \, \overline{g(z)} \, d\lambda(z),$$

for $f, g \in A^2(\Omega)$.
For sake of simplicity we first restrict to domains $\Omega \subseteq \mathbb{C}$. We consider special continuous linear functionals on $A^2(\Omega)$: the point evaluations . Fix $z \in \Omega$. By Cauchy's integral theorem we have

$$f(z) = \frac{1}{\pi r^2} \int_{D(z,r)} f(w) \, d\lambda(w),$$

where $f \in A^2(\Omega)$ and $D(z, r) = \{w \ : \ |w - z| < r\} \subset \Omega$. Then, by Cauchy-Schwarz,

$$\begin{aligned} |f(z)| \ &\leq \tfrac{1}{\pi r^2} \ \textstyle\int_{D(z,r)} 1 \ . \ |f(w)| \, d\lambda(w) \\ &\leq \tfrac{1}{\pi r^2} \ \left(\textstyle\int_{D(z,r)} 1^2 \, d\lambda(w)\right)^{1/2} \ \left(\textstyle\int_{D(z,r)} |f(w)|^2 \, d\lambda(w)\right)^{1/2} \\ &\leq \tfrac{1}{\pi^{1/2} r} \ \left(\textstyle\int_\Omega |f(w)|^2 \, d\lambda(w)\right)^{1/2} \\ &\leq \tfrac{1}{\pi^{1/2} r} \ \|f\|. \end{aligned}$$

If K is a compact subset of Ω, there is an $r(K) > 0$ such that for any $z \in K$ we have $D(z, r(K)) \subset \Omega$ and we get

$$\sup_{z \in K} |f(z)| \leq \frac{1}{\pi^{1/2} r(K)} \, \|f\|.$$

If $K \subset \Omega \subset \mathbb{C}^n$ we can find a polycylinder

$$P(z, r(K)) = \{w \in \mathbb{C}^n \ : \ |w_j - z_j| < r(K), \ j = 1, \ldots, n\}$$

such that for any $z \in K$ we have $P(z, r(K)) \subset \Omega$. Hence by iterating the above Cauchy integrals we get

Proposition 1.1. *Let $K \subset \Omega$ be a compact set. Then there exists a constant $C(K)$, only depending on K such that*

$$\sup_{z \in K} |f(z)| \leq C(K) \, \|f\|, \tag{1.1}$$

for any $f \in A^2(\Omega)$.

Proposition 1.2. *$A^2(\Omega)$ is a Hilbert space.*

Proof. If $(f_k)_k$ is a Cauchy sequence in $A^2(\Omega)$, by (1.1), it is also a Cauchy sequence with respect to uniform convergence on compact subsets of Ω. Hence The sequence $(f_k)_k$ has a holomorphic limit f with respect to uniform convergence on compact subsets of Ω. On the other hand, the original L^2-Cauchy sequence has a subsequence, which converges pointwise almost everywhere to the L^2-limit of the original L^2-Cauchy sequence (see for instance [42]), and so the L^2-limit coincides with the holomorphic function f. Therefore $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ and itself a Hilbert space. $\square$

(1.1) also implies that the mapping $f \mapsto f(z)$ is a continuous linear functional on $A^2(\Omega)$, hence, by the Riesz representation theorem, there is a uniquely determined function $k_z \in A^2(\Omega)$ such that

$$f(z) = (f, k_z) = \int_\Omega f(w)\, \overline{k_z(w)}\, d\lambda(w). \tag{1.2}$$

We set $K(z,w) = \overline{k_z(w)}$. Then $w \mapsto \overline{K(z,w)} = k_z(w)$ is an element of $A^2(\Omega)$, hence the function $w \mapsto K(z,w)$ is antiholomorphic on Ω and we have

$$f(z) = \int_\Omega K(z,w) f(w)\, d\lambda(w) \ , \ f \in A^2(\Omega).$$

The function of two complex variables $(z,w) \mapsto K(z,w)$ is called Bergman kernel of Ω and the above identity represents the reproducing property of the Bergman kernel.
Now we use the reproducing property for the holomorphic function $z \mapsto k_u(z)$, where $u \in \Omega$ is fixed:

$$\begin{aligned} k_u(z) &= \int_\Omega K(z,w) k_u(w)\, d\lambda(w) = \int_\Omega \overline{k_z(w)}\, \overline{K(u,w)}\, d\lambda(w) \\ &= \left(\int_\Omega K(u,w) k_z(w)\, d\lambda(w) \right)^- = \overline{k_z(u)}, \end{aligned}$$

hence we have $k_u(z) = \overline{k_z(u)}$, or $K(z,u) = \overline{K(u,z)}$.
It follows that the Bergman kernel is holomorphic in the first variable and anti-holomorphic in the second variable.

Proposition 1.3. *The Bergman kerrnel is uniquely determined by the properties that it is an element of $A^2(\Omega)$ in z and that it is conjugate symmetric and reproduces $A^2(\Omega)$.*

Proof. To see this let $K'(z,w)$ be another kernel with these properties: Then we have

$$\begin{aligned} K(z,w) &= \int_\Omega K'(z,u) K(u,w)\, d\lambda(u) \\ &= \left(\int_\Omega K(w,u) K'(u,z)\, d\lambda(u) \right)^- \\ &= \overline{K'(w,z)} \\ &= K'(z,w). \end{aligned}$$

□

Now let $\phi \in L^2(\Omega)$. Since $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$ there exists a uniquely determined orthogonal projection $P : L^2(\Omega) \longrightarrow A^2(\Omega)$. For the function $P\phi \in A^2(\Omega)$ we use the reproducing property and obtain

$$P\phi(z) = \int_\Omega K(z,w) P\phi(w)\, d\lambda(w) = (P\phi, k_z) = (\phi, Pk_z) = (\phi, k_z); \tag{1.3}$$

where we still have used that P is a self-adjoint operator and that $Pk_z = k_z$. Hence

$$P\phi(z) = \int_\Omega K(z,w) \phi(w)\, d\lambda(w). \tag{1.4}$$

P is called the Bergman projection.

Proposition 1.4. *Let $K \subset \Omega$ be a compact subset and $\{\phi_j\}$ be a complete orthonormal basis of $A^2(\Omega)$. Then the series*

$$\sum_{j=1}^{\infty} \phi_j(z)\, \overline{\phi_j(w)}$$

sums uniformly on $K \times K$ to the Bergman kernel $K(z,w)$.

Proof. For the proof of this statement we use the Riesz representation theorem to get

$$\begin{aligned} \sup_{z\in K}(\sum_{j=1}^{\infty} |\phi_j(z)|^2)^{1/2} &= \sup\{|\sum_{j=1}^{\infty} a_j\phi_j(z)| \,:\, \sum_{j=1}^{\infty} |a_j|^2 = 1,\, z \in K\} \\ &= \sup\{|f(z)| \,:\, \|f\| = 1,\, z \in K\} \\ &\leq C_K, \end{aligned} \tag{1.5}$$

where we have used (1.1) in the last inequality. Now

$$\sum_{j=1}^{\infty} |\overline{\phi_j(z)}\, \phi_j(w)| \leq (\sum_{j=1}^{\infty} |\phi_j(z)|^2)^{1/2}\, (\sum_{j=1}^{\infty} |\phi_j(w)|^2)^{1/2}$$

with uniform convergence in $z, w \in K$. In addition it follows that $(\phi_j(z))_j \in l^2$ and the function

$$w \mapsto \sum_{j=1}^{\infty} \phi_j(z)\, \overline{\phi_j(w)}$$

belongs to $\overline{A^2(\Omega)}$. Let the sum of the series be denoted by $K'(z,w)$. Notice that $K'(z,w)$ is conjugate symmetric and that for $f \in A^2(\Omega)$ we get

$$\int_\Omega K'(z,w)f(w)\, d\lambda(w) = \sum_{j=1}^{\infty} \int_\Omega f(w)\overline{\phi_j(w)}\, d\lambda(w)\, \phi_j(z) = f(z)$$

with convergence in the Hilbertspace $A^2(\Omega)$. But (1.1) implies uniform convergence on compact subsets of Ω, hence

$$f(z) = \int_\Omega K'(z,w)f(w)\, d\lambda(w),$$

for all $f \in A^2(\Omega)$, so $K'(z,w)$ is a reproducing kernel. By the uniqueness of the Bergman kernel we obtain $K'(z,w) = K(z,w)$. $\square$

We notice that (1.5) implies

$$K(z,z) = \sup\{|f(z)|^2 \;:\; f \in A^2(\Omega)\,, \|f\| = 1\}. \tag{1.6}$$

The functions $\phi_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n$, $n = 0, 1, 2, \dots$ constitute a complete orthonormal system in $A^2(\mathbb{D})$, $\mathbb{D} = \{z \in \mathbb{C} \;:\; |z| < 1\}$.
This follows from

$$\int_{\mathbb{D}} z^n\, \overline{z^m}\, d\lambda(z) = \int_0^{2\pi}\int_0^1 r^n e^{in\theta}\, r^m e^{-im\theta}\, r\, dr\, d\theta = \frac{2\pi}{n+m+2}\, \delta_{n,m}$$

For each $f \in A^2(\mathbb{D})$ with Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we get

$$(f, z^n) = \int_{\mathbb{D}} f(z)\overline{z^n}\, d\lambda(z) = \int_0^1\int_0^{2\pi} f(re^{i\theta})r^n e^{-in\theta} r\, dr\, \theta$$

$$= \int_0^1 \int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} re^{i\theta}\, d\theta\, r^{2n+1}\, dr = 2\pi a_n \int_0^1 r^{2n+1}\, dr = \pi \frac{a_n}{n+1},$$

where we used the fact that

$$a_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(z)}{z^{n+1}}\, dz,$$

for $\gamma_r(\theta) = re^{i\theta}$. Hence, by the uniqueness of the Taylor series expansion, we obtain that $(f, \phi_n) = 0$, for each $n = 0, 1, 2, \ldots$ implies $f \equiv 0$. This means that $(\phi_n)_{n=0}^{\infty}$ constitutes a complete orthonormal system for $A^2(\mathbb{D})$ and we get

$$\|f\|^2 = \sum_{n=0}^{\infty} |(f, \phi_n)|^2,$$

which is equivalent to

$$\|f\|^2 = \pi \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \ , \ f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Hence each $f \in A^2(\mathbb{D})$ can be written in the form $f = \sum_{n=0}^{\infty} c_n\, \phi_n$, where the sum converges in $A^2(\mathbb{D})$, but also uniformly on compact subsets of $\mathbb{D}$. For the coefficients c_n we have : $c_n = (f, \phi_n)$.

Now we compute the Bergman kernel $K(z, w)$ of $\mathbb{D}$. The function $z \mapsto K(z, w)$, with $w \in \mathbb{D}$ fixed, belongs to $A^2(\mathbb{D})$. Hence we get from the above formula that

$$K(z, w) = \sum_{n=0}^{\infty} c_n\, \phi_n(z),$$

where $c_n = (K(., w), \phi_n)$, in other words

$$\overline{c_n} = (\phi_n, K(., w)) = \int_{\mathbb{D}} \phi_n(z) K(w, z)\, d\lambda(z) = \phi_n(w),$$

by the reproducing property of the Bergman kernel. This implies that the Bergman kernel is of the form

$$K(z, w) = \sum_{n=0}^{\infty} \phi_n(z)\, \overline{\phi_n(w)}, \tag{1.7}$$

where the sum converges uniformly in z on all compact subsets of $\mathbb{D}$. (This is true for any complete orthonormal system, as is shown above.) A simple computation now gives

$$K(z, w) = \sum_{n=0}^{\infty} \phi_n(z)\, \overline{\phi_n(w)} = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\overline{w})^n = \frac{1}{\pi} \frac{1}{(1 - z\overline{w})^2}. \tag{1.8}$$

Hence for each $f \in A^2(\mathbb{D})$ we have

$$f(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{(1 - z\overline{w})^2} f(w)\, d\lambda(w),$$

fix $z \in \mathbb{D}$ and set $f(w) = 1/(1 - w\overline{z})^2$, then you get

$$\frac{1}{\pi} \int_{\mathbb{D}} \frac{1}{|1 - z\overline{w}|^4}\, d\lambda(w) = \frac{1}{(1 - |z|^2)^2}.$$

Proposition 1.5. *Let* $\Omega_j \subset \mathbb{C}^{n_j},\ j = 1,2$ *be two bounded domains with Bergman kernels* K_{Ω_1} *and* K_{Ω_2}. *Then the Bergman kernel* K_Ω *of the product domain* $\Omega = \Omega_1 \times \Omega_2$ *is given by*

$$K_\Omega((z_1, z_2), (w_1, w_2)) = K_{\Omega_1}(z_1, w_1)\, K_{\Omega_2}(z_2, w_2) \tag{1.9}$$

for $(z_1, z_2), (w_1, w_2) \in \Omega_1 \times \Omega_2$.

Proof. In order to show this, let F denote the function on the right hand side of (1.9). It is clear that $(z_1, z_2) \mapsto F((z_1, z_2), (w_1, w_2))$ belongs to $A^2(\Omega)$ for each fixed $(w_1, w_2) \in \Omega$ and that F is anti-holomorphic in the second variable. The reproducing property

$$f(z_1, z_2) = \int_{\Omega_1 \times \Omega_2} F((z_1, z_2), (w_1, w_2)) f(w_1, w_2)\ d\lambda(w_1, w_2)$$

is a consequence of Fubini's theorem and the corresponding reproducing properties of K_{Ω_1} and K_{Ω_2}. Hence, by the uniqueness property of the Bergman kernel, Proposition 1.3 we obtain $F = K_\Omega$. $\square$

From this we get that the Bergman kernel of the polycylinder $\mathbb{D}^n$ is given by

$$K_{\mathbb{D}^n}(z, w) = \frac{1}{\pi^n} \prod_{j=1}^{n} \frac{1}{(1 - z_j \overline{w}_j)^2}. \tag{1.10}$$

For the computation of the Bergman kernel $K_{\mathbb{B}^n}$ of the unit ball in $\mathbb{C}^n$ we use the Beta and Gamma function

$$\int_0^1 x^k\, (1 - x)^m\, dx = B(k + 1, m + 1) = \frac{\Gamma(k + 1)\Gamma(m + 1)}{\Gamma(k + m + 2)},$$

where $k, m \in \mathbb{N}$ and that for $0 \le a < 1$,

$$\begin{aligned}
\int_0^{\sqrt{1-a^2}} x^{2k+1} \left(1 - \frac{x^2}{1 - a^2}\right)^{m+1} dx &= \frac{1}{2}\, (1 - a^2)^{k+1} \int_0^1 y^k (1 - y)^{m+1}\, dy \\
&= \frac{1}{2}\, (1 - a^2)^{k+1} B(k + 1, m + 2) \\
&= \frac{1}{2}\, (1 - a^2)^{k+1}\, \frac{\Gamma(k + 1)\Gamma(m + 2)}{\Gamma(k + m + 3)}.
\end{aligned}$$

Now we can normalize the orthogonal basis $\{z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}\}$ in $A^2(\mathbb{B}^n)$ and obtain

$$
\begin{aligned}
\|z^\alpha\|^2 &= \int_{\mathbb{B}^n} |z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n} \, d\lambda(z) \\
&= \frac{\pi}{\alpha_n + 1} \int_{\mathbb{B}^{n-1}} |z_1|^{2\alpha_1} \dots |z_{n-1}|^{2\alpha_{n-1}} (1 - |z_1|^2 - \dots - |z_{n-1}|^2)^{\alpha_n+1} \, d\lambda \\
&= \frac{\pi}{\alpha_n + 1} \int_{\mathbb{B}^{n-1}} |z_1|^{2\alpha_1} \dots |z_{n-2}|^{2\alpha_{n-2}} (1 - |z_1|^2 - \dots - |z_{n-2}|^2)^{\alpha_n+1} \\
&\cdot \ |z_{n-1}|^{2\alpha_{n-1}} \left(1 - \frac{|z_{n-1}|^2}{1 - |z_1|^2 - \dots - |z_{n-2}|^2}\right)^{\alpha_n+1} d\lambda \\
&= \frac{\pi}{\alpha_n + 1} \frac{\pi \Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n + \alpha_{n-1} + 3)} \\
&\cdot \int_{\mathbb{B}^{n-2}} |z_1|^{2\alpha_1} \dots |z_{n-2}|^{2\alpha_{n-2}} (1 - |z_1|^2 - \dots - |z_{n-2}|^2)^{\alpha_n + \alpha_{n-1} + 2} \, d\lambda \\
&= \frac{\pi}{\alpha_n + 1} \frac{\pi \Gamma(\alpha_{n-1}+1)\Gamma(\alpha_n+2)}{\Gamma(\alpha_n + \alpha_{n-1} + 3)} \dots \frac{\pi \Gamma(\alpha_1+1)\Gamma(\alpha_n + \dots + \alpha_2 + n)}{\Gamma(\alpha_n + \dots + \alpha_1 + n + 1)} \\
&= \frac{\pi^n \alpha_1! \dots \alpha_n!}{(\alpha_n + \dots + \alpha_1 + n)!}.
\end{aligned}
$$

Hence the Bergman kernel of the unit ball is given by

$$
\begin{aligned}
K_{\mathbb{B}^n}(z, w) &= \sum_\alpha \frac{(\alpha_n + \dots + \alpha_1 + n)!}{\pi^n \alpha_1! \dots \alpha_n!} \, z^\alpha \overline{w}^\alpha \\
&= \frac{1}{\pi^n} \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{(\alpha_n + \dots + \alpha_1 + n)!}{\alpha_1! \dots \alpha_n!} \, z^\alpha \overline{w}^\alpha \\
&= \frac{1}{\pi^n} \sum_{k=0}^\infty (k+n)(k+n-1)\dots(k+1)(z_1\overline{w}_1 + \dots + z_n\overline{w}_n)^k \\
&= \frac{n!}{\pi^n} \frac{1}{(1 - (z_1\overline{w}_1 + \dots + z_n\overline{w}_n))^{n+1}}.
\end{aligned}
$$

In the sequel we will also consider the Fock space $A^2(\mathbb{C}^n, e^{-|z|^2})$ consisting of all entire functions f such that

$$
\int_{\mathbb{C}^n} |f(z)|^2 \, e^{-|z|^2} \, d\lambda(z) < \infty.
$$

It is clear, that the Fock space is a Hilbert space with the inner product

$$
(f, g) = \int_{\mathbb{C}^n} f(z) \, \overline{g(z)} \, e^{-|z|^2} \, d\lambda(z).
$$

Similar as in beginning of this chapter, setting $n=1$, we obtain for $f \in A^2(\mathbb{C}, e^{-|z|^2})$ that

$$
\begin{aligned}
|f(z)| &\leq \frac{1}{\pi r^2} \int_{D(z,r)} e^{|w|^2/2} \; |f(w)|\, e^{-|w|^2/2}\, d\lambda(w) \\
&\leq \frac{1}{\pi r^2} \left(\int_{D(z,r)} e^{|w|^2}, d\lambda(w) \right)^{1/2} \left(\int_{D(z,r)} |f(w)|^2\, e^{-|w|^2}\, d\lambda(w) \right)^{1/2} \\
&\leq C \left(\int_{\mathbb{C}} |f(w)|^2\, e^{-|w|^2}\, d\lambda(w) \right)^{1/2} \\
&\leq C\|f\|,
\end{aligned}
$$

where C is a constant only depending on z. This implies that the Fock space $A^2(\mathbb{C}^n, e^{-|z|^2})$ has the reproducing property. The monomials $\{z^\alpha\}$ constitute an orthogonal basis and the norms of the monomials are

$$
\begin{aligned}
\|z^\alpha\|^2 &= \int_{\mathbb{C}} |z_1|^{2\alpha_1}\, e^{-|z_1|^2}\, d\lambda(z_1) \dots \int_{\mathbb{C}} |z_n|^{2\alpha_n}\, e^{-|z_n|^2}\, d\lambda(z_n) \\
&= (2\pi)^n \int_0^\infty r^{2\alpha_1+1} e^{-r^2}\, dr \dots \int_0^\infty r^{2\alpha_n+1} e^{-r^2}\, dr \\
&= \pi^n \alpha_1! \dots \alpha_n!.
\end{aligned}
$$

Hence the Bergman kernel of $A^2(\mathbb{C}^n, e^{-|z|^2})$ is of the form

$$
K(z,w) = \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|^2} = \frac{1}{\pi^n} \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{z^\alpha \overline{w}^\alpha}{\alpha_1! \dots \alpha_n!} = \frac{1}{\pi^n} \exp(z_1\overline{w}_1 + \dots + z_n \overline{w}_n). \tag{1.11}
$$

Finally we describe the behavior of the Bergman kernel under biholomorphic maps.

Proposition 1.6. *Let $F : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic map between bounded domains in $\mathbb{C}^n$. Let $f_1, \dots, f_n$ be the components of F and $F'(z) = (\frac{\partial f_j(z)}{\partial z_k})_{j,k=1}^n$. Then*

$$
K_{\Omega_1}(z,w) = detF'(z)\, K_{\Omega_2}(F(z), F(w))\, \overline{detF'(w)}, \tag{1.12}
$$

for all $z, w \in \Omega_1$.

Proof. The substitution formula for integrals implies that for $g \in L^2(\Omega_2)$ we have

$$
\int_{\Omega_2} |g(\zeta)|^2\, d\lambda(\zeta) = \int_{\Omega_1} |g(F(z)|^2\, |\det F'(z)|^2\, d\lambda(z). \tag{1.13}
$$

Hence the map $T_F : g \mapsto (g \circ F) \det F'$ establishes an isometric isomorphism from $L^2(\Omega_2)$ to $L^2(\Omega_1)$, with inverse map $T_{F^{-1}}$, which restricts to an isomorphism between $A^2(\Omega_1)$ and $A^2(\Omega_2)$. Now let $f \in A^2(\Omega_1)$ and apply the reproducing property of K_{Ω_2} to the function $T_{F^{-1}} f = (f \circ F^{-1}) \det(F^{-1})'$, setting $F(z) = u$ we get

$$
\int_{\Omega_2} K_{\Omega_2}(u,v) T_{F^{-1}} f(v)\, d\lambda(v) = T_{F^{-1}} f(u) = f(z) (\det F'(z))^{-1}. \tag{1.14}
$$

Since T_F is an isometry,

$$
\int_{\Omega_2} T_{F^{-1}} f(v) [K_{\Omega_2}(v,u)]^-\, d\lambda(v) = \int_{\Omega_1} f(w) [T_F K_{\Omega_2}(.,u)(w)]^-\, d\lambda(w). \tag{1.15}
$$

From (1.14) and (1.15) we obtain

$$f(z) = \int_{\Omega_1} \det F'(z)\, K_{\Omega_2}(F(z), F(w))\, \overline{\det F'(w)}\, f(w)\, d\lambda(w),$$

which means that the right hand side of (1.12) has the required reproducing property, belongs to $A^2(\Omega_1)$ in the variable z and is anti-holomorphic in the variable w, and hence must agree with $K_{\Omega_1}(z, w)$. $\square$

We derive a useful formula for the coresponding orthogonal projections

$$P_j : L^2(\Omega_j) \longrightarrow A^2(\Omega_j)\ ,\ j = 1, 2.$$

Proposition 1.7. *For all $g \in L^2(\Omega_2)$ one has*

$$P_1(detF'\, g \circ F) = detF'\, (P_2(g) \circ F). \tag{1.16}$$

Proof. The left hand side of (1.16) can be written in the form $P_1(T_F(g))$, hence, by (1.4), we obtain for

$$P_1(T_F(g))(z) = \int_{\Omega_1} K_{\Omega_1}(z, w) T_F(g)(w)\, d\lambda(w)\ ,\ z \in \Omega_1.$$

Now (1.12), together with (1.15), implies that $K_{\Omega_1}(w, z) = [T_F(K_{\Omega_2}(., F(z)))(w)]\, \overline{\det F'(z)}$, so, since T_F is an isometric isomorphism, we get

$$\begin{aligned} P_1(T_F(g))(z) &= \det F'(z) \int_{\Omega_1} T_F(g)(w)\, [T_F(K_{\Omega_2}(., F(z)))(w)]^-\, d\lambda(w) \\ &= \det F'(z) \int_{\Omega_2} g(v)\, [K_{\Omega_2}(v, F(z)))]^-\, d\lambda(v) \\ &= \det F'(z)\, (P_2(g))(F(z)), \end{aligned}$$

which proves (1.16). $\square$

2. The canonical solution operator to $\overline{\partial}$ restricted to spaces of holomorphic functions

We want to solve the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial u}{\partial \overline{z}} = g \quad \text{or} \quad \overline{\partial} u = g,$$

where

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \; z = x + iy \tag{2.1}$$

and $g \in A^2(\mathbb{D})$.

Before we proceed we want to recall some basic facts from operator theory.
Let H_1 and H_2 be separable Hilbert spaces and $A : H_1 \longrightarrow H_2$ a bounded linear operator. The operator A is compact, if the image $A(U)$ of the unit ball U in H_1 is a relatively compact subset of H_2.

Proposition 2.1. *Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator.*
The following properties are equivalent:
(i) A is compact;
(ii) the adjoint operator $A^* : H_2 \longrightarrow H_1$ is compact;
(iii) $A^*A : H_1 \longrightarrow H_1$ is compact.

For a proof see for instance [41].
Let $A : H \longrightarrow H$ be a compact, self-adjoint operator on a separable Hilbert space H. The Spectral Theorem says that there exists a real zero-sequence $(\mu_n)_n$ and an orthonormal system $(e_n)_n$ in H such that for $x \in H$

$$Ax = \sum_{n=0}^{\infty} \mu_n (x, e_n) e_n,$$

where the sum converges in the operator norm, i.e.

$$\sup_{\|x\| \leq 1} \|Ax - \sum_{n=0}^{N} \mu_n (x, e_n) e_n\| \to 0,$$

as $N \to \infty$.

Proposition 2.2. *Let $A : H_1 \longrightarrow H_2$ be a compact operator There exists a decreasing zero-sequence $(s_n)_n$ in $\mathbb{R}^+$ and orthonormal systems $(e_n)_{n\geq 0}$ in H_1 and $(f_n)_{n\geq 0}$ in H_2, such that*

$$Ax = \sum_{n=0}^{\infty} s_n (x, e_n) f_n \; , \; x \in H_1,$$

where the sum converges again in the operator norm.

Proof. In order to show this one applies the spectral theorem for the positive, compact operator $A^*A : H_1 \longrightarrow H_1$ and gets

$$A^*Ax = \sum_{n=0}^{\infty} s_n^2 (x, e_n) e_n, \tag{2.2}$$

where s_n^2 are the eigenvalues of A^*A. If $s_n > 0$, we set $f_n = s_n^{-1} A e_n$ and get

$$(f_n, f_m) = \frac{1}{s_n s_m}(Ae_n, Ae_m) = \frac{1}{s_n s_m}(A^*Ae_n, e_m) = \frac{s_n^2}{s_n s_m}(e_n, e_m) = \delta_{n,m}.$$

For $y \in H_1$ with $y \perp e_n$ for each $n \in \mathbb{N}_0$ we have by (13.1) that

$$\|Ay\|^2 = (Ay, Ay) = (A^*Ay, y) = 0.$$

Hence we have

$$\begin{aligned} Ax &= A\left(x - \sum_{n=0}^{\infty}(x, e_n)e_n\right) + A\left(\sum_{n=0}^{\infty}(x, e_n)e_n\right) \\ &= \sum_{n=0}^{\infty}(x, e_n)Ae_n = \sum_{n=0}^{\infty} s_n(x, e_n)f_n. \end{aligned}$$

□

The numbers s_n are uniquely determined by the operator A, they are the eigenvalues of A^*A, and they are called the s-numbers of A.
Let $0 < p < \infty$. the operator A belongs to the Schatten-class $\mathbf{S}_p$, if its sequence $(s_n)_n$ of s-numbers belongs to l^p. The elements of the Schatten class $\mathbf{S}_2$ are called Hilbert-Schmidt operators. A is a Hilbert-Schmidt operator if and only if $\sum_{n=0}^{\infty} \|Ae_n\|^2 < \infty$ for each complete orthonormal system $(e_n)_n$ in H.
On L^2-spaces Hilbert-Schmidt operators can be described in the following way:
Let $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$ be open sets and $A : L^2(T) \longrightarrow L^2(S)$ a linear mapping. A is a Hilbert-Schmidt operator if and only if there exists $K \in L^2(S \times T)$, such that

$$Af(s) = \int_T K(s,t)f(t)\,dt \quad , \quad f \in L^2(T).$$

For the proof see for instance [41].
The following characterization of compactness is useful for the special operators in the text, see for instance [13]):

Proposition 2.3. *Let H_1 and H_2 be Hilbert spaces, and assume that $S\colon H_1 \to H_2$ is a bounded linear operator. The following three statements are equivalent:*

- *S is compact.*
- *For every $\varepsilon > 0$ there is a $C = C_\varepsilon > 0$ and a compact operator $T = T_\varepsilon\colon H_1 \to H_2$ such that*

$$\|Sv\|_{H_2} \le C\,\|Tv\|_{H_2} + \varepsilon\,\|v\|_{H_1}. \tag{2.3}$$

- *For every $\varepsilon > 0$ there is a $C = C_\varepsilon > 0$ and a compact operator $T = T_\varepsilon\colon H_1 \to H_2$ such that*

$$\|Sv\|_{H_2}^2 \le C\,\|Tv\|_{H_2}^2 + \varepsilon\,\|v\|_{H_1}^2. \tag{2.4}$$

Proof. First we show that (13.2) and (13.3) are equivalent.
Suppose that (13.3) holds. Write (13.3) with ε and C replaced by their squares to obtain

$$\|Sv\|_{H_2}^2 \le C^2\,\|Tv\|_{H_2}^2 + \varepsilon^2\,\|v\|_{H_1}^2 \le (C\,\|Tv\|_{H_2} + \varepsilon\,\|v\|_{H_1})^2,$$

which implies (13.2).

Now suppose that (13.2) holds. Choose η with $\varepsilon = 2\eta^2$ and apply (13.2) with ε replaced by η to get

$$\|Sv\|_{H_2}^2 \leq C^2 \|Tv\|_{H_2}^2 + 2\eta C \|v\|_{H_1} \|Tv\|_{H_2} + \eta^2 \|v\|_{H_1}^2 .$$

It is easily seen (small constant - large constant trick) that there is $C' > 0$ such that

$$2\eta C \|v\|_{H_1} \|Tv\|_{H_2} \leq \eta^2 \|v\|_{H_1}^2 + C' \|Tv\|_{H_2}^2 ,$$

hence

$$\|Sv\|_{H_2}^2 \leq (C^2 + C') \|Tv\|_{H_2}^2 + 2\eta^2 \|v\|_{H_1}^2 = C'' \|Tv\|_{H_2}^2 + \varepsilon \|v\|_{H_1}^2 .$$

To prove the lemma it therefore suffices to prove that (13.2) is equivalent to compactness. When S is known to be compact, we choose $T = S$ and $C = 1$, and (13.2) holds for every positive ε.
For the converse let $(v_n)_n$ be a bounded sequence in H_1. We want to extract a Cauchy subsequence from $(Sv_n)_n$. From (13.2) we have

$$\|Sv_n - Sv_m\|_{H_2} \leq C \|Tv_n - Tv_m\|_{H_2} + \varepsilon \|v_n - v_m\|_{H_1} \tag{2.5}$$

Given a positive integer N, we may choose ε sufficiently small in (13.4) so that the second term on the right-hand side is at most $1/(2N)$. The first term can be made smaller than $1/(2N)$ by extracting a subsequence of $(v_n)_n$ (still labeled the same) for which $(Tv_n)_n$ converges, and then choosing n and m large enough.
Let $(v_n^{(0)})_n$ denote the original bounded sequence. The above argument shows that, for each positive integer N, there is a sequence $(v_n^{(N)})_n$ satisfying : $(v_n^{(N)})_n$ is a subsequence of $(v_n^{(N-1)})_n$, and for any pair v and w in $(v_n^{(N)})_n$ we have $\|Sv - Sw\|_{H_2} \leq 1/N$.
Let $(w_k)_k$ be the diagonal sequence defined by $w_k = v_k^{(k)}$. Then $(w_k)_k$ is a subsequence of $(v_n^{(0)})_n$ and the image sequence under S of $(w_k)_k$ is a Cauchy sequence. Since H_2 is complete, the image sequence converges and S is compact.

□

We return to the inhomogeneous Cauchy-Riemann equation Let

$$S(g)(z) = \int_{\mathbb{D}} K(z,w) g(w) (z-w)^{-} d\lambda(w). \tag{2.6}$$

Then we have

$$S(g)(z) = \overline{z} g(z) - P(\tilde{g})(z),$$

where $P : L^2(\mathbb{D}) \longrightarrow A^2(\mathbb{D})$ is the Bergman projection and $\tilde{g}(w) = \overline{w} g(w)$. We claim that $S(g)$ is a solution of the inhomogeneous Cauchy-Riemann equation:

$$\frac{\partial}{\partial \overline{z}} S(g)(z) = \frac{\partial \overline{z}}{\partial \overline{z}} g(z) + \overline{z}\, \frac{\partial g}{\partial \overline{z}} + \frac{\partial P(\tilde{g})}{\partial \overline{z}} = g(z),$$

because g and $P(\tilde{g})$ are holomorphic functions, therefore $\overline{\partial} S(g) = g$. In addition we have $S(g) \perp A^2(\mathbb{D})$, because for arbitrary $f \in A^2(\mathbb{D})$ we get

$$(Sg, f) = (\tilde{g} - P(\tilde{g}), f) = (\tilde{g}, f) - (P(\tilde{g}), f) = (\tilde{g}, f) - (\tilde{g}, Pf) = (\tilde{g}, f) - (\tilde{g}, f) = 0.$$

The operator $S : A^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ is called the canonical solution operator to $\overline{\partial}$.
Now we want to show that S is a compact operator. For this purpose we consider the adjoint operator S^* and prove that $S^* S$ is compact, which implies that S is compact (for further details see [23]).

For $g \in A^2(\mathbb{D})$ and $f \in L^2(\mathbb{D})$ we have

$$(Sg, f) = \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K(z,w) g(w) (z-w)^{-} \, d\lambda(w) \right) \overline{f(z)} \, d\lambda(z)$$

$$= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} K(w,z)(z-w) f(z) \, d\lambda(z) \right)^{-} g(w) \, d\lambda(w) = (g, S^* f)$$

hence

$$S^*(f)(w) = \int_{\mathbb{D}} K(w,z)(z-w) f(z) \, d\lambda(z). \tag{2.7}$$

Now set

$$c_n^2 = \int_{\mathbb{D}} |z|^{2n} \, d\lambda(z) = \frac{\pi}{n+1},$$

and $\phi_n(z) = z^n / c_n$, $n \in \mathbb{N}_0$, then the Bergman kernel $K(z,w)$ can be expressed in the form

$$K(z,w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}.$$

Next we compute

$$P(\tilde{\phi}_n)(z) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2} \, \overline{w} \, \frac{w^n}{c_n} \, d\lambda(w) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{c_{k-1}^2} \int_{\mathbb{D}} \frac{\overline{w}^k w^n}{c_n} \, d\lambda(w) = \frac{c_n z^{n-1}}{c_{n-1}^2},$$

hence we have

$$S(\phi_n)(z) = \overline{z} \, \phi_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2} \quad , \ n \in \mathbb{N}.$$

Now we apply S^* and get

$$S^* S(\phi_n)(w) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} (z-w) \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) d\lambda(z).$$

The last integral is computed in two steps: first the multiplication by z

$$\int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^{n+1}}{c_n} - \frac{c_n z^n}{c_{n-1}^2} \right) d\lambda(z)$$

$$\begin{aligned}
&= \int_{\mathbb{D}} \frac{z^{n+1}}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2} \, d\lambda(z) - \frac{c_n}{c_{n-1}^2} \int_{\mathbb{D}} z^n \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \, d\lambda(z) \\
&= \frac{w^n}{c_n^3} \int_{\mathbb{D}} |z|^{2n+2} \, d\lambda(z) - \frac{w^n}{c_{n-1}^2 c_n} \int_{\mathbb{D}} |z|^{2n} \, d\lambda(z) \\
&= \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2} \right) w^n.
\end{aligned}$$

Next the multiplication by w

$$w \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) d\lambda(z)$$

$$
\begin{aligned}
&= w \int_{\mathbb{D}} \frac{z^n}{c_n} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^{k+1}}{c_k^2}\, d\lambda(z) - w \int_{\mathbb{D}} \frac{c_n z^{n-1}}{c_{n-1}^2} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2}\, d\lambda(z) \\
&= w \left(\frac{c_n w^{n-1}}{c_{n-1}^2} - \frac{c_n w^{n-1}}{c_{n-1}^2} \right) \\
&= 0,
\end{aligned}
$$

it follows that

$$
S^*S(\phi_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) \phi_n(w) \quad , n = 1, 2, \dots,
$$

for $n = 0$ an analogous computation shows

$$
S^*S(\phi_0)(w) = \frac{c_1^2}{c_0^2}\, \phi_0(w).
$$

Finally we get

Proposition 2.4. *Let $S : A^2(\mathbb{D}) \longrightarrow L^2(\mathbb{D})$ be the canonical solution operator for $\overline{\partial}$ and $(\phi_k)_k$ the normalized monomials. Then*

$$
S^*S\phi = \frac{c_1^2}{c_0^2} (\phi, \phi_0)\phi_0 + \sum_{n=1}^{\infty} \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) (\phi, \phi_n)\phi_n \tag{2.8}
$$

for each $\phi \in A^2(\mathbb{D})$.

Since

$$
\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} = \frac{1}{(n+2)(n+1)} \to 0 \text{ as } n \to \infty,
$$

it follows that S^*S is compact and S too.

We have also shown that the s-numbers of S are $\left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right)^{1/2}$ and since

$$
\sum_{n=0}^{\infty} \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) < \infty
$$

it follows that S is Hilbert-Schmidt.

This can also be shown directly. For this purpose we claim that the function $(z, w) \mapsto K(z, w)(z - w)^-$ belongs to $L^2(\mathbb{D} \times \mathbb{D})$.

We have to prove, that

$$
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\overline{z} - \overline{w}|^2}{|1 - z\overline{w}|^4}\, d\lambda(z)\, d\lambda(w) < \infty.
$$

An easy estimate gives $|z - w| \leq |1 - z\overline{w}|$, for $z, w \in \mathbb{D}$. Hence

$$
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|\overline{z} - \overline{w}|^2}{|1 - z\overline{w}|^4}\, d\lambda(z)\, d\lambda(w) \leq \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\overline{w}|^2}\, d\lambda(z)\, d\lambda(w).
$$

Introducing polar coordinates $z = r\, e^{i\theta}$ and $w = s\, e^{i\phi}$ we can write the last integral in the following form

$$
\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{|1 - z\overline{w}|^2}\, d\lambda(z)\, d\lambda(w) = \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{r\, s\, d\theta\, d\phi\, dr\, ds}{1 - 2\, r\, s\, \cos(\theta - \phi) + r^2\, s^2}
$$

$$= \int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2\, s^2}{1 - 2\, r\, s\, \cos(\theta - \phi) + r^2\, s^2} \, \frac{r\, s}{1 - r^2\, s^2} \, d\theta \, d\phi \, dr \, ds.$$

Integration of the Poisson kernel with respect to θ yields

$$\int_0^{2\pi} \frac{1 - \rho^2}{1 - 2\rho\, \cos(\theta - \phi) + \rho^2} \, d\theta = 2\pi \ , 0 < \rho < 1.$$

Therefore we have

$$\int_0^1 \int_0^1 \int_0^{2\pi} \int_0^{2\pi} \frac{1 - r^2\, s^2}{1 - 2\, r\, s\, \cos(\theta - \phi) + r^2\, s^2} \, \frac{r\, s}{1 - r^2\, s^2} \, d\theta \, d\phi \, dr \, ds$$

$$= (2\pi)^2 \int_0^1 \int_0^1 \frac{r\, s}{1 - r^2\, s^2} \, dr \, ds = -\, (2\pi)^2 \int_0^1 \frac{\log(1 - s^2)}{2s} \, ds < \infty.$$

For further details see [23], [27] and [37].

Now we consider weighted spaces on entire functions

$$A^2(\mathbb{C}, e^{-|z|^m}) = \{ f : \mathbb{C} \longrightarrow \mathbb{C} \ : \ \|f\|_m^2 := \int_{\mathbb{C}} |f(z)|^2 \, e^{-|z|^m} \, d\lambda(z) < \infty \},$$

where $m > 0$. Let

$$c_k^2 = \int_{\mathbb{C}} |z|^{2k} \, e^{-|z|^m} \, d\lambda(z).$$

Then

$$K_m(z, w) = \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{c_k^2}$$

is the reproducing kernel for $A^2(\mathbb{C}, e^{-|z|^m})$.
In the sequel the expression

$$\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2}$$

will become important. Using the integral representation of the $\Gamma-$function one easily sees that the above expression is equal to

$$\frac{\Gamma\left(\frac{2k+4}{m}\right)}{\Gamma\left(\frac{2k+2}{m}\right)} - \frac{\Gamma\left(\frac{2k+2}{m}\right)}{\Gamma\left(\frac{2k}{m}\right)}.$$

For $m = 2$ this expression equals to 1 for each $k = 1, 2, \ldots$. We will be interested in the limit behavior for $k \to \infty$. By Stirlings formula the limit behavior is equivalent to the limit behavior of the expression

$$\left(\frac{2k+2}{m}\right)^{2/m} - \left(\frac{2k}{m}\right)^{2/m},$$

as $k \to \infty$. Hence we have shown the following

Lemma 2.5. *The expression*

$$\frac{\Gamma\left(\frac{2k+4}{m}\right)}{\Gamma\left(\frac{2k+2}{m}\right)} - \frac{\Gamma\left(\frac{2k+2}{m}\right)}{\Gamma\left(\frac{2k}{m}\right)}$$

tends to ∞ for $0 < m < 2$, is equal to 1 for $m = 2$ and tends to zero for $m > 2$ as k tends to ∞.

Let $0 < \rho < 1$, define $f_\rho(z) := f(\rho z)$ and $\tilde{f}_\rho(z) = \overline{z} f_\rho(z)$, for $f \in A^2(\mathbb{C}, e^{-|z|^m})$. Then it is easily seen that $\tilde{f}_\rho \in L^2(\mathbb{C}, e^{-|z|^m})$, but there are functions $g \in A^2(\mathbb{C}, e^{-|z|^m})$ such that $\overline{z} g \notin L^2(\mathbb{C}, e^{-|z|^m})$.
Let $P_m : L^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow A^2(\mathbb{C}, e^{-|z|^m})$ denote the orthogonal projection. Then P_m can be written in the form

$$P_m(f)(z) = \int_{\mathbb{C}} K_m(z,w) f(w) e^{-|w|^m} \, d\lambda(w) \ , \ f \in L^2(\mathbb{C}, e^{-|z|^m}).$$

Proposition 2.6. *Let $m \geq 2$. Then there is a constant $C_m > 0$ depending only on m such that*

$$\int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z),$$

for each $0 < \rho < 1$ and for each $f \in A^2(\mathbb{C}, e^{-|z|^m})$.

Proof. First we observe that for the Taylor expansion of $f(z) = \sum_{k=0}^\infty a_k z^k$ we have

$$\begin{aligned} P_m(\tilde{f}_\rho)(z) &= \int_{\mathbb{C}} \sum_{k=0}^\infty \frac{z^k \overline{w}^k}{c_k^2} \left(\overline{w} \sum_{j=0}^\infty a_j \rho^j w^j \right) e^{-|w|^m} \, d\lambda(w) \\ &= \sum_{k=1}^\infty a_k \, \frac{c_k^2}{c_{k-1}^2} \, \rho^k z^{k-1}. \end{aligned}$$

Now we obtain

$$\begin{aligned} &\int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \\ &= \int_{\mathbb{C}} \left(\overline{z} \sum_{k=0}^\infty a_k \rho^k z^k - \sum_{k=1}^\infty a_k \, \frac{c_k^2}{c_{k-1}^2} \, \rho^k z^{k-1} \right) \\ &\times \left(z \sum_{k=0}^\infty a_k \rho^k \overline{z}^k - \sum_{k=1}^\infty \overline{a_k} \, \frac{c_k^2}{c_{k-1}^2} \, \rho^k \overline{z}^{k-1} \right) e^{-|z|^m} \, d\lambda(z) \\ &= \int_{\mathbb{C}} (\sum_{k=0}^\infty |a_k|^2 \rho^{2k} |z|^{2k+2} - 2 \sum_{k=1}^\infty |a_k|^2 \, \frac{c_k^2}{c_{k-1}^2} \, \rho^{2k} |z|^{2k} \\ &+ \sum_{k=1}^\infty |a_k|^2 \, \frac{c_k^4}{c_{k-1}^4} \, \rho^{2k} |z|^{2k-2}) \, e^{-|z|^m} \, d\lambda(z) \\ &= |a_0|^2 \, c_1^2 + \sum_{k=1}^\infty |a_k|^2 \, c_k^2 \, \rho^{2k} \left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right). \end{aligned}$$

Now the result follows from the fact that

$$\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z) = \sum_{k=0}^\infty |a_k|^2 \, c_k^2,$$

and that the sequence $\left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right)_k$ is bounded. $\square$

Remark 2.7. *Already in the last Proposition the sequence* $\left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2}\right)_k$ *plays an important role and it will turn out that this sequence is the sequence of eigenvalues of the operator* $S_m^* S_m$ *(see below).*

Proposition 2.8. *Let* $m \geq 2$ *and consider an entire function* $f \in A^2(\mathbb{C}, e^{-|z|^m})$ *with Taylor series expansion* $f(z) = \sum_{k=0}^\infty a_k z^k$. *Let*

$$F(z) := \overline{z} \sum_{k=0}^\infty a_k z^k - \sum_{k=1}^\infty a_k \frac{c_k^2}{c_{k-1}^2} z^{k-1}$$

and define $S_m(f) := F$. *Then* $S_m : A^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^m})$ *is a continuous linear operator, representing the canonical solution operator to* $\overline{\partial}$ *restricted to* $A^2(\mathbb{C}, e^{-|z|^m})$, *i.e.* $\overline{\partial} S_m(f) = f$ *and* $S_m(f) \perp A^2(\mathbb{C}, e^{-|z|^m})$.

Proof. By the proof of Proposition 2.6, by Abel's theorem and by Fatou's theorem (see for instance [15]) we have

$$\begin{aligned} \int_{\mathbb{C}} |F(z)|^2 e^{-|z|^m} \, d\lambda(z) &= \int_{\mathbb{C}} \lim_{\rho \to 1} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \\ &\leq \sup_{0<\rho<1} \int_{\mathbb{C}} \left| \tilde{f}_\rho(z) - P_m(\tilde{f}_\rho)(z) \right|^2 e^{-|z|^m} \, d\lambda(z) \\ &\leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z) \end{aligned}$$

and hence the function

$$F(z) := \overline{z} \sum_{k=0}^\infty a_k z^k - \sum_{k=1}^\infty a_k \frac{c_k^2}{c_{k-1}^2} z^{k-1}$$

belongs to $L^2(\mathbb{C}, e^{-|z|^m})$ and satisfies

$$\int_{\mathbb{C}} |F(z)|^2 e^{-|z|^m} \, d\lambda(z) \leq C_m \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^m} \, d\lambda(z). \tag{2.9}$$

The above computation also shows that $\lim_{\rho \to 1} \|\tilde{f}_\rho - P_m(\tilde{f}_\rho)\|_m = \|F\|_m$ and by a standard argument for L^p-spaces (see for instance [15])

$$\lim_{\rho \to 1} \|\tilde{f}_\rho - P_m(\tilde{f}_\rho) - F\|_m = 0.$$

A similar computation as in the case $A^2(\mathbb{D})$ shows that the function F defined above satisfies $\overline{\partial} F = f$. Let $S_m(f) := F$. Then, by the last remarks, $S_m : A^2(\mathbb{C}, e^{-|z|^m}) \longrightarrow L^2(\mathbb{C}, e^{-|z|^m})$ is a continuous linear solution operator for $\overline{\partial}$. For arbitrary $h \in A^2(\mathbb{C}, e^{-|z|^m})$ we have

$$(h, S_m(f))_m = (h, F)_m = \lim_{\rho \to 1} (h, \tilde{f}_\rho - P_m(\tilde{f}_\rho))_m = \lim_{\rho \to 1} (h - P_m(h), \tilde{f}_\rho)_m = 0,$$

where $(.\,,.)_m$ denotes the inner product in $L^2(\mathbb{C}, e^{-|z|^m})$. Hence S_m is the canonical solution operator for $\overline{\partial}$ restricted to $A^2(\mathbb{C}, e^{-|z|^m})$. □

Remark 2.9. *Let*

$$f(z) = \sum_{k=0}^\infty \frac{z^k}{\sqrt{(k+1)!}\sqrt{k+1}}.$$

Then $f \in A^2(\mathbb{C}, e^{-|z|^2})$, *since*

$$\|f\|_2^2 = 2\pi \sum_{k=0}^{\infty} \frac{k!}{(k+1)!(k+1)} = 2\pi \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < \infty.$$

But

$$\|zf\|_2^2 = 2\pi \sum_{k=0}^{\infty} \frac{(k+1)!}{(k+1)!(k+1)} = 2\pi \sum_{k=0}^{\infty} \frac{1}{(k+1)} = \infty,$$

hence $zf \notin L^2(\mathbb{C}, e^{-|z|^2})$.
The expression for the function F *in the last theorem corresponds formally to the expression* $\overline{z}f - P_m(\overline{z}f)$; *in general* $\overline{z}f \notin L^2(\mathbb{C}, e^{-|z|^m})$, *for* $f \in A^2(\mathbb{C}, e^{-|z|^m})$, *but* $f \mapsto F$ *defines a bounded linear operator from* $A^2(\mathbb{C}, e^{-|z|^m})$ *to* $L^2(\mathbb{C}, e^{-|z|^m})$.

Theorem 2.10. *The canonical solution operator to* $\overline{\partial}$ *restricted to the space* $A^2(\mathbb{C}, e^{-|z|^m})$ *is compact if and only if*

$$\lim_{k\to\infty} \left(\frac{c_{k+1}^2}{c_k^2} - \frac{c_k^2}{c_{k-1}^2} \right) = 0.$$

Proof. For a complex polynomial p the canonical solution operator S_m can be written in the form

$$S_m(p)(z) = \int_{\mathbb{C}} K_m(z,w)p(w)(\overline{z} - \overline{w})e^{-|w|^m}\, d\lambda(w),$$

therefore we can express the conjugate S_m^* in the form

$$S_m^*(q)(w) = \int_{\mathbb{C}} K_m(w,z)q(z)(z-w)e^{-|z|^m}\, d\lambda(z),$$

if q is a finite linear combination of the terms $\overline{z}^k\, z^l$. This follows by considering the inner product $(S_m(p), q)_m = (p, S_m^*(q))_m$.
Now we claim that

$$S_m^* S_m(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) u_n(w) \quad , \ n = 1, 2, \dots$$

and

$$S_m^* S_m(u_0)(w) = \frac{c_1^2}{c_0^2}\, u_0(w),$$

where $\{u_n(z) = z^n/c_n, n = 0, 1, \dots\}$ is the standard orthonormal basis of $A^2(\mathbb{C}, e^{-|z|^m})$.
In a similar way as before for the case of $A^2(\mathbb{D})$ we see that

$$S_m(u_n)(z) = \overline{z}u_n(z) - \frac{c_n z^{n-1}}{c_{n-1}^2}, \ n = 1, 2, \dots.$$

Hence

$$\begin{aligned} S_m^* S_m(u_n)(w) &= \int_{\mathbb{C}} K_m(w,z)(z-w) \left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) e^{-|z|^m}\, d\lambda(z) \\ &= \int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} (z-w) \left(\frac{\overline{z}z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) e^{-|z|^m}\, d\lambda(z). \end{aligned}$$

As before we get

$$\int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z}z^{n+1}}{c_n} - \frac{c_n z^n}{c_{n-1}^2} \right) e^{-|z|^m}\, d\lambda(z) = \left(\frac{c_{n+1}^2}{c_n^3} - \frac{c_n}{c_{n-1}^2} \right)\, w^n$$

and

$$w \int_{\mathbb{C}} \sum_{k=0}^{\infty} \frac{w^k \overline{z}^k}{c_k^2} \left(\frac{\overline{z} z^n}{c_n} - \frac{c_n z^{n-1}}{c_{n-1}^2} \right) e^{-|z|^m} \, d\lambda(z) = w \left(\frac{c_n w^{n-1}}{c_{n-1}^2} - \frac{c_n w^{n-1}}{c_{n-1}^2} \right) = 0,$$

which implies that

$$S_m^* S_m(u_n)(w) = \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) u_n(w) \quad , n = 1, 2, \ldots ,$$

the case $n = 0$ follows from an analogous computation.
The last statement says that $S_m^* S_m$ is a diagonal operator with respect to the orthonormal basis $\{u_n(z) = z^n / c_n\}$ of $A^2(\mathbb{C}, e^{-|z|^m})$. Therefore it is easily seen that $S_m^* S_m$ is compact if and only if

$$\lim_{n \to \infty} \left(\frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2} \right) = 0.$$

□

Theorem 2.11. *The canonical solution operator for $\overline{\partial}$ restricted to the space $A^2(\mathbb{C}, e^{-|z|^m})$ is compact, if $m > 2$. The canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is not compact.*

Proof. The first statement follows immediately from Theorem 2.10 and Lemma 2.5 For the second statement we use (2.9) to show that the canonical solution operator is continuous as operator from $A^2(\mathbb{C}, e^{-|z|^2})$ to $L^2(\mathbb{C}, e^{-|z|^2})$.
By Hörmander's L^2-estimate for the solution of the $\overline{\partial}$ equation [30] there is for each $g \in L^2(\mathbb{C}, e^{-|z|^2})$ a function $f \in L^2(\mathbb{C}, e^{-|z|^2})$ such that $\overline{\partial} f = g$ and

$$\int_{\mathbb{C}} |f(z)|^2 \, e^{-|z|^2} \, d\lambda(z) \leq 2 \int_{\mathbb{C}} |g(z)|^2 \, e^{-|z|^2} \, d\lambda(z).$$

(see section 7. Theorem 7.5)
Hence the canonical solution operator for $\overline{\partial}$ as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself is continuous and its restriction to the closed subspace $A^2(\mathbb{C}, e^{-|z|^2})$ fails to be compact by Propositon 2.6 and Lemma 2.5. By the definition of compactness this implies that the canonical solution operator is not compact as operator from $L^2(\mathbb{C}, e^{-|z|^2})$ into itself. □

Remark 2.12. *In the case of the Fock space $A^2(\mathbb{C}, e^{-|z|^2})$ the composition $S_2^* S_2$ equals to the identity on $A^2(\mathbb{C}, e^{-|z|^2})$, which follows from the proof of Theorem 2.10.*

Theorem 2.13. *Let $m \geq 2$. The canonical solution operator for $\overline{\partial}$ restricted to $A^2(\mathbb{C}, e^{-|z|^m})$ fails to be Hilbert Schmidt.*

Proof. By Proposition 2.8 we know that the canonical solution operator is continuous and we can use the techniques from before to get

$$\begin{aligned}
\|S_m(u_n)\|_m^2 &= \frac{1}{c_n^2}\int_{\mathbb{C}} \left| \overline{z}\, z^n - \frac{c_n^2}{c_{n-1}^2}\, z^{n-1}\right|^2 e^{-|z|^m}\, d\lambda(z) \\
&= \frac{1}{c_n^2}\int_{\mathbb{C}} |z|^{2n-2}\left(|z|^4 - \frac{2c_n^2|z|^2}{c_{n-1}^2} + \frac{c_n^4}{c_{n-1}^4}\right) e^{-|z|^m}\, d\lambda(z) \\
&= \frac{1}{c_n^2}\int_{\mathbb{C}} |z|^{2n+2}\, e^{-|z|^m}\, d\lambda(z) - \frac{2}{c_{n-1}^2}\int_{\mathbb{C}} |z|^{2n}\, e^{-|z|^m}\, d\lambda(z) \\
&+ \frac{c_n^2}{c_{n-1}^4}\int_{\mathbb{C}} |z|^{2n-2}\, e^{-|z|^m}\, d\lambda(z) \\
&= \frac{c_{n+1}^2}{c_n^2} - \frac{c_n^2}{c_{n-1}^2}.
\end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} \|S_m(u_n)\|_m^2 < \infty$$

if and only if

$$\lim_{n\to\infty} \frac{c_{n+1}^2}{c_n^2} < \infty.$$

By [41] , 16.8, S_m is a Hilbert Schmidt operator if and only if

$$\sum_{n=0}^{\infty} \|S_m(u_n)\|_m^2 < \infty.$$

(see Appendix A.)
In our case we have

$$\frac{c_{n+1}^2}{c_n^2} = \Gamma\left(\frac{2n+4}{m}\right) / \Gamma\left(\frac{2n+2}{m}\right),$$

which, by Stirling's formula, implies that the corresponding canonical solution operator to $\overline{\partial}$ fails to be Hilbert Schmidt. □

In the case of several variables the corresponding operator S^*S is more complicated, nevertheless, using a suitable orthogonal decomposition, we can generalize the above results, see next section.

3. Spectral properties of the canonical solution operator to $\bar{\partial}$

In this chapter we concentrate on several complex variables and follow [27] to generalize the results of chapter 1 and 2.
For this purpose we introduce the notion of complex differential forms. Let $\Omega \subseteq \mathbb{C}^n$ be an open subset and $f : \Omega \longrightarrow \mathbb{C}$ be a $\mathcal{C}^1$-function. We write $z_j = x_j + iy_j$ and consider for $P \in \Omega$ the differential

$$df_P = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}(P)\, dx_j + \frac{\partial f}{\partial y_j}(P)\, dy_j \right).$$

We use the complex differentials

$$dz_j = dx_j + idy_j \quad , \quad d\overline{z}_j = dx_j - idy_j$$

and the derivatives

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\,\frac{\partial}{\partial y_j}\right) \quad , \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\,\frac{\partial}{\partial y_j}\right)$$

and rewrite the differential df_p in the form

$$df_P = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial z_j}(P)\, dz_j + \frac{\partial f}{\partial \overline{z}_j}(P)\, d\overline{z}_j \right) = \partial f_P + \overline{\partial} f_P.$$

A general differential form is given by

$$\omega = \sum_{|J|=p,|K|=q}{}' \, a_{J,K}\, dz_J \wedge d\overline{z}_K,$$

where the sum is taken only over increasing multiindices $J = (j_1, \dots, j_p),\ K = (k_1, \dots, k_q)$ and

$$dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_p} \quad , \quad d\overline{z}_K = d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}.$$

The derivative $d\omega$ of ω is defined by

$$d\omega = \sum_{|J|=p,|K|=q}{}' \, da_{J,K} \wedge dz_J \wedge d\overline{z}_K = \sum_{|J|=p,|K|=q}{}' \, (\partial a_{J,K} + \overline{\partial} a_{J,K}) \wedge dz_J \wedge d\overline{z}_K,$$

and we set

$$\partial\omega = \sum_{|J|=p,|K|=q}{}' \, \partial a_{J,K} \wedge dz_J \wedge d\overline{z}_K \text{ and } \overline{\partial}\omega = \sum_{|J|=p,|K|=q}{}' \, \overline{\partial} a_{J,K} \wedge dz_J \wedge d\overline{z}_K.$$

We have $d = \partial + \overline{\partial}$ and since $d^2 = 0$ it follows that

$$0 = (\partial + \overline{\partial}) \circ (\partial + \overline{\partial})\omega = (\partial \circ \partial)\omega + (\partial \circ \overline{\partial} + \overline{\partial} \circ \partial)\omega + (\overline{\partial} \circ \overline{\partial})\omega,$$

which implies $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$, by comparing the types of the differential forms involved.

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $A^2_{(0,1)}(\Omega)$ denote the space of all $(0,1)$-forms with holomorphic coefficients belonging to $L^2(\Omega)$. With the same proof as in section 2 one shows that the canonical solution operator $S : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$ has the form

$$S(g)(z) = \int_\Omega K(z,w) < g(w), z - w > \, d\lambda(w), \tag{3.1}$$

where K denotes the Bergman kernel of Ω and

$$< g(w), z - w >= \sum_{j=1}^{n} g_j(w)(\overline{z}_j - \overline{w}_j),$$

for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$.

Let $v(z) = \sum_{j=1}^{n} \overline{z}_j g_j(z)$. Then it follows that

$$\overline{\partial} v = \sum_{j=1}^{n} \frac{\partial v}{\partial \overline{z}_j} d\overline{z}_j = \sum_{j=1}^{n} g_j d\overline{z}_j = g.$$

Hence the canonical solution operator S_1 can be written in the form $S_1(g) = v - P(v)$, where $P : L^2(\Omega) \longrightarrow A^2(\Omega)$ is the Bergman projection. If $\tilde{v}$ is another solution to $\overline{\partial} u = g$, then $v - \tilde{v} \in A^2(\Omega)$ hence $v = \tilde{v} + h$, where $h \in A^2(\Omega)$. Therefore

$$v - P(v) = \tilde{v} + h - P(\tilde{v}) - P(h) = \tilde{v} - P(\tilde{v}).$$

Since $g_j \in A^2(\Omega),\ j = 1, \dots, n$, we have

$$g_j(z) = \int_\Omega K(z, w) g_j(w)\, d\lambda(w).$$

Now we get

$$\begin{aligned} S(g)(z) &= \sum_{j=1}^{n} \overline{z}_j g_j(z) - \int_\Omega K(z, w) \left(\sum_{j=1}^{n} \overline{w}_j g_j(w) \right) d\lambda(w) \\ &= \int_\Omega \left[\left(\sum_{j=1}^{n} \overline{z}_j g_j(w) \right) K(z, w) - \left(\sum_{j=1}^{n} \overline{w}_j g_j(w) \right) K(z, w) \right] d\lambda(w) \\ &= \int_\Omega K(z, w) < g(w), z - w > \ d\lambda(w). \end{aligned}$$

Remark 3.1. *It is pointed out that a $(0,1)$-form $g = \sum_{j=1}^{n} g_j\, d\overline{z}_j$ with holomorphic coefficients is not invariant under the pull back by a holomorphic map $F = (F_1, \dots, F_n) : \Omega_1 \longrightarrow \Omega$. Then*

$$F^* g = \sum_{l=1}^{n} g_l\, d\overline{F}_l = \sum_{j=1}^{n} \left(\sum_{l=1}^{n} g_l \frac{\partial \overline{F}_l}{\partial \overline{z}_j} \right) d\overline{z}_j,$$

where we used the fact that

$$d\overline{F}_l = \partial \overline{F}_l + \overline{\partial}\, \overline{F}_l = \sum_{j=1}^{n} \frac{\partial \overline{F}_l}{\partial z_j}\, dz_j + \sum_{j=1}^{n} \frac{\partial \overline{F}_l}{\partial \overline{z}_j}\, d\overline{z}_j = \sum_{j=1}^{n} \frac{\partial \overline{F}_l}{\partial \overline{z}_j}\, d\overline{z}_j.$$

The expressions $\frac{\partial \overline{F}_l}{\partial \overline{z}_j}$ are not holomorphic.

Nevertheless it is true that $\overline{\partial} u = g$ implies $\overline{\partial}(u \circ F) = F^* g$, which follows from the fact that for a general differential form ω and a holomorphic map F we have

$$\partial(F^* \omega) = F^*(\partial \omega) \text{ and } \overline{\partial}(F^* \omega) = F^*(\overline{\partial} \omega).$$

Now let ω be a holomorphic (n,n)-form, i.e.

$$\omega = \tilde{\omega}\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n,$$

where $\tilde{\omega} \in A^2(\Omega)$. In this case we can express the canonical solution to $\overline{\partial} u = \omega$ in the following form

Proposition 3.2. *Let u be the $(n, n-1)$-form*

$$u = \sum_{j=1}^{n} u_j\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n,$$

where

$$u_j(z) = \frac{(-1)^{n+j-1}}{n} \int_\Omega (\overline{z}_j - \overline{w}_j) K(z,w) \tilde{\omega}(w)\, d\lambda(w).$$

Then $u_j \perp A^2(\Omega)$, $j = 1, \ldots, n$ and $\overline{\partial} u = \omega$.

Proof. It follows that

$$u_j(z) = \frac{(-1)^{n+j-1}}{n} \left(\overline{z}_j \tilde{\omega}(z) - P(\overline{w}_j \tilde{\omega})(z) \right),$$

from this we obtain

$$\frac{\partial u_j}{\partial \overline{z}_k} = \frac{(-1)^{n+j-1}}{n} \left(\frac{\partial \overline{z}_j}{\partial \overline{z}_k} \tilde{\omega} + \overline{z}_j \frac{\partial \tilde{\omega}}{\partial \overline{z}_k} \right) = \frac{(-1)^{n+j-1}}{n}\ \delta_{jk}\ \tilde{\omega},$$

where δ_{jk} is the Kronecker delta symbol. Hence

$$\begin{aligned}
\overline{\partial} u &= \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial u_j}{\partial \overline{z}_k}\ d\overline{z}_k \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \left((-1)^{n+j-1}/n \right)\ \delta_{jk}\ \tilde{\omega}\ d\overline{z}_k \wedge \\
&\quad \wedge dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge [d\overline{z}_j] \wedge \cdots \wedge d\overline{z}_n \\
&= \tilde{\omega}\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n.
\end{aligned}$$

□

Remark 3.3. *The pull back by a holomorphic map F has in this case the form*

$$F^* \omega = \left| \det \frac{\partial F_j}{\partial z_k} \right|^2\ \tilde{\omega}\ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n.$$

For further related results see section 10.

Now we will study boundedness, compactness, and Schatten-class membership of the canonical solution operator to $\overline{\partial}$, restricted to $(0,1)$-forms with holomorphic coefficients, on $L^2(d\mu)$ where μ is a measure with the property that the monomials form an orthogonal family in $L^2(d\mu)$. The characterizations are formulated in terms of moment properties of μ.
This situation covers a number of basic examples:

- Lebesgue measure on bounded domains in $\mathbb{C}^n$ which are invariant under the torus action
$$(\theta_1, \ldots, \theta_n)(z_1, \ldots, z_n) \mapsto (e^{i\theta_1} z_1, \ldots e^{i\theta_n} z_n)$$
(i.e. Reinhardt domains).
- Weighted L^2 spaces with radially symmetric weights (e.g., generalized Fock spaces).
- Weighted L^2 spaces with decoupled radial weights, that is,
$$d\mu = e^{\sum_j \varphi_j(|z_j|^2)} d\lambda,$$
where $\varphi_j \colon \mathbb{R} \to \mathbb{R}$ is a weight function.

We denote by
$$A^2(d\mu) = \overline{\{z^\alpha \colon \alpha \in \mathbb{N}^n\}},$$
the closure of the monomials in $L^2(d\mu)$, and write
$$m_\alpha = c_\alpha^{-1} = \int |z^\alpha|^2 d\mu.$$
We will give necessary and sufficient conditions in terms of these multimoments of the measure μ for the canonical solution operator to $\bar{\partial}$, when restricted to $(0,1)$-forms with coefficients in $A^2(d\mu)$ to be bounded, compact, and to belong to the Schatten class $\mathbf{S}_p$. This is accomplished by presenting a complete diagonalization of the solution operator by orthonormal bases with corresponding estimates.
As usual, for a given function space $\mathcal{F}$, $\mathcal{F}_{(0,1)}$ denotes the space of $(0,1)$-forms with coefficients in $\mathcal{F}$, that is, expressions of the form
$$\sum_{j=0}^{n} f_j d\bar{z}_j, \quad f_j \in \mathcal{F}.$$
The $\bar{\partial}$ operator is the densely defined operator
$$\bar{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j. \tag{3.2}$$
The *canonical solution operator* S assigns to each $\omega \in L^2_{(0,1)}(d\mu)$ the solution to the $\bar{\partial}$ equation which is orthogonal to $A^2(d\mu)$; this solution need not exist, but if the $\bar{\partial}$ equation for ω can be solved, then $S\omega$ is defined, and is given by the unique $f \in L^2(d\mu)$ which satisfies
$$\bar{\partial} f = \omega \text{ in the sense of distributions and } f \perp A^2(d\mu).$$
We will frequently encounter multiindices γ which might have one (but not more than one) entry equal to -1: in that case, we define $c_\gamma = 0$. We will denote the set of these multiindices by Γ. We let $e_j = (0, \cdots, 1, \cdots, 0)$ be the multiindex with a 1 in the jth spot and 0 elsewhere.

Theorem 3.4. *$S \colon A^2_{(0,1)}(d\mu) \to L^2(d\mu)$ is bounded if and only if there exists a constant C such that*
$$\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} < C$$
for all multiindices $\gamma \in \Gamma$ and for all $j = 1, \ldots, n$.

We have a similar criterion for compactness:

Theorem 3.5. *$S\colon A^2_{(0,1)}(d\mu) \to L^2(d\mu)$ is compact if and only if*

$$\lim_{\gamma} \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right) = 0 \tag{3.3}$$

for all $j = 1, \cdots, n$.

In particular, the only if implication of Theorem 3.5 implies several known noncompactness statements for S, e.g. [34], [44], as well as the noncompactness of S on the polydisc. The main interest in these noncompactness statements is that if S fails to be compact, so does the $\bar{\partial}$-Neumann operator N.

The multimoments also lend themselves to characterizing the finer spectral property of being in the Schatten class $\mathbf{S}_p$. Let us recall that an operator $T : H_1 \to H_2$ belongs to the Schatten-class $\mathbf{S}_p$ if the self-adjoint operator T^*T has a sequence of eigenvalues belonging to ℓ^p.

Theorem 3.6. *Let $p > 0$. Then $S\colon A^2_{(0,1)}(d\mu) \to L^2(d\mu)$ is in the Schatten-p-class $\mathbf{S}_p$ if and only if*

$$\sum_{\gamma\in\Gamma} \left(\sum_j \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right) \right)^{\frac{p}{2}} < \infty \tag{3.4}$$

The condition above is substantially easier to check if $p = 2$ (we will show that the sum is actually a telescoping sum then), i.e. for the case of the Hilbert-Schmidt class; we state this as a Theorem:

Theorem 3.7. *The canonical solution operator S is in the Hilbert-Schmidt class if and only if*

$$\lim_{k\to\infty} \sum_{\substack{\gamma\in\mathbb{N}^n, |\gamma|=k \\ 1\le j\le n}} \frac{c_\gamma}{c_{\gamma+e_j}} < \infty. \tag{3.5}$$

Let us apply Theorem 3.4 to the case of decoupled weights, or more generally, of product measures $d\mu = d\mu_1 \times \cdots \times d\mu_n$, where each $d\mu_k$ is a (circle-invariant) measure on $\mathbb{C}$. Note that for such measures, there is *definitely* no compactness by Theorem 3.5. If we denote by

$$c_j^k = \left(\int_{\mathbb{C}} |z|^{2k} d\mu_j \right)^{-1},$$

we have that

$$c_{(\gamma_1,\cdots,\gamma_n)} = \prod_{j=1}^n c_j^{\gamma_j}.$$

We thus obtain the following corollary.

Corollary 3.8. *For a product measure $d\mu = d\mu_1 \times \cdots \times d\mu_n$ as above, the canonical solution operator $S\colon A^2_{(0,1)}(d\mu) \to L^2(d\mu)$ is bounded if and only if there exists a constant C such that*

$$\frac{c_j^{k+1}}{c_j^{k+2}} - \frac{c_j^k}{c_j^{k+1}} < C$$

for all $k \in \mathbb{N}_0$ and for all $j = 1, \cdots, n$. Equivalently, S is bounded if and only if the canonical solution operator $S_j\colon A^2(d\mu_j) \to L^2(d\mu_j)$ is bounded for every $j = 1, \cdots, n$.

To see that (3.3) is not satisfied for product measures consider multiindices $\gamma \in \Gamma$ such that $\gamma_j = -1$: then $c_\gamma = 0$ (by definition) and $c_{\gamma+e_j} \neq 0$, and therefore

$$\left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right) = \frac{c_j^0}{c_j^1} - 0 > \delta > 0,$$

for all multiindices γ with $\gamma_j = -1$.

In the case of a rotation-invariant measure μ, we write

$$m_d = \int_{\mathbb{C}^n} |z|^{2d} d\mu;$$

a computation (see Appendix F and [37, Lemma 2.1]) implies that

$$c_\gamma = \frac{(n + |\gamma| - 1)!}{(n-1)!\gamma!} \frac{1}{m_{|\gamma|}}, \tag{3.6}$$

where $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $\gamma! = \gamma_1! \ldots \gamma_n!$.
In order to express the conditions of our Theorems, we compute (setting $d = |\gamma| + 1$)

$$\sum_j \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right) = \begin{cases} \frac{d+2n-1}{d+n} \frac{m_{d+1}}{m_d} - \frac{m_d}{m_{d-1}} & \gamma_j \neq -1 \text{ for all } j \\ \frac{1}{d+n} \frac{m_{d+1}}{m_d} & \text{else.} \end{cases} \tag{3.7}$$

Note that the Cauchy-Schwarz inequality implies that the first case in (3.7) always dominates the second case for $n \geq 2$; for $n = 1$ we observe that the second case in (3.7) reduces to $\frac{m_1}{m_0}$, compare with Proposition 2.4.
Using this observation and some trivial inequalities, we get the following Corollaries.

Corollary 3.9. *Let μ be a rotation invariant measure on $\mathbb{C}^n$. Then the canonical solution operator to $\bar{\partial}$ is bounded on $A^2_{(0,1)}(d\mu)$ if and only if*

$$\sup_{d \in \mathbb{N}} \left(\frac{(2n + d - 1)m_{d+1}}{(n+d)m_d} - \frac{m_d}{m_{d-1}} \right) < \infty \tag{3.8}$$

Corollary 3.10. *Let μ be a rotation invariant measure on $\mathbb{C}^n$. Then the canonical solution operator to $\bar{\partial}$ is compact on $A^2_{(0,1)}(d\mu)$ if and only if*

$$\lim_{d \to \infty} \left(\frac{(2n + d - 1)m_{d+1}}{(n+d)m_d} - \frac{m_d}{m_{d-1}} \right) = 0. \tag{3.9}$$

Corollary 3.11. *Let μ be a rotation invariant measure on $\mathbb{C}^n$. Then the canonical solution operator to $\bar{\partial}$ is a Hilbert-Schmidt operator on $A^2_{(0,1)}(d\mu)$ if and only if*

$$\lim_{d \to \infty} \binom{n+d-2}{n-1} \frac{m_{d+1}}{m_d} < \infty. \tag{3.10}$$

Remark 3.12. *It follows that the canonical solution operator to $\bar{\partial}$ is a Hilbert-Schmidt operator on $A^2(\mathbb{D})$, but fails to be Hilbert-Schmidt on $A^2(\mathbb{B}^n)$, where $\mathbb{B}^n$ is the unit ball in $\mathbb{C}^n$, for $n \geq 2$.*

Corollary 3.13. *Let μ be a rotation invariant measure on $\mathbb{C}^n$, $p > 0$. Then the canonical solution operator to $\bar{\partial}$ is in the Schatten-class $\mathbf{S}_p$, as an operator from $A^2_{(0,1)}(d\mu)$ to $L^2(d\mu)$ if and only if*

$$\sum_{d=1}^{\infty} \binom{n+d-2}{n-1} \left(\frac{(2n + d - 1)m_{d+1}}{(n+d)m_d} - \frac{m_d}{m_{d-1}} \right)^{\frac{p}{2}} < \infty. \tag{3.11}$$

In particular, Corollary 3.13 improves Theorem C of [37] in the sense that it also covers the case $0 < p < 2$. We would like to note that our techniques can be adapted to the setting of [37] by considering the canonical solution operator on a Hilbert space $\mathcal{H}$ of holomorphic functions endowed with a norm which is comparable to the L^2-norm on each subspace generated by monomials of a fixed degree d, if in addition to the requirements in [37] we also assume that the monomials belong to $\mathcal{H}$; this introduces the additional weights found by [37] in the formulas, as the reader can check. In our setting, the formulas are somewhat "cleaner" by working with $A^2(d\mu)$ (in particular, Corollary 3.11 only holds in this setting).

In what follows, we will denote by

$$u_\alpha = \sqrt{c_\alpha} z^\alpha$$

the orthonormal basis of monomials for the space $A^2(d\mu)$, and by $U_{\alpha,j} = u_\alpha d\bar{z}_j$ the corresponding basis of $A^2_{(0,1)}(d\mu)$. We first note that it is always possible to solve the $\bar\partial$-equation for the elements of this basis; indeed, $\bar\partial \bar{z}_j u_\alpha = U_{\alpha,j}$. The canonical solution operator is also easily determined for forms with monomial coefficients:

Lemma 3.14. *The canonical solution $Sz^\alpha d\bar{z}_j$ for monomial forms is given by*

$$Sz^\alpha d\bar{z}_j = \bar{z}_j z^\alpha - \frac{c_{\alpha-e_j}}{c_\alpha} z^{\alpha-e_j}, \quad \alpha \in \mathbb{N}_0^n. \tag{3.12}$$

Proof. We have $\langle \bar{z}_j z^\alpha, z^\beta\rangle = \langle z^\alpha, z^{\beta+e_j}\rangle$; so this expression is nonzero only if $\beta = \alpha - e_j$ (in particular, if this implies (3.12) for multiindices α with $\alpha_j = 0$; recall our convention that $c_\gamma = 0$ if one of the entries of γ is negative). Thus $Sz^\alpha d\bar{z}_j = \bar{z}_j z^\alpha + cz^{\alpha-e_j}$, and c is computed by

$$0 = \langle \bar{z}_j z^\alpha + cz^{\alpha-e_j}, z^{\alpha-e_j}\rangle = c_\alpha^{-1} + cc_{\alpha-e_j}^{-1},$$

which gives $c = -c_{\alpha-e_j}/c_\alpha$. □

We are going to introduce an orthogonal decomposition

$$A^2_{(0,1)}(d\mu) = \bigoplus_{\gamma\in\Gamma} E_\gamma$$

of $A^2_{(0,1)}(d\mu)$ into at most n-dimensional subspaces E_γ indexed by multiindices $\gamma \in \Gamma$ (we will describe the index set below), and a corresponding sequence of mutually orthogonal finite-dimensional subspaces $F_\gamma \subset L^2(d\mu)$ which diagonalizes S (by this we mean that $SE_\gamma = F_\gamma$). To motivate the definition of E_γ, note that

$$\langle Sz^\alpha d\bar{z}_k, Sz^\beta d\bar{z}_\ell\rangle = \begin{cases} 0 & \beta \neq \alpha + e_\ell - e_k, \\ \frac{1}{c_\alpha}\left(\frac{c_\alpha}{c_{\alpha+e_\ell}} - \frac{c_{\alpha-e_k}}{c_{\alpha+e_\ell-e_k}}\right) & \beta = \alpha + e_\ell - e_k, \end{cases} \tag{3.13}$$

so that $\langle Sz^\alpha d\bar{z}_k, Sz^\beta d\bar{z}_\ell\rangle \neq 0$ if and only if there exists a multiindex γ such that $\alpha = \gamma + e_k$ and $\beta = \gamma + e_\ell$. We thus define

$$E_\gamma = \operatorname{span}\left\{U_{\gamma+e_j,j} \colon 1 \le j \le n\right\} = \operatorname{span}\left\{z^{\gamma+e_j} d\bar{z}_j \colon 1 \le j \le n\right\},$$

and likewise $F_\gamma = SE_\gamma$. Recall that Γ is defined to be the set of all multiindices whose entries are greater or equal to -1 and at most one negative entry. Note that E_γ is 1-dimensional if exactly one entry in γ equals -1, and n-dimensional otherwise. We have already observed that F_γ are mutually orthogonal subspaces of $L^2(d\mu)$ (see 3.13).

Whenever we use multiindices γ and integers $p \in \{1, \cdots, n\}$ as indices, we use the convention that the p run over all p such that $\gamma + e_p \geq 0$; that is, for a fixed multiindex $\gamma \in \Gamma$, either the indices are either all $p \in \{1, \cdots, n\}$ or there is exactly one p such that $\gamma_p = -1$, in which case the index is exactly this one p.

We next observe that we can find an orthonormal basis of E_γ and an orthonormal basis of F_γ such that in these bases $S_\gamma = S|_{E_\gamma} \colon E_\gamma \to F_\gamma$ acts diagonally. First note that it is enough to do this if $\dim E_\gamma = n$ (since an operator between one-dimensional spaces is automatically diagonal). Fixing γ, the functions $U_j := U_{\gamma+e_j,j}$ are an orthonormal basis of E_γ. The operator S_γ is clearly nonsingular on this space, so the functions $SU_j = \Psi_j$ constitute a basis of F_γ. For a basis B of vectors $v^j = \left(v_1^j, \ldots, v_n^j\right)$, $j = 1, \ldots, n$ of $\mathbb{C}^n$ we consider the new basis

$$V_k = \sum_{j=1}^n v_k^j U_j;$$

since the basis given by the U_j is orthonormal, the basis given by the V_k is also orthonormal provided that the vectors $v_k = (v_k^1, \cdots, v_k^n)$ constitute an orthonormal basis for $\mathbb{C}^n$ with the standard hermitian product. Let us write

$$\Phi_k = SV_k = \sum_j v_k^j SU_j.$$

The inner product $\langle \Phi_p, \Phi_q \rangle$ is then given by $\sum_{j,k} v_p^j \bar{v}_q^k \langle SU_j, SU_k \rangle$. We therefore have

$$\begin{pmatrix} \langle \Phi_1, \Phi_1 \rangle & \cdots & \langle \Phi_1, \Phi_n \rangle \\ \vdots & & \vdots \\ \langle \Phi_n, \Phi_1 \rangle & \cdots & \langle \Phi_n, \Phi_n \rangle \end{pmatrix} = \begin{pmatrix} v_1^1 & \cdots & v_1^n \\ \vdots & & \vdots \\ v_n^1 & \cdots & v_n^n \end{pmatrix} \begin{pmatrix} \langle \Psi_1, \Psi_1 \rangle & \cdots & \langle \Psi_1, \Psi_n \rangle \\ \vdots & & \vdots \\ \langle \Psi_n, \Psi_1 \rangle & \cdots & \langle \Psi_n, \Psi_n \rangle \end{pmatrix} \begin{pmatrix} \bar{v}_1^1 & \cdots & \bar{v}_n^1 \\ \vdots & & \vdots \\ \bar{v}_1^n & & \bar{v}_n^n \end{pmatrix}. \tag{3.14}$$

Since the matrix $(\langle \Psi_j, \Psi_k \rangle)_{j,k}$ is hermitian, we can unitarily diagonalize it; that is, we can choose an orthnormal basis B of $\mathbb{C}^n$ such that with this choice of B the vectors $\varphi_{\gamma,k} = V_k = \sum_j v_k^j U_{\gamma+e_j,j}$ of E_γ are orthonormal, and their images $\Phi_k = SV_k$ are orthogonal in F_γ. Therefore, $\Phi_k / \|\Phi_k\|$ is an orthonormal basis of F_γ such that $S_\gamma : E_\gamma \to F_\gamma$ is diagonal when expressed in terms of the bases $\{V_1, \cdots, V_n\} \subset E_\gamma$ and $\{\Phi_1, \cdots, \Phi_n\} \subset F_\gamma$, with entries $\|\Phi_k\|$.

Furthermore, the $\|\Phi_k\|$ are exactly the square roots of the eigenvalues of the matrix $(\langle \Psi_p, \Psi_q \rangle)$ which by (3.13) is given by

$$\begin{aligned} \langle \Psi_p, \Psi_q \rangle &= \langle SU_{\gamma+e_p,p}, SU_{\gamma+e_q,q} \rangle \\ &= \sqrt{c_{\gamma+e_p}} \sqrt{c_{\gamma+e_q}} \langle S\, z^{\gamma+e_p}\, d\bar{z}_p, S\, z^{\gamma+e_q}\, d\bar{z}_q \rangle \\ &= \sqrt{c_{\gamma+e_p} c_{\gamma+e_q}} \frac{1}{c_{\gamma+e_p}} \left(\frac{c_{\gamma+e_p}}{c_{\gamma+e_p+e_q}} - \frac{c_\gamma}{c_{\gamma+e_q}} \right) \\ &= \frac{c_{\gamma+e_p} c_{\gamma+e_q} - c_\gamma c_{\gamma+e_p+e_q}}{c_{\gamma+e_p+e_q} \sqrt{c_{\gamma+e_p} c_{\gamma+e_q}}} \end{aligned} \tag{3.15}$$

Summarizing, we have the following Proposition.

Proposition 3.15. *With μ as above, the canonical solution operator $S\colon A^2_{(0,1)}(d\mu) \to L^2_{(0,1)}(d\mu)$ admits a diagonalization by orthonormal bases. In fact, we have a decomposition $A^2_{(0,1)} = \bigoplus_\gamma E_\gamma$ into mutually orthogonal finite dimensional subspaces E_γ, indexed by the multiindices γ with at most one negative entry (equal to -1), which are of dimension 1 or n, and orthonormal bases $\varphi_{\gamma,j}$ of E_γ, such that $S\varphi_{\gamma,j}$ is a set of mutually orthogonal vectors in $L^2(d\mu)$. For fixed γ, the norms $\|S\varphi_{\gamma,j}\|$ are the square roots of the eigenvalues of the matrix $C_\gamma = (C_{\gamma,p,q})_{p,q}$ given by*

$$C_{\gamma,p,q} = \frac{c_{\gamma+e_p}c_{\gamma+e_q} - c_\gamma c_{\gamma+e_p+e_q}}{c_{\gamma+e_p+e_q}\sqrt{c_{\gamma+e_p}c_{\gamma+e_q}}}. \tag{3.16}$$

In particular, we have that

$$\sum_{j=1}^n \|S\varphi_{\gamma,j}\|^2 = tr(C_{\gamma,p,q})_{p,q} = \sum_{p=1}^n \left(\frac{c_{\gamma+e_p}}{c_{\gamma+2e_p}} - \frac{c_\gamma}{c_{\gamma+e_p}}\right) \tag{3.17}$$

In order to prove Theorem 3.4, we are using Proposition 3.15. We have seen that we have an orthonormal basis $\varphi_{\gamma,j}$, $\gamma \in \Gamma$, $j \in \{1, \cdots, n\}$, such that the images $S\varphi_{\gamma,j}$ are mutually orthogonal. Thus, S is bounded if and only if there exists a constant C such that

$$\|S\varphi_{\gamma,j}\|^2 \leq C$$

for all $\gamma \in \Gamma$ and $j \in \{1, \cdots, \dim E_\gamma\}$. If $\dim E_\gamma = 1$, then γ has exactly one entry (say the jth one) equal to -1; in that case, let us write $\varphi_\gamma = U_{\gamma+e_j} d\bar{z}_j$. We have $S\varphi_\gamma = \sqrt{c_{\gamma+e_j}}\bar{z}_j z^{\gamma+e_j}$, and so

$$\|S\varphi_\gamma\|^2 = \frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}}.$$

On the other hand, if $\dim E_\gamma = n$, we argue as follows: Writing $\|S\varphi_{\gamma,j}\|^2 = \lambda^2_{\gamma,j}$ with $\lambda_{\gamma,j} > 0$, from (3.17) we find that

$$\sum_{j=1}^n \lambda^2_{\gamma,j} = \sum_{j=1}^n \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}}\right).$$

The last 2 equations complete the proof of Theorem 3.4.

In order to prove Theorem 3.5, we use a special characterization of compactness, for the proof see Appendix A.

Lemma 3.16. *Let H_1 and H_2 be Hilbert spaces, and assume that $S\colon H_1 \to H_2$ is a bounded linear operator. The following three statements are equivalent:*

- *S is compact.*
- *For every $\varepsilon > 0$ there is a $C = C_\varepsilon > 0$ and a compact operator $T = T_\varepsilon\colon H_1 \to H_2$ such that*

$$\|Sv\|_{H_2} \leq C\|Tv\|_{H_2} + \varepsilon\|v\|_{H_1}. \tag{3.18}$$

- *For every $\varepsilon > 0$ there is a $C = C_\varepsilon > 0$ and a compact operator $T = T_\varepsilon\colon H_1 \to H_2$ such that*

$$\|Sv\|^2_{H_2} \leq C\|Tv\|^2_{H_2} + \varepsilon\|v\|^2_{H_1}. \tag{3.19}$$

Proof of Theorem 3.5. We first show that (3.3) implies compactness. We will use the notation which was already used in the proof of Theorem 3.4; that is, we write $\|S\varphi_{\gamma,j}\|^2 = \lambda_{\gamma,j}^2$. Let $\varepsilon > 0$. There exists a finite set A_ε of multiindices $\gamma \in \Gamma$ such that for all $\gamma \notin A_\varepsilon$,

$$\sum_{j=1}^{n} \lambda_{\gamma,j}^2 = \sum_{j=1}^{n} \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right) < \varepsilon.$$

Hence, if we consider the finite dimensional (and thus, compact) operator T_ε defined by

$$T_\varepsilon \sum a_{\gamma,j}\varphi_{\gamma,j} = \sum_{\gamma\in A_\varepsilon} a_{\gamma,j} S\varphi_{\gamma,j},$$

for any $v = \sum a_{\gamma,j}\varphi_{\gamma,j} \in A^2_{(0,1)}(d\mu)$ we obtain

$$\begin{aligned}
\|Sv\|^2 &= \|T_\varepsilon v\|^2 + \left\| S \sum_{\gamma\notin A_\varepsilon} a_{\gamma,j}\varphi_{\gamma,j} \right\|^2 \\
&= \|T_\varepsilon v\|^2 + \sum_{\gamma\notin A_\varepsilon} |a_{\gamma,j}|^2 \|S\varphi_{\gamma,j}\|^2 \\
&= \|T_\varepsilon v\|^2 + \sum_{\gamma\notin A_\varepsilon} |a_{\gamma,j}|^2 \lambda_{\gamma,j}^2 \\
&\le \|T_\varepsilon v\|^2 + \varepsilon \sum_{\gamma\notin A_\varepsilon} |a_{\gamma,j}|^2 \\
&\le \|T_\varepsilon v\|^2 + \varepsilon \|v\|^2 .
\end{aligned}$$

Hence, (3.19) holds and we have proved the first implication in Theorem 3.5.
We now turn to the other direction. Assume that (3.3) is not satisfied. Then there exists a $K > 0$ and an infinite family A of multiindices γ such that for all $\gamma \in A$,

$$\sum_{j=1}^{n} \lambda_{\gamma,j}^2 = \sum_{j=1}^{n} \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right) > nK.$$

In particular, for each $\gamma \in A$, there exists a j_γ such that $\lambda^2_{\gamma,j_\gamma} > K$. Thus, we have an infinite orthonormal family $\{\varphi_{\gamma,j_\gamma} : \gamma \in A\}$ of vectors such that their images $S\varphi_{\gamma,j_\gamma}$ are orthogonal and have norm bounded from below by $\sqrt{K}$, which contradicts compactness. $\square$

We keep the notation introduced in the previous sections. We will also need to introduce the usual grading on the index set Γ, that is, we write

$$\Gamma_k = \{\gamma \in \Gamma \colon |\gamma| = k\}, \quad k \ge -1. \tag{3.20}$$

In order to study the membership in the Schatten-class, we need the following elementary Lemma:

Lemma 3.17. *Assume that $p(x)$ and $q(x)$ are continuous, real-valued functions on $\mathbb{R}^N$ which are homogeneous of degree 1 (i.e. $p(tx) = tp(x)$ and $q(tx) = tq(x)$ for $t \in \mathbb{R}$), and $q(x) = 0$ as well as $p(x) = 0$ implies $x = 0$. Then there exists a constant C such that*

$$\frac{1}{C}|q(x)| \le |p(x)| \le C|q(x)|. \tag{3.21}$$

Proof. Note that the set $B_q = \{x\colon q(x) = 1\}$ is compact: it's closed since q is continuous, and since $|q|$ is bounded from below on S^N by some $m > 0$, it is necessarily contained in the closed ball of radius $1/m$. Now, the function $|p|$ is bounded on the compact set B_q; say, by $1/C$ from below and C from above. Thus for all $x \in \mathbb{R}^N$,

$$\frac{1}{C} \leq \left| p\left(\frac{x}{q(x)}\right)\right| \leq C,$$

which proves (3.21). $\square$

Proof of Theorem 3.6. Note that S is in the Schatten-class $\mathbf{S}_p$ if and only if

$$\sum_{\gamma\in\Gamma,\, j} \lambda_{\gamma,j}^p < \infty. \tag{3.22}$$

We rewrite this sum as

$$\sum_{\gamma\in\Gamma}\left(\sum_j \lambda_{\gamma,j}^p\right) =: M \in \mathbb{R}\cup\{\infty\}\,.$$

Lemma 3.17 implies that there exists a constant C such that for every $\gamma\in\Gamma$,

$$\frac{1}{C}\left(\sum_j \lambda_{\gamma,j}^2\right)^{p/2} \leq \sum_j \lambda_{\gamma,j}^p \leq C\left(\sum_j \lambda_{\gamma,j}^2\right)^{p/2}.$$

Hence, $M < \infty$ if and only if

$$\sum_\gamma\left(\sum_j \lambda_{\gamma,j}^2\right)^{p/2} < \infty,$$

which after applying (3.17) becomes the condition (3.4) claimed in Theorem 3.6. $\square$

Proof of Theorem 3.7. S is in the Hilbert-Schmidt class if and only if

$$\sum_{\gamma\in\Gamma, j} \lambda_{\gamma,j}^2 < \infty. \tag{3.23}$$

We will prove that

$$\sum_{\ell=-1}^{k}\sum_{\gamma\in\Gamma_\ell, j} \lambda_{\gamma,j}^2 = \sum_{\substack{\alpha\in\mathbb{N}^n, |\alpha|=k+1\\ 1\leq p\leq n}} \frac{c_\alpha}{c_{\alpha+e_p}}, \tag{3.24}$$

which immediately implies Theorem 3.7. The proof is by induction over k. For $k=-1$, the left hand side of (3.24) is

$$\sum_{j=1}^n \lambda_{-e_j,j}^2 = \sum_{j=1}^n \|z_j\|^2 c_0 = \sum_{j=1}^n \frac{c_0}{c_{e_p}},$$

which is equal to the right hand side. Now assume that the (3.24) holds for $k = K - 1$; we will show that this implies it holds for $k = K$. We write

$$\sum_{\ell=-1}^{K} \sum_{\gamma \in \Gamma_\ell, j} \lambda_{\gamma,j}^2 = \sum_{\substack{\alpha \in \mathbb{N}^n, |\alpha| = K-1 \\ 1 \leq p \leq n}} \frac{c_\alpha}{c_{\alpha+e_p}} + \sum_{\gamma \in \Gamma_K, j} \left(\frac{c_{\gamma+e_j}}{c_{\gamma+2e_j}} - \frac{c_\gamma}{c_{\gamma+e_j}} \right)$$

$$= \sum_{\substack{\alpha \in \mathbb{N}^n, |\alpha| = K \\ 1 \leq p \leq n}} \frac{c_\alpha}{c_{\alpha+e_p}}.$$

This finishes the proof of Theorem 3.7. □

4. The $\overline{\partial}$-complex

Our main task will be to solve the inhomogeneous Cauchy-Riemann equation $\overline{\partial}u = f$, where the right hand side f is given and satisfies the necessary condition $\overline{\partial}f = 0$. For $n > 1$ this is an overdetermined system of partial differential equations, which will be reduced to a system with equal numbers of unknowns and equations.
We demonstrate this method first in its finite dimensional analog: let E, F, G denote finite dimensional vector spaces over $\mathbb{C}$ with inner product. We consider an exact sequence of linear maps

$$E \xrightarrow{S} F \xrightarrow{T} G,$$

which means that $\text{Im}S = \text{Ker}T$, hence $TS = 0$.
Given $f \in \text{Im}S = \text{Ker}T$, we want to solve $Su = f$ with $u \perp \text{Ker}S$, then u will be called the canonical solution.
For this purpose we investigate

$$E \underset{S^*}{\overset{S}{\rightleftarrows}} F \underset{T^*}{\overset{T}{\rightleftarrows}} G$$

and observe that $\text{Ker}T = (\text{Im}T^*)^\perp$ and $\text{Ker}T^* = (\text{Im}T)^\perp$. We claim that the operator

$$SS^* + T^*T : F \longrightarrow F$$

is bijective. Let $(SS^* + T^*T)g = 0$, then $SS^*g = -T^*Tg$, which implies

$$SS^*g \in \text{Im}T^* \cap \text{Im}S = \text{Im}T^* \cap \text{Ker}T = \text{Im}T^* \cap (\text{Im}T^*)^\perp = \{0\},$$

hence $SS^*g = T^*Tg = 0$, but this gives $S^*g \in \text{Ker}S \cap \text{Im}S^* = \text{Ker}S \cap (\text{Ker}S)^\perp = \{0\}$, and $g \in \text{Ker}S^* = (\text{Im}S)^\perp$; from $T^*Tg = 0$ we get $Tg \in \text{Ker}T^* \cap \text{Im}T = (\text{Im}T)^\perp \cap \text{Im}T = \{0\}$ and $g \in \text{Ker}T = \text{Im}S$, therefore we obtain $g \in \text{Im}S \cap (\text{Im}S)^\perp = \{0\}$. So $SS^* + T^*T$ is injective and as F is finite dimensional $SS^* + T^*T$ is bijective.
Let $N = (SS^* + T^*T)^{-1}$. We claim that

$$u = S^*Nf$$

is the canonical solution to $Su = f$. So we have to show that $SS^*Nf = f$ and $S^*Nf \perp \text{Ker}S$. The latter easily follows from the fact that $S^*Nf \in \text{Im}S^* = (\text{Ker}S)^\perp$.
We have

$$f = SS^*Nf + T^*TNf,$$

therefore the assumption $Tf = 0$ implies

$$0 = Tf = TSS^*Nf + TT^*TNf = TT^*TNf,$$

since $TS = 0$. From here we obtain

$$0 = (TT^*TNf, TNf) = (T^*TNf, T^*TNf)$$

and $T^*TNf = 0$, hence $SS^*Nf = f$ and we are done.

In the following we will use this method for the $\overline{\partial}$-operator.

Let $\Omega = \{z \in \mathbb{C}^n \ : \ r(z) < 0\}$, where

$$\nabla_z r := (\frac{\partial r}{\partial z_1}, \dots, \frac{\partial r}{\partial z_n}) \neq 0$$

on $b\Omega = \{z \ : \ r(z) = 0\}$. Without loss of generality we can suppose that $|\nabla_z r| = |\nabla r| = 1$ on $b\Omega$. For $u, v \in \mathcal{C}^\infty(\overline{\Omega})$ and

$$(u, v) = \int_\Omega u(z)\overline{v(z)}\, d\lambda(z)$$

we have

$$(u_{x_k}, v) = -(u, v_{x_k}) + \int_{b\Omega} u(z)\overline{v(z)}\, r_{x_k}(z)\, d\sigma(z),$$

where $d\sigma$ is the surface measure on $b\Omega$.
This follows from the Green-Gauß -theorem: for $\omega \subseteq \mathbb{R}^n$ we have

$$\int_\omega \nabla . F(x)\, d\lambda(x) = \int_{b\omega} (F(x), \nu(x))\, d\sigma(x),$$

where $\nu(x) = \nabla r(x)$ is the normal to $b\omega$ at x, and F is a $\mathcal{C}^1$ vector field on $\overline{\omega}$, and

$$\nabla . F(x) = \sum_{j=1}^n \frac{\partial F_j}{\partial x_j}.$$

For $k = 1$ and $F = (u\overline{v}, 0, \dots, 0)$ one gets

$$(u_{x_1}, v) = -(u, v_{x_1}) + \int_{b\Omega} u(z)\overline{v(z)}\, r_{x_1}(z)\, d\sigma(z),$$

similarly one obtains

$$\left(\frac{\partial u}{\partial z_k}, v\right) = -\left(u, \frac{\partial v}{\partial \overline{z}_k}\right) + \int_{b\Omega} u(z)\,\overline{v(z)}\,\frac{\partial r}{\partial z_k}(z)\, d\sigma(z). \tag{4.1}$$

Definition 4.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let r be a $\mathcal{C}^2$ defining function for Ω. The Hermitian form*

$$i\partial\overline{\partial} r(t,t)(p) = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p)\, t_j \bar{t}_k, \quad p \in b\Omega, \tag{4.2}$$

defined for all $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ with $\sum_{j=1}^n t_j(\partial r/\partial z_j)(p) = 0$ is called the Levi form of the function r at the point p.

The Levi form associated with Ω is independent of the defining function up to a positive factor.
For $p \in b\Omega$, let

$$T_p^{1,0}(b\Omega) = \{t = (t_1, \dots, t_n) \in \mathbb{C}^n \ : \ \sum_{j=1}^n t_j(\partial r/\partial z_j)(p) = 0\}.$$

Then $T_p^{1,0}(b\Omega)$ is the space of type $(1,0)$ vector fields which are tangent to the boundary at the point p.
Analogously

$$T_p^{0,1}(b\Omega) = \{t = (t_1, \dots, t_n) \in \mathbb{C}^n \ : \ \sum_{j=1}^n t_j(\partial r/\partial \overline{z}_j)(p) = 0\},$$

smooth sections in $T_p^{0,1}(b\Omega)$ are the tangential Cauchy-Riemann operators, for instance

$$\frac{\partial r}{\partial \overline{z}_k}\frac{\partial}{\partial \overline{z}_j} - \frac{\partial r}{\partial \overline{z}_j}\frac{\partial}{\partial \overline{z}_k},$$

where $j \neq k$.

Definition 4.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with $n \geq 2$, and let r be a $\mathcal{C}^2$ defining function for Ω. Ω is called (Levi) pseudoconvex at $p \in b\Omega$, if the Levi form*

$$i\partial\overline{\partial}r(t,t)(p) = \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p)\, t_j \bar{t}_k \geq 0$$

for all $t \in T_p^{1,0}(b\Omega)$. The domain Ω is said to be strictly pseudoconvex at p, if the Levi form is strictly positive for all such $t \neq 0$. Ω is called a (Levi) pseudoconvex domain if Ω is (Levi) pseudoconvex at every boundary point of Ω.
A $\mathcal{C}^2$ real valued function φ on Ω is plurisubharmonic, if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z)\, t_j \bar{t}_k \geq 0,$$

for all $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ and all $z \in \Omega$.

A bounded domain Ω in $\mathbb{C}^n$ with $n \geq 2$ with $\mathcal{C}^2$ boundary is pseudoconvex if and only if Ω has a smooth strictly plurisubharmonic exhaustion function φ, i.e. the sets $\{z \in \Omega : \varphi(z) < c\}$ are relatively compact in Ω, for every $c \in \mathbb{R}$.

Definition 4.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain.*

$$L^2_{(0,1)}(\Omega) := \{u = \sum_{j=1}^{n} u_j \, d\overline{z}_j \; : \; u_j \in L^2(\Omega) \; j = 1, \dots, n\}$$

is the space of $(0,1)$- forms with coefficients in L^2, for $u, v \in L^2_{(0,1)}(\Omega)$ we define the inner product by

$$(u, v) = \sum_{j=1}^{n} (u_j, v_j).$$

In this way $L^2_{(0,1)}(\Omega)$ becomes a Hilbert space. $(0,1)$ forms with compactly supported $\mathcal{C}^\infty$ coefficients are dense in $L^2_{(0,1)}(\Omega)$.

Definition 4.4. *Let $f \in \mathcal{C}_0^\infty(\Omega)$ and set*

$$\overline{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j}\, d\overline{z}_j,$$

then

$$\overline{\partial} : \mathcal{C}_0^\infty(\Omega) \longrightarrow L^2_{(0,1)}(\Omega).$$

$\overline{\partial}$ is a densely defined unbounded operator on $L^2(\Omega)$. It does not have closed graph.

Definition 4.5. *The domain dom($\overline{\partial}$) of $\overline{\partial}$ consists of all functions $f \in L^2(\Omega)$ such that $\overline{\partial} f$, in the sense of distributions, belongs to $L^2_{(0,1)}(\Omega)$, i.e. $\overline{\partial} f = g = \sum_{j=1}^n g_j \, d\overline{z}_j$, and for each $\phi \in \mathcal{C}_0^\infty(\Omega)$ we have*

$$\int_\Omega \sum_{j=1}^n f \left(\frac{\partial \phi}{\partial z_j} \right)^- d\lambda = - \int_\Omega \sum_{j=1}^n g_j \, \overline{\phi} \, d\lambda. \tag{4.3}$$

It is clear that $\mathcal{C}_0^\infty(\Omega) \subseteq \text{dom}(\overline{\partial})$, hence $\text{dom}(\overline{\partial})$ is dense in $L^2(\Omega)$. Since differentiation is a continuous operation in distribution theory we have

Lemma 4.6. $\overline{\partial}$: *dom($\overline{\partial}$) $\longrightarrow L^2_{(0,1)}(\Omega)$ has closed graph and Ker$\overline{\partial}$ is a closed subspace of $L^2(\Omega)$.*

Proof. Let $(f_k)_k$ be a sequence in $\text{dom}(\overline{\partial})$ such that $f_k \to f$ in $L^2(\Omega)$ and $\overline{\partial} f_k \to g$ in $L^2_{(0,1)}(\Omega)$. We have to show that $\overline{\partial} f = g$. Let $h \in \mathcal{C}^\infty_{0,(0,1)}(\Omega)$. Then

$$\int_\Omega \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} \, \overline{h}_j \, d\lambda = - \int_\Omega \sum_{j=1}^n f \left(\frac{\partial h_j}{\partial z_j} \right)^- d\lambda$$

$$= - \lim_{k\to\infty} \int_\Omega \sum_{j=1}^n f_k \left(\frac{\partial h_j}{\partial z_j} \right)^- d\lambda = \lim_{k\to\infty} \int_\Omega \sum_{j=1}^n \frac{\partial f_k}{\partial \overline{z}_j} \, \overline{h}_j \, d\lambda = \int_\Omega \sum_{j=1}^n g_j \, \overline{h}_j \, d\lambda,$$

which means that $\overline{\partial} f = g$.
Now we can apply Lemma 13.8 and get that Ker$\overline{\partial}$ is a closed subspace of $L^2(\Omega)$. $\square$

Ker$\overline{\partial}$ coincides with the Bergman space $A^2(\Omega)$ of all holomorphic functions on Ω belonging to $L^2(\Omega)$. This is due to the fact that $\frac{\partial f}{\partial \overline{z}_k} = 0$ in the sense of distributions, implies that f is already a holomorphic function (regularity of the Cauchy-Riemann operator, see for instance [2]).
More general for $q \geq 1$: $\overline{\partial} : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q+1)}(\Omega)$ with domain as before, is again a densely defined, closed operator. In this case Ker$\overline{\partial}$ is a closed subspace of $L^2_{(0,q)}(\Omega)$, which does not mean that all coefficients are holomorphic functions. The $(0,1)$ form $f(z_1, z_2) = \overline{z}_2 \, d\overline{z}_1 + \overline{z}_1 \, d\overline{z}_2$ satisfies $\overline{\partial} f = 0$, but has non-holomorphic coefficients.

Proposition 4.7. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$, with defining function r such that $|\nabla r(z)| = 1$ on $b\Omega$. We set $\mathcal{C}^\infty(\overline{\Omega})$ for the restriction of $\mathcal{C}^\infty(\mathbb{C}^n)$ to $\overline{\Omega}$ and $\mathcal{D}^{0,1} = \mathcal{C}^\infty_{(0,1)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)$. A $(0,1)$-form $u = \sum_{j=1}^n u_j \, d\overline{z}_j$ with coefficients in $\mathcal{C}^\infty(\overline{\Omega})$ belongs to $\mathcal{D}^{0,1}$ if and only if $\sum_{j=1}^n \frac{\partial r}{\partial z_j} u_j = 0$ on $b\Omega$.*

Proof. For $\psi \in \mathcal{C}^\infty(\overline{\Omega}) \subset \text{dom}(\overline{\partial})$ we have

$$\sum_{j=1}^n \left(-\frac{\partial u_j}{\partial z_j}, \psi \right) = \sum_{j=1}^n \left(u_j, \frac{\partial \psi}{\partial \overline{z}_j} \right) - \sum_{j=1}^n \int_{b\Omega} u_j \overline{\psi} \, \frac{\partial r}{\partial z_j} \, d\sigma = (u, \overline{\partial} \psi) - \sum_{j=1}^n \int_{b\Omega} u_j \overline{\psi} \, \frac{\partial r}{\partial z_j} \, d\sigma,$$

if ψ has in addition compact support in Ω, we have

$$(\overline{\partial}^* u, \psi) = (u, \overline{\partial} \psi).$$

Since the compactly supported smooth function are dense in $L^2(\Omega)$, we must have

$$\sum_{j=1}^n \int_{b\Omega} u_j \overline{\psi} \, \frac{\partial r}{\partial z_j} \, d\sigma = 0,$$

for any $\psi \in \mathcal{C}^\infty(\overline{\Omega})$. This implies

$$\sum_{j=1}^n u_j \frac{\partial r}{\partial z_j} = 0$$

on $b\Omega$. $\square$

Now we consider the $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\Omega) \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} L^2_{(0,n)}(\Omega) \xrightarrow{\overline{\partial}} 0\,, \tag{4.4}$$

where $L^2_{(0,q)}(\Omega)$ denotes the space of $(0,q)$-forms on Ω with coefficients in $L^2(\Omega)$. The $\overline{\partial}$-operator on $(0,q)$-forms is given by

$$\overline{\partial}\left(\sum_J{}' a_J\, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J, \tag{4.5}$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices J.
The derivatives are taken in the sense of distributions, and the domain of $\overline{\partial}$ consists of those $(0,q)$-forms for which the right hand side belongs to $L^2_{(0,q+1)}(\Omega)$. So $\overline{\partial}$ is a densely defined closed operator, and therefore has an adjoint operator from $L^2_{(0,q+1)}(\Omega)$ into $L^2_{(0,q)}(\Omega)$ denoted by $\overline{\partial}^*$.
We consider the $\overline{\partial}$-complex

$$L^2_{(0,q-1)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q)}(\Omega) \underset{\overline{\partial}^*}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,q+1)}(\Omega), \tag{4.6}$$

for $1 \le q \le n-1$.
We remark that a $(0,q+1)$-form $u = \sum_J' u_J\, d\overline{z}_J$ belongs to $\mathcal{C}^\infty_{(0,q+1)}(\overline{\Omega}) \cap \operatorname{dom}(\overline{\partial}^*)$ if and only if

$$\sum_{k=1}^n u_{kK} \frac{\partial r}{\partial z_k} = 0 \tag{4.7}$$

on $b\Omega$ for all K with $|K| = q$. To see this let $\alpha \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$

$$\begin{aligned}
(u, \overline{\partial}\alpha) &= \Big(\sum_{|J|=q+1}{}' u_J\, d\overline{z}_J\,,\ \sum_{j=1}^n \sum_{|K|=q}{}' \frac{\partial \alpha_K}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_K\Big) \\
&= \sum_{j=1}^n \sum_{|K|=q}{}' \int_\Omega u_{jK} \frac{\partial \overline{\alpha}_K}{\partial z_j}\, d\lambda \\
&= -\sum_{j=1}^n \sum_{|K|=q}{}' \int_\Omega \frac{\partial u_{jK}}{\partial z_j} \overline{\alpha}_K\, d\lambda + \sum_{j=1}^n \sum_{|K|=q}{}' \int_{b\Omega} u_{jK}\, \overline{\alpha}_K \frac{\partial r}{\partial z_j}\, d\sigma \\
&= \Big(\sum_{|K|=q}{}' \big(-\sum_{j=1}^n \frac{\partial u_{jK}}{\partial z_j}\big)\, d\overline{z}_K\ ,\ \sum_{|K|=q}{}' \alpha_K\, d\overline{z}_K\Big) + \sum_{|K|=q}{}' \int_{b\Omega} \big(\sum_{j=1}^n u_{jK} \frac{\partial r}{\partial z_j}\big)\, \overline{\alpha}_K\, d\sigma \\
&= (\vartheta u, \alpha) - \int_{b\Omega} \langle \theta(\vartheta, dr) u, \alpha \rangle\, d\sigma,
\end{aligned}$$

where

$$\vartheta u = \sum_{|K|=q}{}' \Big(-\sum_{j=1}^{n} \frac{\partial u_{jK}}{\partial z_j}\Big)\, d\overline{z}_K \tag{4.8}$$

and

$$\sum_{|K|=q}{}' \int_{b\Omega} \Big(\sum_{j=1}^{n} u_{jK} \frac{\partial r}{\partial z_j}\Big)\, \overline{\alpha}_K \, d\sigma = -\int_{b\Omega} \langle \theta(\vartheta, \partial r)u, \alpha\rangle \, d\sigma, \tag{4.9}$$

hence we have

$$(\vartheta u, \alpha) = (u, \overline{\partial}\alpha) + \int_{b\Omega} \langle \theta(\vartheta, \partial r)u, \alpha\rangle \, d\sigma, \tag{4.10}$$

where $\theta(\vartheta, dr)u$ denotes the symbol of ϑ in the ∂r direction. Note that for $u \in \mathrm{dom}(\overline{\partial}^*)$ we have $\overline{\partial}^* u = \vartheta u$.

Similarly we have for $u \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ and $\alpha \in \mathcal{C}^\infty_{(0,q+1)}(\overline{\Omega})$:

$$(\overline{\partial} u, \alpha) = (u, \vartheta \alpha) + \int_{b\Omega} \langle \overline{\partial} r \wedge u, \alpha\rangle \, d\sigma, \tag{4.11}$$

where

$$\int_{b\Omega} \langle \overline{\partial} r \wedge u, \alpha\rangle \, d\sigma = \sum_{|K|=q}{}' \sum_{k=1}^{n} \int_{b\Omega} u_K \frac{\partial r}{\partial \overline{z}_k}\, \overline{\alpha}_{kK} \, d\sigma. \tag{4.12}$$

Proposition 4.8. *The complex Laplacian $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}$ defined on*

$$dom(\Box) = \{u \in L^2_{(0,q)}(\Omega) : u \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*)\, , \, \overline{\partial} u \in dom(\overline{\partial}^*) \text{ and } \overline{\partial}^* u \in dom(\overline{\partial})\}$$

acts as an unbounded, densely defined, closed and self-adjoint operator on $L^2_{(0,q)}(\Omega)$, $1 \le q \le n$, which means that $\Box = \Box^*$ and $dom(\Box) = dom(\Box^*)$.

Proof. $\mathrm{dom}(\Box)$ contains all smooth forms with compact support, hence $\Box$ is densely defined. To show that $\Box$ is closed depends on the fact that both $\overline{\partial}$ and $\overline{\partial}^*$ are closed : note that

$$(\Box u, u) = (\overline{\partial}\,\overline{\partial}^* u + \overline{\partial}^*\,\overline{\partial} u, u) = \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2, \tag{4.13}$$

for $u \in \mathrm{dom}(\Box)$. We have to prove that for every sequence $u_k \in \mathrm{dom}(\Box)$ such that $u_k \to u$ in $L^2_{(0,q)}(\Omega)$ and $\Box u_k$ converges, we have $u \in \mathrm{dom}(\Box)$ and $\Box u_k \to \Box u$. It follows from (4.13) that

$$(\Box(u_k - u_\ell), u_k - u_\ell) = \|\overline{\partial}(u_k - u_\ell)\|^2 + \|\overline{\partial}^*(u_k - u_\ell)\|^2,$$

which implies that $\overline{\partial} u_k$ converges in $L^2_{(0,q+1)}(\Omega)$ and $\overline{\partial}^* u_k$ converges in $L^2_{(0,q-1)}(\Omega)$. Since $\overline{\partial}$ and $\overline{\partial}^*$ are closed operators, we get $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ and $\overline{\partial} u_k \to \overline{\partial} u$ in $L^2_{(0,q+1)}(\Omega)$ and $\overline{\partial}^* u_k \to \overline{\partial}^* u$ in $L^2_{(0,q-1)}(\Omega)$.
To show that $\overline{\partial} u \in \mathrm{dom}(\overline{\partial}^*)$ and $\overline{\partial}^* u \in \mathrm{dom}(\overline{\partial})$, we first notice that $\overline{\partial}\,\overline{\partial}^* u_k$ and $\overline{\partial}^*\,\overline{\partial} u_k$ are orthogonal which follows from

$$(\overline{\partial}\,\overline{\partial}^* u_k, \overline{\partial}^*\,\overline{\partial} u_k) = (\overline{\partial}^2\,\overline{\partial}^* u_k, \overline{\partial} u_k) = 0.$$

Therefore the convergence of $\Box u_k = \overline{\partial}\,\overline{\partial}^* u_k + \overline{\partial}^*\,\overline{\partial} u_k$ implies that both $\overline{\partial}\,\overline{\partial}^* u_k$ and $\overline{\partial}^*\,\overline{\partial} u_k$ converge. Now use again that $\overline{\partial}$ and $\overline{\partial}^*$ are closed operators to obtain that $\overline{\partial}\,\overline{\partial}^* u_k \to \overline{\partial}\,\overline{\partial}^* u$ and $\overline{\partial}^*\,\overline{\partial} u_k \to \overline{\partial}^*\,\overline{\partial} u$. This implies that $\Box u_k \to \Box u$. Hence $\Box$ is closed.
In order to show that $\Box$ is self-adjoint we use Lemma 13.11 from the appendix. Define

$$R = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} + I$$

on dom($\Box$). By Lemma 13.11 both $(I+\overline{\partial}\,\overline{\partial}^*)^{-1}$ and $(I+\overline{\partial}^*\,\overline{\partial})^{-1}$ are bounded, self-adjoint operators. Consider

$$L = (I+\overline{\partial}\,\overline{\partial}^*)^{-1} + (I+\overline{\partial}^*\,\overline{\partial})^{-1} - I.$$

Then L is bounded and self-adjoint. We claim that $L = R^{-1}$. Since

$$(I+\overline{\partial}\,\overline{\partial}^*)^{-1} - I = (I-(I+\overline{\partial}\,\overline{\partial}^*))(I+\overline{\partial}\,\overline{\partial}^*)^{-1} = -\overline{\partial}\,\overline{\partial}^*(I+\overline{\partial}\,\overline{\partial}^*)^{-1},$$

we have that the range of $(I+\overline{\partial}\,\overline{\partial}^*)^{-1}$ is contained in $\mathrm{dom}(\overline{\partial}\,\overline{\partial}^*)$. Similarly, we have that the range of $(I+\overline{\partial}^*\,\overline{\partial})^{-1}$ is contained in $\mathrm{dom}(\overline{\partial}^*\,\overline{\partial})$ and we get

$$L = (I+\overline{\partial}^*\,\overline{\partial})^{-1} - \overline{\partial}\,\overline{\partial}^*(I+\overline{\partial}\,\overline{\partial}^*)^{-1}.$$

Since $\overline{\partial}^2 = 0$, we have that the range of L is contained in $\mathrm{dom}(\overline{\partial}^*\,\overline{\partial})$ and

$$\overline{\partial}^*\,\overline{\partial} L = \overline{\partial}^*\,\overline{\partial}(I+\overline{\partial}^*\,\overline{\partial})^{-1}.$$

Similarly, we have that the range of L is contained in $\mathrm{dom}(\overline{\partial}\,\overline{\partial}^*)$ and

$$\overline{\partial}\,\overline{\partial}^* L = \overline{\partial}\,\overline{\partial}^*(I+\overline{\partial}\,\overline{\partial}^*)^{-1}.$$

This implies that the range of L is contained in dom($\Box$). In addition we have

$$RL = \overline{\partial}\,\overline{\partial}^*(I+\overline{\partial}\,\overline{\partial}^*)^{-1} + \overline{\partial}^*\,\overline{\partial}(I+\overline{\partial}^*\,\overline{\partial})^{-1} + L = I.$$

If $Ru = 0$, we get $\Box u = -u$ and $0 \le (\Box u, u) = -(u,u)$, which implies that $u = 0$. Hence R is injective and we have that $L = R^{-1}$. By Lemma 13.11 we know that L is self-adjoint. Apply Lemma 13.10 to get that R is self-adjoint. Therefore $\Box = R - I$ is self-adjoint. $\Box$

In the sequel we will show that for a smoothly bounded pseudoconvex domain Ω we have

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 \ge c\,\|u\|^2, \tag{4.14}$$

for each $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, $c > 0$ (see Theorem 7.1). Since $(\Box u, u) = \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2$, it follows that for a convergent sequence $(\Box u_n)_n$ we get

$$\|\Box u_n - \Box u_m\|\,\|u_n - u_m\| \ge (\Box(u_n - u_m), u_n - u_m) \ge c\|u_n - u_m\|^2,$$

which implies that $(u_n)_n$ is convergent and since $\Box$ is a closed operator we obtain that $\Box$ has closed range. If $\Box u = 0$, we get $\overline{\partial}u = 0$ and $\overline{\partial}^* u = 0$ and by (4.14) also that $u = 0$, hence $\Box$ is injective. By Lemma 13.10 (ii) the image of $\Box$ is dense, therefore $\Box$ is surjective.
We showed that

$$\Box : \mathrm{dom}(\Box) \longrightarrow L^2_{(0,q)}(\Omega)$$

is bijective and has a bounded inverse $N : L^2_{(0,q)}(\Omega) \longrightarrow \mathrm{dom}(\Box)$. (Lemma 13.10 (iv))
For $u \in L^2_{(0,q)}(\Omega)$ and $v \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ we get

$$(u,v) = (\Box N u, v) = ((\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})Nu, v) = (\overline{\partial}^* Nu, \overline{\partial}^* v) + (\overline{\partial}Nu, \overline{\partial}v). \tag{4.15}$$

Let $j : \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*) \longrightarrow L^2_{(0,q)}(\Omega)$ denote the embedding, where $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ is endowed with the graph-norm $u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$, the graph-norm stems from the inner product

$$Q(u,v) = (u,v)_Q = (\Box u, v) = (\overline{\partial}u, \overline{\partial}v) + (\overline{\partial}^* u, \overline{\partial}^* v).$$

The basic estimates (4.14) imply that j is a bounded operator with operator norm

$$\|j\| \leq \frac{1}{\sqrt{c}}.$$

By (4.14) it follows in addition that $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ endowed with the graph-norm $u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$ is a Hilbert space.
Since $(u,v) = (u, jv)$, we have that $(u,v) = (j^* u, v)_Q$. Equation (4.15) suggests that as an operator to $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$, N coincides with j^* and as an operator to $L^2_{(0,q)}(\Omega)$, N should be equal to $j \circ j^*$ (compare with Proposition 13.12). For this purpose set $\tilde{N} = j \circ j^*$, and note that $\tilde{N}^* = (j \circ j^*)^* = j \circ j^* = \tilde{N}$, i.e. $\tilde{N}$ is self-adjoint (of course also bounded). We claim that the range of $\tilde{N}$ is contained in $\mathrm{dom}(\Box)$. To show this we use an approach due to F. Berger (see [3]): since $\Box$ is self-adjoint it suffices to show that $\tilde{N}u \in \mathrm{dom}(\Box^*)$ for all $u \in L^2_{(0,q)}(\Omega)$, which means to show that the functional $v \mapsto (\Box v, \tilde{N}u)$ is bounded on $\mathrm{dom}(\Box)$:

$$\begin{aligned} |(\Box v, \tilde{N}u)| &= |((\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})v, \tilde{N}u)| = |(\overline{\partial}v, \overline{\partial}\tilde{N}u) + (\overline{\partial}^* v, \overline{\partial}^* \tilde{N}u)| \\ &= |(Q(v, j^* u)| = |(jv, u)| = |(v,u)| \leq \|v\|\,\|u\|. \end{aligned}$$

For $v \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ we have

$$(\Box \tilde{N}u, v) = (\tilde{N}u, v)_Q = (j^* u, v)_Q = (u, jv) = (u, v),$$

hence $\Box \tilde{N}u = u$, in a similar way we obtain for $u \in \mathrm{dom}(\Box)$

$$(\tilde{N}\Box u, v) = (\Box u, \tilde{N}v) = (u, \tilde{N}v)_Q = (u, j^* v)_Q = (ju, v) = (u, v),$$

which implies that $\tilde{N}\Box u = u$. Altogether we obtain that $N = \tilde{N}$.
Now we get

$$\|\overline{\partial}Nu\|^2 + \|\overline{\partial}^* Nu\|^2 = (j^* u, j^* u)_Q \leq \|j^*\|^2\,\|u\|^2,$$

for $u \in L^2_{(0,q)}(\Omega)$, which implies that the operators

$$\overline{\partial}N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q+1)}(\Omega) \text{ and } \overline{\partial}^* N : L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q-1)}(\Omega)$$

are both bounded.
Let N_q denote the $\overline{\partial}$-Neumann operator on $L^2_{(0,q)}(\Omega)$ and $u \in \mathrm{dom}(\overline{\partial})$. Then $\overline{\partial}u = \overline{\partial}\overline{\partial}^*\overline{\partial}N_q u$ and

$$N_{q+1}\overline{\partial}u = N_{q+1}\overline{\partial}\,\overline{\partial}^*\overline{\partial}N_q u = N_{q+1}(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\overline{\partial}N_q u = \overline{\partial}N_q u,$$

hence on $\mathrm{dom}(\overline{\partial})$ we have

$$N_{q+1}\overline{\partial} = \overline{\partial}N_q. \tag{4.16}$$

Similarly on $\mathrm{dom}(\overline{\partial}^*)$ we have

$$N_{q-1}\overline{\partial}^* = \overline{\partial}^* N_q. \tag{4.17}$$

Since we already know that both operators $\overline{\partial}N_q$ and $\overline{\partial}^* N_q$ are bounded, we can continue both operators $N_{q+1}\overline{\partial}$ and $N_{q-1}\overline{\partial}^*$ to bounded operators on $L^2_{(0,q)}(\Omega)$.

We remark that $\overline{\partial}^* N_q$ is zero on $(\ker(\overline{\partial})^{\perp}$: let $k \in (\ker\overline{\partial})^{\perp}$ and $u \in \mathrm{dom}(\overline{\partial})$, then

$$(\overline{\partial}^* N_q k, u) = (N_q k, \overline{\partial} u) = (k, N_q \overline{\partial} u) = (k, \overline{\partial} N_{q-1} u) = 0,$$

since $\overline{\partial} N_{q-1} u \in \ker(\overline{\partial})$, which gives $\overline{\partial}^* N_q k = 0$.
For $u \in L^2_{(0,q)}(\Omega)$ we use (4.14) for $N_q u$ to obtain

$$c\|N_q u\|^2 \le \|\overline{\partial} N_q u\|^2 + \|\overline{\partial}^* N_q u\|^2 = (\overline{\partial}^* \overline{\partial} N_q u, N_q u) + (\overline{\partial}\,\overline{\partial}^* N_q u, N_q u) = (u, N_q u) \le \|u\|\,\|N_q u\|,$$

which implies

$$\|N_q u\| \le \frac{1}{c}\,\|u\|. \tag{4.18}$$

Given $\alpha \in L^2_{(0,q)}(\Omega)$, with $\overline{\partial}\alpha = 0$ we get

$$\alpha = \overline{\partial}\,\overline{\partial}^* N_q \alpha + \overline{\partial}^*\, \overline{\partial} N_q \alpha. \tag{4.19}$$

If we apply $\overline{\partial}$ to the last equality we obtain:

$$0 = \overline{\partial}\alpha = \overline{\partial}\,\overline{\partial}^* \overline{\partial} N_q \alpha,$$

since $\overline{\partial} N_q \alpha \in \mathrm{dom}(\overline{\partial}^*)$ we have

$$0 = (\overline{\partial}\,\overline{\partial}^* \overline{\partial} N_q \alpha, \overline{\partial} N_q \alpha) = (\overline{\partial}^* \overline{\partial} N_q \alpha, \overline{\partial}^* \overline{\partial} N_q \alpha) = \|\overline{\partial}^* \overline{\partial} N_q \alpha\|^2.$$

Finally we set $u_0 = \overline{\partial}^* N_q \alpha$ and derive from (4.19) that for $\overline{\partial}\alpha = 0$

$$\alpha = \overline{\partial} u_0,$$

and we see that $u_0 \perp \ker \overline{\partial}$, since for $h \in \ker \overline{\partial}$ we get

$$(u_0, h) = (\overline{\partial}^* N_q \alpha, h) = (N_q \alpha, \overline{\partial} h) = 0.$$

It follows that

$$\|\overline{\partial}^* N_q \alpha\|^2 = (\overline{\partial}\,\overline{\partial}^* N_q \alpha, N_q \alpha) = (\overline{\partial}\,\overline{\partial}^* N_q \alpha, N_q \alpha) + (\overline{\partial}^* \overline{\partial} N_q \alpha, N_q \alpha) = (\alpha, N_q \alpha) \le \|\alpha\|\,\|N_q \alpha\|$$

and using (4.18) we obtain

$$\|\overline{\partial}^* N_q \alpha\| \le c^{-1/2}\,\|\alpha\|, \tag{4.20}$$

hence the canonical solution operator S_q for $\overline{\partial}$ coincides with $\overline{\partial}^* N_q$ as operator on

$$L^2_{(0,q)}(\Omega) \cap \ker\overline{\partial}$$

and is a bounded operator.
Using (4.16) and (4.17) we now show that

$$N_q = S_q^*\, S_q + S_{q+1}\, S_{q+1}^*. \tag{4.21}$$

First note that by 13.3 we have

$$\overline{\partial}^* N_q = \overline{\partial}^* N_q^* = (N_q \overline{\partial})^* \text{ and } (\overline{\partial}^* N_q)^* = N_q \overline{\partial}$$

and

$$\overline{\partial} N_q = \overline{\partial}^{**} N_q^* = (N_q \overline{\partial}^*)^* = (\overline{\partial}^* N_{q+1})^* \text{ and } \overline{\partial}^* N_{q+1} = (\overline{\partial} N_q)^* = N_q \overline{\partial}^*,$$

hence it follows that for $u \in L^2_{(0,q)}(\Omega)$ we have

$$\begin{aligned} N_q u &= N_q(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})N_q u \\ &= (N_q\overline{\partial})(\overline{\partial}^* N_q)u + (N_q\overline{\partial}^*)(\overline{\partial}N_q)u \\ &= (\overline{\partial}^* N_q)^*(\overline{\partial}^* N_q)u + (\overline{\partial}^* N_{q+1})(\overline{\partial}^* N_{q+1})^* u \\ &= S_q^*\, S_q u + S_{q+1}\, S_{q+1}^* u. \end{aligned}$$

Let $P_q : L^2_{(0,q)}(\Omega) \longrightarrow \ker\overline{\partial}$ denote the orthogonal projection, which is the Bergman projection for $q = 0$. We claim that

$$P_q = I - \overline{\partial}^* N_{q+1}\overline{\partial},$$

on $\mathrm{dom}(\overline{\partial})$. First we show that the range of the right hand side, which we denote by $\tilde{P}$, coincides with $\ker\overline{\partial}$: for $u \in \mathrm{dom}(\overline{\partial})$ we have

$$\overline{\partial}u - \overline{\partial}\,\overline{\partial}^* N_{q+1}\overline{\partial}u = \overline{\partial}u - \Box N_{q+1}\overline{\partial}u + \overline{\partial}^*\overline{\partial}N_{q+1}\overline{\partial}u = \overline{\partial}u - \overline{\partial}u = 0,$$

where we used (4.16) to show that $\overline{\partial}N_{q+1}\overline{\partial}u = N_{q+2}\overline{\partial}\,\overline{\partial}u = 0$, and since $u - \overline{\partial}^* N_{n+1}\overline{\partial}u = u$ for $u \in \ker\overline{\partial}$, we have shown the first claim. Now we obtain

$$\tilde{P}^* = (I - \overline{\partial}^* N_{q+1}\overline{\partial})^* = I - \overline{\partial}^* N_{q+1}\overline{\partial}^{**} = \tilde{P},$$

and

$$\begin{aligned} \tilde{P}^2 u &= \tilde{P}u - \overline{\partial}^* N_{q+1}\overline{\partial}\tilde{P}u \\ &= \tilde{P}u - \overline{\partial}^* N_{q+1}\overline{\partial}u + \overline{\partial}^* N_{q+1}\overline{\partial}\,\overline{\partial}^* N_{q+1}\overline{\partial}u \\ &= \tilde{P}u - \overline{\partial}^* N_{q+1}\overline{\partial}u + \overline{\partial}^* N_{q+1}(\Box - \overline{\partial}^*\overline{\partial})N_{q+1}\overline{\partial}u \\ &= \tilde{P}u. \end{aligned}$$

This means that $\tilde{P}$ coincides with P_q on $\mathrm{dom}(\overline{\partial})$.
Finally we remark that $\tilde{P}$ can be extended to a unique bounded operator on $L^2_{(0,q)}(\Omega)$, with coincides with P_q : for $u \in \mathrm{dom}(\overline{\partial})$ we have by (4.16) that $\overline{\partial}^* N_{q+1}\overline{\partial}u = \overline{\partial}^*\overline{\partial}N_q u$ and $u = \Box N_q u = \overline{\partial}\,\overline{\partial}^* N_q u + \overline{\partial}^*\overline{\partial}N_q u$ is an orthogonal decomposition, which follows from

$$(\overline{\partial}\,\overline{\partial}^* N_q u, \overline{\partial}^*\overline{\partial}N_q u) = (\overline{\partial}\,\overline{\partial}\,\overline{\partial}^* N_q u, \overline{\partial}N_q u) = 0.$$

Hence

$$\|\overline{\partial}^* N_{q+1}\overline{\partial}u\| = \|\overline{\partial}^*\overline{\partial}N_q u\| \leq \|u\|,\ u \in \mathrm{dom}(\overline{\partial}),$$

which proves the claim since $\mathrm{dom}(\overline{\partial})$ is dense in $L^2_{(0,q)}(\Omega)$.

Remark 4.9. *If one supposes that*

$$\Box : dom(\Box) \longrightarrow L^2_{(0,1)}(\Omega)$$

is bijective and has a bounded inverse N, the basic estimate (4.14) *must hold; this follows from the spectral theorem, see* [14]*:*
N is self-adjoint and bounded and therefore has a bounded self-adjoint root $N^{1/2}$ which is again injective. By Lemma 13.10 $N^{1/2}$ has a self-adjoint inverse which will be denoted by $N^{-1/2}$. Let $u \in dom(\Box)$. Then there exists $w \in L^2_{(0,1)}(\Omega)$ such that $Nw = u$. Hence

we have $N^{1/2}v = u$, where $v = N^{1/2}w$ and $N^{-1/2}v = w = N^{-1/2}N^{-1/2}u$ is well defined. Now we get

$$\begin{aligned}\|u\|^2 = \|N^{1/2}v\|^2 \le C\|v\|^2 &= C\,(N^{-1/2}u, N^{-1/2}u)\\ &= C\,(N^{-1/2}N^{-1/2}u, u) = C\,(N^{-1/2}N^{-1/2}Nw, Nw)\\ &= C\,(w, Nw) = C\,(\Box u, u)\\ &\le C\,(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2),\end{aligned}$$

which is the basic estimate (4.14).

The two boundary conditions $u \in \mathrm{dom}(\overline{\partial}^*)$ and $\overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*)$ which appear in the definition of $\mathrm{dom}(\Box)$ are called the $\overline{\partial}$-Neumann boundary conditions. They amount to a Dirichlet boundary condition on the normal component of u and to the normal component of $\overline{\partial}u$ respectively, see [46] for more details.

Example. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ with $0 \in b\Omega$. Assume that for some neighborhood U of 0

$$\Omega \cap U = \{z \in \mathbb{C}^n \,:\, \Im z_n = y_n < 0\} \cap U.$$

Let $u = \sum_{j=1}^n u_j\, d\overline{z}_j \in \mathcal{C}^2_{(0,1)}(\overline{\Omega})$ and suppose that the support of u lies in $U \cap \overline{\Omega}$. Then $u \in \mathrm{dom}(\Box)$ if and only if

$$u_n = 0 \ \text{ on } b\Omega \cap U, \tag{4.22}$$

$$\frac{\partial u_j}{\partial \overline{z}_n} = 0 \ \text{ on } b\Omega \cap U,\ j = 1, \dots, n-1. \tag{4.23}$$

(4.22) follows from (4.7), which means that $u \in \mathrm{dom}(\overline{\partial}^*)$, and $\overline{\partial}u \in \mathrm{dom}(\overline{\partial}^*)$ is equivalent to

$$\frac{\partial u_j}{\partial \overline{z}_n} - \frac{\partial u_n}{\partial \overline{z}_j} = 0 \ \text{ on } b\Omega \cap U,\ j = 1, \dots, n-1,$$

again by (4.7). Since $\frac{\partial u_n}{\partial \overline{z}_j} = 0$ on $b\Omega \cap U,\ j = 1, \dots, n-1$, we get (4.23).
It is the second boundary condition which makes the system non-coercive.

We continue investigating the boundary conditions:

Proposition 4.10. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$, with defining function r such that $|\nabla r(z)| = 1$ on $b\Omega$. Then, if $u \in \mathcal{D}^{0,1}$, we have*

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 = \sum_{j,k=1}^n \left\| \frac{\partial u_j}{\partial \overline{z}_k} \right\|^2 + \int_{b\Omega} \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}\, u_j\, \overline{u}_k\, d\sigma. \tag{4.24}$$

Proof. For $u \in \mathcal{D}^{0,1}$ we have

$$\overline{\partial}u = \sum_{j<k} \left(\frac{\partial u_k}{\partial \overline{z}_j} - \frac{\partial u_j}{\partial \overline{z}_k} \right) d\overline{z}_j \wedge d\overline{z}_k,$$

and

$$\overline{\partial}^* u = -\sum_{j=1}^n \frac{\partial u_j}{\partial z_j}.$$

For the norms we get

$$\|\overline{\partial} u\|^2 = \sum_{j<k} \left\| \frac{\partial u_k}{\partial \overline{z}_j} - \frac{\partial u_j}{\partial \overline{z}_k} \right\|^2 = \sum_{j,k=1}^{n} \left\| \frac{\partial u_k}{\partial \overline{z}_j} \right\|^2 - \sum_{j,k=1}^{n} \left(\frac{\partial u_k}{\partial \overline{z}_j}, \frac{\partial u_j}{\partial \overline{z}_k} \right),$$

and

$$\|\overline{\partial}^* u\|^2 = \left\| \sum_{j=1}^{n} \frac{\partial u_j}{\partial z_j} \right\|^2 = \left(\sum_{j=1}^{n} \frac{\partial u_j}{\partial z_j}, \sum_{k=1}^{n} \frac{\partial u_k}{\partial z_k} \right).$$

Note that the commutator $\left[\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \overline{z}_j}\right] = 0$ and integrate by parts

$$\begin{aligned}
-\sum_{j,k=1}^{n} \left(\frac{\partial u_k}{\partial \overline{z}_j}, \frac{\partial u_j}{\partial \overline{z}_k} \right) &= \sum_{j,k=1}^{n} \left\{ \left(\frac{\partial}{\partial z_k} \frac{\partial u_k}{\partial \overline{z}_j}, u_j \right) - \int_{b\Omega} \frac{\partial r}{\partial z_k} \frac{\partial u_k}{\partial \overline{z}_j} \overline{u}_j \, d\sigma \right\} \\
&= \sum_{j,k=1}^{n} \left\{ \left(\frac{\partial}{\partial \overline{z}_j} \frac{\partial u_k}{\partial z_k}, u_j \right) - \int_{b\Omega} \frac{\partial r}{\partial z_k} \frac{\partial u_k}{\partial \overline{z}_j} \overline{u}_j \, d\sigma \right\} \\
&= \sum_{j,k=1}^{n} \left\{ - \left(\frac{\partial u_k}{\partial z_k}, \frac{\partial u_j}{\partial z_j} \right) + \int_{b\Omega} \frac{\partial r}{\partial \overline{z}_j} \frac{\partial u_k}{\partial z_k} \overline{u}_j \, d\sigma - \int_{b\Omega} \frac{\partial r}{\partial z_k} \frac{\partial u_k}{\partial \overline{z}_j} \overline{u}_j \, d\sigma \right\} \\
&= -\|\overline{\partial}^* u\|^2 + \sum_{j,k=1}^{n} \int_{b\Omega} \frac{\partial r}{\partial \overline{z}_j} \frac{\partial u_k}{\partial z_k} \overline{u}_j \, d\sigma - \sum_{j,k=1}^{n} \int_{b\Omega} \frac{\partial r}{\partial z_k} \frac{\partial u_k}{\partial \overline{z}_j} \overline{u}_j \, d\sigma.
\end{aligned}$$

Since $u \in \mathcal{D}^{0,1}$, which means that $\sum_{j=1}^{n} \frac{\partial r}{\partial z_j} u_j = 0$ on $b\Omega$, the second term of the last line is 0.
Also since $u \in \mathcal{D}^{0,1}$, the vector field $X := \sum_{k=1}^{n} u_k \frac{\partial}{\partial z_k}$ is tangent to $b\Omega$. Thus , if g is any function vanishing on $b\Omega$, then $X(g) = 0$ on $b\Omega$. The function $g_0 = \sum_{j=1}^{n} \frac{\partial r}{\partial \overline{z}_j} \overline{u}_j = 0$ on $b\Omega$. Hence we get

$$X(g_0) = \sum_{k=1}^{n} u_k \frac{\partial}{\partial z_k} \left(\sum_{j=1}^{n} \frac{\partial r}{\partial \overline{z}_j} \overline{u}_j \right) = \sum_{j,k=1}^{n} u_k \frac{\partial^2 r}{\partial \overline{z}_j \partial z_k} \overline{u}_j + \sum_{j,k=1}^{n} u_k \frac{\partial r}{\partial \overline{z}_j} \frac{\partial \overline{u}_j}{\partial z_k} = 0.$$

Taking complex conjugates we see that

$$\sum_{j,k=1}^{n} \frac{\partial r}{\partial z_k} \frac{\partial u_k}{\partial \overline{z}_j} \overline{u}_j = -\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k,$$

and since the right side is real this implies that

$$\|\overline{\partial} u\|^2 = \sum_{j,k=1}^{n} \left\| \frac{\partial u_k}{\partial \overline{z}_j} \right\|^2 - \|\overline{\partial}^* u\|^2 + \int_{b\Omega} \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, d\sigma.$$

□

Corollary 4.11. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, with defining function r such that $|\nabla r(z)| = 1$ on $b\Omega$. Then, if $u \in \mathcal{D}^{0,1}$, we have*

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \geq \sum_{j,k=1}^{n} \left\| \frac{\partial u_j}{\partial \overline{z}_k} \right\|^2. \tag{4.25}$$

Proof. By Definition 4.2 the Levi form

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(p)\, u_j(p)\overline{u_k(p)} \geq 0,$$

for all $p \in b\Omega$ and for all $u \in \mathcal{D}^{0,1}$. Hence the result follows immediately from 4.10. $\square$

The following density result is crucial for the whole analysis.

Proposition 4.12. *If $b\Omega$ is $\mathcal{C}^{k+1}$, then $\mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)$ is dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$. The statement also holds with $k+1$ and k replaced by ∞.*

Before we begin with the proof of this important approximation result we mention a few consequences of it.

Remark 4.13. *(a) From Proposition 4.12 it follows that $\mathcal{D}^{0,q}$ is dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$. If (4.14) holds, we can take $u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$ instead of $u \mapsto (\|u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$.*

(b) It is also useful to know that $dom(\overline{\partial}^*)$ is preserved under multiplication by a function in $\mathcal{C}^1(\overline{\Omega})$: let $u \in dom(\overline{\partial}^*)$, $v \in dom(\overline{\partial})$ and $\psi \in \mathcal{C}^1(\overline{\Omega})$. Then

$$(\overline{\partial}v, \psi u) = (\overline{\psi}\, \overline{\partial}v, u) = (\overline{\partial}(\overline{\psi}v), u) - (\overline{\partial}\, \overline{\psi} \wedge v, u) = (\overline{\psi}v, \overline{\partial}^* u) - (\overline{\partial}\, \overline{\psi} \wedge v, u).$$

The right-hand side is bounded by $\|v\|$, hence $\psi u \in dom(\overline{\partial}^*)$, (see [46] for more details).

(c) Compactly supported forms are not dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm: for compactly supported forms Proposition 4.10 gives

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2 = \sum_{j,k=1}^{n} \left\| \frac{\partial u_j}{\partial \overline{z}_k} \right\|^2,$$

and integration by parts also shows that in this case

$$\left\| \frac{\partial u_j}{\partial z_k} \right\|^2 = \left\| \frac{\partial u_j}{\partial \overline{z}_k} \right\|^2.$$

Hence

$$\|u\|_1^2 \leq 2\, (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2),$$

where $\|u\|_1^2$ denotes the standard Sobolev-1 norm of u on Ω. Therefore the closure of the compactly supported forms in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm is contained in the Sobolev space $W_0^1(\Omega)$ for forms that are $\mathcal{C}^\infty$ on $\overline{\Omega}$, this means that they are zero on the boundary, which is stronger than the condition

$$\sum_{j=1}^{n} \frac{\partial r}{\partial z_j}\, u_j = 0$$

on $b\Omega$ from Proposition 4.7.

(d) If Ω is a smoothly bounded pseudoconvex domain, then $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ is a Hilbert space in the graph norm $u \mapsto (\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$. This follows from (4.14).

We follow the reasoning in [9] to prove Proposition 4.12.

Lemma 4.14. *Let Ω be as in Proposition 4.12. Then $\mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ is dense in dom$(\overline{\partial}) \cap$ dom$(\overline{\partial}^*)$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$.*

Proof. By this we mean that if $u \in \text{dom}(\overline{\partial}) \cap \text{dom}(\overline{\partial}^*)$, one can construct a sequence $u_m \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ such that $u_m \to u$, $\overline{\partial}u_m \to \overline{\partial}f$ and $\vartheta u_m \to \vartheta u$ in $L^2(\Omega)$.
We use a method closely related to Friedrichs' Lemma 16.3 and use the notation from there.
Let $(\chi_\epsilon)_\epsilon$ be an approximation of the identity and $(\delta_\nu)_\nu$ a sequence of small positive numbers with $\delta_\nu \to 0$, and define

$$\Omega_{\delta_\nu} = \{z \in \Omega \, : \, r(z) < -\delta_\nu\}.$$

Then Ω_{δ_ν} is a sequence of relatively compact open subsets of Ω with union equal to Ω. The forms $u_\epsilon = u * \chi_\epsilon$ belong to $\mathcal{C}^\infty_{(0,q)}(\overline{\Omega}_{\delta_\nu})$ and $u_\epsilon \to u$, $\overline{\partial}u_\epsilon \to \overline{\partial}u$ and $\vartheta u_\epsilon \to \vartheta u$ in $L^2(\Omega_{\delta_\nu})$, see Lemma 16.2 and Lemma 16.3.
To see that this can be done up to the boundary, we first assume that the domain Ω is star-shaped and $0 \in \Omega$ is a center. Let $\Omega^\epsilon = \{(1+\epsilon)z \, : \, z \in \Omega\}$ and

$$u^\epsilon(z) = u\left(\frac{z}{1+\epsilon}\right),$$

where the dilation is performed for each coefficient of u. Then $\Omega \subset\subset \Omega^\epsilon$ and $u^\epsilon \in L^2(\Omega^\epsilon)$. Also, by the dominated convergence theorem, $u^\epsilon \to u$, $\overline{\partial}u^\epsilon \to \overline{\partial}u$ and $\vartheta u^\epsilon \to \vartheta u$ in $L^2(\Omega)$. Now we regularize u^ϵ defining

$$u_{(\epsilon)} = u^\epsilon * \chi_{\delta_\epsilon}, \tag{4.26}$$

where $\delta_\epsilon \to 0$ as $\epsilon \to 0$ and δ_ϵ is chosen sufficiently small. Then $u_{(\epsilon)} \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ and $u_{(\epsilon)} \to u$, $\overline{\partial}u_{(\epsilon)} \to \overline{\partial}u$ and $\vartheta u_{(\epsilon)} \to \vartheta u$ in $L^2(\Omega)$. Thus, $\mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ is dense in the graph norm when Ω is star-shaped. The general case follows by using a partition of unity since we assume that our domain has at least $\mathcal{C}^2$ boundary. □

Lemma 4.15. *Let Ω be as in Proposition 4.12. Then compactly supported smooth forms are dense in dom$(\overline{\partial}^*)$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$.*

Proof. We remark that if $u \in \text{dom}(\overline{\partial}^*)$ and if we extend u to $\tilde{u}$ on the whole space $\mathbb{C}^n$ by setting $\tilde{u}$ to be zero outside of Ω, then $\vartheta\tilde{u} \in L^2(\mathbb{C}^n)$ in the distribution sense: in fact for $u \in \text{dom}(\overline{\partial}^*)$ we have

$$\vartheta\tilde{u} = \widetilde{\vartheta u}$$

where $\widetilde{\vartheta u} = \vartheta u$ in Ω and $\widetilde{\vartheta u} = 0$ outside of Ω. This can be checked from the definition of $\overline{\partial}^*$, since for any $v \in \mathcal{C}^\infty_{(0,q-1)}(\mathbb{C}^n)$,

$$(\tilde{u}, \overline{\partial}v)_{L^2(\mathbb{C}^n)} = (u, \overline{\partial}v)_{L^2(\Omega)} = (\vartheta u, v)_{L^2(\Omega)} = (\widetilde{\vartheta u}, v)_{L^2(\mathbb{C}^n)}.$$

We assume again without loss of generality that Ω is star-shaped with 0 as a center. We first approximate $\tilde{u}$ by

$$\tilde{u}^{-\epsilon}(z) = \tilde{u}\left(\frac{z}{1-\epsilon}\right).$$

Now we have forms $\tilde{u}^{-\epsilon}$ with compact support in Ω and $\vartheta\tilde{u}^{-\epsilon} \to \vartheta\tilde{u}$ in $L^2(\mathbb{C}^n)$. Regularizing $\tilde{u}^{-\epsilon}$ as before, we define

$$u_{(-\epsilon)} = \tilde{u}^{-\epsilon} * \chi_{\delta_\epsilon}. \tag{4.27}$$

Then the $u_{(-\epsilon)}$ are $(0,q)$-forms with coefficients in $\mathcal{C}_0^\infty(\Omega)$ such that $u_{(-\epsilon)} \to u$ and $\vartheta u_{(-\epsilon)} \to \vartheta u$ in $L^2(\Omega)$. $\square$

However, compactly supported smooth forms are not dense in $\mathrm{dom}(\overline{\partial})$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2)^{1/2}$. Nevertheless, we have

Lemma 4.16. *Let Ω be as in Proposition 4.12. Then $\mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap dom(\overline{\partial}^*)$ is dense in $dom(\overline{\partial})$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2)^{1/2}$.*

Proof. By Lemma 4.14 it suffices to show that for any $u \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ one can find a sequence $u_m \in \mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$ such that $u_m \to u$ and $\overline{\partial} u_m \to \overline{\partial} u$ in $L^2(\Omega)$.
Let r be a $\mathcal{C}^{k+1}$ defining function such that $|dr| = 1$ on $b\Omega$. We now introduce some special vector fields and $(1,0)$-forms associated with $b\Omega$. Near a point $p \in b\Omega$ we choose fields $L_1, L_2, \dots, L_{n-1}$ of type $(1,0)$ that are orthonormal and span $T^{1,0}_p(b\Omega)$. This can be done by choosing a basis, and then using the Gram-Schmidt process. To this collection add L_n, the complex normal, normalized to have length 1. So L_n is a smooth multiple of

$$\sum_{j=1}^n \frac{\partial r}{\partial \overline{z}_j} \frac{\partial}{\partial z_j}.$$

Now denote by $w_1, w_2, \dots w_2$ the $(1,0)$-forms such that $w_j(L_k) = \delta_{jk}$. L_n is defined globally, in contrast to $L_1, \dots, L_{n-1}$. The w^j 's then form an orthonormal basis for the $(1,0)$-forms near p. The $(1,0)$-form w_n is a smooth multiple of $\sum_{j=1}^n \frac{\partial r}{\partial z_j}\, dz_j$, and is again globally defined. Taking wedge products of the w^j's yields (local) orthonormal bases for the $(1,0)$-forms.
We will regularize near a boundary point $p \in b\Omega$. Let U be a small neighborhood of p. By a partition of unity, we may assume that $\Omega \cap U$ is star-shaped and u is supported in $U \cap \overline{\Omega}$. Shrinking U if necessary, we can choose a special boundary chart $(t_1, t_2, \dots, t_{2n-1}, r)$, where $(t_1, t_2, \dots, t_{2n-1}, 0)$ are coordinates on $b\Omega$ near p. Let $\overline{w}_1, \dots, \overline{w}_n$ be an orthonormal basis for the $(0,1)$-forms on U such that $\overline{\partial} r = \overline{w}_n$.
Let $L_j = \sum_{s=1}^n a_{js} \frac{\partial}{\partial z_s}$, $w_j = \sum_{s=1}^n b_{js} dz_s$, $1 \le j \le n$. Then

$$\delta_{jk} = w_j(L_k) = \sum_{s=1}^n b_{js}\, dz_s\, (\sum_{\ell=1}^n a_{k\ell} \frac{\partial}{\partial z_\ell}\,) = \sum_{s=1}^n b_{js} a_{ks}.$$

Consequently, if f is a function,

$$\overline{\partial} f = \sum_{s=1}^n \frac{\partial f}{\partial \overline{z}_s}\, d\overline{z}_s = \sum_{j,k,s=1}^n \overline{a^{sk}} (\overline{L}_k f)\, \overline{b^{sj}}\, \overline{w}_j = \sum_{j=1}^n (\overline{L}_j f)\, \overline{w}_j, \tag{4.28}$$

where the superscripts denote the entries of the inverses of the corresponding matrices with subscripts. Since multiplication by functions in $\mathcal{C}^1(\overline{\Omega})$ preserves $\mathrm{dom}(\overline{\partial}^*)$, we may assume that the form u is supported in a special boundary chart. So $u = \sum_{|J|=q}{}' u_J \overline{w}_J$, where $\overline{w}_J = \overline{w}_{j_1} \wedge \cdots \wedge \overline{w}_{j_q}$ and each u_J is a function in $\mathcal{C}^k(\overline{\Omega})$. Then, in view of (4.28)

$$\overline{\partial} u = \overline{\partial} (\sum_{|J|=q}{}' u_J \overline{w}_J) = \sum_{|J|=q}{}' (\overline{\partial} u_J \wedge \overline{w}_J + u_J \overline{\partial}\, \overline{w}_J) \tag{4.29}$$

$$= \sum_{|J|=q}{}' \sum_{j=1}^n (\overline{L}_j u_J)\, \overline{w}_j \wedge \overline{w}_J + \sum_{|J|=q}{}' u_J \overline{\partial}\, \overline{w}_J. \tag{4.30}$$

Using the special boundary chart we get from (4.7) that

$$u \in \operatorname{dom}(\overline{\partial}^*) \Longleftrightarrow u_J = 0 \text{ on } b\Omega \text{ when } n \in J. \tag{4.31}$$

Indeed, the only boundary terms that arise when proving (4.7) come from integrating

$$\int_\Omega \overline{L}_n \alpha_K \, \overline{u}_{nK} \, d\lambda$$

by parts and they equal

$$\int_{b\Omega} \alpha_K \, \overline{u}_{nK} (\overline{L}_n r) \, d\sigma.$$

Since α_K can be arbitrary on $b\Omega$ and $\overline{L}_n r \neq 0$ on $b\Omega$, we conclude that $u_{nK} = 0$ on $b\Omega$ for all K. To see that the condition is sufficient, note that the computation to prove (4.10) shows that $(u, \overline{\partial}\alpha) = (\vartheta u, \alpha)$ when (4.7) holds and $\alpha \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$. In view of Lemma 4.14 $\mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ is dense in $\operatorname{dom}(\overline{\partial})$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2)^{1/2}$. Hence, $(u, \overline{\partial}\alpha) = (\vartheta u, \alpha)$ for all $\alpha \in \operatorname{dom}(\overline{\partial})$, which implies $u \in \operatorname{dom}(\overline{\partial}^*)$ and $\overline{\partial}^* u = \vartheta u$.
These arguments also give a formula for ϑ and $\overline{\partial}^*$ in special boundary frames:

$$\vartheta u = \vartheta(\sum_{|J|=q}{}' u_J \overline{w}_J) = - \sum_{|K|=q-1}{}' (\sum_{j=1}^n L_j u_{jK}) \, \overline{w}_K + \text{0-th order}(u). \tag{4.32}$$

0-th order(u) indicates terms that contain no derivatives of the u_J's.

We note that both $\overline{\partial}$ and ϑ are first order differential operators with variable coefficients in $\mathcal{C}^k(\overline{\Omega})$ when computed in the special frame $\overline{w}_1, \dots, \overline{w}_n$. We write

$$u = u^\tau + u^\nu,$$

where

$$u^\tau = \sum_{|J|=q, n \notin J}{}' u_J \overline{w}_J \ , \ u^\nu = \sum_{|J|=q, n \in J}{}' u_J \overline{w}_J.$$

u^τ is the complex tangential part of u, and u^ν is the complex normal part of u.
Our arguments from above imply that

$$u \in \mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \operatorname{dom}(\overline{\partial}^*) \iff u^\nu = 0 \text{ on } b\Omega.$$

For $u \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ and $\alpha \in \mathcal{C}^\infty_{(0,q+1)}(\overline{\Omega})$ we have by (4.11)

$$(\overline{\partial} u, \alpha) = (u, \vartheta\alpha) + \int_{b\Omega} \langle \overline{\partial} r \wedge u, \alpha \rangle \, d\sigma$$

and $\overline{\partial} r \wedge u = \overline{\partial} r \wedge u^\tau$ on $b\Omega$, which follows from the representation in special boundary charts:

$$\overline{\partial} r \wedge u^\nu = c \overline{w}_n \wedge \sum_{|J|=q, n \in J}{}' u_J \overline{w}_J = 0. \tag{4.33}$$

In order to approximate a form $u \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ by forms in $\mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \operatorname{dom}(\overline{\partial}^*)$ we only change the complex normal part u^ν and leave the complex tangential part u^τ unchanged: for $u \in \mathcal{C}^\infty_{(0,q)}(\overline{\Omega})$ it follows that $u^\tau \in \mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \operatorname{dom}(\overline{\partial}^*)$ and we denote by $\tilde{u}^\nu$ the extension of u^ν to $\mathbb{C}^n$ by setting $\tilde{u}^\nu$ equal to zero outside of Ω. We approximate $\tilde{u}^\nu$ as in Lemma 4.15 by

$$u^\nu_{(-\epsilon)} = (\tilde{u}^\nu)^{-\epsilon} * \chi_{\delta_\epsilon}.$$

Then $u^\nu_{(-\epsilon)}$ is smooth and supported in a compact subset of $\Omega\cap U$. By this, we approximate u^ν by $u^\nu_{(-\epsilon)} \in \mathcal{C}^\infty_0(\Omega \cap U)$ in the L^2-norm. Furthermore, by extending $\overline{\partial} u^\nu$ to be zero outside $\Omega \cap U$ and denoting the extension by $\widetilde{\overline{\partial} u^\nu}$, we have

$$\overline{\partial} \tilde{u}^\nu = \widetilde{\overline{\partial} u^\nu}$$

in $L^2(\mathbb{C}^n)$ in the sense of distributions. This follows from (4.11) and (4.33), since $u^\nu \in \mathcal{C}^k_{(0,q)}(\overline{\Omega})$ and for $\alpha \in \mathcal{C}^\infty_{(0,q+1)}(\mathbb{C}^n)$ we have

$$(\tilde{u}^\nu, \vartheta\alpha)_{L^2(\mathbb{C}^n)} = (\overline{\partial} u^\nu, \alpha)_{L^2(\Omega)} - \int_{b\Omega} \langle \overline{\partial} r \wedge u^\nu, \alpha \rangle \, d\sigma = (\widetilde{\overline{\partial} u^\nu}, \alpha)_{L^2(\mathbb{C}^n)}.$$

Since $\overline{\partial}$ is a first order differential operator with variable coefficients, we get from Friedrichs' Lemma 16.4

$$\overline{\partial} u^\nu_{(-\epsilon)} \to \overline{\partial} \tilde{u}^\nu \text{ in } L^2(\mathbb{C}^n). \tag{4.34}$$

We set

$$u_{(-\epsilon)} = u^\tau + u^\nu_{(-\epsilon)}.$$

It follows that $u_{(-\epsilon)} \in \mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$, since each coefficient of $u^\tau, u^\nu_{(-\epsilon)}$ and w_j is in $\mathcal{C}^k(\overline{\Omega \cap U})$. Therefore we get $u_{(-\epsilon)} \in \mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$ and $u_{(-\epsilon)} \to u$ in $L^2(\Omega)$.
To see that $\overline{\partial} u_{(-\epsilon)} \to \overline{\partial} u$ in the $L^2(\Omega)$-norm, using (4.34), we find that

$$\overline{\partial} u_{(-\epsilon)} = \overline{\partial} u^\tau + \overline{\partial} u^\nu_{(-\epsilon)} \to \overline{\partial} u$$

in $L^2(\Omega)$ as $\epsilon \to 0$. $\square$

To finish the proof of Proposition 4.12 we consider an arbitrary $u \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ and use a partition of unity and the same notation as before to regularize u in each small star-shaped neighborhood near the boundary. We regularize the complex tangential and normal part separately by setting

$$u_{((\epsilon))} = u^\tau_{(\epsilon)} + u^\nu_{(-\epsilon)},$$

this means that we first consider $u_{(\epsilon)}$ as it was defined in (4.26) and take then the tangential components $u^\tau_{(\epsilon)}$, then we consider $u_{(-\epsilon)}$ as it is defined in (4.27) and then take the normal components $u^\nu_{(-\epsilon)}$. It follows that for sufficiently small $\epsilon > 0$, $u^\nu_{(-\epsilon)}$ has coefficients in $\mathcal{C}^\infty_0(\Omega)$ and $u^\tau_{(\epsilon)}$ has coefficients in $\mathcal{C}^\infty(\overline{\Omega})$.
Thus we see that $u_{((\epsilon))} \in \mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$. We get from Lemma 4.14 that $u_{(\epsilon)} \to u$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$, hence $u^\tau_{(\epsilon)} \to u^\tau$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$. From Lemma 4.15 we obtain $u_{(-\epsilon)} \to u$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$, hence $u^\nu_{(-\epsilon)} \to u^\nu$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$. Finally, we use Lemma 4.16, in particular formula (4.34), and see that $\overline{\partial} u^\nu_{(-\epsilon)} \to \overline{\partial} \tilde{u}^\nu$ in $L^2(\mathbb{C}^n)$, hence $u_{((\epsilon))} \to u$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$.
This shows that $\mathcal{C}^k_{(0,q)}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$ is dense in $\mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*)$ in the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$.

5. The weighted $\overline{\partial}$-complex

Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function and define the space

$$L^2(\mathbb{C}^n, e^{-\varphi}) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-\varphi} \, d\lambda < \infty\},$$

where λ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, e^{-\varphi})$ and the space $L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi})$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, e^{-\varphi})$. Let

$$(f,g)_\varphi = \int_{\mathbb{C}^n} f\, \overline{g} e^{-\varphi} \, d\lambda$$

denote the inner product and

$$\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda$$

the norm in $L^2(\mathbb{C}^n, e^{-\varphi})$.
We consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, e^{-\varphi}) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) J \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi}), \tag{5.1}$$

where $\overline{\partial}^*_\varphi$ is the adjoint operator to $\overline{\partial}$ with respect to the weighted inner product. For $u = \sum_{j=1}^n u_j d\overline{z}_j \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ one has

$$\overline{\partial}^*_\varphi u = - \sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) u_j. \tag{5.2}$$

The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_\varphi := \overline{\partial}\, \overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},$$

where the symbol $\Box_\varphi$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $\mathcal{C}^\infty_0$, i.e., the space of smooth functions with compact support.
$\Box_\varphi$ is a selfadjoint and positive operator, which means that

$$(\Box_\varphi f, f)_\varphi \geq 0 \ , \ \text{for } f \in dom(\Box_\varphi).$$

The associated Dirichlet form is denoted by

$$Q_\varphi(f,g) = (\overline{\partial} f, \overline{\partial} g)_\varphi + (\overline{\partial}^*_\varphi f, \overline{\partial}^*_\varphi g)_\varphi, \tag{5.3}$$

for $f, g \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. The weighted $\overline{\partial}$-Neumann operator N_φ is - if it exists - the bounded inverse of $\Box_\varphi$. For further details see [22].

There is an interesting connection between $\overline{\partial}$ and the theory of Schrödinger operators with magnetic fields, see for example [10], [4], [19] and [11] for recent contributions exploiting this point of view.
In the weighted space $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ we can give a simple characterization of dom $(\overline{\partial}^*_\varphi)$ (see [20]):

Proposition 5.1. *Let $f = \sum f_j d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$. Then $f \in dom(\overline{\partial}^*_\varphi)$ if and only if*

$$e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right) \in L^2(\mathbb{C}^n, e^{-\varphi}), \tag{5.4}$$

where the derivative is to be taken in the sense of distributions.

Proof. Suppose first that $e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right) \in L^2(\mathbb{C}^n, e^{-\varphi})$. We have to show that there exists a constant C such that $|(\overline{\partial} g, f)_\varphi| \leq C\|g\|_\varphi$ for all $g \in dom(\overline{\partial})$. To this end let $(\chi_R)_{R\in\mathbb{N}}$ be a family of radially symmetric smooth cutoff funtions, which are identically one on $\mathbb{B}_R$, the ball with radius R, such that the support of χ_R is contained in $\mathbb{B}_{R+1}$, $supp(\chi_R) \subset \mathbb{B}_{R+1}$, and such that furthermore all first order derivatives of all functions in this family are uniformly bounded by a constant M. Then for all $g \in \mathcal{C}^\infty_0(\mathbb{C}^n)$:

$$(\overline{\partial} g, \chi_R f)_\varphi = \sum_{j=1}^{n} (\frac{\partial g}{\partial \overline{z}_j}, \chi_R f_j)_\varphi = -\int_{\mathbb{C}^n} \sum_{j=1}^{n} g \frac{\partial}{\partial \overline{z}_j} \left(\chi_R \overline{f}_j e^{-\varphi} \right) d\lambda,$$

by integration by parts, which in particular means

$$|(\overline{\partial} g, f)_\varphi| = \lim_{R\to\infty} |(\overline{\partial} g, \chi_R f)_\varphi| = \lim_{R\to\infty} \left| \int_{\mathbb{C}^n} \sum_{j=1}^{n} g \frac{\partial}{\partial \overline{z}_j} \left(\chi_R \overline{f}_j e^{-\varphi} \right) \, d\lambda \right|.$$

Now we use the triangle inequality, afterwards Cauchy – Schwarz, to get

$$\begin{aligned}
&\left| \int_{\mathbb{C}^n} \sum_{j=1}^{n} g \frac{\partial}{\partial \overline{z}_j} \left(\chi_R \overline{f}_j e^{-\varphi} \right) \, d\lambda \right| \\
\leq &\left| \int_{\mathbb{C}^n} \chi_R \, g \sum_{j=1}^{n} \frac{\partial}{\partial \overline{z}_j} \left(\overline{f}_j e^{-\varphi} \right) \, d\lambda \right| + \left| \int_{\mathbb{C}^n} \sum_{j=1}^{n} \overline{f}_j g \frac{\partial \chi_R}{\partial \overline{z}_j} e^{-\varphi} \, d\lambda \right| \\
\leq &\|\chi_R \, g\|_\varphi \left\| e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right) \right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi \\
= &\|g\|_\varphi \left\| e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right) \right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi .
\end{aligned}$$

Hence by assumption,

$$|(\overline{\partial} g, f)_\varphi| \leq \|g\|_\varphi \left\| e^{\varphi} \sum_{j=1}^{n} \frac{\partial}{\partial z_j} \left(f_j e^{-\varphi} \right) \right\|_\varphi + M\|g\|_\varphi \|f\|_\varphi \leq C\|g\|_\varphi$$

for all $g \in \mathcal{C}^\infty_0(\mathbb{C}^n)$, and by density of $\mathcal{C}^\infty_0(\mathbb{C}^n)$ this is true for all $g \in dom(\overline{\partial})$.
Conversely, let $f \in dom(\overline{\partial}^*_\varphi)$, which means that there exists a uniquely determined element $\overline{\partial}^*_\varphi f \in L^2(\mathbb{C}^n, e^{-\varphi})$ such that for each $g \in dom(\overline{\partial})$ we have

$$(\overline{\partial} g, f)_\varphi = (g, \overline{\partial}^*_\varphi f)_\varphi .$$

Now take $g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. Then $g \in dom(\overline{\partial})$ and

$$\begin{aligned}(g, \overline{\partial}^*_\varphi f)_\varphi =& (\overline{\partial} g, f)_\varphi \\ =& \sum_{j=1}^n (\frac{\partial g}{\partial \overline{z}_j}, f_j)_\varphi \\ =& -(g, \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right))_{L^2} \\ =& -(g, e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right))_\varphi .\end{aligned}$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $L^2(\mathbb{C}^n, e^{-\varphi})$, we conclude that

$$\overline{\partial}^*_\varphi f = -e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right),$$

which in particular implies that $e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right) \in L^2(\mathbb{C}^n, e^{-\varphi})$. □

The following lemma will be important for our considerations.

Lemma 5.2. *Forms with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ are dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ in the graph norm $f \mapsto (\|f\|_\varphi^2 + \|\overline{\partial} f\|_\varphi^2 + \|\overline{\partial}^*_\varphi f\|_\varphi^2)^{\frac{1}{2}}$.*

Proof. First we show that compactly supported L^2-forms are dense in the graph norm. So let $\{\chi_R\}_{R\in\mathbb{N}}$ be a family of smooth radially symmetric cutoffs identically one on $\mathbb{B}_R$ and supported in $\mathbb{B}_{R+1}$, such that all first order derivatives of the functions in this family are uniformly bounded in R by a constant M.
Let $f \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. Then, clearly, $\chi_R f \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ and $\chi_R f \to f$ in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ as $R \to \infty$. As observed in Proposition 5.1, we have

$$\overline{\partial}^*_\varphi f = -e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(f_j e^{-\varphi}\right),$$

hence

$$\overline{\partial}^*_\varphi (\chi_R f) = -e^\varphi \sum_{j=1}^n \frac{\partial}{\partial z_j}\left(\chi_R f_j e^{-\varphi}\right).$$

We need to estimate the difference of these expressions

$$\overline{\partial}^*_\varphi f - \overline{\partial}^*_\varphi(\chi_R f) = \overline{\partial}^*_\varphi f - \chi_R \overline{\partial}^*_\varphi f + \sum_{j=1}^n \frac{\partial \chi_R}{\partial z_j} f_j,$$

which is by the triangle inequality

$$\|\overline{\partial}^*_\varphi f - \overline{\partial}^*_\varphi(\chi_R f)\|_\varphi \le \|\overline{\partial}^*_\varphi f - \chi_R \overline{\partial}^*_\varphi f\|_\varphi + M \sum_{j=1}^n \int\limits_{\mathbb{C}^n \setminus \mathbb{B}_R} |f_j|^2 e^{-\varphi}\, d\lambda.$$

Now both terms tend to 0 as $R \to \infty$, and one can see similarly that also $\overline{\partial}(\chi_R f) \to \overline{\partial} f$ as $R \to \infty$.
So we have density of compactly supported forms in the graph norm, and density of forms with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ will follow by applying Friedrichs' lemma, see 16.4. □

As in the case of bounded domains, the canonical solution operator to $\overline{\partial}$, which we denote by S_φ, is given by $\overline{\partial}^*_\varphi N_\varphi$. Existence and compactness of N_φ and S_φ are closely related.

Remark 5.3. *In order to prove a basic estimate for the weighted $\overline{\partial}$-complex we now assume that the lowest eigenvalue μ_φ of the Levi matrix*

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}$$

satisfies

$$\mu_\varphi(z) > \epsilon, \ \textit{for all}\, z \in \mathbb{C}^n, \tag{5.3}$$

for some $\epsilon > 0$.
Using methods from real analysis, one can replace (6.5) by the the weaker assumption that

$$\liminf_{|z| \to \infty} \mu_\varphi(z) > 0. \tag{5.3}$$

For this purpose we follow the reasoning in [26]. First we notice that

$$e^{-\varphi/2} \, \Box_\varphi \, e^{\varphi/2} = \triangle_\varphi^{(0,1)},$$

where $\triangle_\varphi^{(0,1)}$ is the Witten-Laplcian. Condition (5.3') implies that $\triangle_\varphi^{(0,1)}$ is injective and that the bottom of the essential spectrum $\sigma_e(\triangle_\varphi^{(0,1)})$ is positive (Persson's Theorem). By the spectral theorem for unbounded self-adjoint operators, one derives that $\triangle_\varphi^{(0,1)}$ has a bounded inverse, hence $\Box_\varphi$ has a bounded inverse N_φ and so the square root $N_\varphi^{1/2}$ is also bounded, which gives the basic estimate for the weighted $\overline{\partial}$-complex .

Proposition 5.4.
*For a plurisubharmonic weight function φ satisfying (**??**), there is a $C > 0$ such that*

$$\|u\|_\varphi^2 \leq C(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2) \tag{5.5}$$

for each $(0,1)$-form $u \in$dom$\,(\overline{\partial}) \cap$ dom$\,(\overline{\partial}^*_\varphi)$.

Proof. By Lemma 5.2 and the assumption on φ it suffices to show that

$$\int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \, u_j \overline{u}_k \, e^{-\varphi} \, d\lambda \leq \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2, \tag{5.6}$$

for each $(0,1)$-form $u = \sum_{k=1}^n u_k \, d\overline{z}_k$ with coefficients $u_k \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, for $k = 1, \dots, n$. For this purpose we set $\delta_k = \frac{\partial}{\partial z_k} - \frac{\partial \varphi}{\partial z_k}$ and get since

$$\overline{\partial} u = \sum_{j<k} \left(\frac{\partial u_j}{\partial \overline{z}_k} - \frac{\partial u_k}{\partial \overline{z}_j} \right) d\overline{z}_j \wedge d\overline{z}_k$$

that

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2 = \int_{\mathbb{C}^n} \sum_{j<k} \left| \frac{\partial u_j}{\partial \overline{z}_k} - \frac{\partial u_k}{\partial \overline{z}_j} \right|^2 e^{-\varphi} \, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n \delta_j u_j \, \overline{\delta_k u_k} \, e^{-\varphi} \, d\lambda$$

$$= \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi} \, d\lambda + \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left(\delta_j u_j \, \overline{\delta_k u_k} - \frac{\partial u_j}{\partial \overline{z}_k} \, \overline{\frac{\partial u_k}{\partial \overline{z}_j}} \right) e^{-\varphi} \, d\lambda$$

$$= \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left[\delta_j, \frac{\partial}{\partial \overline{z}_k} \right] u_j\, \overline{u}_k\, e^{-\varphi}\, d\lambda,$$

where we used the fact that for $f, g \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ we have

$$\left(\frac{\partial f}{\partial \overline{z}_k}, g \right)_\varphi = -(f, \delta_k g)_\varphi$$

and hence

$$\left(\left[\delta_j, \frac{\partial}{\partial \overline{z}_k} \right] u_j, u_k \right)_\varphi = - \left(\frac{\partial u_j}{\partial \overline{z}_k}, \frac{\partial u_k}{\partial \overline{z}_j} \right)_\varphi + (\delta_j u_j, \delta_k u_k)_\varphi.$$

Since

$$\left[\delta_j, \frac{\partial}{\partial \overline{z}_k} \right] = \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k},$$

we have

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 = \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi}\, d\lambda + \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k\, e^{-\varphi}\, d\lambda \tag{5.7}$$

and since φ satisfies (**??**) we are done (see also [30]). □

At this stage we first generalize formula (5.7) for $(0,q)$-forms $u = \sum_{|J|=q}' u_J\, d\overline{z}_J$ with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$. We notice that

$$\overline{\partial} u = \sum_{|J|=q}{}' \sum_{j=1}^{n} \frac{\partial u_J}{\partial \overline{z}_j}\, d\overline{z}_j \wedge d\overline{z}_J,$$

and

$$\overline{\partial}_\varphi^* u = - \sum_{|K|=q-1}{}' \sum_{k=1}^{n} \delta_k u_{kK}\, d\overline{z}_K.$$

We obtain

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 = \sum_{|J|=|M|=q}{}' \sum_{j,k=1}^{n} \epsilon_{jJ}^{kM} \int_{\mathbb{C}^n} \frac{\partial u_J}{\partial \overline{z}_j} \overline{\frac{\partial u_M}{\partial \overline{z}_k}}\, e^{-\varphi}\, d\lambda$$
$$+ \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \delta_j u_{jK} \overline{\delta_k u_{kK}}\, e^{-\varphi}\, d\lambda,$$

where $\epsilon_{jJ}^{kM} = 0$ if $j \in J$ or $k \in M$ or if $k \cup M \neq j \cup J$, and equals the sign of the permutation $\binom{kM}{jJ}$ otherwise. The right-hand side of the last formula can be rewritten as

$$\sum_{|J|=q}{}' \sum_{j=1}^{n} \left\| \frac{\partial u_J}{\partial \overline{z}_j} \right\|_\varphi^2 + \sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \left(\delta_j u_{jK} \overline{\delta_k u_{kK}} - \frac{\partial u_{jK}}{\partial \overline{z}_k} \overline{\frac{\partial u_{kK}}{\partial \overline{z}_j}} \right) e^{-\varphi}\, d\lambda, \tag{5.8}$$

see [46] Proposition 2.4. Consider first the (nonzero) terms where $j = k$ (and hence $M = J$). These terms result in the portion of the first sum in (5.8) where $j \notin J$. On the other hand, when $j \neq k$, then $j \in M$ and $k \in J$, and deletion of j from M and k from J results in the strictly increasing multi-index K of length $q-1$. Consequently, these terms can be collected into the second sum in (5.8) (the part with the minus sign, we have interchanged the summation indices j and k). In this sum, the terms where

$j = k$ compensate for the terms in the first sum where $j \in J$. Now one can use the same reasoning as in the last proof to get

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 = \sum_{|J|=q}{}' \sum_{j=1}^n \left\| \frac{\partial u_J}{\partial \overline{z}_j} \right\|_\varphi^2 + \sum_{|K|=q-1}{}' \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_{jK} \overline{u}_{kK} \, e^{-\varphi} \, d\lambda. \tag{5.9}$$

Proposition 5.5. *For a plurisubharmonic weight function φ satisfying (**??**), there exists a uniquely determined bounded linear operator*

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}),$$

such that $\square_\varphi \circ N_\varphi u = u$, for any $u \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. If $u \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ satisfies $\overline{\partial} u = 0$, then $\overline{\partial}_\varphi^* N_\varphi u$ is the canonical solution of $\overline{\partial} f = u$, which means that $\overline{\partial}_\varphi^* N_\varphi u \perp A^2(\mathbb{C}^n, \varphi)$, where

$$A^2(\mathbb{C}^n, e^{-\varphi}) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \textit{ entire} : \ f \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})\}.$$

Proof. First we mention that $\square_\varphi$ is a self-adjoint operator, which is proved in a similar way as in the case without weight in Chapter 4.

For a given $v \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ consider the linear functional L on $\operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}_\varphi^*)$ given by $L(u) = (u, v)_\varphi$. Notice that $\operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}_\varphi^*)$ is a Hilbert space with the inner product Q_φ. Since we have by Proposition 5.4

$$|L(u)| = |(u, v)_\varphi| \leq \|u\|_\varphi \, \|v\|_\varphi \leq C Q_\varphi(u, u)^{1/2} \, \|v\|_\varphi.$$

Hence by the Riesz representation theorem there exists a uniquely determined $(0, 1)$-form $N_\varphi v \in \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}_\varphi^*)$ such that

$$(u, v)_\varphi = Q_\varphi(u, N_\varphi v) = (\overline{\partial} u, \overline{\partial} N_\varphi v)_\varphi + (\overline{\partial}_\varphi^* u, \overline{\partial}_\varphi^* N_\varphi v)_\varphi,$$

and we claim that $N_\varphi v \in \operatorname{dom}(\square_\varphi) = \operatorname{dom}(\square_\varphi^*)$, for which we have to show that $w \mapsto (\square_\varphi w, N_\varphi v)_\varphi$ is bounded on $\operatorname{dom}(\square_\varphi)$:

$$\begin{aligned} |(\square_\varphi w, N_\varphi v)_\varphi| &= |(\overline{\partial} w, \overline{\partial} N_\varphi v)_\varphi + (\overline{\partial}_\varphi^* w, \overline{\partial}_\varphi^* N_\varphi v)_\varphi| \\ &= |Q_\varphi(w, N_\varphi v)| = |(w, v)_\varphi| \leq \|w\|_\varphi \|v\|_\varphi, \end{aligned}$$

now we get

$$(u, v)_\varphi = Q_\varphi(u, N_\varphi v) = (u, \square_\varphi N_\varphi v)_\varphi$$

hence $\square_\varphi N_\varphi v = v$, for any $v \in L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. If we set $u = N_\varphi v$ we get again from 5.4

$$\begin{aligned} \|\overline{\partial} N_\varphi v\|_\varphi^2 + \|\overline{\partial}_\varphi^* N_\varphi v\|_\varphi^2 &= Q_\varphi(N_\varphi v, N_\varphi v) = (N_\varphi v, v)_\varphi \leq \|N_\varphi v\|_\varphi \, \|v\|_\varphi \\ &\leq C_1 (\|\overline{\partial} N_\varphi v\|_\varphi^2 + \|\overline{\partial}_\varphi^* N_\varphi v\|_\varphi^2)^{1/2} \, \|v\|_\varphi, \end{aligned}$$

hence

$$(\|\overline{\partial} N_\varphi v\|_\varphi^2 + \|\overline{\partial}_\varphi^* N_\varphi v\|_\varphi^2)^{1/2} \leq C_2 \|v\|_\varphi$$

and finally again by 5.4

$$\|N_\varphi v\|_\varphi \leq C_3 (\|\overline{\partial} N_\varphi v\|_\varphi^2 + \|\overline{\partial}_\varphi^* N_\varphi v\|_\varphi^2)^{1/2} \leq C_4 \|v\|_\varphi,$$

where $C_1, C_2, C_3, C_4 > 0$ are constants. Hence we get that N_φ is a continuous linear operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself (see also [30] or [9]). The rest is clear from the remarks made for the unweighted $\overline{\partial}$- Neumann operator. ☐

In this case one can also show that $N_\varphi = j_\varphi \circ j_\varphi^*$, where

$$j_\varphi : dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$$

is the embedding and $dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$ is endowed with the graph norm

$$u \mapsto (\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}.$$

Remark 5.6. *(a) If condition (??) is satisfied, we can replace the graph norm*

$$u \mapsto (\|u\|^2_\varphi + \|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}$$

by

$$u \mapsto (\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}.$$

(b) If $\Box_\varphi$ has a bounded inverse N_φ, then the basic estimate (5.5) *holds for $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$. This follows from the spectral theorem (see for instance* [48]*): N_φ is a positive, self-adjoint operator which has a uniquely determined bounded root $N_\varphi^{1/2}$, this implies:*

$$\|N_\varphi^{1/2} v\|^2_\varphi \le C\|v\|^2_\varphi,$$

for all $v \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$. Now let $u \in dom\,\Box_\varphi$. It follows that $u \in dom\,\Box_\varphi^{1/2}$ and $v = \Box_\varphi^{1/2} u \in dom\,\Box_\varphi^{1/2}$, see [14] *and we obtain*

$$\|N_\varphi^{1/2} v\|^2_\varphi = \|u\|^2_\varphi \le C\,(\Box_\varphi^{1/2}u, \Box_\varphi^{1/2}u)_\varphi = C\,(\Box_\varphi u, u)_\varphi = C(\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi),$$

for all $u \in dom\,\Box_\varphi$ and by Lemma 5.2 also for $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$.

Proposition 5.7. *Let $1 \le q \le n$ and suppose that the sum s_q of any q (equivalently: the smallest q) eigenvalues of M_φ satisfies*

$$\liminf_{|z|\to\infty} s_q(z) > 0. \tag{5.10}$$

Then there exists a uniquely determined bounded linear operator

$$N_{\varphi,q} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}),$$

such that $\Box_\varphi \circ N_{\varphi,q} u = u$, for any $u \in L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$.

Proof. Let $\mu_{\varphi,1} \le \mu_{\varphi,2} \le \cdots \le \mu_{\varphi,n}$ denote the eigenvalues of M_φ and suppose that M_φ is diagonalized. Then, in a suitable basis,

$$\begin{aligned}
\sum_{|K|=q-1}{}' \sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial\overline{z}_k} u_{jK}\overline{u}_{kK} &= \sum_{|K|=q-1}{}' \sum_{j=1}^n \mu_{\varphi,j}|u_{jK}|^2 \\
&= \sum_{J=(j_1,\dots,j_q)}{}' (\mu_{\varphi,j_1} + \cdots + \mu_{\varphi,j_q})|u_J|^2 \\
&\ge s_q|u|^2
\end{aligned}$$

The last equality results as follows: for $J = (j_1, \dots, j_q)$ fixed, $|u_J|^2$ occurs precisely q times in the second sum, once as $|u_{j_1K_1}|^2$, once as $|u_{j_2K_2}|^2$, etc. At each occurence, it is multiplied by μ_{φ,j_ℓ}. For the rest of the proof proceed as in the proof of Proposition 5.5. ∎

Remark 5.8. *For the $\bar{\partial}$-Neumann operator $N_{\varphi,q}$ on $(0,q)$-forms one obtains in a similar way that $N_{\varphi,q} = j_{\varphi,q} \circ j^*_{\varphi,q}$, where*

$$j_{\varphi,q} : dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_{\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}).$$

6. The twisted $\overline{\partial}$-complex

We will consider the twisted $\overline{\partial}$-complex

$$L^2(\Omega) \xrightarrow{T} L^2_{(0,1)}(\Omega) \xrightarrow{S} L^2_{(0,2)}(\Omega) \tag{6.1}$$

for operators $T = \overline{\partial} \circ \sqrt{\tau}$ and $S = \sqrt{\tau} \circ \overline{\partial}$, where $\tau \in \mathcal{C}^2(\overline{\Omega})$ and $\tau > 0$ on Ω. For further details see [40] or [46].
First we prove a general result about operators like T and S.

Proposition 6.1. *Let H_1, H_2, H_3 be Hilbert spaces and $T : H_1 \longrightarrow H_2$ and $S : H_2 \longrightarrow H_3$ densely defined linear operators, such that $S(T(f)) = 0$, for each $f \in dom(T)$, and let $P : H_2 \longrightarrow H_2$ be a positive invertible operator such that*

$$\|Pu\|_2^2 \leq \|T^*u\|_1^2 + \|Su\|_3^2, \tag{6.2}$$

for all $u \in dom(S) \cap dom(T^*)$, where

$$dom(T^*) = \{u \in H_2 : |(u, Tf)_2| \leq C\,\|f\|_1, \text{ for all } f \in dom(T)\}.$$

Suppose (6.2) holds and let $\alpha \in H_2$, such that $S\alpha = 0$. Then there exists $\sigma \in H_1$, such that (i) $T(\sigma) = \alpha$ and (ii) $\|\sigma\|_1^2 \leq \|P^{-1}\alpha\|_2^2$.

Proof. Since P is positive, it follows that $P = P^*$. Now let $\alpha \in H_2$ be such that $S\alpha = 0$. We consider the linear functional $T^*u \mapsto (u, \alpha)_2$ for $u \in \mathrm{dom}(T^*)$: if $u \in \mathrm{Ker}S$, then

$$\begin{aligned} |(u,\alpha)_2| &= |(Pu, P^{-1}\alpha)_2| \leq \|Pu\|_2\, \|P^{-1}\alpha\|_2 \leq \left(\|T^*u\|_1^2 + \|Su\|_3^2\right)^{1/2}\, \|P^{-1}\alpha\|_2 \\ &= \|T^*u\|_1\, \|P^{-1}\alpha\|_2, \end{aligned}$$

if $u \perp_2 \mathrm{Ker}S$, then $(u, \alpha)_2 = 0$. It also holds that $T^*w = 0$ for all $w \perp_2 \mathrm{Ker}S$, this follows from the assumption that $Tf \in \mathrm{Ker}S$, so $0 = (w, Tf)_2 \leq C\|f\|_1$, which means that $w \in \mathrm{dom}(T^*)$ and $T^*w = 0$, since $(T^*w, f)_1 = (w, Tf)_2 = 0$ for all $f \in \mathrm{dom}(T)$. If $T^*u = 0$, it follows from the above estimate that $(u, \alpha)_2 = 0$.
We apply the Hahn-Banach theorem, where we keep the constant for the estimate of the functional and the Riesz representation theorem to get $\sigma \in H_1$, such that $(T^*u, \sigma)_1 = (u, \alpha)_2$, which implies that $(u, T\sigma)_2 = (u, \alpha)_2$. Hence $T\sigma = \alpha$ and, again by the above estimate $\|\sigma\|_1 \leq \|P^{-1}\alpha\|_2$.

□

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, with defining function r such that $|\nabla r(z)| = 1$ on $b\Omega$. Let $\tau \in \mathcal{C}^2(\overline{\Omega})$ and $\tau > 0$ on Ω. For $f \in \mathcal{C}^\infty(\overline{\Omega})$ we define

$$Tf = (\overline{\partial} \circ \sqrt{\tau})f = \sum_{k=1}^n \frac{\partial}{\partial \overline{z}_k}(\sqrt{\tau} f)\, d\overline{z}_k, \tag{6.3}$$

and for $u = \sum_{j=1}^n u_j\, d\overline{z}_j$ with coefficients u_j in $\mathcal{C}^\infty(\overline{\Omega})$, we will write $u \in \Lambda^{0,1}(\overline{\Omega})$, we define

$$Su = \sum_{j<k} \sqrt{\tau} \left(\frac{\partial u_k}{\partial \overline{z}_j} - \frac{\partial u_j}{\partial \overline{z}_k} \right) d\overline{z}_j \wedge d\overline{z}_k. \tag{6.4}$$

We call τ a twist factor. But we also introduce a weight factor φ : for $f \in \mathcal{C}^\infty(\overline{\Omega})$ and $u \in \mathcal{D}^{0,1}$ (see Proposition 4.7),

$$(Tf, u)_\varphi = (\overline{\partial}(\sqrt{\tau} f), u)_\varphi = (\overline{\partial}(\sqrt{\tau} f), e^{-\varphi} u) = (\sqrt{\tau} f, \overline{\partial}^*(e^{-\varphi} u)) = (f, \sqrt{\tau}\, e^{\varphi}(\overline{\partial}^*(e^{-\varphi} u)))_\varphi,$$

which implies that

$$T^* u = \sqrt{\tau}\, \overline{\partial}^*_\varphi u,$$

where (see Proposition 5.1)

$$\overline{\partial}^*_\varphi u = -\sum_{\ell=1}^n e^{\varphi} \frac{\partial}{\partial z_\ell} \left(e^{-\varphi}\, u_\ell\right) =: -\sum_{\ell=1}^n \delta_\ell\, u_\ell.$$

In the sequel we will use the following equations: let $f, g \in \mathcal{C}^\infty(\overline{\Omega})$

(A) $$(\delta_\ell f, g)_\varphi = -(f, \frac{\partial g}{\partial \overline{z}_\ell})_\varphi + \int_{b\Omega} f\, \overline{g}\, \frac{\partial r}{\partial z_\ell}\, e^{-\varphi}\, d\sigma$$

(B) $$(\frac{\partial f}{\partial \overline{z}_k}, g)_\varphi = -(f, \delta_k g)_\varphi + \int_{b\Omega} f\, \overline{g}\, \frac{\partial r}{\partial \overline{z}_k}\, e^{-\varphi}\, d\sigma$$

(C) $$[\delta_\ell, \frac{\partial}{\partial \overline{z}_k}]\, f = \frac{\partial^2 \varphi}{\partial \overline{z}_k \partial z_\ell}\, f$$

(D) $$\delta_\ell(f\, g) = f(\delta_\ell g) + \frac{\partial f}{\partial z_\ell}\, g$$

We introduce the notations

(6.5) $$i\partial\overline{\partial} g(\xi, \xi)(p) := \sum_{k,\ell=1}^n \frac{\partial^2 g}{\partial \overline{z}_k \partial z_\ell}(p) \overline{\xi}_k \xi_\ell$$

and

(6.6) $$\langle \partial g, \xi \rangle(p) := \sum_{k=1}^n \frac{\partial g}{\partial z_k}(p)\, \xi_k,$$

for $g \in \mathcal{C}^\infty(\Omega)$ and $\xi \in \mathbb{C}^n$.
If $u \in \mathcal{D}^{0,1}$, then we set

(6.7) $$\|\sqrt{\tau} u\|^2_{\varphi, \overline{z}} := \sum_{j,k=1}^n \|\sqrt{\tau}\, \frac{\partial u_j}{\partial \overline{z}_k}\|^2_\varphi.$$

Now we prove the a priori basic estimates

Theorem 6.2. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, with defining function r such that $|\nabla r(z)| = 1$ on $b\Omega$. Let $\tau, \varphi, A \in \mathcal{C}^2(\overline{\Omega})$ and $\tau, A > 0$ on Ω and let $u \in \mathcal{D}^{0,1}$. Then*

(6.8)
$$\|\sqrt{\tau + A}\, \overline{\partial}^*_\varphi u\|^2_\varphi + \|\sqrt{\tau}\, \overline{\partial} u\|^2_\varphi \geq \|\sqrt{\tau} u\|^2_{\varphi, \overline{z}} + \int_\Omega \Theta(u, u)\, e^{-\varphi}\, d\lambda + \int_{b\Omega} \tau i\partial\overline{\partial} r(u, u)\, e^{-\varphi}\, d\sigma,$$

where

(6.9) $$\Theta(u, u) = \tau i\partial\overline{\partial}\varphi(u, u) - i\partial\overline{\partial}\tau(u, u) - \frac{|\langle \partial\tau, u\rangle|^2}{A}.$$

Proof. Like in the untwisted case we get

$$\|\sqrt{\tau}\,\overline{\partial}u\|_\varphi^2 = \sum_{j<k} \|\sqrt{\tau}\left(\frac{\partial u_k}{\partial \overline{z}_j} - \frac{\partial u_j}{\partial \overline{z}_k}\right)\|_\varphi^2 = \sum_{j,k=1}^n \|\sqrt{\tau}\,\frac{\partial u_k}{\partial \overline{z}_j}\|_\varphi^2 - \sum_{j,k=1}^n (\sqrt{\tau}\,\frac{\partial u_k}{\partial \overline{z}_j}, \sqrt{\tau}\,\frac{\partial u_j}{\partial \overline{z}_k})_\varphi$$

and

$$\|\sqrt{\tau}\,\overline{\partial}^*_\varphi u\|_\varphi^2 = \sum_{j,k=1}^n (\sqrt{\tau}\,\delta_k u_k, \sqrt{\tau}\,\delta_j u_j)_\varphi,$$

hence

$$\|\sqrt{\tau}\,\overline{\partial}u\|_\varphi^2 + \|\sqrt{\tau}\,\overline{\partial}^*_\varphi u\|_\varphi^2 = \|\sqrt{\tau}u\|_{\varphi,\overline{z}}^2 + \sum_{j,k=1}^n \left\{(\tau\,\delta_k u_k, \delta_j u_j)_\varphi - (\tau\,\frac{\partial u_k}{\partial \overline{z}_j}, \frac{\partial u_j}{\partial \overline{z}_k})_\varphi\right\},$$

using (A) and (B) and integrating by parts

$$= \|\sqrt{\tau}u\|_{\varphi,\overline{z}}^2 + \sum_{j,k=1}^n \left\{-(\frac{\partial}{\partial \overline{z}_j}(\tau\,\delta_k u_k), u_j)_\varphi + (\delta_k(\tau\,\frac{\partial u_k}{\partial \overline{z}_j}), u_j)_\varphi\right\} + T_1 + T_2,$$

where

$$T_1 = \sum_{j,k=1}^n \int_{b\Omega} \frac{\partial r}{\partial \overline{z}_j}\,\tau(\delta_k u_k)\,\overline{u}_j\,e^{-\varphi}\,d\sigma,$$

and

$$T_2 = -\sum_{j,k=1}^n \int_{b\Omega} \frac{\partial r}{\partial z_k}\,\tau\frac{\partial u_k}{\partial \overline{z}_j}\,\overline{u}_j\,e^{-\varphi}\,d\sigma.$$

Next we obtain by (C) and (D)

$$\|\sqrt{\tau}\,\overline{\partial}u\|_\varphi^2 + \|\sqrt{\tau}\,\overline{\partial}^*_\varphi u\|_\varphi^2$$

$$= \|\sqrt{\tau}u\|_{\varphi,\overline{z}}^2 + \sum_{j,k=1}^n (\tau\delta_k(\frac{\partial u_k}{\partial \overline{z}_j}) + \frac{\partial \tau}{\partial z_k}\frac{\partial u_k}{\partial \overline{z}_j}, u_j)_\varphi - \sum_{j,k=1}^n (\tau\,\frac{\partial}{\partial \overline{z}_j}(\delta_k u_k) + \frac{\partial \tau}{\partial \overline{z}_j}\,\delta_k u_k, u_j)_\varphi + T_1 + T_2$$

$$= \|\sqrt{\tau}u\|_{\varphi,\overline{z}}^2 + \sum_{j,k=1}^n (\tau[\delta_k, \frac{\partial}{\partial \overline{z}_j}]\,u_k, u_j)_\varphi + \sum_{j,k=1}^n (\frac{\partial \tau}{\partial z_k}\frac{\partial u_k}{\partial \overline{z}_j} - \frac{\partial \tau}{\partial \overline{z}_j}\,\delta_k u_k, u_j)_\varphi + T_1 + T_2$$

$$= \underbrace{\|\sqrt{\tau}u\|_{\varphi,\overline{z}}^2}_{(A1)} + \underbrace{\int_\Omega \tau\, i\partial\overline{\partial}\varphi(u,u)\; e^{-\varphi}\,d\lambda}_{(A2)} - \sum_{j,k=1}^n (\delta_k u_k, \frac{\partial \tau}{\partial z_j}\,u_j)_\varphi + \sum_{j,k=1}^n (\frac{\partial u_k}{\partial \overline{z}_j}, \frac{\partial \tau}{\partial \overline{z}_k}\,u_j)_\varphi + \underbrace{T_1 + T_2}_{(A3)}$$

$$= (A1) + (A2) + (A3) - \sum_{j,k=1}^n \left\{(\delta_k u_k, \frac{\partial \tau}{\partial z_j}\,u_j)_\varphi + (u_k, \delta_j(\frac{\partial \tau}{\partial \overline{z}_k}\,u_j))_\varphi\right\} + T_3$$

where

$$T_3 = \sum_{j,k=1}^n \int_{b\Omega} \frac{\partial r}{\partial \overline{z}_j}\,u_k\,\frac{\partial \tau}{\partial z_k}\,\overline{u}_j\,e^{-\varphi}\,d\sigma.$$

Now we get

$$\|\sqrt{\tau}\,\overline{\partial}u\|_\varphi^2 + \|\sqrt{\tau}\,\overline{\partial}^*_\varphi u\|_\varphi^2$$

$$= (A1) + (A2) + (A3) + T_3 - \left\{\sum_{j,k=1}^n (\delta_k u_k, \frac{\partial \tau}{\partial z_j}\,u_j)_\varphi + \sum_{j,k=1}^n (u_k, \frac{\partial \tau}{\partial \overline{z}_k}\,\delta_j u_j + \frac{\partial^2 \tau}{\partial \overline{z}_k \partial z_j}\,u_j)_\varphi\right\}$$

$$= (A1) + (A2) + (A3) + T_3 - \int_\Omega i\partial\overline{\partial}\tau(u,u)\, e^{-\varphi}\, d\lambda - 2\Re \sum_{j,k=1}^n (\delta_k u_k, \frac{\partial \tau}{\partial z_j}\, u_j)_\varphi.$$

We now estimate the last term :

$$\left| -2\Re \sum_{j,k=1}^n (\delta_k u_k, \frac{\partial \tau}{\partial z_j}\, u_j)_\varphi \right| = \left| -2\Re \int_\Omega \sqrt{A}\, \overline{\partial}^*_\varphi u\, \frac{\langle \partial\tau, u\rangle}{\sqrt{A}}\, e^{-\varphi}\, d\lambda \right|$$

$$\leq 2\|\sqrt{A}\, \overline{\partial}^*_\varphi u\|_\varphi\, \|\langle \partial\tau, u\rangle/\sqrt{A}\|_\varphi \leq \|\sqrt{A}\, \overline{\partial}^*_\varphi u\|^2_\varphi + \|\langle \partial\tau, u\rangle/\ \sqrt{A}\|^2_\varphi,$$

which means that

$$-2\Re \sum_{j,k=1}^n (\delta_k u_k, \frac{\partial \tau}{\partial z_j}\, u_j)_\varphi \geq -\|\sqrt{A}\, \overline{\partial}^*_\varphi u\|^2_\varphi - \|\langle \partial\tau, u\rangle/\ \sqrt{A}\|^2_\varphi,$$

now we move the first term in the last expression to the other side and get the desired result, since

$$\|\sqrt{\tau + A}\, \overline{\partial}^*_\varphi u\|^2_\varphi = \int_\Omega (\tau + A)|\overline{\partial}^*_\varphi u|^2\, e^{-\varphi}\, d\lambda,$$

and $T_1 = T_3 = 0$ for $u \in \mathcal{D}^{0,1}$, and

$$T_2 = \sum_{j,k=1}^n \int_{b\Omega} \tau\, \frac{\partial^2 r}{\partial \overline{z}_j \partial z_k}\, \overline{u}_j u_k\, e^{-\varphi}\, d\sigma = \int_{b\Omega} \tau\, i\, \partial\overline{\partial} r(u,u)\, e^{-\varphi}\, d\sigma.$$

$\square$

Here we apply the basic estimates to get the Hörmander L^2-estimates for the $\overline{\partial}$-equation (see [30]) and a result of Shigekawa on the dimension of weighted spaces of entire functions ([45]).

Theorem 7.1. *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain such that $\Omega \subseteq B(0,R)$. Then for each $u \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ we have*

$$\|\overline{\partial}^* u\|^2 + \|\overline{\partial} u\|^2 \geq \frac{1}{4R^2}\,\|u\|^2. \tag{7.1}$$

Proof. Since Ω is pseudoconvex, the boundary integral in Theorem 6.2 is ≥ 0. Take $\varphi = 0$ and $\tau(z) = R^2 - |z|^2$, then $-i\,\partial\overline{\partial}\tau(u,u) = |u|^2$ and $|\langle \partial\tau, u\rangle|^2 = |\sum_{j=1}^n \overline{z}_j\, u_j|^2 \leq |z|^2\,|u|^2$. We choose $A = 2|z|^2$, then $|\langle \partial\tau, u\rangle|^2/A \leq \frac{1}{2}\,|u|^2$. Hence we get from Theorem 6.2 :

$$\|\sqrt{R^2+|z|^2}\,\overline{\partial}^* u\|^2 + \|\sqrt{R^2-|z|^2}\,\overline{\partial} u\|^2 \geq \frac{1}{2}\,\|u\|^2,$$

for $u \in \mathcal{D}^{0,1}$. Now, the result follows from Proposition 4.12.

□

Corollary 7.2. *Let Ω be as in Theorem 7.1 and let $\alpha \in L^2_{(0,1)}(\Omega)$ such that $\overline{\partial}\alpha = 0$. Then there exists $s \in L^2(\Omega)$ such that $\overline{\partial} s = \alpha$ and*

$$\int_\Omega |s|^2\,d\lambda \leq 4R^2 \int_\Omega |\alpha|^2\,d\lambda. \tag{7.2}$$

Proof. Apply Proposition 6.1 for $T = \overline{\partial} \circ \sqrt{R^2+|z|^2}$ and $S = \sqrt{R^2-|z|^2} \circ \overline{\partial}$, and set $P = 1/\sqrt{2}\,\mathrm{Id}$. Then we have $T^* = \sqrt{R^2+|z|^2} \circ \overline{\partial}^*$ and Theorem 7.1 gives

$$\|Pu\|^2 \leq \|T^*u\|^2 + \|Su\|^2,$$

by Proposition 6.1 we obtain $\sigma \in L^2(\Omega)$ such that $T\sigma = \overline{\partial}(\sqrt{R^2+|z|^2}\,\sigma) = \alpha$ and $\|\sigma\|^2 \leq 2\|\alpha\|^2$. Now set $s = \sqrt{R^2+|z|^2}\,\sigma$, then $\overline{\partial} s = \alpha$ and

$$\int_\Omega \frac{|s|^2}{R^2+|z|^2}\,d\lambda \leq 2\int_\Omega |\alpha|^2\,d\lambda,$$

so we get

$$\frac{1}{2R^2}\int_\Omega |s|^2\,d\lambda \leq 2\int_\Omega |\alpha|^2\,d\lambda.$$

□

Theorem 7.3. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a real valued function in $\mathcal{C}^2(\mathbb{C}^n)$ such that*

$$c(z)\sum_{j=1}^n |w_j|^2 \leq \sum_{j,k=1}^n \frac{\partial^2\varphi(z)}{\partial z_j \partial\overline{z}_k}\,w_j\overline{w}_k, \quad z \in \mathbb{C}^n,\ w \in \mathbb{C}^n,$$

where c is a positive continuous function in $\mathbb{C}^n$. If $g \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ and $\overline{\partial} g = 0$, it follows that one can find $f \in L^2(\mathbb{C}^n, e^{-\varphi})$ with $\overline{\partial} f = g$ and

$$\int_{\mathbb{C}^n} |f(z)|^2\,e^{-\varphi(z)}\,d\lambda(z) \leq \int_{\mathbb{C}^n} |g(z)|^2\,\frac{e^{-\varphi(z)}}{c(z)}\,d\lambda(z), \tag{7.3}$$

provided that the right hand side is finite.

Proof. From Proposition 5.4 we have

$$\int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi} \, d\lambda \leq \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2,$$

for each $(0,1)$-form $u = \sum_{k=1}^n u_k \, d\overline{z}_k$ with coefficients $u_k \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, for $k = 1, \dots, n$.
Let $P : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ be the multiplication operator by the function $\sqrt{c}$. Then it follows from the assumption that

$$\|Pu\|_\varphi^2 \leq \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2.$$

By Proposition 6.1 we get a function $f \in L^2(\mathbb{C}^n, e^{-\varphi})$ with $\overline{\partial} f = g$ and $\|f\|_\varphi \leq \|P^{-1} g\|_\varphi$. $\square$

Theorem 7.4. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a plurisubharmonic function in $\mathcal{C}^2(\mathbb{C}^n)$. If $g \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ and $\overline{\partial} g = 0$, it follows that one can find a solution u of $\overline{\partial} u = g$ such that*

$$2 \int_{\mathbb{C}^n} |u(z)|^2 \, e^{-\varphi(z)} \, (1 + |z|^2)^{-2} \, d\lambda(z) \leq \int_{\mathbb{C}^n} |g(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z). \tag{7.4}$$

Proof. We apply Theorem 7.3 with φ replaced by $\varphi + 2\log(1 + |z|^2)$ and use that

$$\sum_{j,k=1}^n w_j \overline{w}_k \, \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log(1 + |z|^2) = (1 + |z|^2)^{-2}(|w|^2(1 + |z|^2) - |(w, z)|^2) \geq (1 + |z|^2)^{-2} |w|^2,$$

so we can take $c(z) = 2(1 + |z|^2)^{-2}$ to obtain the desired result. $\square$

Theorem 7.5. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi : \Omega \longrightarrow \mathbb{R}$ be a real valued function in $\mathcal{C}^2(\Omega)$ such that*

$$c(z) \sum_{j=1}^n |w_j|^2 \leq \sum_{j,k=1}^n \frac{\partial^2 \varphi(z)}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k, \quad z \in \Omega, \ w \in \mathbb{C}^n,$$

where c is a positive continuous function on Ω. If $g \in L^2_{(0,1)}(\Omega, e^{-\varphi})$ and $\overline{\partial} g = 0$, it follows that one can find $f \in L^2(\Omega, e^{-\varphi})$ with $\overline{\partial} f = g$ and

$$\int_\Omega |f(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z) \leq 2 \int_\Omega |g(z)|^2 \, \frac{e^{-\varphi(z)}}{c(z)} \, d\lambda(z), \tag{7.5}$$

provided that the right hand side is finite.

Proof. We use Theorem 6.2 for $\tau = A = 1$ and get

$$\int_\Omega \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi} \, d\lambda \leq \|\overline{\partial} u\|_\varphi^2 + 2\|\overline{\partial}_\varphi^* u\|_\varphi^2,$$

for $u \in \mathcal{D}^{0,1}$.
Let $P : L^2_{(0,1)}(\Omega, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\Omega, e^{-\varphi})$ be the multiplication operator by the function $\sqrt{c}$. Then it follows from the assumption that

$$\|Pu\|_\varphi^2 \leq \|\overline{\partial} u\|_\varphi^2 + 2\|\overline{\partial}_\varphi^* u\|_\varphi^2.$$

By Proposition 6.1 we get a function $f \in L^2(\Omega, e^{-\varphi})$ with $\overline{\partial} f = g$ and $\|f\|_\varphi \leq \sqrt{2} \|P^{-1} g\|_\varphi$. $\square$

For a positive $\psi = \sum_{j,k=1}^n \psi_{j,k}\, dz_j \wedge d\overline{z}_k \in \Lambda^{1,1}(\Omega)$ and $\alpha = \sum_{j=1}^n \alpha_j\, d\overline{z}_j \in \Lambda^{0,1}(\Omega)$ we set

$$|\alpha|_\psi^2 := \sum_{j,k=1}^n \psi^{j,k}\, \alpha_j \overline{\alpha}_k, \tag{7.6}$$

where $\psi^{j,k} = (\psi_{\ell,m})^{-1}_{j,k}$. Then we get a more general result

Theorem 7.6. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi : \Omega \longrightarrow \mathbb{R}$ a strictly plurisubharmonic function belonging to $\mathcal{C}^2(\Omega)$. If $\alpha \in L^2_{(0,1)}(\Omega, e^{-\varphi})$ satisfies $\overline{\partial}\alpha = 0$, then one can find $u \in L^2(\Omega, \varphi)$ such that $\overline{\partial}u = \alpha$ and*

$$\int_\Omega |u(z)|^2\, e^{-\varphi(z)}\, d\lambda(z) \leq 2 \int_\Omega |\alpha(z)|^2_{i\partial\overline{\partial}\varphi}\, e^{-\varphi(z)}\, d\lambda(z), \tag{7.7}$$

provided that the right hand side is finite.

We return again to $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ and remark that the Kohn-Morrey formula from Proposition 5.4 can be written in the form

$$(M_\varphi u, u)_\varphi \leq (\Box_\varphi u, u)_\varphi$$

for a $(0,1)$-form $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$. So, under the assumptions of Theorem 7.3, we obtain using Ruelle's lemma (see Appendix E) that

$$(N_\varphi u, u)_\varphi \leq (M_\varphi^{-1} u, u)_\varphi,$$

setting $\overline{\partial}v = u$ we get

$$\|v\|_\varphi^2 = (v, v)_\varphi = (v, \overline{\partial}^*_\varphi N_\varphi u)_\varphi = (\overline{\partial}v, N_\varphi u)_\varphi = (u, N_\varphi u)_\varphi \leq (M_\varphi^{-1}\overline{\partial}v, \overline{\partial}v)_\varphi$$

for each $v \in dom\,(\overline{\partial})$ orthogonal to $ker\,(\overline{\partial})$.
This gives a different proof of Hörmander's L^2-estimates similar to the Brascamp-Lieb inequality (see [26] and [32]):

$$\int_{\mathbb{C}^n} |v(z)|^2\, e^{-\varphi(z)}\, d\lambda(z) \leq \int_{\mathbb{C}^n} |\overline{\partial}v(z)|^2_{i\partial\overline{\partial}\varphi}\, e^{-\varphi(z)}\, d\lambda(z), \tag{7.8}$$

for each $v \in dom\,(\overline{\partial})$ orthogonal to $ker\,(\overline{\partial})$.

Let $1 \leq q \leq n$. If u is a $(0,q)$-form in $dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$, we get by (5.9)

$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^n \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, u_{jK}\, \overline{u}_{kK}\, e^{-\varphi}\, d\lambda \leq (\Box_\varphi u, u)_\varphi.$$

The left hand side can be written in the form $(\tilde{M}_\varphi u, u)_\varphi$. We suppose that $\tilde{M}_\varphi$ is invertible and get as above

$$\|v\|_\varphi^2 \leq (\tilde{M}_\varphi^{-1}\overline{\partial}v, \overline{\partial}v)_\varphi \tag{7.9}$$

for each $(0, q-1)$-form $v \in dom\,(\overline{\partial})$ orthogonal to $ker\,(\overline{\partial})$. For a differential geometric interpretation of $\tilde{M}_\varphi$ see Appendix B.

Theorem 7.7. *Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi : \Omega \longrightarrow \mathbb{R}$ be a plurisubharmonic function in $\mathcal{C}^2(\Omega)$. For every $g \in L^2_{(0,1)}(\Omega, e^{-\varphi})$ with $\overline{\partial} g = 0$ there is a solution $u \in L^2(\Omega, loc)$ of the equation $\overline{\partial} u = g$ such that*

$$\int_\Omega |u(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{-2} \, d\lambda(z) \le \int_\Omega |g(z)|^2 e^{-\varphi(z)} \, d\lambda(z). \tag{7.10}$$

Proof. We apply Theorem7.5 with φ replaced by $\varphi + 2\log(1+|z|^2)$ and use that

$$\sum_{j,k=1}^n w_j \overline{w}_k \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log(1+|z|^2) = (1+|z|^2)^{-2}(|w|^2(1+|z|^2) - |(w,z)|^2) \ge (1+|z|^2)^{-2}|w|^2,$$

so we can take $c(z) = 2(1+|z|^2)^{-2}$ to obtain the desired result.

□

Theorem 7.8. *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi : \Omega \longrightarrow \mathbb{R}$ be a plurisubharmonic function in $\mathcal{C}^2(\Omega)$. If $z_0 \in \Omega$ and $e^{-\varphi}$ is integrable in a neighborhood of z_0 one can find a holomorphic function u in Ω such that $u(z_0) = 1$ and*

$$\int_\Omega |u(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{-3n} \, d\lambda(z) < \infty. \tag{7.11}$$

Proof. We may assume that $z_0 = 0$. Choose a polydisc

$$D = \{z \,:\, |z_j| < r, \, j = 1, \dots, n\} \subset \Omega$$

where $e^{-\varphi}$ is integrable, and define

$$\Omega_k := \{z \in \Omega \,:\, |z_j| < r \text{ for } j > k\},$$

for $k = 0, 1, \dots, n$.

We shall prove inductively that for every k there is a holomorphic function u_k in Ω_k with $u_k(z_0) = 1$ and

$$\int_{\Omega_k} |u_k(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{-3k} \, d\lambda(z) < \infty.$$

When $k = 0$ we can take $u_0(z) \equiv 1$, and u_n will have the desired properties.

Assume that $0 < k \le n$ and that u_{k-1} has already been constructed. Choose $\psi \in \mathcal{C}_0^\infty(\mathbb{C})$ so that $\psi(z_k) = 0$ when $|z_k| > r/2$ and $\psi(z_k) = 1$ when $|z_k| < r/3$, and set

$$u_k(z) := \psi(z_k) u_{k-1}(z) - z_k v(z),$$

notice that $\psi(z_k) u_{k-1}(z) = 0$ in $\Omega_k \setminus \Omega_{k-1}$. To make u_k holomorphic we must choose v as a solution of the equation $\overline{\partial} v = z_k^{-1} u_{k-1} \overline{\partial} \psi = f$. By the inductive hypothesis we have

$$\int_{\Omega_k} |f(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{-3(k-1)} \, d\lambda(z) < \infty.$$

Hence it follows from Theorem 7.7 that v can be found so that

$$\int_{\Omega_k} |v(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{1-3k} \, d\lambda(z) < \infty.$$

Together with the inductive hypothesis on u_{k-1} this implies that

$$\int_{\Omega_k} |u_k(z)|^2 e^{-\varphi(z)} (1+|z|^2)^{-3k} \, d\lambda(z) < \infty.$$

Since $\overline{\partial} v = 0$ in a neighborhood of 0, v is a $\mathcal{C}^\infty$-function there and we have $u_k(0) = u_{k-1}(0) = 1$ so u_k has the required properties.

$\square$

Lemma 7.9. *Let $\zeta \in \mathbb{C}^n$ and $K > 0$ and define $g(z) = \log(1 + K|z-\zeta|^2)$. Then for each $w \in \mathbb{C}^n$ we have*

$$\frac{K}{(1+K|z-\zeta|^2)^2}\,|w|^2 \le i\partial\overline{\partial}g(w,w)(z) \le \frac{K}{1+K|z-\zeta|^2}\,|w|^2. \tag{7.12}$$

Proof. An easy computation shows

$$\begin{aligned}\frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(z) &= -\frac{K^2(\overline{z}_j - \overline{\zeta}_j)(z_k - \zeta_k)}{(1+K|z-\zeta|^2)^2} + \frac{K\delta_{jk}}{1+K|z-\zeta|^2} \\ &= \frac{K}{(1+K|z-\zeta|^2)^2}\,[(1+K|z-\zeta|^2)\,\delta_{jk} - K(\overline{z}_j - \overline{\zeta}_j)(z_k-\zeta_k)]\end{aligned}$$

This implies

$$i\partial\overline{\partial}g(w,w)(z) = \frac{K}{(1+K|z-\zeta|^2)^2}\,[(1+K|z-\zeta|^2)\,|w|^2 - K|(w,z-\zeta)|^2]$$

and hence

$$\begin{aligned}&\frac{K}{(1+K|z-\zeta|^2)^2}\,[(1+K|z-\zeta|^2)\,|w|^2 - K|w|^2|z-\zeta)|^2] \\ &\qquad \le i\partial\overline{\partial}g(w,w)(z) \le \frac{K}{1+K|z-\zeta|^2}\,|w|^2.\end{aligned}$$

$\square$

We are now able to show the following

Theorem 7.10. *Let $W : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^\infty$ function and let $\mu(z)$ denote the lowest eigenvalue of the Levi matrix*

$$i\partial\overline{\partial}W(z) = \left(\frac{\partial^2 W(z)}{\partial z_j \partial \overline{z_k}}\right)_{j,k=1}^n.$$

Suppose that

$$\lim_{|z|\to\infty} |z|^2\mu(z) = \infty. \tag{7.13}$$

Then the Hilbertspace $A^2(\mathbb{C}^n, e^{-2W})$ of all entire functions f such that

$$\int_{\mathbb{C}^n} |f(z)|^2 \exp(-2W(z))\,d\lambda(z) < \infty,$$

is of infinite dimension.

Proof. Assumption (7.13) implies that there exists a constant $K > 0$ such that

$$i\partial\overline{\partial}W(w,w)(z) \ge -K|w|^2,$$

for all $z, w \in \mathbb{C}^n$, and that $i\partial\overline{\partial}W(z)$ is strictly positive for large $|z|$.
From Lemma 7.9 we have

$$i\partial\overline{\partial}g(w,w)(z) \ge \frac{8K}{(1+8K|z-\zeta|^2)^2}\,|w|^2,$$

where $g(z) = \log(1 + 8K|z-\zeta|^2)$.
Hence, for $|z-\zeta| \le 1/\sqrt{8K}$, we have

$$i\partial\overline{\partial}g(w,w)(z) \ge 2K|w|^2.$$

Since $i\partial\overline{\partial}W(w,w)(z)$ is negative on a compact set in $\mathbb{C}^n$ there exist finitely many points $\zeta_1,\dots,\zeta_M \in \mathbb{C}^n$ such that this compact set is covered by the balls $\{z \,:\, |z-\zeta_l| < 1/\sqrt{8K}\}$. Hence

$$\tilde{\varphi}(z) := 2W(z) + \sum_{l=1}^{M} g_l(z)$$

is strictly plurisubharmonic, where $g_l(z) = \log(1 + 8K|z-\zeta_l|^2)\ ,\ l = 1,\dots,M$.
Let $\tilde{\mu}(z)$ be the least eigenvalue of $i\partial\overline{\partial}\tilde{\varphi}$. Then, by assumption (7.13), we have

$$\lim_{|z|\to\infty} |z|^2 \tilde{\mu}(z) = \infty.$$

For each $N \in \mathbb{N}$ there exists $R > 0$ such that

$$\tilde{\mu}(z) \geq \frac{N+M+1}{|z|^2} \quad , \text{ for } |z| > R.$$

Let $\tilde{\mu}_0 := \inf\{\tilde{\mu}(z) \,:\, |z| \leq R\}$. Then $\tilde{\mu}_0 > 0$. Set

$$\kappa = \frac{\tilde{\mu}_0}{2(N+M)}$$

and

$$\varphi(z) := 2W(z) + \sum_{l=1}^{M} g_l(z) - (N+M)\log(1+\kappa|z|^2).$$

It follows that $e^{-\varphi}$ is locally integrable.
Next we claim that φ is strictly plurisubharmonic. Notice that

$$i\partial\overline{\partial}\varphi(w,w)(z) \geq |w|^2 \left(\tilde{\mu}(z) - \frac{(N+M)\kappa}{1+\kappa|z|^2} \right).$$

For $|z| \leq R$ we have

$$\tilde{\mu}(z) - \frac{(N+M)\kappa}{1+\kappa|z|^2} \geq \tilde{\mu}_0 - (N+M)\kappa = \tilde{\mu}_0 - \frac{(N+M)\tilde{\mu}_0}{2(N+M)} = \frac{\tilde{\mu}_0}{2} > 0$$

and for $|z| > R$ we have

$$\tilde{\mu}(z) - \frac{(N+M)\kappa}{1+\kappa|z|^2} \geq \frac{N+M+1}{|z|^2} - \frac{N+M}{|z|^2} = \frac{1}{|z|^2},$$

which implies that φ is strictly plurisubharmonic.
Therefore we can apply Theorem 7.8 and get an entire function f with $f(0) = 1$ and

$$\int_{\mathbb{C}^n} |f(z)|^2 (1+|z|^2)^{-3n}\, e^{-\varphi(z)}\, d\lambda(z) < \infty.$$

Now we set $\tilde{N} = N - 3n$ and we get

$$
\begin{aligned}
&\int_{\mathbb{C}^n} |f(z)|^2 (1+|z|^2)^{\tilde{N}} \, e^{-2W(z)} \, d\lambda(z) \\
&\qquad = \int_{\mathbb{C}^n} \frac{\prod_{l=1}^{M}(1+8K|z-\zeta_l|^2)}{(1+\kappa|z|^2)^{N+M}} \, |f(z)|^2 (1+|z|^2)^{\tilde{N}} \, e^{-\varphi(z)} \, d\lambda(z) \\
&\qquad \le \sup_{z\in\mathbb{C}^n} \left\{ \frac{(1+|z|^2)^N \prod_{l=1}^{M}(1+8K|z-\zeta_l|^2)}{(1+\kappa|z|^2)^{N+M}} \right\} \int_{\mathbb{C}^n} |f(z)|^2 (1+|z|^2)^{-3n} \, e^{-\varphi(z)} \, d\lambda(z) \\
&\qquad < \infty.
\end{aligned}
$$

Hence $fp \in A^2(\mathbb{C}^n, e^{-2W})$ for any polynomial p of degree $< \tilde{N}$, and since $N = \tilde{N} + 3n$ was arbitrary, we are done.

□

The following example in $\mathbb{C}^2$ shows that 7.10 is not sharp.
Let $\varphi(z,w) = |z|^2|w|^2 + |w|^4$. In this case we have that $A^2(\mathbb{C}^2, e^{-\varphi})$ contains all the functions $f_k(z,w) = w^k$ for $k \in \mathbb{N}$, since

$$
\begin{aligned}
\int_0^\infty \int_0^\infty r_2^{2k} \, e^{-(r_1^2 r_2^2 + r_2^4)} \, r_1 r_2 \, dr_1 \, dr_2 &= \int_0^\infty \left(\int_0^\infty r_1 r_2^2 \, e^{-r_1^2 r_2^2} \, dr_1 \right) r_2^{2k-1} \, e^{-r_2^4} \, dr_2 \\
= \int_0^\infty \left(\frac{1}{2} \int_0^\infty e^{-s} \, ds \right) r_2^{2k-1} \, e^{-r_2^4} \, dr_2 &= \frac{1}{2} \int_0^\infty r_2^{2k-1} \, e^{-r_2^4} \, dr_2 < \infty.
\end{aligned}
$$

The Levi matrix of φ has the form

$$
i\partial\overline{\partial}\varphi = \begin{pmatrix} |w|^2 & \overline{z}w \\ \overline{w}z & |z|^2 + 4|w|^2 \end{pmatrix}
$$

hence φ is plurisubharmonic and the lowest eigenvalue has the form

$$
\begin{aligned}
\mu_\varphi(z,w) &= \frac{1}{2} \left(5|w|^2 + |z|^2 - \sqrt{9|w|^4 + 10|z|^2|w|^2 + |z|^4} \right) \\
&= \frac{16|w|^4}{2\left(5|w|^2 + |z|^2 + \sqrt{9|w|^4 + 10|z|^2|w|^2 + |z|^4} \right)},
\end{aligned}
$$

hence

$$
\lim_{|z|\to\infty} |z|^2 \mu_\varphi(z,0) = 0,
$$

which implies that condition (7.13) of Theorem 7.10 is not satisfied.

Let φ be a subharmonic $\mathcal{C}^2$-function. We want to solve $\overline{\partial} u = f$ for $f \in L^2(\mathbb{C}, e^{-\varphi})$. The canonical solution operator to $\overline{\partial}$ gives a solution with minimal $L^2(\mathbb{C}, e^{-\varphi})$-norm. We substitute $v = u\, e^{-\varphi/2}$ and $g = f\, e^{-\varphi/2}$ and the equation becomes

$$\overline{D} v = g \ ,$$

where

$$\overline{D} = e^{-\varphi/2} \, \frac{\partial}{\partial \overline{z}} \, e^{\varphi/2}. \tag{8.1}$$

u is the minimal solution to the $\overline{\partial}$-equation in $L^2(\mathbb{C}, e^{-\varphi})$ if and only if v is the solution to $\overline{D} v = g$ which is minimal in $L^2(\mathbb{C})$.
The formal adjoint of $\overline{D}$ is $\overline{D}^* = -e^{\varphi/2} \frac{\partial}{\partial z} e^{-\varphi/2}$. We define $\mathrm{dom}(\overline{D}) = \{f \in L^2(\mathbb{C}) \ : \ \overline{D} f \in L^2(\mathbb{C})\}$ and likewise for $\overline{D}^*$. Then $\overline{D}$ and $\overline{D}^*$ are closed unbounded linear operators from $L^2(\mathbb{C})$ to itself. Further we define $\mathrm{dom}(\overline{D}\,\overline{D}^*) = \{u \in \mathrm{dom}(\overline{D}^*) \ : \ \overline{D}^* u \in \mathrm{dom}(\overline{D})\}$ and we define $\overline{D}\,\overline{D}^*$ as $\overline{D} \circ \overline{D}^*$ on this domain. Any function of the form $e^{\varphi/2}\, g$, with $g \in \mathcal{C}^2_0(\mathbb{C})$ belongs to $\mathrm{dom}(\overline{D}\,\overline{D}^*)$ and hence $\mathrm{dom}(\overline{D}\,\overline{D}^*)$ is dense in $L^2(\mathbb{C})$. Since $\overline{D} = \frac{\partial}{\partial \overline{z}} + \frac{1}{2} \frac{\partial \varphi}{\partial \overline{z}}$ and $\overline{D}^* = -\frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial \varphi}{\partial z}$ we see that

$$\mathcal{S} = \overline{D}\,\overline{D}^* = -\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{1}{2} \frac{\partial \varphi}{\partial \overline{z}} \frac{\partial}{\partial z} + \frac{1}{2} \frac{\partial \varphi}{\partial z} \frac{\partial}{\partial \overline{z}} + \frac{1}{4} \left| \frac{\partial \varphi}{\partial z} \right|^2 + \frac{1}{2} \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \ . \tag{8.2}$$

For further details see [10] and [24], [26]. It is easily seen that $\mathcal{S}$ is a Schrödinger operator with magnetic field :

$$\mathcal{S} = \frac{1}{4} \, (-\Delta_A + B), \tag{8.3}$$

where the 1-form $A = A_1 \, dx + A_2 \, dy$ is related to the weight φ by

$$A_1 = -\partial_y \varphi / 2 \, , \ A_2 = \partial_x \varphi / 2 \, ,$$

$$\Delta_A = \left(\frac{\partial}{\partial x} - i A_1 \right)^2 + \left(\frac{\partial}{\partial y} - i A_2 \right)^2 \ , \tag{8.4}$$

and the magnetic field $B dx \wedge dy$ satisfies

$$B(x, y) = \frac{1}{2} \Delta \varphi(x, y) \ . \tag{8.5}$$

Both operators $\overline{D}\,\overline{D}^*$ and $\overline{D}^*\,\overline{D}$ are non-negative, self-adjoint operators, see Lemma 13.10 and Lemma 13.11.
Since $4 \overline{D}\,\overline{D}^* = -\Delta_A + \frac{1}{2} \Delta \varphi$, it follows that $((-\Delta_A + \frac{1}{2} \Delta \varphi) f, f) \geq 0$, for $f \in \mathcal{C}^2_0(\mathbb{C})$. Similarly one shows that $4 \overline{D}^*\,\overline{D} = -\Delta_A - \frac{1}{2} \Delta \varphi$, and this implies, using the standard comparison between self-adjoint operators ($T \geq S$, if $(Tf, f) \geq (Sf, f)$):

$$-2 \Delta_A \geq -\Delta_A + \frac{1}{2} \Delta \varphi \geq -\Delta_A \, . \tag{8.6}$$

It follows that

$$\overline{D}^*\,\overline{D} = e^{-\varphi/2} \, \overline{\partial}^*_{\varphi} \, \overline{\partial} \ e^{\varphi/2}, \tag{8.7}$$

and that

$$\overline{D}\,\overline{D}^* = e^{-\varphi/2}\,\overline{\partial}\,\overline{\partial}^*_\varphi\; e^{\varphi/2}, \tag{8.8}$$

where $\overline{\partial}^*_\varphi = -\frac{\partial}{\partial z} + \frac{\partial\varphi}{\partial z}$. For $n = 1$ we have

$$\Box_\varphi = \overline{\partial}\,\overline{\partial}^*_\varphi, \tag{8.9}$$

which means that

$$\overline{D}\,\overline{D}^* = e^{-\varphi/2}\,\Box_\varphi\; e^{\varphi/2}. \tag{8.10}$$

Now we can apply 5.5 to get

Theorem 8.1. *Let φ be a subharmonic $\mathcal{C}^2$-function on $\mathbb{C}$ such that*

$$\liminf_{|z|\to\infty} \Delta\varphi(z) > 0.$$

Then the Schrödinger operator

$$\mathcal{S} = \overline{D}\,\overline{D}^* = \frac{1}{4}(-\Delta_A + \frac{1}{2}\Delta\varphi) \tag{8.11}$$

has a bounded inverse on $L^2(\mathbb{C})$

$$(\overline{D}\,\overline{D}^*)^{-1} = e^{-\varphi/2}\,N_\varphi\; e^{\varphi/2}, \tag{8.12}$$

where $N_\varphi = \Box_\varphi^{-1}$.

For several complex variables the situation is more complicated.
Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-weight function. We consider the $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\overline{\partial}} L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \xrightarrow{\overline{\partial}} L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi})\,. \tag{8.13}$$

For $v \in L^2(\mathbb{C}^n)$, let

$$\overline{D}_1 v = \sum_{k=1}^n \left(\frac{\partial v}{\partial \overline{z}_k} + \frac{1}{2}\frac{\partial\varphi}{\partial\overline{z}_k}\, v\right) d\overline{z}_k \tag{8.14}$$

and for $g = \sum_{j=1}^n g_j\, d\overline{z}_j \in L^2_{(0,1)}(\mathbb{C}^n)$, let

$$\overline{D}_1^* g = \sum_{j=1}^n \left(\frac{1}{2}\frac{\partial\varphi}{\partial z_j}\, g_j - \frac{\partial g_j}{\partial z_j}\right), \tag{8.15}$$

where the derivatives are taken in the sense of distributions.
It is easy to see that $\overline{\partial}u = f$ for $u \in L^2(\mathbb{C}^n, e^{-\varphi})$ and $f \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ if and only if $\overline{D}_1 v = g$, where $v = u\,e^{-\varphi/2}$ and $g = f\,e^{-\varphi/2}$. It is also clear that the necessary condition $\overline{\partial}f = 0$ for solvability holds if and only if $\overline{D}_2 g = 0$ holds. Here

$$\overline{D}_2 g = \sum_{j,k=1}^n \left(\frac{\partial g_j}{\partial\overline{z}_k} + \frac{1}{2}\frac{\partial\varphi}{\partial\overline{z}_k}\, g_j\right) d\overline{z}_k \wedge d\overline{z}_j. \tag{8.16}$$

and

$$\overline{D}_2^* h = \sum_{j,k=1}^n \left(\frac{1}{2}\frac{\partial\varphi}{\partial z_k}\, h_{kj} - \frac{\partial h_{kj}}{\partial z_k}\right) d\overline{z}_j. \tag{8.17}$$

for a suitable $(0,2)$-form $h = \sum'_{|J|=2} h_J\, d\overline{z}_J$.

We consider the corresponding $\overline{D}$-complex :

$$L^2(\mathbb{C}^n) \underset{\overline{D}_1^*}{\overset{\overline{D}_1}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n) \underset{\overline{D}_2^*}{\overset{\overline{D}_2}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n)\ . \tag{8.18}$$

The so-called Witten Laplacians(see [28]) $\Delta_\varphi^{(0,0)}$ and $\Delta_\varphi^{(0,1)}$ are defined by

$$\begin{aligned} \Delta_\varphi^{(0,0)} &= \overline{D}_1^*\overline{D}_1\ , \\ \Delta_\varphi^{(0,1)} &= \overline{D}_1\overline{D}_1^* + \overline{D}_2^*\overline{D}_2\ . \end{aligned} \tag{8.19}$$

A computation shows that

$$\overline{D}_1^*\overline{D}_1 v = \sum_{j=1}^n \left(\frac{1}{2}\frac{\partial\varphi}{\partial z_j}\frac{\partial v}{\partial \overline{z}_j} + \frac{1}{4}\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial \overline{z}_j} v - \frac{1}{2}\frac{\partial\varphi}{\partial \overline{z}_j}\frac{\partial v}{\partial z_j} - \frac{1}{2}\frac{\partial^2\varphi}{\partial z_j\partial \overline{z}_j} v - \frac{\partial^2 v}{\partial z_j \partial \overline{z}_j}\right)$$

and that

$$(\overline{D}_1\overline{D}_1^* + \overline{D}_2^*\overline{D}_2)g = \sum_{k=1}^n \left[\sum_{j=1}^n \left(\frac{1}{2}\frac{\partial\varphi}{\partial z_j}\frac{\partial g_k}{\partial \overline{z}_j} + \frac{1}{4}\frac{\partial\varphi}{\partial z_j}\frac{\partial\varphi}{\partial \overline{z}_j} g_k \right.\right. \tag{8.20}$$

$$\left.\left. -\frac{1}{2}\frac{\partial\varphi}{\partial \overline{z}_j}\frac{\partial g_k}{\partial z_j} - \frac{1}{2}\frac{\partial^2\varphi}{\partial z_j\partial \overline{z}_j} g_k - \frac{\partial^2 g_k}{\partial z_j\partial \overline{z}_j} + \frac{\partial^2\varphi}{\partial z_j\partial \overline{z}_k} g_j \right)\right] d\overline{z}_k. \tag{8.21}$$

More general, we set $Z_k = \frac{\partial}{\partial \overline{z}_k} + \frac{1}{2}\frac{\partial\varphi}{\partial \overline{z}_k}$ and $Z_k^* = -\frac{\partial}{\partial z_k} + \frac{1}{2}\frac{\partial\varphi}{\partial z_k}$ and we consider $(0,q)$-forms $h = \sum_{|J|=q}' h_J\, d\overline{z}_J$, where $\sum'$ means that we sum up only increasing multiindices $J = (j_1, \dots, j_q)$ and where $d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$. We define

$$\overline{D}_{q+1} h = \sum_{k=1}^n \sum_{|J|=q}{}' Z_k(h_J)\, d\overline{z}_k \wedge d\overline{z}_J \tag{8.22}$$

and

$$\overline{D}_q^* h = \sum_{k=1}^n \sum_{|J|=q}{}' Z_k^*(h_J)\, d\overline{z}_k \rfloor d\overline{z}_J, \tag{8.23}$$

where $d\overline{z}_k \rfloor d\overline{z}_J$ denotes the contraction, or interior multiplication by $d\overline{z}_k$, i.e. we have

$$\langle \alpha, d\overline{z}_k \rfloor d\overline{z}_J\rangle = \langle d\overline{z}_k \wedge \alpha, d\overline{z}_J\rangle$$

for each $(0, q-1)$-form α.
The complex Witten-Laplacian on $(0,q)$-forms is then given by

$$\Delta_\varphi^{(0,q)} = \overline{D}_q\, \overline{D}_q^* + \overline{D}_{q+1}^*\, \overline{D}_{q+1}, \tag{8.24}$$

for $q = 1, \dots, n-1$.
The general $\overline{D}$-complex has the form

$$L^2_{(0,q-1)}(\mathbb{C}^n) \underset{\overline{D}_q^*}{\overset{\overline{D}_q}{\rightleftarrows}} L^2_{(0,q)}(\mathbb{C}^n) \underset{\overline{D}_{q+1}^*}{\overset{\overline{D}_{q+1}}{\rightleftarrows}} L^2_{(0,q+1)}(\mathbb{C}^n)\ . \tag{8.25}$$

It follows that

$$\overline{D}_{q+1}\, \Delta_\varphi^{(0,q)} = \Delta_\varphi^{(0,q+1)}\, \overline{D}_{q+1} \quad\text{and}\quad \overline{D}_{q+1}^*\, \Delta_\varphi^{(0,q+1)} = \Delta_\varphi^{(0,q)}\, \overline{D}_{q+1}^*. \tag{8.26}$$

We remark that

$$\overline{D}_q^* h = \sum_{k=1}^n \sum_{|J|=q}{}' \, Z_k^*(h_J)\, d\overline{z}_k \rfloor d\overline{z}_J = \sum_{|K|=q-1}{}' \sum_{k=1}^n Z_k^*(h_{kK})\, d\overline{z}_K. \tag{8.27}$$

In particular we get for a function $v \in L^2(\mathbb{C}^n)$

$$\Delta_\varphi^{(0,0)} v = \overline{D}_1^* \, \overline{D}_1 v = \sum_{j=1}^n Z_j^* Z_j(v), \tag{8.28}$$

and for a $(0,1)$-form $g = \sum_{\ell=1}^n g_\ell \, d\overline{z}_\ell \in L^2_{(0,1)}(\mathbb{C}^n)$ we obtain

$$\begin{aligned}
\Delta_\varphi^{(0,1)} g &= (\overline{D}_1 \, \overline{D}_1^* + \overline{D}_2^* \, \overline{D}_2) g \\
&= \sum_{j,k,\ell=1}^n \{ Z_j(Z_k^*(g_\ell))\, d\overline{z}_j \wedge (d\overline{z}_k \rfloor d\overline{z}_\ell) + Z_k^*(Z_j(g_\ell))\, d\overline{z}_k \rfloor (d\overline{z}_j \wedge d\overline{z}_\ell) \} \\
&= \sum_{j,k,\ell=1}^n \{ Z_k^*(Z_j(g_\ell))\, (d\overline{z}_j \wedge (d\overline{z}_k \rfloor d\overline{z}_\ell) + d\overline{z}_k \rfloor (d\overline{z}_j \wedge d\overline{z}_\ell)) \\
&+ [Z_j, Z_k^*](g_\ell)\, d\overline{z}_j \wedge (d\overline{z}_k \rfloor d\overline{z}_\ell) \} \\
&= \sum_{j,\ell=1}^n Z_j^* Z_j(g_\ell)\, d\overline{z}_\ell + \sum_{j,k,\ell=1}^n \frac{\partial^2 \varphi}{\partial \overline{z}_j \partial z_k} \, g_\ell \, \delta_{k\ell} d\overline{z}_j \\
&= (\Delta_\varphi^{(0,0)} \otimes I) g + M_\varphi g,
\end{aligned}$$

where we used that for $(0,1)$-forms α, a, b we have

$$\alpha \rfloor (a \wedge b) = (\alpha \rfloor a) \wedge b - a \wedge (\alpha \rfloor b),$$

which implies that

$$\begin{aligned}
d\overline{z}_j \wedge (d\overline{z}_k \rfloor d\overline{z}_\ell) \; &+ \; d\overline{z}_k \rfloor (d\overline{z}_j \wedge d\overline{z}_\ell) \\
&= d\overline{z}_j \wedge (d\overline{z}_k \rfloor d\overline{z}_\ell) + (d\overline{z}_k \rfloor d\overline{z}_j) \wedge d\overline{z}_\ell - d\overline{z}_j \wedge (d\overline{z}_k \rfloor d\overline{z}_\ell) \\
&= (d\overline{z}_k \rfloor d\overline{z}_j) \wedge d\overline{z}_\ell = \delta_{k\ell} \, d\overline{z}_\ell,
\end{aligned}$$

and where we set

$$M_\varphi g = \sum_{j=1}^n \left(\sum_{k=1}^n \frac{\partial^2 \varphi}{\partial z_k \partial \overline{z}_j} \, g_k \right) d\overline{z}_j$$

and

$$(\Delta_\varphi^{(0,0)} \otimes I)\, g = \textstyle\sum_{k=1}^n \Delta_\varphi^{(0,0)} \, g_k \, d\overline{z}_k.$$

For more details see [24], [26] and [20]. By 5.5 we obtain now

Theorem 8.2. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^2$-plurisubharmonic function and suppose that the lowest eigenvalue μ_φ of the Levi - matrix M_φ of φ satisfies*

$$\liminf_{|z| \to \infty} \mu_\varphi(z) > 0.$$

Then the operator $\Delta_\varphi^{(0,1)}$ has a bounded inverse on $L^2_{(0,1)}(\mathbb{C}^n)$

$$(\overline{D}_1 \overline{D}_1^* + \overline{D}_2^* \overline{D}_2)^{-1} = (\Delta_\varphi^{(0,1)})^{-1} = e^{-\varphi/2} \, N_\varphi \;\; e^{\varphi/2}, \tag{8.29}$$

where $N_\varphi = \Box_\varphi^{-1}$.

There is an interesting connection to Dirac and Pauli operators: recall (8.4) and (8.5) and define the Dirac operator $\mathcal{D}$ by

$$\mathcal{D} = (-i\frac{\partial}{\partial x} - A_1)\,\sigma_1 + (-i\frac{\partial}{\partial y} - A_2)\,\sigma_2 = \mathcal{A}_1\sigma_1 + \mathcal{A}_2\sigma_2, \tag{8.30}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Hence we can write

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{A}_1 - i\mathcal{A}_2 \\ \mathcal{A}_1 + i\mathcal{A}_2 & 0 \end{pmatrix}.$$

We remark that $i(\mathcal{A}_2\mathcal{A}_1 - \mathcal{A}_1\mathcal{A}_2) = B$ and hence it turns out that the square of $\mathcal{D}$ is diagonal with the Pauli operators $P_\pm$ on the diagonal:

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{A}_1^2 - i(\mathcal{A}_2\mathcal{A}_1 - \mathcal{A}_1\mathcal{A}_2) + \mathcal{A}_2^2 & 0 \\ 0 & \mathcal{A}_1^2 + i(\mathcal{A}_2\mathcal{A}_1 - \mathcal{A}_1\mathcal{A}_2) + \mathcal{A}_2^2 \end{pmatrix} = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix}, \tag{8.31}$$

where

$$P_\pm = \left(-i\,\frac{\partial}{\partial x} - A_1\right)^2 + \left(-i\,\frac{\partial}{\partial y} - A_2\right)^2 \pm B \ = -\Delta_A \pm B. \tag{8.32}$$

By Lemma 13.10 and Lemma 13.11 the Pauli operators $P_\pm$ are non-negative self-adjoint operators.
It follows that

$$4\mathcal{S} = P_+ \tag{8.33}$$

is the Schrödinger operator with magnetic field and that

$$4\,\Delta_\varphi^{(0,0)} = P_- \ . \tag{8.34}$$

In addition we obtain that $\mathcal{D}^2$ is self-adjoint and likewise $\mathcal{D}$ by the spectral theorem.

Finally we consider decoupled weights $\varphi(z_1, \dots, z_n) = \sum_{j=1}^n \varphi_j(z_j)$. In this case the operator $\Delta_\varphi^{(0,1)}$ acts diagonally on $(0,1)$-forms, each component E_k of the diagonal being

$$E_k = \frac{1}{4}\,P_+^{(k)} + \frac{1}{4}\sum_{j\neq k} P_-^{(j)}, \tag{8.35}$$

where

$$P_\pm^{(\ell)} = \left(-i\,\frac{\partial}{\partial x_\ell} - A_1^{(\ell)}\right)^2 + \left(-i\,\frac{\partial}{\partial y_\ell} - A_2^{(\ell)}\right)^2 \pm B^{(\ell)} \tag{8.36}$$

with $z_\ell = x_\ell + iy_\ell$, $A_1^{(\ell)} = -\frac{1}{2}\frac{\partial \varphi_\ell}{\partial y_\ell}$, $A_2^{(\ell)} = \frac{1}{2}\frac{\partial \varphi_\ell}{\partial x_\ell}$, and $B^{(\ell)} = \frac{1}{2}\Delta\varphi_\ell$, $\ell = 1, \dots, n$. This follows from (8.20) for a decoupled weight and from (8.32).

For further details see [12] and [29].

9. Compactness

We define an appropriate Sobolev space and prove compactness of the corresponding embedding.

Definition 9.1. *Let*

$$\mathcal{W}^{Q_\varphi} = \{u \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \ : \ \|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi < \infty\}$$

with norm

$$\|u\|_{Q_\varphi} = (\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}. \tag{9.1}$$

Remark: $\mathcal{W}^{Q_\varphi}$ coincides with the form domain $dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ of Q_φ (see Proposition 5.5).

Proposition 9.2. *Suppose that the weight function φ is plurisubharmonic and that the lowest eigenvalue μ_φ of the Levi - matrix M_φ satisfies*

$$\lim_{|z|\to\infty} \mu_\varphi(z) = +\infty\,. \tag{9.2}$$

Then the embedding

$$j_\varphi : \mathcal{W}^{Q_\varphi} \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \tag{9.3}$$

is compact.

Proof. For $u \in \mathcal{W}^{Q_\varphi}$ we have by Proposition 5.4

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \geq (M_\varphi u, u)_\varphi.$$

This implies

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \geq \int_{\mathbb{C}^n} \mu_\varphi(z)\, |u(z)|^2\, e^{-\varphi(z)}\, d\lambda(z). \tag{9.4}$$

We show that the unit ball in $\mathcal{W}^{Q_\varphi}$ is relatively compact in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$. For this purpose we use a characterization of compact subsets in L^2-spaces (see Appendix C):
A bounded subset $\mathcal{A}$ of $L^2(\Omega)$ is precompact in $L^2(\Omega)$ if and only if the following two conditions are satisfied:
(i) for every $\epsilon > 0$ and for each $\omega \subset\subset \Omega$ there exists a number $\delta > 0$ such that for every $u \in \mathcal{A}$ and $h \in \mathbb{R}^n$ with $|h| < \delta$ the following inequality holds:

$$\int_\omega |\tilde{u}(x+h) - \tilde{u}(x)|^2\, dx < \epsilon^2; \tag{9.5}$$

(ii) for every $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that for every $u \in \mathcal{A}$

$$\int_{\Omega\setminus\overline{\omega}} |u(x)|^2\, dx < \epsilon^2. \tag{9.6}$$

An analogous result holds in weighted spaces $L^2(\mathbb{C}^n, \varphi)$.
First we show that condition (i) is satisfied in our situation. Let $u = \sum_{j=1}^n u_j\, dz_j$ be a $(0,1)$-form with coefficients in $\mathcal{C}^\infty_0$. For each u_j and for $t \in \mathbb{R}$ and $h = (h_1, \dots, h_n) \in \mathbb{C}^n$ let

$$v_j(t) := u_j(z + th).$$

Note that

$$|v_j'(t)| \leq |h| \left[\sum_{k=1}^{n} \left(\left| \frac{\partial u_j}{\partial x_k}(z+th) \right|^2 + \left| \frac{\partial u_j}{\partial y_k}(z+th) \right|^2 \right) \right]^{1/2},$$

where $z_k = x_k + iy_k$, for $k = 1, \dots, n$. By the fact that

$$u_j(z+h) - u_j(z) = v_j(1) - v_j(0) = \int_0^1 v_j'(t)\, dt$$

we can now estimate for $|h| < R$

$$\int_{\mathbb{B}_R} |\tau_h u_j(z) - u_j(z)|^2 e^{-\varphi(z)}\, d\lambda(z) = \int_{\mathbb{B}_R} |\tau_h(\chi_R u_j)(z) - \chi_R u_j(z)|^2 e^{-\varphi(z)}\, d\lambda(z)$$

$$\leq |h|^2 \int_{\mathbb{B}_R} \left[\int_0^1 \sum_{k=1}^{n} \left(\left| \frac{\partial(\chi_R u_j)}{\partial x_k}(z+th) \right|^2 + \left| \frac{\partial(\chi_R u_j)}{\partial y_k}(z+th) \right|^2 \right) dt \right] e^{-\varphi(z)}\, d\lambda(z)$$

$$\leq C_{\varphi,R}\, |h|^2 \int_{\mathbb{B}_{3R}} \sum_{k=1}^{n} \left(\left| \frac{\partial(\chi_R u_j)}{\partial x_k}(z) \right|^2 + \left| \frac{\partial(\chi_R u_j)}{\partial y_k}(z) \right|^2 \right) e^{-\varphi(z)}\, d\lambda(z)$$

for $j = 1, \dots, n$ where χ_R is a $\mathcal{C}^\infty$ cutoff function which is identically 1 on $\mathbb{B}_{2R}$ and zero outside $\mathbb{B}_{3R}$. It is clear that the corresponding Dirichlet form of $\Box_\varphi$ satisfies the assumptions of Theorem 16.6 in $\mathbb{B}_{3R}$, so by Gårding's inequality for $\mathbb{B}_{3R}$, see Appendix D

$$\begin{aligned} \|\chi_R u\|_{1,\varphi}^2 &\leq C'_{\varphi,R} \left(\|\overline{\partial}(\chi_R u)\|_\varphi^2 + \|\overline{\partial}_\varphi^*(\chi_R u)\|_\varphi^2 + \|\chi_R u\|_\varphi^2 \right) \\ &\leq C''_{\varphi,R} \left(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 + \|u\|_\varphi^2 \right) \end{aligned}$$

we can control the last integral by the norm $\|u\|_{Q_\varphi}^2$. Since we started from the unit ball in $\mathcal{W}^{Q_\varphi}$ we get that condition (15.2) is satisfied.

Condition (15.3) is satisfied for the unit ball of $\mathcal{W}^{Q_\varphi}$ since we have

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \leq \int_{\mathbb{C}^n \setminus \mathbb{B}_R} \frac{\mu_\varphi(z)|u(z)|^2}{\inf\{\mu_\varphi(z) : |z| \geq R\}} e^{-\varphi(z)} d\lambda(z).$$

So formula (9.4) together with assumption (9.2) shows that

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \leq \frac{\|u\|_{Q_\varphi}^2}{\inf\{\mu_\varphi(z) \,:\, |z| \geq R\}} < \epsilon, \tag{9.7}$$

if R is big enough.

$\square$

We are now able to give a short proof of the main result in [26] or [22], see [25] for further details.

Proposition 9.3. *Let φ be a plurisubharmonic $\mathcal{C}^2$- weight function. If the lowest eigenvalue $\mu_\varphi(z)$ of the Levi - matrix M_φ satisfies (9.2), then N_φ is compact.*

Proof. By Proposition 9.2, the embedding $\mathcal{W}^{Q_\varphi} \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ is compact. The inverse N_φ of $\Box_\varphi$ is continuous as an operator from $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ into $\mathcal{W}^{Q_\varphi}$, this follows from Proposition 5.5. Therefore we have that N_φ is compact as an operator from $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ into itself.

$\square$

Proposition 9.4. *Let φ be a plurisubharmonic $\mathcal{C}^2$- weight function. Let $1 \le q \le n$ and suppose that the sum s_q of any q (equivalently: the smallest q) eigenvalues of M_φ satisfies*

$$\lim_{|z|\to\infty} s_q(z) = +\infty. \tag{9.8}$$

Then $N_{\varphi,q} : L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,q)}(\mathbb{C}^n, e^{-\varphi})$ is compact.

Proof. For $(0,q)$ forms one has by (5.9) and Proposition 5.7 that

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \ge \int_{\mathbb{C}^n} s_q(z)\, |u(z)|^2\, e^{-\varphi(z)}\, d\lambda(z). \tag{9.9}$$

Now one can continue as in the proof of Proposition 9.2. $\square$

Example: We consider the plurisubharmonic weight function $\varphi(z,w) = |z|^2|w|^2 + |w|^4$ on $\mathbb{C}^2$. The Levi matrix of φ has the form

$$\begin{pmatrix} |w|^2 & \overline{z}w \\ \overline{w}z & |z|^2 + 4|w|^2 \end{pmatrix}$$

and the eigenvalues are

$$\mu_{\varphi,1}(z,w) = \frac{1}{2}\left(5|w|^2 + |z|^2 - \sqrt{9|w|^4 + 10|z|^2|w|^2 + |z|^4}\right)$$

and

$$\mu_{\varphi,2}(z,w) = \frac{1}{2}\left(5|w|^2 + |z|^2 + \sqrt{9|w|^4 + 10|z|^2|w|^2 + |z|^4}\right).$$

It follows that (9.2) fails, but

$$s_2(z,w) = \frac{1}{4}\,\Delta\varphi(z,w) = |z|^2 + 5|w|^2,$$

hence (9.8) is satisfied.

Notice that

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$$

can be written in the form

$$N_\varphi = j_\varphi \circ j^*_\varphi\ ,$$

where

$$j^*_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow \mathcal{W}^{Q_\varphi}$$

is the adjoint operator to j_φ (see [46] and Proposition 13.12).

This means that N_φ is compact if and only if j_φ is compact and summarizing the above results we get the following

Theorem 9.5. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function . The $\overline{\partial}$-Neumann operator*

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$$

is compact if and only if for each $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z) \leq \epsilon(\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi) \tag{9.10}$$

for each $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*_\varphi)$.

For a further study of compactness we define weighted Sobolev spaces and prove, under suitable conditions, a Rellich - Lemma for these weighted Sobolev spaces. We will also have to consider their dual spaces, which already appeared in [6] and [33].

Definition 9.6.
For $k \in \mathbb{N}$ let

$$W^k(\mathbb{C}^n, e^{-\varphi}) := \{f \in L^2(\mathbb{C}^n, e^{-\varphi}) \ : \ D^\alpha f \in L^2(\mathbb{C}^n, e^{-\varphi}) \text{ for } |\alpha| \leq k\},$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_{2n}} y_n}$ for $(z_1, \dots, z_n) = (x_1, y_1, \dots, x_n, y_n)$ with norm

$$\|f\|^2_{k,\varphi} = \sum_{|\alpha| \leq k} \|D^\alpha f\|^2_\varphi .$$

We will also need weighted Sobolev spaces with negative exponent. But it turns out that for our purposes it is more reasonable to consider the dual spaces of the following spaces.

Definition 9.7.
Let

$$X_j = \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \text{ and } Y_j = \frac{\partial}{\partial y_j} - \frac{\partial \varphi}{\partial y_j},$$

for $j = 1, \dots, n$ and define

$$W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi) = \{f \in L^2(\mathbb{C}^n, e^{-\varphi}) \ : X_j f, \ Y_j f \in L^2(\mathbb{C}^n, e^{-\varphi}), j = 1, \dots, n\},$$

with norm

$$\|f\|^2_{\varphi,\nabla\varphi} = \|f\|^2_\varphi + \sum_{j=1}^n (\|X_j f\|^2_\varphi + \|Y_j f\|^2_\varphi).$$

In the next step we will analyze the dual space of $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$.
By the mapping $f \mapsto (f, X_j f, Y_j f)$, the space $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ can be identified with a closed product of $L^2(\mathbb{C}^n, e^{-\varphi})$, hence each continuous linear functional L on $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ is represented (in a non-unique way) by

$$L(f) = \int_{\mathbb{C}^n} f(z) g_0(z) e^{-\varphi(z)} \, d\lambda(z) + \sum_{j=1}^n \int_{\mathbb{C}^n} (X_j f(z) g_j(z) + Y_j f(z) h_j(z)) e^{-\varphi(z)} \, d\lambda(z),$$

for some $g_j, h_j \in L^2(\mathbb{C}^n, e^{-\varphi})$.
For $f \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ it follows that

$$L(f) = \int_{\mathbb{C}^n} f(z) g_0(z) e^{-\varphi(z)} \, d\lambda(z) - \sum_{j=1}^n \int_{\mathbb{C}^n} f(z) \left(\frac{\partial g_j(z)}{\partial x_j} + \frac{\partial h_j(z)}{\partial y_j} \right) e^{-\varphi(z)} \, d\lambda(z).$$

Since $\mathcal{C}^\infty_0(\mathbb{C}^n)$ is dense in $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ we have shown

Lemma 9.8.
Each element $u \in (W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi))' = W^{-1}(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ *can be represented in a non-unique way by*

$$u = g_0 + \sum_{j=1}^n \left(\frac{\partial g_j}{\partial x_j} + \frac{\partial h_j}{\partial y_j} \right),$$

where $g_j, h_j \in L^2(\mathbb{C}^n, e^{-\varphi})$.
The dual norm $\|u\|_{-1,\varphi,\nabla\varphi} := \sup\{|u(f)| \, : \, \|f\|_{\varphi,\nabla\varphi} \leq 1\}$ *can be expressed in the form*

$$\|u\|^2_{-1,\varphi,\nabla\varphi} = \inf\{\|g_0\|^2 + \sum_{j=1}^n (\|g_j\|^2 + \|h_j\|^2)\},$$

where the infimum is taken over all families (g_j, h_j) *in* $L^2(\mathbb{C}^n, e^{-\varphi})$ *representing the functional* u.

(see for instance [47])
In particular each function in $L^2(\mathbb{C}^n, e^{-\varphi})$ can be indentified with an element of $W^{-1}(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$.

Proposition 9.9.
Suppose that the weight function satisfies

$$\lim_{|z|\to\infty} (\theta |\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty,$$

for some $\theta \in (0,1)$, *where*

$$|\nabla\varphi(z)|^2 = \sum_{k=1}^n \left(\left| \frac{\partial\varphi}{\partial x_k} \right|^2 + \left| \frac{\partial\varphi}{\partial y_k} \right|^2 \right).$$

Then the embedding of $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ *into* $L^2(\mathbb{C}^n, e^{-\varphi})$ *is compact.*

Proof. We adapt methods from [6] or [32], Proposition 6.2., or [33]. For the vector fields X_j from 9.7 and their formal adjoints $X_j^* = -\frac{\partial}{\partial x_j}$ we have

$$(X_j + X_j^*)f = -\frac{\partial\varphi}{\partial x_j} f \text{ and } [X_j, X_j^*]f = -\frac{\partial^2\varphi}{\partial x_j^2} f,$$

for $f \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, and

$$\langle [X_j, X_j^*]f, f\rangle_\varphi = \|X_j^* f\|^2_\varphi - \|X_j f\|^2_\varphi,$$
$$\|(X_j + X_j^*)f\|^2_\varphi \leq (1 + 1/\epsilon)\|X_j f\|^2_\varphi + (1+\epsilon)\|X_j^* f\|^2_\varphi$$

for each $\epsilon > 0$. Similar relations hold for the vector fields Y_j. Now we set

$$\Psi(z) = |\nabla\varphi(z)|^2 + (1+\epsilon)\triangle\varphi(z).$$

It follows that

$$\langle \Psi f, f\rangle_\varphi \leq (2 + \epsilon + 1/\epsilon) \sum_{j=1}^n (\|X_j f\|^2_\varphi + \|Y_j f\|^2_\varphi).$$

Since $\mathcal{C}_0^\infty(\mathbb{C}^n)$ is dense in $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ by definition, this inequality holds for all $f \in W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$.
If $(f_k)_k$ is a sequence in $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ converging weakly to 0, then $(f_k)_k$ is a bounded sequence in $W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ and our assumption implies that

$$\Psi(z) = |\nabla\varphi(z)|^2 + (1+\epsilon)\triangle\varphi(z)$$

is positive in a neighborhood of ∞. So we obtain

$$\int\limits_{\mathbb{C}^n} |f_k(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \le \int\limits_{|z|<R} |f_k(z)|^2 e^{-\varphi(z)}\, d\lambda(z) + \int\limits_{|z|\ge R} \frac{\Psi(z)|f_k(z)|^2}{\inf\{\Psi(z)\,:\,|z|\ge R\}}\, e^{-\varphi(z)}\, d\lambda(z)$$

$$\le C_{\varphi,R}\, \|f_k\|^2_{L^2(B(0,R))} + \frac{C_\epsilon\, \|f_k\|^2_{\varphi,\nabla\varphi}}{\inf\{\Psi(z)\,:\,|z|\ge R\}}.$$

Hence the assumption and the fact that the injection $W^1(B(0,R)) \hookrightarrow L^2(B(0,R))$ is compact (see for instance [47]) show that a subsequence of $(f_k)_k$ tends to 0 in $L^2(\mathbb{C}^n, e^{-\varphi})$. □

Remark 9.10. *It follows that the adjoint to the above embedding, the embedding of $L^2(\mathbb{C}^n, e^{-\varphi})$ into $(W^1(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi))' = W^{-1}(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$ (in the sense of 9.8) is also compact.*

Remark 9.11. *Note that one does not need plurisubharmonicity of the weight function in Proposition 9.9. If the weight function is plurisubharmonic, one can drop θ in the assumptions of Proposition 9.9.*

The following Proposition reformulates the compactness condition for the case of a bounded pseudoconvex domain in $\mathbb{C}^n$, see [5], [46]. The difference to the compactness estimate for a bounded pseudoconvex domain is that here we have to assume a condition on the weight function implying a corresponding Rellich - Lemma.

Proposition 9.12.
*Suppose that the weight function φ satisfies (**??**) and*

$$\lim_{|z|\to\infty} (\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty,$$

for some $\theta \in (0,1)$, then the following statements are equivalent.

(1) *The $\overline{\partial}$-Neumann operator $N_{1,\varphi}$ is a compact operator from $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ into itself.*
(2) *The embedding of the space dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$, provided with the graph norm $u \mapsto (\|u\|^2_\varphi + \|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}$, into $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ is compact.*
(3) *For every positive ϵ there exists a constant C_ϵ such that*

$$\|u\|^2_\varphi \le \epsilon(\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi) + C_\epsilon \|u\|^2_{-1,\varphi,\nabla\varphi},$$

for all $u \in$ dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$.
(4) *For every positive ϵ there exists $R > 0$ such that*

$$\int_{\mathbb{C}^n\setminus\mathbb{B}_R} |u(z)|^2\, e^{-\varphi(z)}\, d\lambda(z) \le \epsilon(\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)$$

for all $u \in$ dom $(\overline{\partial})$ $\cap$ dom $(\overline{\partial}^*_\varphi)$.
(5) *The operators*

$$\overline{\partial}^*_\varphi N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \cap ker(\overline{\partial}) \longrightarrow L^2(\mathbb{C}^n, e^{-\varphi}) \quad \textit{and}$$

$$\overline{\partial}^*_\varphi N_{2,\varphi} : L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi}) \cap ker(\overline{\partial}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$$

are both compact.

Proof. (1) and (4) are equivalent by Theorem 9.5. Next we show that (1) and (5) are equivalent: suppose that $N_{1,\varphi}$ is compact. For $f \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ it follows that

$$\|\overline{\partial}^*_\varphi N_{1,\varphi} f\|^2_\varphi \leq \langle f, N_{1,\varphi} f\rangle_\varphi \leq \epsilon \|f\|^2_\varphi + C_\epsilon \|N_{1,\varphi} f\|^2_\varphi.$$

Hence, by Lemma 13.1, $\overline{\partial}^*_\varphi N_{1,\varphi}$ is compact. Applying the formula

$$N_{1,\varphi} - (\overline{\partial}^*_\varphi N_{1,\varphi})^* (\overline{\partial}^*_\varphi N_{1,\varphi}) = (\overline{\partial}^*_\varphi N_{2,\varphi})(\overline{\partial}^*_\varphi N_{2,\varphi})^*,$$

(see for instance [9]), we get that also $\overline{\partial}^*_\varphi N_{2,\varphi}$ is compact. The converse follows easily from the same formula.
Now we show (5) $\implies$ (3) $\implies$ (2) $\implies$ (1). We follow the lines of [46], where the case of a bounded pseudoconvex domain is handled.
Assume (5): if (3) does not hold, then there exists $\epsilon_0 > 0$ and a sequence $(u_n)_n$ in dom $(\overline{\partial}) \cap$dom $(\overline{\partial}^*_\varphi)$ with $\|u_n\|_\varphi = 1$ and

$$\|u_n\|^2_\varphi \geq \epsilon_0 (\|\overline{\partial} u_n\|^2_\varphi + \|\overline{\partial}^*_\varphi u_n\|^2_\varphi) + n\|u_n\|^2_{-1,\varphi,\nabla\varphi}$$

for each $n \geq 1$, which implies that $u_n \to 0$ in $W^{-1}_{(0,1)}(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$. Since u_n can be written in the form

$$u_n = (\overline{\partial}^*_\varphi N_{1,\varphi})^* \, \overline{\partial}^*_\varphi u_n + (\overline{\partial}^*_\varphi N_{2,\varphi}) \, \overline{\partial} u_n,$$

(5) implies there exists a subsequence of $(u_n)_n$ converging in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ and the limit must be 0, which contradicts $\|u_n\|_\varphi = 1$.
To show that (3) implies (2) we consider a bounded sequence in dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$. By Proposition 5.4 this sequence is also bounded in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$. Now Proposition 9.9 implies that it has a subsequence converging in $W^{-1}_{(0,1)}(\mathbb{C}^n, e^{-\varphi}, \nabla\varphi)$. Finally use (3) to show that this subsequence is a Cauchy sequence in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$, therefore (2) holds.
Assume (2) : by Proposition 5.4 and the basic facts about $N_{1,\varphi}$, it follows that

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow \text{dom}\,(\overline{\partial}) \cap \text{dom}\,(\overline{\partial}^*_\varphi)$$

is continuous in the graph topology, hence

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow \text{dom}\,(\overline{\partial}) \cap \text{dom}\,(\overline{\partial}^*_\varphi) \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$$

is compact.

□

Remark 9.13. *If*

$$\lim_{|z| \to \infty} \mu_\varphi(z) = +\infty,$$

then the condition of the Rellich - Lemma 9.9 is satisfied.
This follows from the fact that we have for the trace $tr(M_\varphi)$ of the Levi - matrix

$$tr(M_\varphi) = \frac{1}{4} \triangle \varphi,$$

and since for any invertible $(n \times n)$-matrix T

$$tr(M_\varphi) = tr(T M_\varphi T^{-1}),$$

it follows that $tr(M_\varphi)$ equals the sum of all eigenvalues of M_φ. Hence our assumption on the lowest eigenvalue μ_φ of the Levi - matrix implies that the assumption of Proposition 9.9 is satisfied.

Remark 9.14. *We mention that for the weight $\varphi(z) = |z|^2$ the $\overline{\partial}$-Neumann operator fails to be compact (see Chapter 12), but the condition*

$$\lim_{|z|\to\infty} (\theta|\nabla\varphi(z)|^2 + \triangle\varphi(z)) = +\infty$$

of the Rellich - Lemma is satisfied.

Remark 9.15. *Let $A^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ denote the space of $(0,1)$-forms with holomorphic coefficients belonging to $L^2(\mathbb{C}^n, e^{-\varphi})$.*
We point out that assuming (9.3) implies directly – without use of Sobolev spaces – that the embedding of the space

$$A^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \cap dom\,(\overline{\partial}^*_\varphi)$$

provided with the graph norm $u \mapsto (\|u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)^{1/2}$ into $A^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ is compact. Compare 9.12 (2).

For this purpose let $u \in A^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \cap dom\,(\overline{\partial}^*_\varphi)$. Then we obtain from the proof of 5.4 that

$$\|\overline{\partial}^*_\varphi u\|^2_\varphi = \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\, u_j \overline{u}_k\, e^{-\varphi}\, d\lambda.$$

Let us for $u = \sum_{j=1}^n u_j\, d\overline{z}_j$ indentify $u(z)$ with the vector $(u_1(z), \dots, u_n(z)) \in \mathbb{C}^n$. Then, if we denote by $\langle ., . \rangle$ the standard inner product in $\mathbb{C}^n$, we have

$$\langle u(z), u(z)\rangle = \sum_{j=1}^n |u_j(z)|^2 \text{ and } \langle M_\varphi u(z), u(z)\rangle = \sum_{j,k=1}^n \frac{\partial^2\varphi(z)}{\partial z_j \partial \overline{z}_k}\, u_j(z)\overline{u_k(z)}.$$

Note that the lowest eigenvalue μ_φ of the Levi - matrix M_φ can be expressed as

$$\mu_\varphi(z) = \inf_{u(z)\neq 0} \frac{\langle M_\varphi u(z), u(z)\rangle}{\langle u(z), u(z)\rangle}.$$

So we get

$$\begin{aligned}
\int_{\mathbb{C}^n} \langle u, u\rangle e^{-\varphi}\, d\lambda &\leq \int_{\mathbb{B}_R} \langle u, u\rangle e^{-\varphi}\, d\lambda + [\inf_{\mathbb{C}^n\setminus\mathbb{B}_R} \mu_\varphi(z)]^{-1} \int_{\mathbb{C}^n\setminus\mathbb{B}_R} \mu_\varphi(z)\, \langle u, u\rangle e^{-\varphi}\, d\lambda \\
&\leq \int_{\mathbb{B}_R} \langle u, u\rangle e^{-\varphi}\, d\lambda + [\inf_{\mathbb{C}^n\setminus\mathbb{B}_R} \mu_\varphi(z)]^{-1} \int_{\mathbb{C}^n} \langle M_\varphi u, u\rangle e^{-\varphi}\, d\lambda.
\end{aligned}$$

For a given $\epsilon > 0$ choose R so large that

$$[\inf_{\mathbb{C}^n\setminus\mathbb{B}_R} \mu_\varphi(z)]^{-1} < \epsilon,$$

and use the fact that for Bergman spaces of holomorphic functions the embedding of $A^2(\mathbb{B}_{R_1})$ into $A^2(\mathbb{B}_{R_2})$ is compact for $R_2 < R_1$. So the desired conclusion follows.

Inspired by a result on Schrödinger operators with magnetic field of Iwatsuka [31] we point out another characterization of compactness, which will be used later, for further details see [21].

Proposition 9.16. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function . The $\overline{\partial}$-Neumann operator*

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$$

is compact if and only if there is a smooth function $\Lambda : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that $\Lambda(z) \to \infty$ as $|z| \to \infty$ and

$$(\Box_\varphi u, u)_\varphi \geq \int_{\mathbb{C}^n} \Lambda\, |u|^2\, e^{-\varphi}\, d\lambda \tag{9.11}$$

for each $u \in \mathcal{W}^{Q_\varphi}$.

Proof. Suppose (9.11) holds. Then for each $\epsilon > 0$ there exists a number $R > 0$ such that $\Lambda \geq 1/\epsilon$ on $\mathbb{C}^n \setminus \mathbb{B}_R$. This implies

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi = (\Box_\varphi u, u)_\varphi \geq \int_{\mathbb{C}^n} \Lambda\, |u|^2\, e^{-\varphi}\, d\lambda \geq \frac{1}{\epsilon} \int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u|^2\, e^{-\varphi}\, d\lambda,$$

which means that (9.10) holds.
We indicate that the condition of Theorem 9.5 can be written in the form : for each $\epsilon > 0$ there exists $R(\epsilon) > 0$ such that

$$\|u\|_{L^2_{(0,1)}(\mathbb{C}^n \setminus \mathbb{B}_{R(\epsilon)}, \varphi)} \leq \epsilon \|u\|_{Q_\varphi}.$$

Hence for all $u \in \mathcal{W}^{Q_\varphi}$ and for $j \in \mathbb{N}$ we have

$$2^j \int_{\mathbb{C}^n \setminus \mathbb{B}_{R(1/2^j)}} |u|^2\, e^{-\varphi}\, d\lambda \leq \frac{1}{2^j}\, \|u\|^2_{Q_\varphi}$$

and hence

$$\begin{aligned}\int_{\mathbb{C}^n} |u|^2\, e^{-\varphi}\, d\lambda &\leq \int_{\mathbb{B}_{R(1/2)}} 1 \cdot |u|^2\, e^{-\varphi}\, d\lambda + \int_{\mathbb{B}_{R(1/4)} \setminus \mathbb{B}_{R(1/2)}} 2 \cdot |u|^2\, e^{-\varphi}\, d\lambda \\ &+ \int_{\mathbb{B}_{R(1/8)} \setminus \mathbb{B}_{R(1/4)}} 4 \cdot |u|^2\, e^{-\varphi}\, d\lambda + \cdots \\ &\leq (C+1)\|u\|^2_{Q_\varphi}.\end{aligned}$$

Now it is easy to define a smooth function Λ tending to ∞ as $|z|$ tends to ∞ such that (9.11) holds. ∎

Finally we investigate compactness of the $\overline{\partial}$-Neumann operator of a bounded pseudoconvex domain.
Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Ω satisfies property (P), if for each $M > 0$ there exists a a neighborhood U of $\partial\Omega$ and a plurisubharmonic function $\varphi_M \in \mathcal{C}^2(U)$ with $0 \leq \varphi_M \leq 1$ on U such that

$$\sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t}_k \geq M \|t\|^2,$$

for all $p \in \partial\Omega$ and for all $t \in \mathbb{C}^n$.

Ω satisfies property $(\tilde{\mathrm{P}})$ if the following holds: there is a constant C such that for all $M > 0$ there exists a $\mathcal{C}^2$ function φ_M in a neighborhood U (depending on M) of $\partial\Omega$ with
(i) $\left|\sum_{j=1}^n \frac{\partial \varphi_M}{\partial z_j}(z) t_j\right|^2 \leq C \sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k$

and
(ii) $\sum_{j=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \bar{t}_k \geq M\|t\|^2$,
for all $z \in U$ and for all $t \in \mathbb{C}^n$.

In [8] Catlin showed that property (P) implies compactness of the $\overline{\partial}$-operator N on $L^2_{(0,1)}(\Omega)$ and McNeal ([39]) showed that property $(\tilde{\mathrm{P}})$ also implies compactness of the $\overline{\partial}$-operator N on $L^2_{(0,1)}(\Omega)$. It is not difficult to show that property (P) implies property $(\tilde{\mathrm{P}})$: if (φ_M) is the family of functions from the definition of property (P), then (e^{φ_M}) will work for $(\tilde{\mathrm{P}})$, see also [46].

We can now use a similar approach as before to prove Catlin's result. For this purpose we use again 8.2.
In order to show that the unit ball in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$ satisfies condition (i) of 8.2 we first remark that compactly supported forms are not dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$, but forms in $dom(\overline{\partial}^*)$ with coefficients in $\mathcal{C}^\infty(\overline{\Omega})$ are dense (see [46]). So if $\omega \subset\subset \Omega$, we choose $\omega \subset\subset \omega_1 \subset\subset \omega_2 \subset\subset \Omega$ and a cut-off function ψ with $\psi(z) = 1$ for $z \in \omega_1$ and $\psi(z) = 0$ for $z \in \Omega \setminus \omega_2$. For $u \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ we define $\tilde{u} = \psi u$ and remark that the domain of $\overline{\partial}^*$ is preserved under multiplication by a function in $\mathcal{C}^1(\overline{\Omega})$ (see [46]), therefore $\tilde{u}$ has compactly supported coefficients and belongs to $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$. The graph norm of $\tilde{u}$ is bounded by a constant C depending only on $\omega, \omega_1, \omega_2, \Omega$, if u belongs to the unit ball in the graph norm. By construction we have

$$\|\tau_h u - u\|_{L^2(\omega)} = \|\tau_h \tilde{u} - \tilde{u}\|_{L^2(\omega)},$$

if $|h|$ is small enough, hence we can use Gårding's inequality for $\omega \subset\subset \Omega$ to show that condition (i) holds.

To verify condition (ii) we use property (P) and the following version of the Kohn-Morrey formula

$$\int_\Omega \sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi_M} \, d\lambda \leq \|\overline{\partial} u\|^2_{\varphi_M} + \|\overline{\partial}^*_{\varphi_M} u\|^2_{\varphi_M}, \tag{9.12}$$

here we used that Ω is pseudoconvex, which means that the boundary terms in the Kohn-Morrey formula can be neglected. Now we point out that the weighted $\overline{\partial}$-complex is equivalent to the unweighted one and that the expression $\sum_{j=1}^n \frac{\partial \varphi_M}{\partial z_j} u_j$ which appears in $\overline{\partial}^*_{\varphi_M} u$, can be controlled by the complex Hessian $\sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k$, which follows from the fact that property (P) implies property $(\tilde{\mathrm{P}})$. Of course we also use that the weight φ_M is bounded on $\Omega \subset\subset \mathbb{C}^n$. In this way the same reasoning as in the weighted case shows that property (P) implies condition (15.3). Therefore condition (P) gives that the unit ball of $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$ is relatively compact in $L^2_{(0,1)}(\Omega)$ and hence that the $\overline{\partial}$-Neumann operator is compact.

Now let

$$j : dom(\overline{\partial}) \cap dom(\overline{\partial}^*) \hookrightarrow L^2_{(0,1)}(\Omega)$$

denote the embedding. It follows from [46] that

$$N = j \circ j^*.$$

Hence N is compact if and only if j is compact, where $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ is endowed with the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$.

Theorem 9.17. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. The $\overline{\partial}$-Neumann operator N is compact if and only if for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that*

$$\int_{\Omega\setminus\omega} |u(z)|^2 \, d\lambda(z) \leq \epsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)$$

for each $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*)$.

This follows from the above remarks about the embedding j and the fact that the two conditions (15.2) and (15.3) are also necessary for a bounded set in L^2 to be relatively compact.

In a similar way as for Proposition 9.12 one obtains compactness estimates for the $\overline{\partial}$-Neumann operator on a smoothly bounded domain. Here we use the standard Sobolev spaces $W^1(\Omega)$ and the classical Rellich - Lemma without weights.

Proposition 9.18.
Let Ω be a smoothly bounded pseudoconvex domain. Then the following statements are equivalent.

(1) *The $\overline{\partial}$-Neumann operator N_1 is a compact operator from $L^2_{(0,1)}(\Omega)$ into itself.*
(2) *The embedding of the space $dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*)$, provided with the graph norm $u \mapsto (\|u\|^2 + \|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)^{1/2}$, into $L^2_{(0,1)}(\Omega)$ is compact.*
(3) *For every positive ϵ there exists a constant C_ϵ such that*

$$\|u\|^2 \leq \epsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2) + C_\epsilon \|u\|^2_{-1},$$

for all $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*)$.
(4) *For every positive ϵ there exists $\omega \subset\subset \Omega$ such that*

$$\int_{\Omega\setminus\omega} |u(z)|^2 \, d\lambda(z) \leq \epsilon(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2)$$

for all $u \in dom\,(\overline{\partial}) \cap dom\,(\overline{\partial}^*)$.
(5) *The operators*

$$\overline{\partial}^* N_1 : L^2_{(0,1)}(\Omega) \cap ker(\overline{\partial}) \longrightarrow L^2(\Omega) \quad \textit{and}$$

$$\overline{\partial}^* N_2 : L^2_{(0,2)}(\Omega) \cap ker(\overline{\partial}) \longrightarrow L^2_{(0,1)}(\Omega)$$

are both compact.

10. The $\overline{\partial}$-Neumann operator and commutators of the Bergman projection and multiplication operators.

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $A^2_{(0,1)}(\Omega)$ denote the space of all $(0,1)$-forms with holomorphic coefficients belonging to $L^2(\Omega)$. With the same proof as in section 2 one shows that the canonical solution operator $S : A^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega)$ has the form

$$S(g)(z) = \int_\Omega K(z,w) < g(w), z-w > \, d\lambda(w), \tag{10.1}$$

where K denotes the Bergman kernel of Ω and

$$< g(w), z-w >= \sum_{j=1}^n g_j(w)(\overline{z}_j - \overline{w}_j),$$

for $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$.
In this chapter we investigate the connection between the $\overline{\partial}$-Neumann operator and commutators of the Bergman projection with multiplication operators. In [7] it is shown that compactness of the $\overline{\partial}$-Neumann operator N on $L^2_{(0,1)}(\Omega)$ implies compactness of the commutator $[P, M]$, where P is the Bergman projection and M is pseudodifferential operator of order 0. Here we show that compactness of the $\overline{\partial}$-Neumann operator N restricted to $(0,1)$-forms with holomorphic coefficients is equivalent to compactness of the commutator $[P, M]$ defined on the whole $L^2(\Omega)$. In addition we derive a formula for the $\overline{\partial}$-Neumann operator restricted to $(0,1)$ forms with holomorphic coefficients expressed by commutators of the Bergman projection and the multiplications operators by z and $\bar{z}$.
The restriction of the canonical solution operator to forms with holomorphic coefficients has many interesting aspects, which in most cases correspond to certain growth properties of the Bergman kernel. It is also of great interest to clarify to what extent compactness of the restriction already implies compactness of the original solution operator to $\overline{\partial}$. This is the case for convex domains, see [17]. There are many other examples of non-compactness where the obstruction already occurs for forms with holomorphic coefficients (see [36], [35]).
We define the following operator

$$T : L^2_{(0,1)}(\Omega) \longrightarrow L^2(\Omega),$$

by

$$Tf(z) = \int_\Omega K(z,w)\langle f(w), z-w\rangle \, d\lambda(w), \tag{10.2}$$

where $f = \sum_{k=1}^n f_k \, d\overline{z}_k$ and $\langle f(w), z-w\rangle = \sum_{k=1}^n f_k(w)(\overline{z}_k - \overline{w}_k)$.
The operator T can be written as a sum of commutators

$$Tf = \sum_{k=1}^n [\overline{M_k}, P] f_k, \;\; f = \sum_{k=1}^n f_k \, d\overline{z}_k \tag{10.3}$$

where $\overline{M_k} v(z) = \overline{z}_k v(z), \; v \in L^2(\Omega), \; k = 1, \dots, n.$

Let $\mathcal{P} : L^2_{(0,1)}(\Omega) \longrightarrow A^2_{(0,1)}(\Omega)$ be the orthogonal projection on the space of $(0,1)$-forms with holomorphic coefficients. We claim that

$$Tf = T\mathcal{P}f \ , \ f \in L^2_{(0,1)}(\Omega).$$

It suffices to show that $Tg = 0$, for $g \perp A^2_{(0,1)}(\Omega)$:

$$\begin{aligned} Tg(z) &= -\sum_{k=1}^n P\overline{M_k} g_k(z) = -\sum_{k=1}^n \int_\Omega K(z,w)\overline{w}_k g_k(w)\, d\lambda(w) \\ &= -\sum_{k=1}^n \int_\Omega g_k(w) \ \ [K(w,z)w_k]^- \ d\lambda(w) = 0, \end{aligned}$$

because $w \mapsto K(w,z)w_k$ is holomorphic and $g_k \perp A^2(\Omega)$, for $k = 1, \dots, n$.

Now, let S denote the canonical solution operator to $\overline{\partial}$ restricted to $A^2_{(0,1)}(\Omega)$. From (10.1) we have for $f \in L^2_{(0,1)}(\Omega)$

$$S(\mathcal{P}f) = T(\mathcal{P}f) = Tf. \tag{10.4}$$

Hence we have proved the following

Theorem 10.1. *If $f \in L^2_{(0,1)}(\Omega)$, then $T(\mathcal{P}f) = Tf$. The operator S is compact as an operator from $A^2_{(0,1)}(\Omega)$ to $L^2(\Omega)$, if and only if the operator T is compact as an operator from $L^2_{(0,1)}(\Omega)$ to $L^2(\Omega)$.*

Remark 10.2. *The adjoint operator $T^* : L^2(\Omega) \longrightarrow L^2_{(0,1)}(\Omega)$ is given by*

$$T^*(g) = \sum_{k=1}^n [P, M_k]\, g\, d\overline{z}_k, \ \ g \in L^2(\Omega), \tag{10.5}$$

where $M_k v(z) = z_k v(z)$.
Here we have

$$T^*(I - P)(g) = T^*(g),$$

since

$$[P, M_k]\, Pg = PM_kPg - M_kPg = 0.$$

In a similar way the following results can be proved

Lemma 10.3. (1) $PM_jP = M_jP$,

(2) $P\overline{M_j}P = P\overline{M_j}$.

Let

$$B^2_{(0,1)}(\Omega) = \{f \in L^2_{(0,1)}(\Omega) \ : \ f \in \ker\overline{\partial}\}.$$

Now suppose that Ω is bounded pseudoconvex domain in $\mathbb{C}^n$. The $\overline{\partial}$-Neumann operator N can be viewed as an operator from $B^2_{(0,1)}(\Omega)$ to $B^2_{(0,1)}(\Omega)$. The operator

$$\overline{\partial}^* N : B^2_{(0,1)}(\Omega) \longrightarrow A^2(\Omega)^\perp$$

is the canonical solution operator to $\overline{\partial}$ (see [9]).

Theorem 10.4. *If $f = \sum_{k=1}^n f_k d\overline{z}_k \in B^2_{(0,1)}(\Omega)$, then*

$$\mathcal{P}N\mathcal{P}f = \sum_{k=1}^n \left(\sum_{j=1}^n (PM_k\overline{M}_jPf_j - M_kP\overline{M}_jf_j) \right) d\overline{z}_k. \tag{10.6}$$

If $f = \sum_{k=1}^n f_k d\overline{z}_k \in A^2_{(0,1)}(\Omega)$, then

$$\mathcal{P}Nf = \sum_{k=1}^n [P, M_k] \left(\sum_{j=1}^n \overline{M}_j f_j \right) d\overline{z}_k. \tag{10.7}$$

Proof. First we observe that for $f \in B^2_{(0,1)}(\Omega)$ we have

$$N\overline{\partial}\,\overline{\partial}^* Nf = N(I - \overline{\partial}^*\overline{\partial}N)f = Nf,$$

where we used the fact that

$$N : B^2_{(0,1)}(\Omega) \longrightarrow B^2_{(0,1)}(\Omega).$$

If $f \in A^2_{(0,1)}(\Omega)$, then by Theorem 10.1 it follows that

$$\overline{\partial}^* Nf = Tf.$$

Let $f \in A^2_{(0,1)}(\Omega)$ and $g \in B^2_{(0,1)}(\Omega)$ with orthogonal decompostion $g = h + \tilde{h}$, where $h \in A^2_{(0,1)}(\Omega)$ and $\tilde{h} = (I - \mathcal{P})g$, then

$$\begin{aligned}(g, N\overline{\partial}\,\overline{\partial}^* Nf) &= (\overline{\partial}^* N(h + \tilde{h}), Tf) = (\overline{\partial}^* Nh, Tf) + (\overline{\partial}^* N\tilde{h}, Tf) \\ &= (Th, Tf) + (\overline{\partial}^* N\tilde{h}, Tf) = (Tg, Tf) + (\overline{\partial}^* N\tilde{h}, Tf) \\ &= (g, T^*Tf) + (\overline{\partial}^* N\tilde{h}, Tf).\end{aligned}$$

Since

$$(\overline{\partial}^* N\tilde{h}, Tf) = (N\tilde{h}, \overline{\partial}Tf) = (N\tilde{h}, f) = (\tilde{h}, Nf),$$

we obtain

$$\begin{aligned}(g, Nf) &= (g, N\overline{\partial}\,\overline{\partial}^* Nf) = (g, T^*Tf) + (\tilde{h}, Nf) \\ &= (g, T^*Tf) + ((I - \mathcal{P})g, Nf) = (g, T^*Tf) + (g, (I - \mathcal{P})Nf).\end{aligned}$$

Now, since $g \in B^2_{(0,1)}(\Omega)$ was arbitrary, we get

$$Nf = T^*Tf + Nf - \mathcal{P}Nf,$$

and therefore

$$\mathcal{P}Nf = T^*Tf.$$

If we take into account, that for $f \in B^2_{(0,1)}(\Omega)$ we have $Tf = T\mathcal{P}f$, we can now apply the last formula for $\mathcal{P}f$ and get

$$\mathcal{P}N\mathcal{P}f = T^*Tf.$$

It remains to compute T^*T. If $f \in B^2_{(0,1)}(\Omega)$, then

$$T^*Tf = \sum_{k=1}^n [P, M_k] \left(\sum_{j=1}^n [\overline{M}_j, P] f_j \right) d\overline{z}_k$$

$$= \sum_{k=1}^n \left(\sum_{j=1}^n (PM_k\overline{M}_jP - M_kP\overline{M}_jP - PM_kP\overline{M}_j + M_kP\overline{M}_j) f_j \right) d\overline{z}_k$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} (PM_k \overline{M}_j P f_j - M_k P \overline{M}_j f_j) \right) d\overline{z}_k,$$

where we used Lemma 10.3.
If $f \in A^2_{(0,1)}(\Omega)$, then

$$Pf_j = f_j$$

and we obtain the second formula in Theorem 10.4. $\square$

Using the last results we get the criterion for compactness of the commutators $[P, M_k]$:

Theorem 10.5. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then the following conditions are equivalent:*

(1) $N \mid_{A^2_{(0,1)}(\Omega)}$ *is compact;*
(2) $\overline{\partial}^* N \mid_{A^2_{(0,1)}(\Omega)}$ *is compact;*
(3) $[P, M_k]$ *is compact on* $L^2(\Omega)$ *for* $k = 1, \dots, n$;
(4) $(I - P)\overline{M}_k P$ *is compact on* $L^2(\Omega)$ *for* $k = 1, \dots, n$;
(5) $[M_\varphi, P]$ *is compact on* $L^2(\Omega)$ *for each continuous function* φ *on* $\overline{\Omega}$.

Proof. Let $S_1 = \overline{\partial}^* N_1 : B^2_{(0,1)}(\Omega) \longrightarrow A^2(\Omega)^\perp$ be the canonical solution operator to $\overline{\partial}$ and similarly $S_2 = \overline{\partial}^* N_2 : B^2_{(0,2)}(\Omega) \longrightarrow B^2_{(0,1)}(\Omega)^\perp$, then

$$N_1 = S_1^* S_1 + S_2 S_2^*.$$

(see for instance [9] or [18]). Since $S_2^* \mid_{A^2_{(0,1)}(\Omega)} = 0$, we have

$$N_1 \mid_{A^2_{(0,1)}(\Omega)} = S_1^* S_1 \mid_{A^2_{(0,1)}(\Omega)},$$

and (1) is equivalent to (2).

Now suppose that (2) holds. Then, since the restriction of $\overline{\partial}^* N$ to $A^2_{(0,1)}(\Omega)$ is of the form

$$\overline{\partial}^* N f = \sum_{k=1}^{n} [\overline{M_k}, P] f_k,$$

where $f = \sum_{k=1}^n f_k \, dz_k \in A^2_{(0,1)}(\Omega)$, then by Theorem 10.1 it follows that the operators $[\overline{M_k}, P]$ are compact on $L^2(\Omega)$. Since $[\overline{M_k}, P]^* = [P, M_k]$, we obtain property (3).
It is also clear by Theorem 10.1 that (3) implies (2).

Now suppose that (3) holds. It follows that $[\overline{M_k}, P]P$ is also compact, and since

$$[\overline{M_k}, P]P = \overline{M_k} P - P\overline{M_k} P = (I - P)\overline{M_k} P,$$

the Hankel operators $(I - P)\overline{M_k} P$ are compact. So we have shown that (3) implies (4).

Suppose that (4) holds. The Hankel operators $H_{z_j \overline{z_k}}$ with symbol $z_j \overline{z_k}$ can be written in the form

$$H_{z_j \overline{z_k}} = (I - P) M_j (P + (I - P)) \overline{M_k} P = (I - P) M_j (I - P) \overline{M_k} P,$$

hence it follows that $H_{z_j \overline{z_k}}$ is compact. Similarly one can show that for any polynomial

$$p(z, \overline{z}) = \sum_{|\alpha| \le N} \lambda_\alpha z^{\alpha_1} \overline{z}^{\alpha_2},$$

where $\alpha = (\alpha_1, \alpha_2)$ in a multiindex in $\mathbb{N}^{2n}$, the corresponding Hankel operator $H_p = (I-P)M_pP$ is compact. Now let $\varphi \in \mathcal{C}(\overline{\Omega})$. Then, by the Stone- Weierstraß Theorem, there exists a polynomial p of the above form such that

$$\|\varphi - p\|_\infty < \epsilon.$$

hence

$$\|H_\varphi - H_p\| = \|(I-P)M_{\varphi - p}P\| \leq \|\varphi - p\|_\infty.$$

Since the compact operators form a closed twosided ideal in the operator norm and since for $g = g_1 + g_2$ where $g_1 \in A^2(\Omega)$ and $g_2 \in A^2(\Omega)^\perp$ we have

$$[M_\varphi, P]g = -H^*_{\overline{\varphi}}g_2 + H_\varphi g_1,$$

it follows that $[M_\varphi, P]$ is compact. $\square$

Remark 10.6. *If Ω is a bounded convex domain, then compactness of $\overline{\partial}^* N \mid_{A^2_{(0,1)}(\Omega)}$ implies already compactness of $\overline{\partial}^* N$ on all of $L^2_{(0,1)}(\Omega)$ (see [17]), hence , in this case property (1) of Theorem 10.5 can be replaced by N being compact on $L^2_{(0,1)}(\Omega)$ and property (2) of Theorem 10.5 can be replaced by $\overline{\partial}^* N$ being compact on $L^2_{(0,1)}(\Omega)$.*

11. Differential operators in $\mathbb{R}^2$

Next we characterize compactness of the $\overline{\partial}$-Neumann operator N_φ, which was originally done in [26] using methods from Schrödinger operators, and later in [38] using estimates of the Bergman kernel in $A^2(\mathbb{C}, e^{-\varphi})$. Here we give a direct proof using methods of chapter 9.

First we give a sufficient condition for compactness of the $\overline{\partial}$-Neumann operator N_φ on $L^2(\mathbb{C}, e^{-\varphi})$. Then we describe a characterization of compactness of the $\overline{\partial}$-Neumann operator N_φ on $L^2(\mathbb{C}, e^{-\varphi})$ as it is done in [26] using methods from real analysis.

Theorem 11.1. *Let φ be a subharmonic $\mathcal{C}^2$-function such that*

$$\triangle\varphi(z) \to +\infty \tag{11.1}$$

as $|z| \to \infty$. Then the $\overline{\partial}$-Neumann operator N_φ is compact on $L^2(\mathbb{C}, e^{-\varphi})$.

Proof. Suppose that $\triangle\varphi(z) \to \infty$ as $|z| \to \infty$. We already showed that $\Box_\varphi = e^{\varphi/2}\,\overline{D}\,\overline{D}^*\,e^{-\varphi/2}$ and that $\overline{D}\,\overline{D}^* = -\frac{1}{4}\triangle_A + \frac{1}{8}\triangle\varphi$. We also proved that $-\triangle_A \geq \frac{1}{2}\triangle\varphi$, which implies that $-\frac{1}{4}\triangle_A + \frac{1}{8}\triangle\varphi \geq \frac{1}{4}\triangle\varphi$ and hence for $f \in \mathcal{C}_0^\infty(\mathbb{C})$ we obtain

$$\begin{aligned}
(\Box_\varphi f, f)_\varphi &= (e^{\varphi/2}\,\overline{D}\,\overline{D}^*\,e^{-\varphi/2} f, f)_\varphi \\
&= (e^{-\varphi/2}\,\overline{D}\,\overline{D}^*\,e^{-\varphi/2} f, f) \\
&= (\overline{D}\,\overline{D}^*\,e^{-\varphi/2} f, e^{-\varphi/2} f)
\end{aligned}$$

and setting $g = e^{-\varphi/2} f$ we get

$$(\Box_\varphi f, f)_\varphi = (\overline{D}\,\overline{D}^*\,g, g) \geq \frac{1}{4}\,(\triangle\varphi\, g, g) = \frac{1}{4}\,(\triangle\varphi\, f, f)_\varphi$$

and we can apply Proposition 9.16. to see that N_φ is compact. $\square$

Remark 11.2. *In the following we describe a charcterization of compactness in the complex one-dimensional case, see [26].*
The reverse Hölder class $B_2(\mathbb{R}^2)$ consists of L^2 positive and almost non zero everywhere functions V for which there exists a constant $C > 0$ such that

$$\left(\frac{1}{|Q|}\int_Q V^2\,d\lambda\right)^{\frac{1}{2}} \leq C\left(\frac{1}{|Q|}\int_Q V\,d\lambda\right) \tag{11.2}$$

for any ball Q in $\mathbb{R}^2$.
Note that any positive (non zero) polynomial is in B_2.
Using different methods of real analysis one can now show the following characterization (see [26], for the details):
Let φ be a subharmonic $\mathcal{C}^2$- function on $\mathbb{R}^2$ such that

$$\triangle\varphi \in B_2(\mathbb{R}^2)\ . \tag{11.3}$$

Then the $\overline{\partial}$-Neumann operator N_φ is compact on $L^2(\mathbb{C}, e^{-\varphi})$ if and only if

$$\lim_{|z|\to\infty} \int_{D(z,1)} \triangle\varphi(w)\,d\lambda(w) = +\infty\ , \tag{11.4}$$

where $D(z,1) = \{w \in \mathbb{C} : |w - z| < 1\}$.

That (11.4) is necessary for compactness follows from a result of Iwatsuka [31]. The sufficiency of (11.4) is derived form the diamagnetic inequality and a special form of Fefferman-Phong inequality.

Under the same assumptions on φ as in Remark 11.2, we can express the last result in the following way: The Schrödinger operator with magnetic field

$$\mathcal{S} = \frac{1}{4}\,(-\Delta_A + B), \tag{11.5}$$

where

$$\Delta_A = \left(\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial}{\partial y} - \frac{i}{2}\frac{\partial \varphi}{\partial x}\right)^2 \quad \textit{and} \quad B = \frac{1}{2}\triangle\varphi$$

has compact resolvent if and only if (11.4) holds.

We return to the Dirac and Pauli operators related with the weight function φ :

$$\mathcal{D} = (-i\frac{\partial}{\partial x} - A_1)\,\sigma_1 + (-i\frac{\partial}{\partial y} - A_2)\,\sigma_2,$$

where $A_1 = -\frac{1}{2}\frac{\partial\varphi}{\partial y}$, $A_2 = \frac{1}{2}\frac{\partial\varphi}{\partial x}$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

The square of $\mathcal{D}$ is diagonal with the Pauli operators $P_\pm$ on the diagonal:

$$\mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix},$$

where

$$P_\pm = \left(-i\,\frac{\partial}{\partial x} - A_1\right)^2 + \left(-i\,\frac{\partial}{\partial y} - A_2\right)^2 \pm B \ = -\Delta_A \pm B,$$

where $B = \frac{1}{2}\Delta\varphi$.

Theorem 11.3. *Suppose that $|z|^2\Delta\varphi(z) \to +\infty$ as $|z| \to \infty$. Then the corresponding Dirac operator $\mathcal{D}$ has non-compact resolvent.*

Proof. By 13.12 $\mathcal{D}^2$ has compact resolvent, if and only if $\mathcal{D}$ has compact resolvent. Suppose that $\mathcal{D}$ has compact resolvent. Since

$$\mathcal{D}^2 = \begin{pmatrix} P_- & 0 \\ 0 & P_+ \end{pmatrix},$$

this would imply that both $P_\pm$ have compact resolvent.
We know from (8.7) that

$$P_- = 4\overline{D}^*\,\overline{D} = 4e^{-\varphi/2}\,\overline{\partial}^*_\varphi\overline{\partial}\ e^{\varphi/2}$$

and that P_- is non-negative self-adjoint operator. It follows from Theorem 7.10 that the space of entire functions $A^2(\mathbb{C}, e^{-\varphi})$ is of infinite dimension. This means that 0 belongs to the essential spectrum of P_-. Hence, by Proposition 13.13, P_- fails to have compact resolvent and we arrive at a contradiction. □

For further results see [20] and [21].

A similar conclusion can be drawn in several variables for the Witten Laplacian

$$\Delta_\varphi^{(0,0)} = \overline{D}_1^* \overline{D}_1 = e^{-\varphi/2}\, \overline{\partial}_\varphi^*\, \overline{\partial}\; e^{\varphi/2},$$

if $\lim_{|z|\to\infty} |z|^2 \mu_\varphi(z) = +\infty$, then $\Delta_\varphi^{(0,0)}$ fails to have compact resolvent. (μ_φ is the lowest eigenvalue of the Levi matrix M_φ.)

In this chapter we give some examples of domains or weights, for which the corresponding $\overline{\partial}$-Neumann operator or the canonical solution operator to $\overline{\partial}$ fails to be compact.

First we consider the the canonical solution operator to $\overline{\partial}$ for the bidisc $\mathbb{D} \times \mathbb{D}$ (see [35] for the details):
We know from section 1 that the monomials

$$\varphi_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n \ , \ n = 0, 1, 2, \dots$$

constitute a complete orthonormal system in $A^2(\mathbb{D})$. Consider the following $(0,1)$-forms α_n in $L^2_{(0,1)}(\mathbb{D} \times \mathbb{D})$ with holomorphic coefficients:

$$\alpha_n(z_1, z_2) = \varphi_n(z_1)\, d\overline{z}_2.$$

They are $\overline{\partial}$-closed and their norms in $L^2_{(0,1)}(\mathbb{D} \times \mathbb{D})$ are

$$\|\alpha_n\| = \sqrt{\pi} \ , \ n = 0, 1, 2, \dots$$

The canonical solution to $\overline{\partial} u = \alpha_n$ is given by

$$u_n(z_1, z_2) = \varphi_n(z_1)\, \overline{z}_2,$$

this means that $u_n \in A^2(\mathbb{D} \times \mathbb{D})^\perp$, which follows by the fact that for each $h \in A^2(\mathbb{D} \times \mathbb{D})$ we have

$$\int_{\mathbb{D}\times\mathbb{D}} u_n(z_1, z_2)\, \overline{h(z_1, z_2)}\, d\lambda(z_1, z_2) = \int_{\mathbb{D}} \varphi_n(z_1) \left(\int_{\mathbb{D}} z_2 h(z_1, z_2)\, d\lambda(z_2) \right)^{-} d\lambda(z_1) = 0,$$

where the inner integral vanishes by Cauchy's theorem applied to the holomorphic function $z_2 \mapsto z_2 h(z_1, z_2)$. Finally,

$$\|u_n\| = \sqrt{\frac{\pi}{2}} \ \text{ and } \ u_n \perp u_m \ \text{ if } n \neq m,$$

which follows from

$$(u_n, u_m) = \int_{\mathbb{D}} |z_2|^2\, d\lambda(z_2) \int_{\mathbb{D}} \varphi_n(z_1) \overline{\varphi_m(z_1)}\, d\lambda(z_1)$$

and $(\varphi_n, \varphi_m) = \delta_{n,m}$.
Thus $\{u_n\}$ has no convergent subsequence in $L^2(\mathbb{D} \times \mathbb{D})$. This shows that the canonical solution operator $\overline{\partial}^* N$ to $\overline{\partial}$ fails to be compact.
Further obstructions to compactness can be found in [17], [18] and [46].

We continue to calculate the integrals in (9.10) for the weight $\varphi(z) = |z|^\alpha$ in $\mathbb{C}$. We set $\beta = \alpha/2$ and $u_k(z) = z^k$ for $k \in \mathbb{N}$. The left hand side of (9.10) is

$$\int_{\mathbb{C}\setminus B_R} |u_k(z)|^2\, e^{-|z|^\alpha}\, d\lambda(z) = 2\pi \int_R^\infty r^{2k+1}\, e^{-r^\alpha}\, dr,$$

we indicate that

$$\int_0^\infty r^{2k+1}\, e^{-r^\alpha}\, dr = \frac{1}{2\beta}\, \Gamma\left(\frac{k}{\beta} + \frac{1}{\beta} \right).$$

The right hand side of (9.10) reads

$$\begin{aligned}\int_{\mathbb{C}} |\overline{\partial}^*_\varphi u_k(z)|^2 \, e^{-|z|^\alpha} \, d\lambda(z) &= \int_{\mathbb{C}} | - kz^{k-1} + \beta z^{\beta+k-1} \overline{z}^\beta |^2 \, e^{-|z|^\alpha} \, d\lambda(z) \\ &= 2\pi \int_0^\infty (k^2 r^{2k-1} - 2k\beta \, r^{2\beta+2k-1} + \beta^2 \, r^{4\beta+2k-1}) \, e^{-r^{2\beta}} \, dr \\ &= 2\pi \left[\frac{k^2}{2\beta} \Gamma\left(\frac{k}{\beta}\right) - k \, \Gamma\left(\frac{k}{\beta}+1\right) + \frac{\beta}{2} \Gamma\left(\frac{k}{\beta}+2\right) \right] \\ &= \pi\beta \, \Gamma\left(\frac{k}{\beta}+1\right).\end{aligned}$$

If $\alpha = 2$, it follows that condition (9.10) is not satisfied. For this purpose we consider the integral

$$\int_R^\infty r^{2k+1} \, e^{-r^2} \, dr$$

and substitue $r^2 = s$ obtaining

$$\int_R^\infty r^{2k+1} \, e^{-r^2} \, dr = \int_{R^2}^\infty s^k e^{-s} \, ds.$$

Now we apply k-times partial integration and get

$$\int_{R^2}^\infty s^k e^{-s} \, ds = e^{-R^2} \, R^{2k} + k \int_{R^2}^\infty s^{k-1} e^{-s} \, ds = e^{-R^2} \, k! \sum_{j=0}^k \frac{R^{2j}}{j!}.$$

Observe that for $\beta = 1$ we have

$$\Gamma\left(\frac{k}{\beta}+1\right) = \Gamma\left(\frac{k}{\beta}+\frac{1}{\beta}\right) = k!$$

and as there is $\epsilon_0 > 0$ such that for each $R > 0$ there exists $k \in \mathbb{N}$ such that

$$e^{-R^2} \sum_{j=0}^k \frac{R^{2j}}{j!} > \epsilon_0,$$

condition (9.10) is not satisfied for $\alpha = 2$. This means that $\overline{\partial}$-Neumann N_φ operator on $L^2(\mathbb{C}, e^{-|z|^2})$ fails to be compact, and as $N_\varphi = S^* S$, where S is the canonical solution operator to $\overline{\partial}$, the canonical solution operator S also fails to be compact (compare Theorem 2.11).
Another proof for this fact uses spectral theory: from (8.9) we know that

$$\Box_\varphi u = \overline{\partial} \, \overline{\partial}^*_\varphi u = -\frac{\partial^2 u}{\partial z \partial \overline{z}} + \overline{z} \, \frac{\partial u}{\partial \overline{z}} + u,$$

hence it follows immediately that the whole space $A^2(\mathbb{C}, e^{-|z|^2})$ is a subspace of the eigenspace to the eigenvalue 1 of the operator $\Box_\varphi$, which means that the essential spectrum of $\Box_\varphi$ is nonempty and N_φ fails to be compact by 13.13.

In the next examples we consider decoupled $\mathcal{C}^2$ weights

$$\varphi(z_1, z_2, \ldots, z_n) = \varphi(z_1) + \varphi(z_2) + \cdots + \varphi(z_n)$$

and follow an idea of G. Schneider ([44]).

Theorem 12.1. *Suppose that $n \geq 2$ and that there exists ℓ such that $A^2(\mathbb{C}, e^{-\varphi_\ell})$ is infinite dimensional. Suppose also that $1 \in L^2(\mathbb{C}, e^{-\varphi_j})$ for all j. Suppose finally that for some $k \neq \ell$, $\overline{z}_k \in L^2(\mathbb{C}, e^{-\varphi_k})$. Then the canonical solution operator to $\overline{\partial}$ fails to be compact even on the space $A^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$.*

Proof. Let P_k denote the Bergman projection from $L^2(\mathbb{C}, e^{-\varphi_k})$ onto $A^2(\mathbb{C}, e^{-\varphi_k})$. It is clear that the function $(\overline{z}_k - P_k\overline{z}_k)$ is not zero. Let $(f_\nu)_\nu$ be an infinite orthonormal system in $A^2(\mathbb{C}, e^{-\varphi_\ell})$ and define

$$h_\nu(z) := f_\nu(z_\ell)(\overline{z}_k - P_k\overline{z}_k).$$

Then $(h_\nu)_\nu$ is an orthogonal family in $A^2(\mathbb{C}^n, e^{-\varphi})^\perp$. To see this let $g \in A^2(\mathbb{C}^n, e^{-\varphi})$ and consider

$$\begin{aligned}(g, h_\nu)_\varphi = &\int_{\mathbb{C}} \cdots \int_{\mathbb{C}} g(z) z_k \, e^{-\varphi_k(z_k)} \, d\lambda(z_k) \dots \overline{f_\nu(z_\ell)} e^{-\varphi_\ell(z_\ell)} \, d\lambda(z_\ell) \dots e^{-\varphi_n(z_n)} \, d\lambda(z_n) \\ &- \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} g(z) z_k \, e^{-\varphi_k(z_k)} \, d\lambda(z_k) \dots \overline{f_\nu(z_\ell)} e^{-\varphi_\ell(z_\ell)} \, d\lambda(z_\ell) \dots e^{-\varphi_n(z_n)} \, d\lambda(z_n) = 0,\end{aligned}$$

where we used that $(v, P_k\overline{z}_k)_{\varphi_k} = (v, \overline{z}_k)_{\varphi_k}$ for $v \in A^2(\mathbb{C}, e^{-\varphi_k})$.
In addition we have $\overline{\partial} h_\nu = f_\nu(z_\ell) d\overline{z}_k$.
Hence $\left(\overline{\partial} h_\nu\right)_\nu$ constitutes a bounded sequence in $A^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$, and for the canonical solution operator S we have $S(f_\nu(z_\ell) d\overline{z}_k) = h_\nu$ and since $(h_\nu)_\nu$ is an orthogonal family, it has no convergent subsequence, which implies the result. $\square$

Remark 12.2. *If the conditions of Theorem 12.1 are satisfied, then the corresponding $\overline{\partial}$-Neumann operator $N_{\varphi,1}$ also fails to be compact, which follows from Proposition 9.12.*

In the following example we consider the $\overline{\partial}$-Neumann operator $N_{\varphi,1}$ for a decoupled weight φ :
Example. Let $\varphi(z_1, z_2) = |z_1|^2 + |z_2|^2$ and consider the corresponding $\overline{\partial}$-Neumann operator $N_{\varphi,1}$. We will investigate the following sequence of $(0,1)$-forms

$$u_k(z_1, z_2) = \psi_k(z_1) \, d\overline{z}_2,$$

where $\psi_k(z_1) = \frac{z_1^k}{\sqrt{\pi k!}}$, for $k \in \mathbb{N}$. It follows that $\overline{\partial} u_k = 0$ for each $k \in \mathbb{N}$ and

$$\overline{\partial}^*_\varphi u_k(z_1, z_2) = \overline{z}_2 \psi_k(z_1).$$

This implies

$$\Box_{\varphi,1} \, u_k = u_k \quad \text{and} \quad N_{\varphi,1} \, u_k = u_k$$

for each $k \in \mathbb{N}$. The set $\{u_k \, : \, k \in \mathbb{N}\}$ is a bounded set of mutually orthogonal $(0,1)$-forms in $L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$. As $N_{\varphi,1} \, u_k = u_k$, it follows that $N_{\varphi,1}$ fails to be compact.
The following computation shows that condition (9.10) is not satisfied for the $(0,1)$-forms u_k, where we consider $\int_{\mathbb{C}^2 \setminus Q_R}$ instead of $\int_{\mathbb{C}^2 \setminus \mathbb{B}_R}$, where

$$Q_R = \{(z_1, z_2) \, : \, |z_1| < R \, , \, |z_2| < R\}.$$

We have

$$\begin{aligned}\int_{\mathbb{C}^2 \setminus Q_R} |u_k(z_1, z_2)|^2 e^{-|z_1|^2 - |z_2|^2} \, d\lambda(z_1, z_2) &= \frac{4\pi}{k!} \int_0^R \left(\int_R^\infty r_1^{2k+1} e^{-r_1^2} \, dr_1 \right) r_2 \, e^{-r_2^2} \, dr_2 \\ &+ \frac{4\pi}{k!} \int_R^\infty \left(\int_0^\infty r_1^{2k+1} e^{-r_1^2} \, dr_1 \right) r_2 \, e^{-r_2^2} \, dr_2.\end{aligned}$$

After the substitution $r_1^2 = s$ the first integral is equal to

$$\frac{2\pi}{k!}\int_0^R \left(\int_{R^2}^\infty s^k e^{-s}\, ds\right) r_2\, e^{-r_2^2}\, dr_2.$$

As in the example from above we get

$$\int_{R^2}^\infty s^k e^{-s}\, ds = e^{-R^2}\, k! \sum_{j=0}^k \frac{R^{2j}}{j!},$$

and finally substituting $r_2^2 = t$

$$\begin{aligned}\frac{2\pi}{k!}\int_0^R \left(\int_{R^2}^\infty s^k e^{-s}\, ds\right) r_2\, e^{-r_2^2}\, dr_2 &= \pi e^{-R^2}\sum_{j=0}^k \frac{R^{2j}}{j!}\int_0^{R^2} e^{-t}\, dt \\ &= \pi e^{-R^2}\sum_{j=0}^k \frac{R^{2j}}{j!}\,(1-e^{-R^2}).\end{aligned}$$

On the right hand side of (9.10) we only have the term

$$\frac{1}{\pi k!}\int_{\mathbb{C}^2} |z_1|^{2k}\, |\overline{z}_2|^2 e^{-|z_1|^2-|z_2|^2}\, d\lambda(z_1,z_2) = \frac{4\pi}{k!}\int_0^\infty r_1^{2k+1} e^{-r_1^2}\, dr_1 \int_0^\infty r_2^3 e^{-r_2^2}\, dr_2 = \pi.$$

This implies

$$\int_{\mathbb{C}^2\setminus Q_R} |u_k(z_1,z_2)|^2 e^{-|z_1|^2-|z_2|^2}\, d\lambda(z_1,z_2) \geq \pi e^{-R^2}\sum_{j=0}^k \frac{R^{2j}}{j!}\,(1-e^{-R^2}).$$

As there is $\epsilon_0 > 0$ such that for each $R > 0$ there exists $k \in \mathbb{N}$ such that

$$e^{-R^2}\sum_{j=0}^k \frac{R^{2j}}{j!}\,(1-e^{-R^2}) > \epsilon_0,$$

condition (9.10) is not satisfied.

Finally we discuss compactness of $N_{\varphi,1}$ and $N_{\varphi,2}$ in $\mathbb{C}^2$ for a more general setting: let $\varphi(z_1,z_2) = \varphi_1(z_1) + \varphi_2(z_2)$. The eigenvalues of the Levi matrix are $\frac{\partial^2\varphi_1}{\partial z_1\partial\overline{z}_1}$ and $\frac{\partial^2\varphi_2}{\partial z_2\partial\overline{z}_2}$. If the $(0,1)$-form $u = u_1 d\overline{z}_1 + u_2 d\overline{z}_2$ belongs to $\mathrm{dom}(\Box_{\varphi,1})$, then

$$\begin{aligned}\Box_{\varphi,1}u = &\left(-\frac{\partial^2 u_1}{\partial z_1\partial\overline{z}_1} - \frac{\partial^2 u_1}{\partial z_2\partial\overline{z}_2} + \frac{\partial\varphi_1}{\partial z_1}\frac{\partial u_1}{\partial\overline{z}_1} + \frac{\partial\varphi_2}{\partial z_2}\frac{\partial u_1}{\partial\overline{z}_2} + \frac{\partial^2\varphi_1}{\partial z_1\partial\overline{z}_1}u_1\right) d\overline{z}_1 \\ &+\left(-\frac{\partial^2 u_2}{\partial z_1\partial\overline{z}_1} - \frac{\partial^2 u_2}{\partial z_2\partial\overline{z}_2} + \frac{\partial\varphi_1}{\partial z_1}\frac{\partial u_2}{\partial\overline{z}_1} + \frac{\partial\varphi_2}{\partial z_2}\frac{\partial u_2}{\partial\overline{z}_2} + \frac{\partial^2\varphi_2}{\partial z_2\partial\overline{z}_2}u_2\right) d\overline{z}_2\end{aligned}$$

and for $V = v\, d\overline{z}_1\wedge d\overline{z}_2 \in dom(\Box_{\varphi,2})$ we have

$$\Box_{\varphi,2}V = \left(-\frac{\partial^2 v}{\partial z_1\partial\overline{z}_1} - \frac{\partial^2 v}{\partial z_2\partial\overline{z}_2} + \frac{\partial\varphi_1}{\partial z_1}\frac{\partial v}{\partial\overline{z}_1} + \frac{\partial\varphi_2}{\partial z_2}\frac{\partial v}{\partial\overline{z}_2} + \frac{\partial^2\varphi_1}{\partial z_1\partial\overline{z}_1}v + \frac{\partial^2\varphi_2}{\partial z_2\partial\overline{z}_2}v\right) d\overline{z}_1\wedge d\overline{z}_2.$$

Now suppose that $A^2(\mathbb{C}, e^{-\varphi_1})$ is infinite dimensional, that $1 \in L^2(\mathbb{C}, e^{-\varphi_j})$ for $j = 1,2$, that $\overline{z}_2 \in L^2(\mathbb{C}, e^{-\varphi_2})$ and finally that

$$\frac{\partial^2\varphi_1(z_1)}{\partial z_1\partial\overline{z}_1} + \frac{\partial^2\varphi_2(z_2)}{\partial z_2\partial\overline{z}_2} \to \infty \quad \text{as} \quad |z_1|^2 + |z_2|^2 \to \infty.$$

Then $N_{\varphi,2}$ is compact, but $N_{\varphi,1}$ fails to be compact.

Our assumptions imply that $N_{\varphi,2}$ is compact by 9.4. In addition we have that

$$N_{\varphi,2} = S_2^* S_2,$$

where S_2 is the canonical solution operator for $\overline{\partial}$ for $(0,2)$-forms. Hence S_2 is also compact. Now suppose that $N_{\varphi,1}$ is compact. Since

$$N_{\varphi,1} = S_1^* S_1 + S_2 S_2^*$$

this would imply that S_1 is compact, contradicting 12.1. We get the same conclusion if we apply Proposition 9.12.
The above assumptions are all satisfied for instance for the weightfunctions

$$\varphi(z_1, z_2) = |z_1|^{2k} + |z_2|^{2k}, \ k = 2, 3, \ldots.$$

13. Appendix A: Spectral theory

Here we describe some properties of compact operators on separble Hilbert spaces which are used in the text, see [41] for the details. In addition we include elements of unbounded self-adjoint operators and discuss some properties of non-negative self-adjoint operators with compact resolvent, see [14].
Let $A : H \longrightarrow H$ be a compact, self-adjoint operator on a separable Hilbert space H. The Spectral Theorem says that there exists a real zero-sequence $(\mu_n)_n$ and an orthonormal system $(e_n)_n$ in H such that for $x \in H$

$$Ax = \sum_{n=0}^{\infty} \mu_n (x, e_n) e_n,$$

where the sum converges in the operator norm, i.e.

$$\sup_{\|x\| \leq 1} \|Ax - \sum_{n=0}^{N} \mu_n (x, e_n) e_n\| \to 0,$$

as $N \to \infty$.
Now let H_1 and H_2 be separable Hilbert spaces and $A : H_1 \longrightarrow H_2$ a compact operator. First we indicate that A is compact if and only if A^*A is compact.
There exists a decreasing zero-sequence $(s_n)_n$ in $\mathbb{R}^+$ and orthonormal systems $(e_n)_{n\geq 0}$ in H_1 and $(f_n)_{n\geq 0}$ in H_2, such that

$$Ax = \sum_{n=0}^{\infty} s_n (x, e_n) f_n \ , \ x \in H_1,$$

where the sum converges again in the operator norm. In order to show this one applies the spectral theorem for the positive, compact operator $A^*A : H_1 \longrightarrow H_1$ and gets

$$A^*Ax = \sum_{n=0}^{\infty} s_n^2 (x, e_n) e_n, \tag{13.1}$$

where s_n^2 are the eigenvalues of A^*A. If $s_n > 0$, we set $f_n = s_n^{-1} A e_n$ and get

$$(f_n, f_m) = \frac{1}{s_n s_m} (Ae_n, Ae_m) = \frac{1}{s_n s_m} (A^*Ae_n, e_m) = \frac{s_n^2}{s_n s_m} (e_n, e_m) = \delta_{n,m}.$$

For $y \in H_1$ with $y \perp e_n$ for each $n \in \mathbb{N}_0$ we have by (13.1) that

$$\|Ay\|^2 = (Ay, Ay) = (A^*Ay, y) = 0.$$

Hence we have

$$\begin{aligned} Ax &= A \left(x - \sum_{n=0}^{\infty} (x, e_n) e_n \right) + A \left(\sum_{n=0}^{\infty} (x, e_n) e_n \right) \\ &= \sum_{n=0}^{\infty} (x, e_n) A e_n = \sum_{n=0}^{\infty} s_n (x, e_n) f_n. \end{aligned}$$

The numbers s_n are uniquely determined by the operator A, they are the eigenvalues of A^*A, and they are called the s-numbers of A.
Let $0 < p < \infty$. the operator A belongs to the Schatten-class $\mathbf{S}_p$, if its sequence $(s_n)_n$ of s-numbers belongs to l^p. The elements of the Schatten class $\mathbf{S}_2$ are called Hilbert-Schmidt

operators. A is a Hilbert-Schmidt operator if and only if $\sum_{n=0}^{\infty} \|Ae_n\|^2 < \infty$ for each complete orthonormal system $(e_n)_n$ in H.
On L^2-spaces Hilbert-Schmidt operators can be described in the following way:
Let $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$ be open sets and $A : L^2(T) \longrightarrow L^2(S)$ a linear mapping. A is a Hilbert-Schmidt operator if and only if there exists $K \in L^2(S \times T)$, such that

$$Af(s) = \int_T K(s,t) f(t)\, dt \quad , \; f \in L^2(T).$$

The following characterization of compactness is useful for the special operators in the text, see for instance [13]):

Lemma 13.1. *Let H_1 and H_2 be Hilbert spaces, and assume that $S\colon H_1 \to H_2$ is a bounded linear operator. The following three statements are equivalent:*

- *S is compact.*
- *For every $\varepsilon > 0$ there is a $C = C_\varepsilon > 0$ and a compact operator $T = T_\varepsilon\colon H_1 \to H_2$ such that*

$$\|Sv\|_{H_2} \le C \|Tv\|_{H_2} + \varepsilon \|v\|_{H_1} . \tag{13.2}$$

- *For every $\varepsilon > 0$ there is a $C = C_\varepsilon > 0$ and a compact operator $T = T_\varepsilon\colon H_1 \to H_2$ such that*

$$\|Sv\|_{H_2}^2 \le C \|Tv\|_{H_2}^2 + \varepsilon \|v\|_{H_1}^2 . \tag{13.3}$$

Proof. First we show that (13.2) and (13.3) are equivalent.
Suppose that (13.3) holds. Write (13.3) with ε and C replaced by their squares to obtain

$$\|Sv\|_{H_2}^2 \le C^2 \|Tv\|_{H_2}^2 + \varepsilon^2 \|v\|_{H_1}^2 \le (C \|Tv\|_{H_2} + \varepsilon \|v\|_{H_1})^2,$$

which implies (13.2).
Now suppose that (13.2) holds. Choose η with $\varepsilon = 2\eta^2$ and apply (13.2) with ε replaced by η to get

$$\|Sv\|_{H_2}^2 \le C^2 \|Tv\|_{H_2}^2 + 2\eta C \|v\|_{H_1} \|Tv\|_{H_2} + \eta^2 \|v\|_{H_1}^2 .$$

It is easily seen (small constant - large constant trick) that there is $C' > 0$ such that

$$2\eta C \|v\|_{H_1} \|Tv\|_{H_2} \le \eta^2 \|v\|_{H_1}^2 + C' \|Tv\|_{H_2}^2 ,$$

hence

$$\|Sv\|_{H_2}^2 \le (C^2 + C') \|Tv\|_{H_2}^2 + 2\eta^2 \|v\|_{H_1}^2 = C'' \|Tv\|_{H_2}^2 + \varepsilon \|v\|_{H_1}^2 .$$

To prove the lemma it therefore suffices to prove that (13.2) is equivalent to compactness. When S is known to be compact, we choose $T = S$ and $C = 1$, and (13.2) holds for every positive ε.
For the converse let $(v_n)_n$ be a bounded sequence in H_1. We want to extract a Cauchy subsequence from $(Sv_n)_n$. From (13.2) we have

$$\|Sv_n - Sv_m\|_{H_2} \le C \|Tv_n - Tv_m\|_{H_2} + \varepsilon \|v_n - v_m\|_{H_1} \tag{13.4}$$

Given a positive integer N, we may choose ε sufficiently small in (13.4) so that the second term on the right-hand side is at most $1/(2N)$. The first term can be made smaller than $1/(2N)$ by extracting a subsequence of $(v_n)_n$ (still labeled the same) for which $(Tv_n)_n$ converges, and then choosing n and m large enough.

Let $(v_n^{(0)})_n$ denote the original bounded sequence. The above argument shows that, for each positive integer N, there is a sequence $(v_n^{(N)})_n$ satisfying : $(v_n^{(N)})_n$ is a subsequence of $(v_n^{(N-1)})_n$, and for any pair v and w in $(v_n^{(N)})_n$ we have $\|Sv - Sw\|_{H_2} \leq 1/N$.
Let $(w_k)_k$ be the diagonal sequence defined by $w_k = v_k^{(k)}$. Then $(w_k)_k$ is a subsequence of $(v_n^{(0)})_n$ and the image sequence under S of $(w_k)_k$ is a Cauchy sequence. Since H_2 is complete, the image sequence converges and S is compact.

$\square$

In the sequel we develop elements of unbounded self-adjoint operators which are used for the $\overline{\partial}$- complex.

Definition 13.2. *Let H_1, H_2 be Hilbert spaces and $T : dom(T) \longrightarrow H_2$ be a densely defined linear operator. Let $dom(T^*)$ be the space of all $y \in H_2$ such that $x \mapsto (Tx, y)_2$ defines a continuous linear functional on $dom(T)$. Since $dom(T)$ is dense in H_1 there exists a uniquely determined element $T^*y \in H_1$ such that $(Tx, y)_2 = (x, T^*y)_1$ (Riesz representation theorem!). The map $y \mapsto T^*y$ is linear and $T^* : dom(T^*) \longrightarrow H_1$ is the adjoint operator to T.*
T is a closed operator, if the graph $\mathcal{G}(T) = \{(f, Tf) \in H_1 \times H_2 \, : \, f \in dom(T)\}$ is a closed subspace of $H_1 \times H_2$. The inner product in $H_1 \times H_2$ is $((x, y), (u, v)) = (x, u)_1 + (y, v)_2$.

Remark 13.3. *If $dom(T)$ is a closed subspace of H_1, then, by the closed graph theorem; T is bounded if and only if T is closed.*
Let $T_1 : dom(T_1) \longrightarrow H_2$ be a densely defined operator and $T_2 : H_2 \longrightarrow H_3$ be a bounded operator. Then $(T_2 T_1)^* = T_1^* T_2^*$.
Let T be a densely defined operator on H and let S be a bounded operator on H. Then $(T + S)^* = T^* + S^*$.

Lemma 13.4. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined linear operator and define $V : H_1 \times H_2 \longrightarrow H_2 \times H_1$ by $V((x, y)) = (y, -x)$. Then*

$$\mathcal{G}(T^*) = [V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp);$$

in particular T^* is always closed.

Proof. $(y, z) \in \mathcal{G}(T^*) \Leftrightarrow (Tx, y)_2 = (x, z)_1$ for each $x \in \mathrm{dom}(T) \;\Leftrightarrow ((x, Tx), (-z, y)) = 0$ for each $x \in \mathrm{dom}(T) \;\Leftrightarrow V^{-1}((y, z)) = (-z, y) \in \mathcal{G}(T)^\perp$. Hence $\mathcal{G}(T^*) = V(\mathcal{G}(T)^\perp)$ and since V is unitary we have $V^* = V^{-1}$ and $[V(\mathcal{G}(T))]^\perp = V(\mathcal{G}(T)^\perp)$.

$\square$

Lemma 13.5. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then*

$$H_2 \times H_1 = V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*).$$

Proof. $\mathcal{G}(T)$ is closed, therefore, by Lemma 13.4: $\mathcal{G}(T^*)^\perp = V(\mathcal{G}(T))$. $\square$

Lemma 13.6. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $dom(T^*)$ is dense in H_2 and $T^{**} = T$.*

Proof. Let $z \perp \mathrm{dom}(T^*)$. Hence $(z, y)_2 = 0$ for each $y \in \mathrm{dom}(T^*)$. We have

$$V^{-1} : H_2 \times H_1 \longrightarrow H_1 \times H_2$$

where $V^{-1}((y, x)) = (-x, y)$, and $V^{-1}V = \mathrm{Id}$. Now, by Lemma 13.5, we have

$$H_1 \times H_2 \cong V^{-1}(H_2 \times H_1) = V^{-1}(V(\mathcal{G}(T)) \oplus \mathcal{G}(T^*)) \cong \mathcal{G}(T) \oplus V^{-1}(\mathcal{G}(T^*)).$$

Hence $(z, y)_2 = 0 \Leftrightarrow ((0, z), (-T^*y, y)) = 0$ for each $y \in \text{dom}(T^*)$ implies $(0, z) \in \mathcal{G}(T)$ and therefore $z = T(0) = 0$, which means that $\text{dom}(T^*)$ is dense in H_2.
Since T and T^* are densely defined and closed we have by Lemma 13.4

$$\mathcal{G}(T) = \mathcal{G}(T)^{\perp\perp} = [V^{-1}\mathcal{G}(T^*)]^{\perp} = \mathcal{G}(T^{**}),$$

where $-V^{-1}$ corresponds to V in considering operators from H_2 to H_1. $\square$

Lemma 13.7. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined linear operator. Then $KerT^* = (ImT)^{\perp}$, which means that $KerT^*$ is closed.*

Proof. Let $v \in \text{Ker}T^*$ and $y \in \text{Im}T$, which means that there exists $u \in \text{dom}(T)$ such that $Tu = y$. Hence

$$(v, y)_2 = (v, Tu)_2 = (T^*v, u)_1 = 0,$$

and $\text{Ker}T^* \subseteq (\text{Im}T)^{\perp}$.
And if $y \in (\text{Im}T)^{\perp}$, then $(y, Tu)_2 = 0$ for each $u \in \text{dom}(T)$, which implies that $y \in \text{dom}(T^*)$ and $(y, Tu)_2 = (T^*y, u)_1$ for each $u \in \text{dom}(T)$. Since each $\text{dom}(T)$ is dense in H_1 we obtain $T^*y = 0$ and $(\text{Im}T)^{\perp} \subseteq \text{Ker}T^*$. $\square$

Lemma 13.8. *Let $T : dom(T) \longrightarrow H_2$ be a densely defined, closed linear operator. Then $KerT$ is a closed linear subspace of H_1.*

Proof. We use Lemma 13.7 for T^* and get $\text{Ker}T^{**} = (\text{Im}T^*)^{\perp}$. Since, by Lemma 13.6, $T^{**} = T$ we obtain $\text{Ker}T = (\text{Im}T^*)^{\perp}$ and that $\text{Ker}T$ is a closed linear subspace of H_1. $\square$

Definition 13.9. *Let $T : dom(T) \longrightarrow H$ be a densely defined linear operator. T is symmetric if $(Tx, y) = (x, Ty)$ for all $x, y \in dom(T)$. We say that T is self-adjoint if T is symmetric and $dom(T) = dom(T^*)$. This is equivalent to requiring that $T = T^*$ and implies that T is closed.*

Lemma 13.10. *Let T be a densely defined, symmetric operator.*
(i) If $dom(T) = H$, then T is self-adjoint and T is bounded.
(ii) If T is self-adjoint and injective, then $Im(T)$ is dense in H, and T^{-1} is self-adjoint.
(iii) If $Im(T)$ is dense in H, then T is injective.
(iv) If $Im(T) = H$, then T is self-adjoint, and T^{-1} is bounded.

Proof. (i) By assumption $\text{dom}(T) \subseteq \text{dom}(T^*)$. If $\text{dom}(T) = H$, it follows that T is self-adjoint, therefore also closed (Lemma 13.4) and continuous by the closed graph theorem.
(ii) Suppose $y \perp \text{Im}(T)$. Then $x \mapsto (Tx, y) = 0$ is continuous on $\text{dom}(T)$, hence $y \in \text{dom}(T^*) = \text{dom}(T)$, and $(x, Ty) = (Tx, y) = 0$ for all $x \in \text{dom}(T)$. Thus $Ty = 0$ and since T is assumed to be injective, it follows that $y = 0$. This proves that $\text{Im}(T)$ in dense in H.
T^{-1} is therefore densely defined, with $\text{dom}(T^{-1}) = \text{Im}(T)$, and $(T^{-1})^*$ exists. Now let $U : H \times H \longrightarrow H \times H$ be defined by $U((x, y)) = (-y, x)$. It easily follows that $U^2 = -I$ and $U^2(M) = M$ for any subspace M of $H \times H$, and we get $\mathcal{G}(T^{-1}) = U(\mathcal{G}(-T))$ and $U(\mathcal{G}(T^{-1})) = \mathcal{G}(-T))$. Being self-adjoint, T is closed; hence $-T$ is closed and T^{-1} is closed. By Lemma 13.5 applied to T^{-1} and to $-T$ we get the orthogonal decompositions

$$H \times H = U(\mathcal{G}(T^{-1})) \oplus \mathcal{G}((T^{-1})^*)$$

and

$$H \times H = U(\mathcal{G}(-T)) \oplus \mathcal{G}(-T)) = \mathcal{G}(T^{-1}) \oplus U(\mathcal{G}(T^{-1})).$$

Consequently

$$\mathcal{G}((T^{-1})^*) = [U(\mathcal{G}(T^{-1}))]^\perp = \mathcal{G}(T^{-1}),$$

which shows that $(T^{-1})^* = T^{-1}$.
(iii) Suppose $Tx = 0$. Then $(x, Ty) = (Tx, y) = 0$ for each $y \in \mathrm{dom}(T)$. Thus $x \perp \mathrm{Im}(T)$, and therefore $x = 0$.
(iv) Since $\mathrm{Im}(T) = H$, (iii) implies that T is injective, $\mathrm{dom}(T^{-1}) = H$. If $x, y \in H$, then $x = Tz$ and $y = Tw$, for some $z \in \mathrm{dom}(T)$ and $w \in \mathrm{dom}(T)$, so that

$$(T^{-1}x, y) = (z, Tw) = (Tz, w) = (x, T^{-1}y).$$

Hence T^{-1} is symmetric. (i) implies that T^{-1} is self-adjoint (and bounded), and now it follows from (ii) that $T = (T^{-1})^{-1}$ is also self-adjoint. □

Lemma 13.11. *Let T be a densely defined closed operator, dom$(T) \subseteq H_1$ and T : dom$(T) \longrightarrow H_2$. Then $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are everywhere defined and bounded,* $\|B\| \leq 1, \quad \|C\| \leq 1$; *in addition B is self-adjoint and positive.*

Proof. Let $h \in H_1$ be an arbitrary element and consider $(h, 0) \in H_1 \times H_2$. Form the proof of Lemma 13.6 we get

$$H_1 \times H_2 = \mathcal{G}(T) \oplus V^{-1}(\mathcal{G}(T^*)), \tag{13.5}$$

which implies that $(h, 0)$ can be written in a unique way as

$$(h, 0) = (f, Tf) + (-T^*(-g), -g),$$

for $f \in \mathrm{dom}(T)$ and $g \in \mathrm{dom}(T^*)$, which gives $h = f + T^*g$ and $0 = Tf - g$. We set $Bh := f$ and $Ch := g$. In this way we get two linear operators B and C everywhere defined on H_1. The two equations from above can now be written as

$$I = B + T^*C, \quad 0 = TB - C,$$

which gives

$$C = TB \quad \text{and} \quad I = B + T^*TB = (I + T^*T)B. \tag{13.6}$$

The decomposition in (13.5) is orthogonal, therefore we obtain

$$\|h\|^2 = \|(h, 0)\|^2 = \|(f, Tf)\|^2 + \|(T^*g, -g)\|^2 = \|f\|^2 + \|Tf\|^2 + \|T^*g\|^2 + \|g\|^2,$$

and hence

$$\|Bh\|^2 + \|Ch\|^2 = \|f\|^2 + \|g\|^2 \leq \|h\|^2,$$

which implies $\|B\| \leq 1$ and $\|C\| \leq 1$.
For each $u \in \mathrm{dom}(T^*T)$ we get

$$((I + T^*T)u, u) = (u, u) + (Tu, Tu) \geq (u, u)$$

hence, if $(I + T^*T)u = 0$ we get $u = 0$. Therefore $(I + T^*T)^{-1}$ exists and (13.6) implies that $(I + T^*T)^{-1}$ is defined everywhere and $B = (I + T^*T)^{-1}$. Finally let $u, v \in H_1$. Then

$$\begin{aligned}(Bu, v) &= (Bu, (I + T^*T)Bv) = (Bu, Bv) + (Bu, T^*TBv) \\ &= (Bu, Bv) + (T^*TBu, Bv) = ((I + T^*T)Bu, Bv) = (u, Bv)\end{aligned}$$

and

$$(Bu, u) = (Bu, (I + T^*T)Bu) = (Bu, Bu) + (TBu, TBu) \geq 0,$$

which proves the lemma. □

Next we mention some facts from spectral theory of unbounded operators on Hilbert spaces, see for instance [14].
If A is a linear operator on a Hilbert space H with domain $\mathrm{dom}(A)$, then its spectrum $\mathrm{Spec}(A)$ is defined as follows. We say a complex number z does not lie in $\mathrm{Spec}(A)$ if the operator $(z - A)$ maps $\mathrm{dom}(A)$ one-one onto H, and the inverse (or resolvent) operator, which we shall denote by $R(z, A)$ or $(z - A)^{-1}$, is bounded.
For $z, w \notin \mathrm{Spec}(A)$ we have

$$R(z, A) - R(w, A) = -(z - w)R(z, A)R(w, A). \tag{13.7}$$

Using the spectral theorem for non-negative self-adjoint operators A (i.e. $(Af, f) \geq 0$, for each $f \in \mathrm{dom}(A)$) one gets that the spectrum of A is contained in $[0, \infty)$. There exists a self-adjoint square root $A^{1/2}$ of A and $\mathrm{dom}A^{1/2} = \mathrm{dom}A$. In addition $\mathrm{dom}A$ endowed with the norm

$$\|f\|_{\mathcal{D}} := (\|A^{1/2}f\|^2 + \|f\|^2)^{1/2}$$

becomes a Hilbert space, see [14], Chapter 4. The norm $\|.\|_{\mathcal{D}}$ stems from the inner product $(f, g)_{\mathcal{D}} = (A^{1/2}f, A^{1/2}g) + (f, g)$.

Proposition 13.12. *Let A be a non-negative self-adjoint operator on H. Let domA be endowed with the norm $\|.\|_{\mathcal{D}}$. A has compact resolvent if and only if the canonical imbedding*

$$j : domA \longrightarrow H$$

is a compact linear operator.
Furthermore, A has compact resolvent if and only if $A^{1/2}$ has compact resolvent.

Proof. Since $-1 \notin \mathrm{Spec}(A)$, we know that $(A + 1)^{-1}$ is a bounded operator on H. From (13.7) we get that $R(-1, A) = (A+1)^{-1}$ is compact if and only if $R(z, A)$ is compact for any $z \notin \mathrm{Spec}(A)$.
Let $u \in H$ and $v \in \mathrm{dom}A$. Then

$$\begin{aligned}
(j^*u, v)_{\mathcal{D}} &= (u, jv) = (u, v) = ((A + 1)(A + 1)^{-1}u, v) = ((A + 1)^{-1}u, (A + 1)v) \\
&= ((A + 1)^{-1}u, Av) + ((A + 1)^{-1}u, v) \\
&= (A^{1/2}(A + 1)^{-1}u, A^{1/2}v) + ((A + 1)^{-1}u, v) \\
&= ((A + 1)^{-1}u, v)_{\mathcal{D}},
\end{aligned}$$

This implies that $j^* = (A+1)^{-1}$ as operator on $\mathrm{dom}A$ and $j \circ j^* = (A+1)^{-1}$ as operator on H. So we get the first statement by the fact that j is compact if and only if $j \circ j^*$ is compact.
The second statement follows from $(A^{1/2} + i)^* = A^{1/2} - i$ and

$$(A + 1) = (A^{1/2} + i)(A^{1/2} - i).$$

□

The point spectrum of A is by definition the set of all of its eigenvalues. The discrete spectrum is defined as the set of all eigenvalues μ of finite multiplicity which are isolated in the sense that $(\mu - \epsilon, \mu)$ and $(\mu, \mu + \epsilon)$ are disjoint from the spectrum for some $\epsilon > 0$. The non-discrete part of the spectrum of A is called the essential spectrum. The next proposition follows from the spectral theorem of unbounded, self-adjoint operators (see [14]).

Proposition 13.13. *Let A be a non-negative self-adjoint operator on H. Then the following conditions are equivalent:*
(i) The resolvent operator $(A+1)^{-1}$ is compact.
(ii) The operator A has empty essential spectrum.

14. Appendix B: Some differential geometric aspects

Let (M, ω) be a Kähler manifold with fundamental form ω and (E, M, π) a holomorphic vector bundle over M.
Let
$$\nabla : \Gamma(T^{\mathbb{C}}(M)) \times \Gamma(E) \longrightarrow \Gamma(E)$$
be the uniquely determined connection on E that is both holomorphic and compatible with the metric. The operator
$$\Theta := \nabla^2$$
is called the curvature of the connection ∇.
We consider the weighted $\overline{\partial}$ complex on $\mathbb{C}^n$ with fundamental form
$$\omega = i \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k.$$
The weight factor $e^{-\varphi}$ can be interpreted as a metric on the trivial line bundle over $\mathbb{C}^n$ and
$$\Theta = \partial\overline{\partial}\varphi = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, dz_j \wedge d\overline{z}_k,$$
see [49] for the details. Let Λ denote the interior multiplication with the fundamental form ω :
$$(\Lambda\alpha, w) = (\alpha, \omega \wedge w),$$
for suitable differential forms α and w.
Let $u = \sum_{|J|=q}' u_J \, d\overline{z}_J$ be a $(0, q)$-form with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$, we want to interprete the term
$$\sum_{|K|=q-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, u_{jK} \overline{u}_{kK}\, e^{-\varphi}\, d\lambda.$$
of (5.9) by the curvature Θ and the operator Λ. For this purpose we consider (n, q)-forms
$$\xi = \sum_{|I|=q}{}' \xi_I \, dz \wedge d\overline{z}_I,$$
instead of $(0, q)$-forms, where $dz = dz_1 \wedge \cdots \wedge dz_n$. We use the notation
$$d\hat{z}_j := dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n,$$
which means that dz_j is excluded. It follows that
$$\Lambda\xi = i \sum_{j=1}^{n} \sum_{|J|=q-1}{}' \xi_{jJ} d\hat{z}_j \wedge d\overline{z}_J$$
and since $\Theta\xi = 0$ we obtain for the commutator $[\Theta, \Lambda]$ that
$$(i[\Theta, \Lambda]\xi, \xi)_\varphi = \sum_{|J|=q-1}{}' \sum_{j,k=1}^{n} \int_{\mathbb{C}^n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, \xi_{jJ} \overline{\xi}_{kJ}\, e^{-\varphi}\, d\lambda.$$
The commutator $[\Theta, \Lambda]$ appears in the Nakano vanishing theorem, see [49].

15. Appendix C: Compact subsets in L^2-spaces

A set A is precompact (i.e. $\overline{A}$ is compact) in a Banach space X if and only if for every positive number ϵ there is a finite subset N_ϵ of points of X such that $A \subset \bigcup_{y\in N_\epsilon} B_\epsilon(y)$. A set N_ϵ with this property is called a finite ϵ-net for A.

We recall the Arzela-Ascoli theorem: Let Ω be a bounded domain in $\mathbb{R}^n$. A subset K of $\mathcal{C}(\overline{\Omega})$ is precompact in $\mathcal{C}(\overline{\Omega})$ if the following two conditions hold:
(i) There exists a constant M such that $|\phi(x)| \le M$ holds for every $\phi \in K$ and $x \in \Omega$. (Boundedness)
(ii) For every $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \in K,\ x, y \in \Omega$, and $|x-y| < \delta$, then $|\phi(x) - \phi(y)| < \epsilon$. (Equicontinuity)

Let J be a nonnegative, real-valued function belonging to $\mathcal{C}_0^\infty(\mathbb{R}^n)$ and having the properties $J(x) = 0$ if $|x| \ge 1$, and $\int_{\mathbb{R}^n} J(x)\,dx = 1$ and let $J_\epsilon(x) = \epsilon^{-n} J(x/\epsilon)$ for $\epsilon > 0$. Consider the convolution

$$J_\epsilon * u(x) = \int_{\mathbb{R}^n} J_\epsilon(x-y) u(y)\,dy,$$

defined for functions u for which the right side makes sense.
$J_\epsilon * u$ is called a mollification of u. We have $J_\epsilon * u \in \mathcal{C}^\infty(\mathbb{R}^n)$, if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$.
If Ω is a domain in $\mathbb{R}^n$ and $u \in L^2(\Omega)$, then $J_\epsilon * u \in L^2(\Omega)$ and

$$\|J_\epsilon * u\|_2 \le \|u\|_2 \quad , \quad \lim_{\epsilon\to 0+} \|J_\epsilon * u - u\|_2 = 0$$

(see [1] for further details).

Let $\Omega \subseteq \mathbb{R}^n$ be a domain and u a complex-valued function on Ω. Let

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

Theorem 15.1. *A bounded subset $\mathcal{A}$ of $L^2(\Omega)$ is precompact in $L^2(\Omega)$ if and only if for every $\epsilon > 0$ there exists a number $\delta > 0$ and a subset $\omega \subset\subset \Omega$ such that for every $u \in \mathcal{A}$ and $h \in \mathbb{R}^n$ with $|h| < \delta$ both of the following inequalities hold:*

$$\int_\Omega |\tilde{u}(x+h) - \tilde{u}(x)|^2\,dx < \epsilon^2 \quad , \quad \int_{\Omega\setminus\overline{\omega}} |u(x)|^2\,dx < \epsilon^2. \tag{15.1}$$

Proof. Let $\tau_h u(x) = u(x+h)$ denote the translate of u by h. First assume that $\mathcal{A}$ is precompact. Since $\mathcal{A}$ has a finite $\epsilon/6$- net, and since $\mathcal{C}_0(\Omega)$ is dense in $L^2(\Omega)$, there exists a finite set $S \subset \mathcal{C}_0(\Omega)$, such that for each $u \in \mathcal{A}$ there exists $\phi \in S$ satisfying $\|u - \phi\|_2 < \epsilon/3$. Let ω be the union of the supports of the finitely many functions in S. Then $\omega \subset\subset \Omega$ and the second inequality follows immediately. To prove the first inequality choose a closed ball B_r of radius r centered at the origin and containing ω. Note that $(\tau_h\phi - \phi)(x) = \phi(x+h) - \phi(x)$ is uniformly continuous and vanishes outside B_{r+1} provided $|h| < 1$. Hence

$$\lim_{|h|\to 0} \int_{\mathbb{R}^n} |\tau_h\phi(x) - \phi(x)|^2\,dx = 0,$$

the convergence being uniform for $\phi \in S$. For $|h|$ sufficiently small, we have $\|\tau_h\phi-\phi\|_2 < \epsilon/3$. If $\phi \in S$ satisfies $\|u-\phi\|_2 < \epsilon/3$, then also $\|\tau_h\tilde{u}-\tau_h\phi\|_2 < \epsilon/3$. Hence we have for $|h|$ sufficiently small (independent of $u \in \mathcal{A}$),

$$\|\tau_h\tilde{u}-\tilde{u}\|_2 \leq \|\tau_h\tilde{u}-\tau_h\phi\|_2 + \|\tau_h\phi-\phi\|_2 + \|\phi-u\|_2 < \epsilon$$

and the first inequality follows.

It is sufficient to prove the converse for the special case $\Omega = \mathbb{R}^n$, as it follows for general Ω from its application in this special case to the set $\tilde{\mathcal{A}} = \{\tilde{u} \,:\, u \in \mathcal{A}\}$.

Let $\epsilon > 0$ be given and choose $\omega \subset\subset \mathbb{R}^n$ such that for all $u \in \mathcal{A}$

$$\int_{\mathbb{R}^n\setminus\overline{\omega}} |u(x)|^2\,dx < \frac{\epsilon}{3}.$$

For any $\eta > 0$ the function $J_\eta * u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and in particular it belongs to $\mathcal{C}(\overline{\omega})$. If $\phi \in \mathcal{C}_0(\mathbb{R}^n)$, then by Hölder's inequality

$$\begin{aligned}|J_\eta * \phi(x)-\phi(x)|^2 &= \left|\int_{\mathbb{R}^n} J_\eta(y)(\phi(x-y)-\phi(x))\,dy\right|^2\\ &\leq \int_{B_\eta} J_\eta(y)|\tau_{-y}\phi(x)-\phi(x)|^2\,dy\end{aligned}$$

Hence

$$\|J_\eta*\phi-\phi\|_2 \leq \sup_{h\in B_\eta}\|\tau_h\phi-\phi\|_2.$$

If $u \in L^2(\mathbb{R}^n)$, let $(\phi_j)_j$ be a sequence in $\mathcal{C}_0(\mathbb{R}^n)$ converging to u in L^2 norm. Then $(J_\eta*\phi_j)_j$ is a sequence converging to $J_\eta*u$ in $L^2(\mathbb{R}^n)$. Since also $\tau_h\phi_j \to \tau_h u$ in $L^2(\mathbb{R}^n)$, we have

$$\|J_\eta*u-u\|_2 \leq \sup_{h\in B_\eta}\|\tau_h u-u\|_2.$$

From the first inequality in our assumption we derive that $\lim_{|h|\to0}\|\tau_h u-u\|_2 = 0$ uniformly for $u \in \mathcal{A}$. Hence $\lim_{\eta\to0}\|J_\eta*u-u\|_2 = 0$ uniformly for $u \in \mathcal{A}$. Fix $\eta > 0$ so that

$$\int_{\overline{\omega}} |J_\eta*u(x)-u(x)|^2\,dx < \frac{\epsilon}{6}$$

for all $u \in \mathcal{A}$.

We show that $\{J_\eta*u \,:\, u\in\mathcal{A}\}$ satisfies the conditions of the Arzela-Ascoli theorem on $\overline{\omega}$ and hence is precompact in $\mathcal{C}(\overline{\omega})$. We have

$$|J_\eta*u(x)| \leq \left(\sup_{y\in\mathbb{R}^n} J_\eta^2(y)\right)^{1/2}\|u\|_2,$$

which is bounded uniformly for $x \in \mathbb{R}^n$ and $u \in \mathcal{A}$ since $\mathcal{A}$ is bounded in $L^2(\mathbb{R}^n)$ and η is fixed. Similarly

$$|J_\eta*u(x+h)-J_\eta*u(x)| \leq \left(\sup_{y\in\mathbb{R}^n} J_\eta^2(y)\right)^{1/2}\|\tau_h u-u\|_2$$

and so $\lim_{|h|\to0} J_\eta*u(x+h) = J_\eta*u(x)$ uniformly for $x \in \mathbb{R}^n$ and $u \in \mathcal{A}$. Thus $\{J_\eta*u \,:\, u\in\mathcal{A}\}$ is precompact in $\mathcal{C}(\overline{\omega})$ and there exists a finite set $\{\psi_1,\ldots,\psi_m\}$ of functions in $\mathcal{C}(\overline{\omega})$ such that if $u \in \mathcal{A}$, then for some j, $1 \leq j \leq m$, and all $x \in \overline{\omega}$ we have

$$|\psi_j(x) - J_\eta * u(x)| < \frac{\epsilon}{6|\overline{\omega}|}.$$

This together with the inequality $(|a| + |b|)^2 \leq 2(|a|^2 + |b|^2)$ implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} |u(x) - \tilde{\psi}_j(x)|^2\, dx &= \int_{\mathbb{R}^n \setminus \overline{\omega}} |u(x)|^2\, dx + \int_{\overline{\omega}} |u(x) - \psi_j(x)|^2\, dx \\
&< \frac{\epsilon}{3} + 2 \int_{\overline{\omega}} (|u(x) - J_\eta * u(x)|^2 + |J_\eta * (x) - \psi_j(x)|^2)\, dx \\
&< \frac{\epsilon}{3} + 2 \left(\frac{\epsilon}{6} + \frac{\epsilon}{6.|\overline{\omega}|}\, |\overline{\omega}| \right) = \epsilon.
\end{aligned}$$

Hence $\mathcal{A}$ has a finite ϵ-net in $L^2(\mathbb{R}^n)$ and is therefore precompact in $L^2(\mathbb{R}^n)$.

□

Remark 15.2. *(a) With the same proof one gets:*
A bounded subset $\mathcal{A}$ of $L^2(\Omega)$ is precompact in $L^2(\Omega)$ if and only if the following two conditions are satisfied:
(i) for every $\epsilon > 0$ and for each $\omega \subset\subset \Omega$ there exists a number $\delta > 0$ such that for every $u \in \mathcal{A}$ and $h \in \mathbb{R}^n$ with $|h| < \delta$ the following inequality holds:

$$\int_\omega |\tilde{u}(x+h) - \tilde{u}(x)|^2\, dx < \epsilon^2; \tag{15.2}$$

(ii) for every $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that for every $u \in \mathcal{A}$

$$\int_{\Omega \setminus \overline{\omega}} |u(x)|^2\, dx < \epsilon^2. \tag{15.3}$$

(b) An analogous result holds in weighted spaces $L^2(\mathbb{C}^n, \varphi)$.

16. Appendix D: Friedrichs' lemma and Gårding's inequality Sobolev spaces and Rellich's lemma

First we approximate solutions of a first order differential operator by regularization using convolutions. To begin with we define for a function f on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ the function f_x to be $f_x(y) = f(x+y)$.

Lemma 16.1. *If* $1 \leq p < \infty$ *and* $f \in L^p(\mathbb{R}^n)$, *then* $\lim_{x\to 0} \|f_x - f\|_p = 0$.

Proof. If g is continuous with compact support, then g is uniformly continuous, so $g_x \to g$ uniformly as $x \to 0$. Since g_x and g are supported in a common compact set for $|x| \leq 1$, it follows that $\|g_x - g\|_p \to 0$. Given $f \in L^p(\mathbb{R}^n)$ and $\epsilon > 0$, choose a continuous function g with compact support such that $\|f - g\|_p < \epsilon/3$. Then also $\|f_x - g_x\|_p < \epsilon/3$, so

$$\|f_x - f\|_p \leq \|f_x - g_x\|_p + \|g_x - g\|_p + \|g - f\|_p < \|g_x - g\|_p + 2\epsilon/3.$$

For $|x|$ sufficiently small, $\|g_x - g\|_p < \epsilon/3$, hence $\|f_x - f\|_p < \epsilon$.

□

Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function with support in the unit ball such that $\chi \geq 0$ and

$$\int_{\mathbb{R}^n} \chi(x)\, dx = 1.$$

We define $\chi_\epsilon(x) = \epsilon^{-n}\chi(x/\epsilon)$ for $\epsilon > 0$. Let f be an L^2 function on $\mathbb{R}^n$ and define for $x \in \mathbb{R}^n$

$$f_\epsilon(x) = (f*\chi_\epsilon)(x) = \int_{\mathbb{R}^n} f(x')\chi_\epsilon(x-x')\, dx' = \int_{\mathbb{R}^n} f(x-x')\chi_\epsilon(x')\, dx' = \int_{\mathbb{R}^n} f(x-\epsilon x')\chi(x')\, dx'.$$

In the first integral we can differentiate under the integral sign to show that $f_\epsilon \in \mathcal{C}^\infty(\mathbb{R}^n)$. The family of functions $(\chi_\epsilon)_\epsilon$ is called an approximation to the identity.

Lemma 16.2. $\|f_\epsilon - f\|_p \to 0$ *as* $\epsilon \to 0$.

Proof.

$$f_\epsilon(x) - f(x) = \int_{\mathbb{R}^n} [f(x - \epsilon x') - f(x)]\, \chi(x')\, dx'.$$

We use Minkowski's inequality

$$\left[\int \left(\int |F(x',x)|\, dx'\right)^p dx\right]^{1/p} \leq \int \left(\int |F(x',x)|^p\, dx\right)^{1/p} dx' \tag{16.1}$$

to get

$$\|f_\epsilon - f\|_p \leq \int_{\mathbb{R}^n} \|f_{-\epsilon x'} - f\|_p\, |\chi(x')|\, dx'.$$

But $\|f_{-\epsilon x'} - f\|_p$ is bounded by $2\|f\|_p$ and tends to 0 as $\epsilon \to 0$ by Lemma 16.1. The desired result follows from the dominated convergence theorem.

□

If $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

$$D_j(u * \chi_\epsilon) = (D_j u) * \chi_\epsilon,$$

where $D_j = \partial/\partial x_j$. This also true, if $u \in L^2(\mathbb{R}^n)$ and $D_j u$ is defined in the sense of distributions.

We are now ready to prove

Lemma 16.3 (Friedrichs' lemma)**.** *If $v \in L^2(\mathbb{R}^n)$ with compact support and a is a $\mathcal{C}^1$-function in a neighborhood of the support of v, it follows that*

$$\|aD_j(v * \chi_\epsilon) - (aD_jv) * \chi_\epsilon\|_2 \to 0 \quad \textit{as } \epsilon \to 0,$$

where $D_j = \partial/\partial x_j$ and aD_jv is defined in the sense of distributions.

Proof. If $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, we have

$$D_j(v * \chi_\epsilon) = (D_jv) * \chi_\epsilon \to D_jv \quad , \quad (aD_jv) * \chi_\epsilon \to aD_jv,$$

with uniform convergence. We claim that

$$\|aD_j(v * \chi_\epsilon) - (aD_jv) * \chi_\epsilon\|_2 \leq C\|v\|_2, \tag{16.2}$$

where $v \in L^2(\mathbb{R}^n)$ and C is some positive constant independent of ϵ and v. Since $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the lemma will follow from 16.2 and the dominated convergence theorem.

To show (16.2) we may assume that $a \in \mathcal{C}_0^1(\mathbb{R}^n)$, since v has compact support. We have for $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
& a(x)D_j(v * \chi_\epsilon)(x) - ((aD_jv) * \chi_\epsilon)(x) \\
= \; & a(x)\, D_j \int v(x-y)\chi_\epsilon(y)\,dy - \int a(x-y)\frac{\partial v}{\partial x_j}(x-y)\chi_\epsilon(y)\,dy \\
= \; & \int (a(x) - a(x-y))\,\frac{\partial v}{\partial x_j}(x-y)\chi_\epsilon(y)\,dy \\
= \; & -\int (a(x) - a(x-y))\,\frac{\partial v}{\partial y_j}(x-y)\chi_\epsilon(y)\,dy \\
= \; & \int (a(x) - a(x-y))v(x-y)\,\frac{\partial}{\partial y_j}\chi_\epsilon(y)\,dy - \int \left(\frac{\partial}{\partial y_j}a(x-y)\right) v(x-y)\chi_\epsilon(y)\,dy.
\end{aligned}$$

Let M be the Lipschitz constant for a such that $|a(x)-a(x-y)| \leq M|y|$, for all $x, y \in \mathbb{R}^n$. Then

$$|a(x)D_j(v * \chi_\epsilon)(x) - ((aD_jv) * \chi_\epsilon)(x)| \leq M \int |v(x-y)|(\chi_\epsilon(y) + |y\frac{\partial}{\partial y_j}\chi_\epsilon(y)|)\,dy.$$

By Minkowski's inequality (16.1) we obtain

$$\begin{aligned}
\|aD_j(v * \chi_\epsilon) - (aD_jv) * \chi_\epsilon\|_2 \;&\leq\; M\,\|v\|_2 \int (\chi_\epsilon(y) + |y\frac{\partial}{\partial y_j}\chi_\epsilon(y)|)\,dy \\
&=\; M(1 + m_j)\|v\|_2,
\end{aligned}$$

where

$$m_j = \int |y\frac{\partial}{\partial y_j}\chi_\epsilon(y)|\,dy = \int |y\frac{\partial}{\partial y_j}\chi(y)|\,dy.$$

This shows (16.2) when $v \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. Snce $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we have proved (16.2) and the lemma. ☐

Lemma 16.4. *Let*

$$L = \sum_{j=1}^n a_jD_j + a_0$$

be a first order differential operator with variable coefficients where $a_j \in \mathcal{C}^1(\mathbb{R}^n)$ and $a_0 \in \mathcal{C}(\mathbb{R}^n)$. If $v \in L^2(\mathbb{R}^n)$ with compact support and $Lv = f \in L^2(\mathbb{R}^n)$ where Lv is defined in *the distribution sense, the convolution* $v_\epsilon = v * \chi_\epsilon$ *is in* $\mathcal{C}_0^\infty(\mathbb{R}^n)$ *and* $v_\epsilon \to v,\ Lv_\epsilon \to f$ *in* $L^2(\mathbb{R}^n)$ *as* $\epsilon \to 0$.

Proof. Since $a_0 v \in L^2(\mathbb{R}^n)$, we have

$$\lim_{\epsilon\to 0} a_0(v * \chi_\epsilon) = \lim_{\epsilon\to 0}(a_0 v * \chi_\epsilon) = a_0 v$$

in $L^2(\mathbb{R}^n)$. Using Friedrichs' lemma 16.3, we have

$$Lv_\epsilon - Lv * \chi_\epsilon = Lv_\epsilon - f * \chi_\epsilon \to 0$$

in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$. The lemma follows easily since $f * \chi_\epsilon \to f$ in $L^2(\mathbb{R}^n)$.

□

In the following we will prove a simple version of Gårding's inequality (coercive estimate), which will be used to investigate compactness of the $\overline{\partial}$-Neumann operator, for a comprehensive treatment of Gårding's inequality see for instance [16] or [9].

Definition 16.5. *If* Ω *is a bounded open set in* $\mathbb{R}^n$, *we define the Sobolev space* $H^k(\Omega)$ *for* k *a nonnegative integer to be the completion of* $\mathcal{C}^\infty(\overline{\Omega})$ *with respect to the norm*

$$\|f\|_{k,\Omega} = \left[\sum_{|\alpha|\le k}\int_\Omega |\partial^\alpha f|^2\, d\lambda\right]^{1/2}, \tag{16.3}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ *is a multiindex ,* $|\alpha| = \sum_{j=1}^n \alpha_j$ *and*

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\dots \partial x_n^{\alpha_n}}.$$

If Ω is a domain with a $\mathcal{C}^1$ boundary, then $H_k(\Omega)$ coincides with

$$W^k(\Omega) = \{f \in L^2(\Omega) \,:\, \partial^\alpha f \in L^2(\Omega),\, |\alpha| \le k\},$$

where the derivatives are taken in the sense of distributions. (See [16].)

Theorem 16.6. *Let* D *be a Dirichlet form of order* 1 *given by*

$$D(u,v) = \sum_{j,k=1}^n (\partial_j u, b_{jk}\partial_k v) + \sum_{k=1}^n (\partial_k u, b_k v) + \sum_{k=1}^n (u, b_k'\partial_k v) + (u, bv), \tag{16.4}$$

where $\partial_j = \frac{\partial}{\partial x_j}$ *and* b_{jk}, b_k, b_k', b *are* $\mathcal{C}^\infty$ *coefficients and the* b_{jk} *are real-valued. Suppose that there exists a constant* $C_0 > 0$ *such that*

$$\Re \sum_{j,k=1}^n b_{jk}(x)\xi_j\xi_k \ge C_0|\xi|^2 \quad ,\xi \in \mathbb{R}^n\ ,\ x \in \overline{\Omega}, \tag{16.5}$$

we say that D *is strongly elliptic on* Ω.
Then there exist constants $C > 0$ *and* $M \ge 0$ *such that*

$$\Re D(u,u) \ge C\|u\|_{1,\Omega}^2 - M\|u\|_{0,\Omega}^2\ ,\ u \in H_1(\Omega), \tag{16.6}$$

we say that D *is coercive over* $H_1(\Omega)$ *(Gårding's inequality).*

Proof. We first set $a_{jk} = \frac{1}{2}(b_{jk} + b_{kj})$. Since the b_{jk} 's are real, strong ellipticity means that for some constant $C_0 > 0$,

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k = \sum_{j,k=1}^{n} b_{jk}\xi_j\xi_k \geq C_0|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. Thus (a_{jk}) is positive definite ($a_{jk} = a_{kj}$), so if ξ is any complex n-vector,

$$\Re \sum_{j,k=1}^{n} b_{jk}\xi_j\overline{\xi}_k = \sum_{j,k=1}^{n} a_{jk}\xi_j\overline{\xi}_k \geq C_0|\xi|^2.$$

Setting $\xi = \nabla u$, where $u \in H_1(\Omega)$, we obtain

$$\Re \sum_{j,k=1}^{n} b_{jk}(\partial_j u)(\partial_k \overline{u}) \geq C_0 \sum_{k=1}^{n} |\partial_k u|^2,$$

so an integration over Ω yields

$$\Re \sum_{j,k=1}^{n} (\partial_j u, b_{jk}\partial_k v) \geq C_0 \sum_{k=1}^{n} \|\partial_k u\|_{0,\Omega}^2 = C_0(\|u\|_{1,\Omega}^2 - \|u\|_{0,\Omega}^2).$$

Also, for some $C_1 > 0$ (independent of u) we have

$$|(\partial_k u, b_k u)| \leq \|u\|_{1,\Omega}\|b_k u\|_{0,\Omega} \leq C_1\|u\|_{1,\Omega}\|u\|_{0,\Omega},$$
$$|(u, b'_k\partial_k u)| \leq \|u\|_{0,\Omega}\|b'_k\partial_k u\|_{0,\Omega} \leq C_1\|u\|_{1,\Omega}\|u\|_{0,\Omega},$$
$$|(u, bu)| \leq C_1\|u\|_{0,\Omega}^2 \leq C_1\|u\|_{1,\Omega}\|u\|_{0,\Omega}.$$

If we set $C_2 = (2n+1)C_1$, we have

$$\Re D(u,u) \geq C_0(\|u\|_{1,\Omega}^2 - \|u\|_{0,\Omega}^2) - C_2\|u\|_{1,\Omega}\|u\|_{0,\Omega}.$$

But since $cd \leq \frac{1}{2}(c^2 + d^2)$ for all $c, d > 0$,

$$C_2\|u\|_{1,\Omega}\|u\|_{0,\Omega} \leq \frac{C_0}{2}\,\|u\|_{1,\Omega}^2 + \frac{C_2^2}{2C_0}\,\|u\|_{0,\Omega}^2,$$

so

$$\Re D(u,u) \geq \frac{C_0}{2}\,\|u\|_{1,\Omega}^2 - \frac{2C_0^2 + C_2^2}{2C_0}\,\|u\|_{0,\Omega}^2, \tag{16.7}$$

which proves Gårding's inequality. □

17. Appendix E: Ruelle's lemma

Let H be a separable Hilbert space with inner product $(.,.)$. We consider nonnegative self-adjoint operators T and S and we write $T \leq S$, if and only if $\mathrm{dom}S \subseteq \mathrm{dom}T$ and $(Tf, f) \leq (Sf, f)$ for each $f \in \mathrm{dom}S$. By the spectral theorem for unbounded self-adjoint operators (see for instance [48]), the square roots of T and S exist and are themselves nonnegative, self-adjoint operators.
For each $f \in \mathrm{dom}S^{1/2} = \mathrm{dom}S$ we have $(T^{1/2}f, T^{1/2}f) \leq (S^{1/2}f, S^{1/2}f)$ if and only if $(Tf, f) \leq (Sf, f)$. So we get that $T \leq S$ if and only if $\mathrm{dom}S^{1/2} \subseteq \mathrm{dom}T^{1/2}$ and $\|T^{1/2}f\| \leq \|S^{1/2}f\|$ for each $f \in \mathrm{dom}S^{1/2}$.

Lemma 17.1 (Ruelle's lemma)**.** *Let T and S be nonnegative, self-adjoint operators. Suppose that $0 \in \rho(T)$ which means that T^{-1} exists and is a bounded operator. Then $T \leq S$ if and only if $S^{-1} \leq T^{-1}$.*

Proof. First we show that $T \leq S$ if and only if $\|T^{1/2}S^{-1/2}\| \leq 1$. For this purpose we notice that for each $g \in H$ we have $S^{-1/2}g \in \mathrm{dom}S^{1/2} \subseteq \mathrm{dom}T^{1/2}$.
Hence $T \leq S \Leftrightarrow \|T^{1/2}f\| \leq \|S^{1/2}f\|$ for each $f \in \mathrm{dom}S^{1/2} \Leftrightarrow \|T^{1/2}S^{-1/2}g\| \leq \|g\|$ for each $g \in H \Leftrightarrow \|T^{1/2}S^{-1/2}\| \leq 1$.

In the next step we show that $\|T^{1/2}S^{-1/2}\| \leq 1 \Leftrightarrow \|S^{-1/2}T^{1/2}f\| \leq \|f\|$ for each $f \in \mathrm{dom}T^{1/2}$. First suppose that $\|T^{1/2}S^{-1/2}\| \leq 1$ and let $f \in \mathrm{dom}T^{1/2}$. Then

$$\begin{aligned}\|S^{-1/2}T^{1/2}f\|^2 &= (S^{-1/2}T^{1/2}f, S^{-1/2}T^{1/2}f) = (T^{1/2}S^{-1/2}S^{-1/2}T^{1/2}f, f)\\ &\leq \|T^{1/2}S^{-1/2}\|\,\|S^{-1/2}T^{1/2}f\|\,\|f\| \leq \|S^{-1/2}T^{1/2}f\|\,\|f\|,\end{aligned}$$

this implies $\|S^{-1/2}T^{1/2}f\| \leq \|f\|$ for each $f \in \mathrm{dom}T^{1/2}$. If we suppose that $\|S^{-1/2}T^{1/2}f\| \leq \|f\|$ for each $f \in \mathrm{dom}T^{1/2}$ we get

$$\begin{aligned}\|T^{1/2}S^{-1/2}g\|^2 &= (T^{1/2}S^{-1/2}g, T^{1/2}S^{-1/2}g) = (S^{-1/2}T^{1/2}T^{1/2}S^{-1/2}g, g)\\ &\leq \|S^{-1/2}T^{1/2}T^{1/2}S^{-1/2}g\|\,\|g\| \leq \|T^{1/2}S^{-1/2}g\|\,\|g\|,\end{aligned}$$

for each $g \in H$, which implies that $\|T^{1/2}S^{-1/2}\| \leq 1$.

Finally we show : $\|S^{-1/2}T^{1/2}f\| \leq \|f\|$ for each $f \in \mathrm{dom}T^{1/2} \Leftrightarrow S^{-1} \leq T^{-1}$. If $\|S^{-1/2}T^{1/2}f\| \leq \|f\|$ for each $f \in \mathrm{dom}T^{1/2}$ we set $g = T^{1/2}f$ and obtain $\|S^{-1/2}g\| \leq \|T^{-1/2}g\|$ for each $g \in H$, which implies that $(S^{-1/2}g, S^{-1/2}g) \leq (T^{-1/2}g, T^{-1/2}g)$ and $(S^{-1}g, g) \leq (T^{-1}g, g)$ for each $g \in H$. In the last reasoning all steps can be reversed, which finishes the proof.

$\square$

18. Appendix F: Some special integrals

Let μ be a rotation-invariant measure on $\mathbb{C}^n$, let $\mathcal{U}$ be the unitary group consisting of all $n \times n$ unitary matrices and let dU denote the Haar probability measure on $\mathcal{U}$. Let σ be the rotation-invariant probability measure on the unit sphere $\mathbb{S}$ in $\mathbb{C}^n$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we define $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ and $\alpha! = \alpha_1! \dots \alpha_n!$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. Due to the invariance of μ it follows by Fubini's theorem that

$$\begin{aligned} \int_{\mathbb{C}^n} z^\alpha \, \overline{z}^\beta \, d\mu(z) &= \int_{\mathcal{U}} \int_{\mathbb{C}^n} (Uz)^\alpha \, (\overline{Uz})^\beta \, d\mu(z) \, dU \\ &= \int_{\mathbb{C}^n} \int_{\mathcal{U}} (Uz)^\alpha \, (\overline{Uz})^\beta \, dU \, d\mu(z) \\ &= \int_{\mathbb{C}^n} |z|^{|\alpha|+|\beta|} \int_{\mathbb{S}} \zeta^\alpha \, \overline{\zeta}^\beta \, d\sigma(\zeta) \, d\mu(z), \end{aligned} \tag{18.1}$$

where we used the fact that for a continuous function $f \in \mathcal{C}(\mathbb{S})$ we have

$$\int_{\mathbb{S}} f(\zeta) \, d\sigma(\zeta) = \int_{\mathcal{U}} f(U\eta) \, dU,$$

for any $\eta \in \mathbb{S}$ (see [43], Proposition 1.4.7.).
It is clear that for $\alpha \neq \beta$ we have

$$\int_{\mathbb{S}} \zeta^\alpha \, \overline{\zeta}^\beta \, d\sigma(\zeta) = 0.$$

Next we claim that for any multi-index γ

$$\int_{\mathbb{S}} |\zeta^\gamma|^2 \, d\sigma(\zeta) = \frac{(n-1)! \, \gamma!}{(n-1+|\gamma|)!} \tag{18.2}$$

To prove (18.2) we use the integral

$$I = \int_{\mathbb{C}^n} |z^\gamma|^2 \, \exp(-|z|^2) \, d\lambda_{2n}(z) = \prod_{j=1}^n \int_{\mathbb{C}} |w|^{2\gamma_j} \, \exp(-|w|^2) \, d\lambda_2(w),$$

where λ_{2n} is the Lebesgue measure on $\mathbb{R}^{2n}$. It follows easily that $I = \pi^n \, \gamma!$. Now we apply integration in polar coordinates to I and get

$$\pi^n \, \gamma! = 2n \, c_n \int_0^\infty r^{2|\gamma|+2n-1} \, e^{-r^2} \, dr \int_{\mathbb{S}} |\zeta^\gamma|^2 \, d\sigma(\zeta),$$

where c_n is the volume of the unit ball in $\mathbb{C}^n$.
Hence

$$\int_{\mathbb{S}} |\zeta^\gamma|^2 \, d\sigma(\zeta) = \frac{\pi^n \, \gamma!}{(n-1+|\gamma|)! \, n c_n},$$

taking $\gamma = 0$ we get $c_n = \pi^n/n!$, which proves (18.2).

For $d \in \mathbb{N}$ we set

$$m_d = \int_{\mathbb{C}^n} |z|^{2d} \, d\mu \quad \text{, and} \quad c_\gamma^{-1} = \int_{\mathbb{C}^n} |z^\gamma|^2 \, d\mu$$

and obtain from (18.1) and (18.2)

$$c_\gamma = \frac{(n-1+|\gamma|)!}{(n-1)! \, \gamma! \, m_{|\gamma|}}. \tag{18.3}$$

References

1. R.A. Adams and J.J.F. Fournier, *Sobolev spaces*, Pure and Applied Math. Vol. 140, Academic Press, 2006.
2. C.A. Berenstein and R. Gay, *Complex variables, an introduction*, Graduate Texts in Mathematics, Springer Verlag, 1991.
3. F. Berger, *The inhomogeneous cauchy-riemann equation*, Bachelorarbeit, Universitaet Wien, 2012.
4. B. Berndtsson, *$\overline{\partial}$ and Schrödinger operators*, Math. Z. **221** (1996), 401–413.
5. H.P. Boas and E.J. Straube, *Global regularity of the $\overline{\partial}$-Neumann problem: a survey of the L^2-Sobolev theory* , Several Complex Variables (M. Schneider and Y.-T. Siu, eds.) MSRI Publications, Cambridge University Press **37** (1999), 79–111.
6. P. Bolley, M. Dauge, and B. Helffer, *Conditions suffisantes pour l'injection compacte d'espace de Sobolev à poids* , Séminaire équation aux dérivées partielles (France), Université de Nantes **1** (1989), 1–14.
7. D. Catlin and J. D'Angelo, *Positivity conditions for bihomogeneous polynomials* , Math. Res. Lett. **4** (1997), 555–567.
8. D.W. Catlin, *Global regularity of the $\overline{\partial}$-Neumann operator* , Proc. Symp. Pure Math. **41** (1984), 39–49.
9. So-Chin Chen and Mei-Chi Shaw, *Partial differential equations in several complex variables*, Studies in Advanced Mathematics, Vol. 19, Amer. Math. Soc., 2001.
10. M. Christ, *On the $\overline{\partial}$ equation in weighted L^2 norms in $\mathbb{C}^1$* , J. of Geometric Analysis **1** (1991), 193–230.
11. M. Christ and S. Fu, *Compactness in the $\overline{\partial}$-Neumann problem, magnetic Schrödinger operators, and the Aharonov-Bohm effect*, Adv. Math. **197** (2005), 1–40.
12. H.L. Cycon, R.G. Froese, W. Kirsch, and B. Simon, *Schrödinger operators with applications to quantum mechanics and global geometry*, Springer - Verlag, 1987.
13. J. P. D'Angelo, *Inequalities from complex analysis*, Carus Mathematical Monographs, vol. 28, Mathematical Association of America, Washington, DC, 2002.
14. E.B. Davies, *Spectral theory and differential operators*, Cambridge studies in advanced mathematics, vol. 42, Cambridge University Press, Cambridge, 1995.
15. J. Elstrodt, *Maß- und Integrationstheorie*, Springer Verlag, Berlin, 1996.
16. G.B. Folland, *Introduction to partial differential equations*, Princeton University Press, Princeton, 1995.
17. S. Fu and E.J. Straube, *Compactness of the $\overline{\partial}-$Neumann problem on convex domains*, J. of Functional Analysis **159** (1998), 629–641.
18. ———, *Compactness in the $\overline{\partial}-$Neumann problem*, Complex Analysis and Geometry (J.McNeal, ed.), Ohio State Math. Res. Inst. Publ. **9** (2001), 141–160.
19. ———, *Semi-classical analysis of Schrödinger operators and compactness in the $\overline{\partial}$ Neumann problem* , J. Math. Anal. Appl. **271** (2002), 267–282.
20. K. Gansberger, *Compactness of the $\overline{\partial}$-Neumann operator*, Dissertation, University of Vienna, 2009.
21. ———, *On the resolvent of the Dirac operator in $\mathbb{R}^2$* , arXiv: **1003.5124** (2010).
22. K. Gansberger and F. Haslinger, *Compactness estimates for the $\overline{\partial}$- Neumann problem in weighted L^2 - spaces*, Complex Analysis (P. Ebenfelt, N. Hungerbühler, J.J. Kohn, N. Mok, E.J. Straube, eds.), Trends in Mathematics, Birkhäuser (2010), 159–174.
23. F. Haslinger, *The canonical solution operator to $\overline{\partial}$ restricted to Bergman spaces*, Proc. Amer. Math. Soc. **129** (2001), 3321–3329.
24. ———, *Schrödinger operators with magnetic fields and the $\overline{\partial}$-equation*, J. Math. Kyoto Univ. **46** (2006), 249–257.
25. ———, *Compactness for the $\overline{\partial}$- Neumann problem- a functional analysis approach* , Collectanea Math. **62** (2011), 121–129.
26. F. Haslinger and B. Helffer, *Compactness of the solution operator to $\overline{\partial}$ in weighted L^2 - spaces*, J. of Functional Analysis **243** (2007), 679–697.
27. F. Haslinger and B. Lamel, *Spectral properties of the canonical solution operator to $\overline{\partial}$* , J. of Functional Analysis **255** (2008), 13–24.
28. B. Helffer, *Semiclassical Analysis, Witten Laplacians, and Statistical Mechanics*, World Scientific, 2002.

29. B. Helffer, J. Nourrigat, and X.P. Wang, *Sur le spectre de l'équation de Dirac (dans $\mathbb{R}^2$ ou $\mathbb{R}^3$) avec champ magnétique* , Annales scientifiques de l'E.N.S. **22** (1989), 515–533.
30. L. Hörmander, *An introduction to complex analysis in several variables*, North Holland, 1990.
31. A. Iwatsuka, *Magnetic Schrödinger operators with compact resolvent* , J. Math. Kyoto Univ. **26** (1986), 357–374.
32. J. Johnsen, *On the spectral properties of Witten Laplacians, their range projections and Brascamp-Lieb's inequality* , Integral Equations Operator Theory **36** (2000), 288–324.
33. J.-M. Kneib and F. Mignot, *Equation de Schmoluchowski généralisée* , Ann. Math. Pura Appl. (IV) **167** (1994), 257–298.
34. W. Knirsch and G. Schneider, *Continuity and Schatten-von Neumann p-class membership of Hankel operators with anti-holomorphic symbols on (generalized) Fock spaces*, J. Math. Anal. Appl. **320** (2006), no. 1, 403–414.
35. St. Krantz, *Compactness of the $\overline{\partial}-$Neumann operator* , Proc. Amer. Math. Soc. **103** (1988), 1136–1138.
36. E. Ligocka, *The regularity of the weighted bergman projections*, Lecture Notes in Math. **1165**, Seminar on deformations, Proceedings, Lodz-Warsaw, 1982/84, Springer-Verlag, Berlin, 1985, pp. 197–203.
37. S. Lovera and E.H. Youssfi, *Spectral properties of the $\bar{\partial}$-canonical solution operator*, J. Functional Analysis **208** (2004), 360–376.
38. J. Marzo and J. Ortega-Cerdá, *Pointwise estimates for the Bergman kernel of the weighted Fock space* , J. Geom. Anal. **19** (2009), 890–910.
39. J.D. McNeal, *A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator* , J. of Functional Analysis **195** (2002), 190–205.
40. ———, *L^2 estimates on twisted Cauchy-Riemann complexes* , 150 years of mathematics at Washington University in St. Louis. Sesquicentennial of mathematics at Washington University, St. Louis, MO, USA, October 3–5, 2003. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics **395** (2006), 83–103.
41. R. Meise and D. Vogt, *Einführung in die Funktionalanalysis* , Vieweg Studium, 1992.
42. W. Rudin, *Real and complex analysis* , Springer Verlag, 1974.
43. ———, *Function Theory in the Unit Ball in $\mathbb{C}^n$* , McGraw-Hill, 1980.
44. G. Schneider, *Non-compactness of the solution operator to $\overline{\partial}$ on the Fock-space in several dimensions*, Math. Nachr. **278** (2005), no. 3, 312–317.
45. I. Shigekawa, *Spectral properties of Schrödinger operators with magnetic fields for a spin* 1/2 *particle* , J. of Functional Analysis **101** (1991), 255–285.
46. E. Straube, *The L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem* , ESI Lectures in Mathematics and Physics, EMS, 2010.
47. F. Treves, *Basic linear partial differential equations* , Dover Books on Mathematics, Dover Publications, Inc., Mineola, NY, 2006.
48. J. Weidmann, *Lineare Operatoren in Hilberträumen* , B.G. Teubner, Stuttgart, 1976.
49. R. O. Wells, *Differential Analysis on Complex Manifolds* , Prentice-Hall, 1973.

Index