An Isometric Embedding of the Impossible Triangle into the Euclidean Space of Lowest Dimension *

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Abstract. The impossible triangle, invented independently by Oscar Reutersvärd and Roger Penrose in 1934 and 1957, is a famous geometry configuration that can not be realized in our living space. Many people admitted that this object could be constructed in the four dimensional Euclidean space without rigorous proof. In this paper, we proved that the isometric embedding problem can be decided by finite points on the configuration, then applying Menger and Blumenthal's classi classical method of Euclidean embedding of finite metric space we determined the lowest Euclidean dimension, and finally using the symbolic algebraic computation we obtained the coordinates of the isometric embedding. Our investigation shows that the impossible triangle is impossible to be isometrically embedded in the four dimension Euclidean space, but there is an isometric embedding of the impossible triangle to the five dimension space.

Keywords: isometric embedding · impossible triangle · Euclidean space · simplex.

1 Introduction

Impossible triangle was firstly painted in 1934 by the Swedish painter Oscar Reutersvärd who was born in 1915 in Stockholm and was trained in arts by Russian immigrant professor of Academy of Arts in St.Petersburg at that time. Oscar Reutersvärd drew his version of traingle as a set of cubes in parallel

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projection, as shown in Figure 1(left). Actually, he started this figure by placing a perfect six-pointed star shape in the middle, and around the star, he added nine cubes, filling the empty spaces between the stars points for creating the 3D illusion. He soon realized that what he'd drawn was paradoxical: something that couldn't be built in the real world. (See [1]).

Reutersvärd was diagnosed with dyslexia at a young age, which prevented him from accurately estimating the size and distance of objects, but he was determined to follow in the footsteps of his artistic family. He continued to design thousands of impossible figures throughout his life. Reutersvärd's achievements were honoured in 1982 by a series of three Swedish postage stamps.

A different version of this impossible triangle was independently created by the English physicist and mathematician Rogers Penrose in 1954. Unlike Reutersvärd's figure, he painted triangle as three bars connected with right angles (later known as the Penrose tribar or Penrose triangle), as shown in Fiure 1(right).

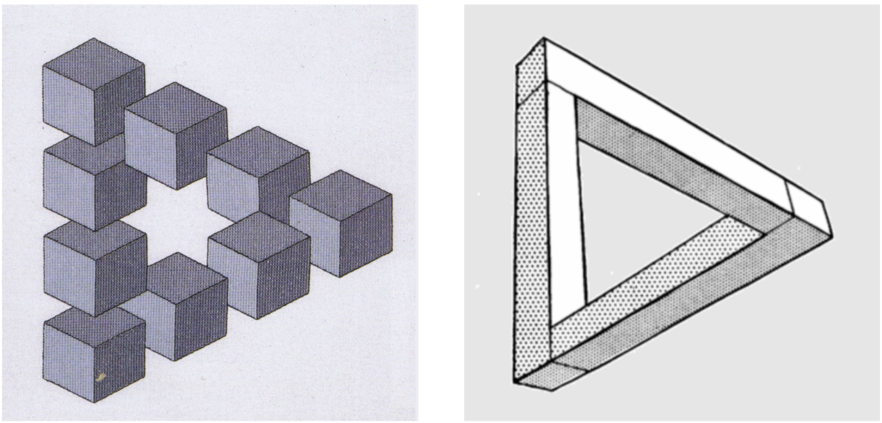


Fig. 1. (*Left*): The impossible figure drawn by Oscar Reutersvärd. (*Right*). The impossible structure in L. S. Penrose and R. Penrose's article published in 1958 in the British Journal of Psychology [2].

Penroses sent a copy of the article to M.C. Escher. Note, neither Penrose nor Escher had not knew about artworks by Reutersvärd at that time (cf. [3]). Escher created his famous lithographs "Ascending and Descending" in 1960 and "Waterfall" in 1961. M.C. Escher provided many popular examples of impossible figures in his drawings and woodcuts. Perhaps the most weird structure to mathematicians is the impossible cube in "Belvedere" (1958) as shown in Figure 2(left). In order to understand impossible figures, we need first to understand two-dimensional representations of the three-dimensional objects. A simple line drawing, such as the *Necker cube* illustrated in Figure 2(right), could be interpreted in two ambiguous ways. The Belvedere' toy cube can be regarded as a version of the Necker cube where the edges cross in inconsistent ways.

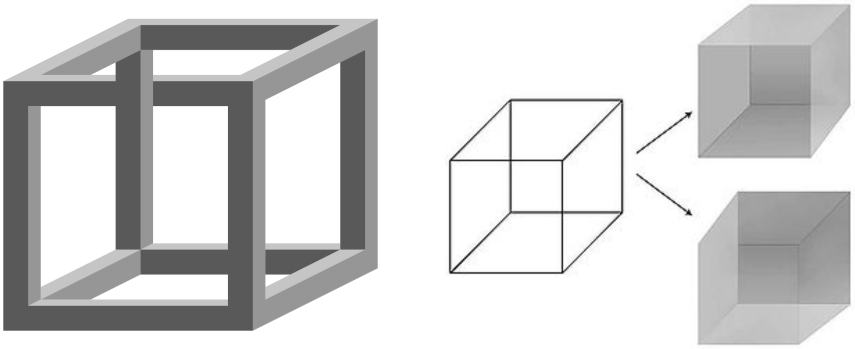


Fig. 2. (*Left*): The toy cube held in hands of the boy in M.C. Escher's lithograph "Belvedere" (1958). (*Right*): The Necker Cube is first published as a rhomboid in 1832 by Swiss crystallographer Louis Albert Necker. It is a wire-frame drawing of a cube in isometric perspective. When two lines cross, the picture does not show which is in front and which is behind. It makes the picture it can be interpreted two different ways.

Impossible figures are helpful to psychology research on human visual perception (see [7]). The Gestalt psychologists used impossible triangle and cube to explain the Law of Pragnanz, that human mind loves to simplify, and quickly make sense of objects, and therefore, human sees the whole image, before the sum of its parts. This theory emphasizes that human perceives objects as wholes rather than as parts. On the reverse, they say that the Gestalt approach to psychology reveals some interesting insights about impossible figures and why they are so captivating.

Some artists find mind-bending ways to bring the Penrose Triangle and other impossible figures into three-dimensional reality. They create clever design on certain objects so that they look like the proper impossible figure when viewed from the correct angle. An example is the Impossible Triangle sculpture in Perth, Australia, shown in the Figure 3, created in the November 1999 by artist Brian McKay in collaboration with architect Ahmad Abas. Another impossible triangle is located in Ophoven, Belgium. This one was achieved differently (due to the great fire-wall and/or intellectual property reason we are not able to provide further information on this one).

As we have seen, for the Impossible Triangle sculpture, there are only two appropriate positions from where could see the proper Penrose triangle. It is curious to ask that if people there are other ways to install certain structure, in higher dimensional space when it is impossible to do this in the usual three dimensional space, so that people can see the impossible figure in a larger viewing angle? The question can be rephrased more precisely as below: if any body can

Fig. 3. An Impossible Triangle sculpture was designed by artist Brian McKay and architect Ahmad Abas, which was built in Claisebrook Square in East Perth, Australia. It is 13.5 metres high, and has remained an East Perth landmark for 20 years.

build a geometric configuration in four or higher dimensional Euclidean space so that people in the real world could actually view an image of the impossible triangle in the real world?

Some people argued (cf. [4]) that since each local part of the Penrose triangle is 3-dimensional, so lies in some 3-dimensional subspace, and because that the edges are straight lines, every piece lies on the same 3-dimensional subspace, so if we don't allow the edges to bend, then the figure is also not possible in higher dimensions when it is not possible in the 3-dimensions.

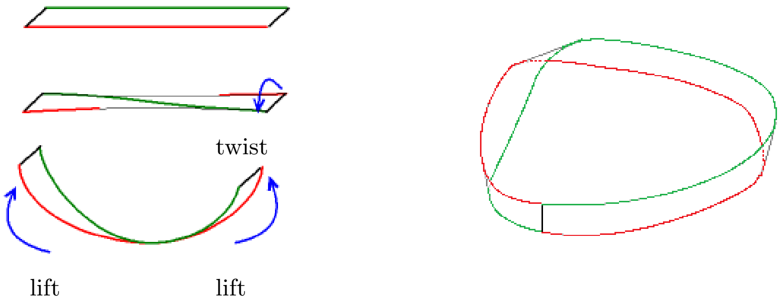
An opposing viewpoint is that figures like the Penrose triangle which seems impossible in our three-dimensional space might be possible in fourth dimension. For example, Blue Sam [5]) indicated as the surface of the Penrose triangle is (up to taking a smooth approximation at the edges) a smooth 2-manifold, so by the Whitney Embedding Theorem, must embed in 5-dimensional space, and it's easy to remove the smooth approximation then. Sam also said that if we are not bothered about keeping the edges straight, the embedding can be done in three dimensions, and he was confident to the four dimension space when it is allow with straight edges. Vlad Alexeev [6] claimed that the bars of four-dimensional impossible triangle can be connected at right angles and it will not be distorted from any point of view as distinct from three-dimensional impossible triangle. However, to the best of our knowledge, we have not found a rigorous proof of the embedding neither to 5 nor to 4 dimensional space.

In this paper, we devote to construct an explicit embedding of the Penrose triangle in the Euclidean space. We will prove that the minimal n for constructing an isometric embedding of the Penrose triangle in $\mathbb{R}^n$ is $n = 5$. The paper is organized as follows: In § 2 we show that the embedding problem can be reduced to a set of finite points, and the finite points can be isometrically embedded into the Euclidean space $\mathbb{R}^n$ with minimal $n = 5$, in § 3 we solve an equation system derived from the reduced embedding problem, so to give an explicit embedding of the Penrose triangle into $\mathbb{R}^n$, in § 4 we present an intuitive explanation to the five dimensional configuration of the Penrose triangle, and show an analogue isometric embedding of the Möbius band to $\mathbb{R}^4$.

2 The Minimal Isometric Embedding

In view of mathematics, making an imagined object in higher dimensional Euclidean space $\mathbb{R}^n$ means to construct points set $\mathfrak{S}$ that is isometric to the configuration in imagination, or topologically looks like the imagined object, and to see a point set of higher dimensional space from the usual $\mathbb{R}^3$ just means to compute the image of certain function $f : \mathbb{R}^n \to \mathbb{R}^3$ that acts like a camera we use everyday for taking $2D$ images of $3D$ objects. As we often take several photos of a three-dimensional object from several different view angles to obtain more information of the $3D$ shape, it is also necessary to take several $3D$ images of one higher dimensional object to percept its whole structure.

The idea for lifting the impossible figures in higher dimensional space is very natural. As we all know, any $2D$ animal living in a plane world $\mathbb{R}^2$ is not able to build a Möbius trip from a paper band, since any movement of the paper in the plane world just can not twist the paper band, and lift it as depicted in the Figure 4. Though the task to twist and lift up a paper band is an impossible mission for any $2D$ animals, human in $\mathbb{R}^3$ can do this job very easily, and mathematicians even can write down the coordinates of points on the Möbius band as in [8].



paper band

Möbius band

Fig. 4. When making a Möbius band from a paper band, the twist and lift operations must be done in the three dimensional space.

For constructing the Penrose triangle, we may start from three equal copies of an L-shape object L, formed by two perpendicular cylinders in the three dimensional space, as depicted in the Figure 5. Just like that in the plane world $2D$ animals can not twist or lift up a paper band, in our living $3D$ space we can connect L_1, L_2 at point A without problem, but we can not connect L_1, L_3 at C and L_2, L_3 at B simultaneously. So the real difficulty for making the Penrose triangle is that we are not able to *move* the three objects *out* of the $3D$ space where we are living.

Now imaging that some of us (say, Jog) happened to know the gate to higher dimensional world, then he could build the Penrose triangle from the $3D$ components L_1, L_2, L_3 in some higher dimensional space which is invisible to us. To show that he had done the craft correctly, Jog could also cut his product again into several pieces $3D$ figures, possibly new ones, as shown in Figure 6, and brought them back to the three dimensional world where we are living, as an evidence of his work in the higher dimensional space.

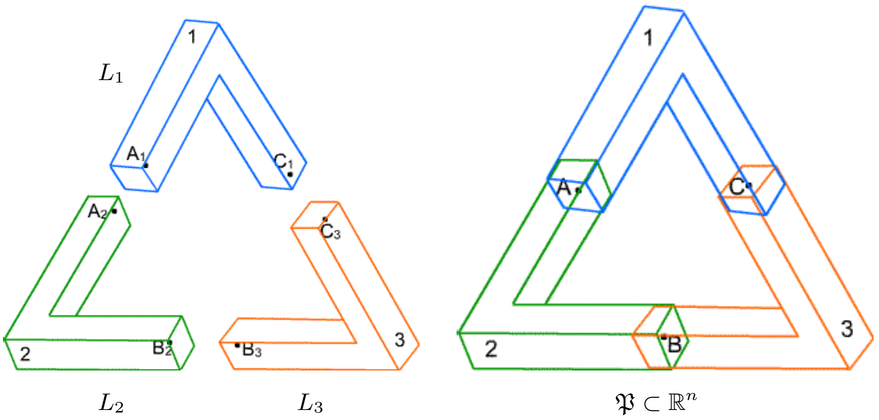


Fig. 5. L_1, L_2, L_3 in the left are three equal copies of $3D$ materials for making the Penrose triangle. If any body knows the gate to the higher dimension, he could take L_1, L_2, L_3 out of the three dimensional space and construct a right Penrose triangle in some higher dimensional space with the $3D$ components.

In mathematics, the above above actions can be understood as decomposing the an object that is impossible figure into finite many $3D$ components, and constructing the impossible configuration (an isometric embedding) in $\mathbb{R}^n$ for some $n > 3$ with the components, and then partitioning the point set into some disjoint parts that can be displayed in the three dimensional space (a piecewise isometric immersion of the impossible configuration in $\mathbb{R}^3$).

For doing the isometric embedding and piecewise immersion, we may decompose the Penrose triangle into the union of three regular cubes and three right cylinders as depicted in the Figure 7. Where as altogether the cubes and cylinders have 24 vertices, we use the 24 lower case Greek letters to denote them. The cubes and cylinders are constructed as follows. Notice that the Penrose is a non-convex polytope, and its convex hull $\mathfrak{P}$ (i.e., the smallest convex set that contains the Penrose triangle) has six extremals. Here a point P is called an extremal point of a convex set $K \subset \mathbb{R}^n$, if there exists no $P_1, P_2 \in K$ such that $P_1 \neq P_2 \in K$ and $P = c \cdot P_1 + (1 - c)P_2$ for some $c \in (0, 1)$. Let $\alpha, \beta, \lambda, o, \phi, \omega$ denote the six extremals of $\mathfrak{P}$ Let C_1, C_2, C_3 be the maximal cylinders contained

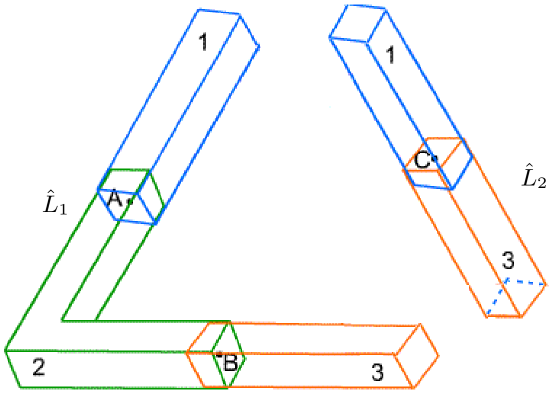


Fig. 6. If any body had built a proper Penrose triangle in higher dimensional space $\mathbb{R}^n$ from three equal $3D$ objects L_1, L_2, L_3, he could cut his product into two $3D$ objects $\hat{L}_1, \hat{L}_2$ and brought back to the real world.

in the Penrose triangle. It is clear that

$$A_1 = C_2 \cap C_3, \quad A_2 = C_3 \cap C_1, \quad A_3 = C_1 \cap C_2,$$

are three disjoint regular cubes contained in the Penrose triangle,

$$B_i = C_i \setminus (A_1 \cup A_2 \cup A_3), i = 1, 2, 3,$$

are three right cylinders contained in the Penrose triangle, and the three cubes and three cylinders form a disjoint partition of the Penrose triangle. We may write

$$A_1 = \alpha\beta\gamma\delta\varepsilon\zeta\eta\theta, A_2 = \iota\kappa\lambda\mu\nu\xi o\pi, A_3 = \rho\sigma\varsigma\tau\upsilon\phi\chi\psi\omega, \tag{1}$$

and

$$B_2 = \gamma\eta\theta\delta\text{-}\nu\iota\kappa\xi, B_1 = \nu\iota\mu\pi\text{-}\chi\upsilon\tau\psi, C_3 = \rho\upsilon\tau\sigma\text{-}\varepsilon\zeta\eta\theta. \tag{2}$$

Without loss of generality, we may assume that the cubes are isometric to $[0,1]\times[0,1]\times[0,1]$, and the cylinders are isometric to $[0,1]\times[0,1]\times(0,a)$, for appropriate $a \geq 3$. Note that The last requirement $a \geq 3$ is pre-assumed according to the most of drawings of Reutersvärd or Penrose's impossible figures. For convenient, we shall denote the Penrose triangle with $\alpha\beta = 1, \gamma\kappa = a$ by notation $\Delta(a)$.

Note also that the Penrose triangle $\Delta(a)$ is contained in the polytype shell formed by removing the convex hull (we shall denote it by $\mathfrak{Q}$) of the six points $\{\eta, \theta, \iota, \nu, \tau, \upsilon\}$ from $\mathfrak{P}$, the convex hull of $\{\alpha, \beta, \lambda, o, \phi, \omega\}$. That is,

$$\Delta(a) \subset \mathfrak{P} \setminus \mathfrak{Q} \quad (a \geq 3). \tag{3}$$

For convenience, we shall call the extremals of $\mathfrak{P}$ and $\mathfrak{Q}$ the extremal point of $\Delta(a)$. As we have seen from Figure 7, the decomposition

$$\Delta(a) = (A_1 \cup A_2 \cup A_3) \cup (\overline{B}_1 \cup \overline{B}_2 \cup \overline{B}_3), \tag{4}$$

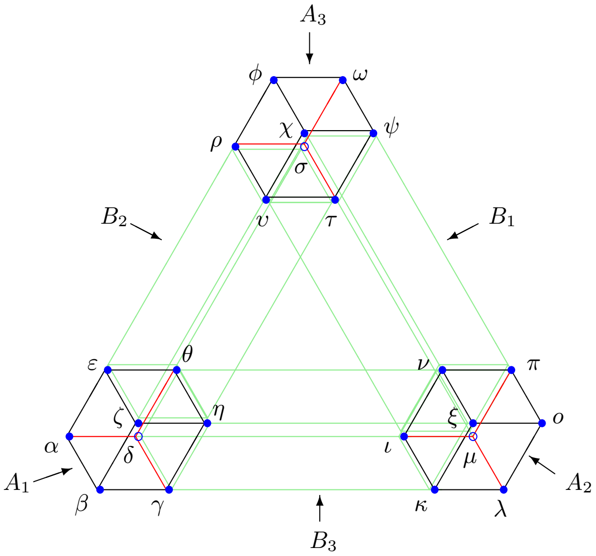


Fig. 7. The Penrose triangle $\Delta(a)$ is decomposed into the disjoint union of three regular cubes A_1, A_2, A_3 and three right (open) cylinders B_1, B_2, B_3.

where $\overline{B}_i\,(i = 1, 2, 3)$ are the closure of B_i, shows that $\Delta(a)$ is a polyhedral-complex (cf. [9]). Therefore, if we can isometrically embed the vertices of the cubes A_1, A_2, A_3 and cylinders B_1, B_2, B_3, that is, the 24 points $\alpha, \beta, \gamma, \cdots, \omega \in \Delta(a)$, into any Euclidean space $\mathbb{R}^n$, then we can construct a Penrose triangle configuration in that space, too. Indeed, we can prove that, the extremals of $\Delta(a)$ are essential for constructing such isometric embedding. Namely, we have the following result.

Theorem 1. *Assume that $\Delta(a)$ is the Penrose triangle as in Figure 7, so that $\alpha\beta = 1, \gamma\kappa = a \geq 3$, and*

$$F : \{\alpha, \beta, \eta, \theta; \iota, \lambda, o, \nu; \tau, \upsilon, \phi, \omega\} \to \mathbb{R}^n$$

is any isometric embedding, then F can be extended to an isometric mapping:

$$\hat{F} : \Delta(a) \to \mathbb{R}^n.$$

Proof(outline) Without loss of generality we may assume that

$$\begin{array}{l} X_1 = F(\alpha),\, X_2 = F(\beta),\, Y_1 = F(\theta),\, Y_2 = F(\eta); \\ X_3 = F(\lambda),\, X_4 = F(o),\, Y_3 = F(\iota),\, Y_4 = F(\nu); \\ X_5 = F(\omega),\, X_6 = F(\phi),\, Y_5 = F(\tau),\, Y_6 = F(\upsilon). \end{array} \tag{5}$$

Let $G, H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be functions defined by

$$G(V, W) = \frac{a+1}{a+2}V + \frac{1}{a+2}W, H(V, W) = \frac{a}{a+1}V + \frac{1}{a+1}W.$$

Then we can construct points $Y_i, Z_i, U_i (i = 1, 2, \cdots, 6)$ in the space $\mathbb{R}^n$ as follows:

$$\begin{array}{l} Z_1 = G(X_1, X_6), Z_2 = G(X_2, X_3), Z_3 = G(X_3, X_2), \\ Z_4 = G(X_4, X_5), Z_5 = G(X_5, X_4), Z_6 = G(X_6, X_1), \end{array} \tag{6}$$

$$\begin{array}{l} U_2 = H(X_2, Y_6), U_4 = H(X_4, Y_2), U_6 = H(X_6, Y_4), \\ U_1 = H(X_1, Y_3), U_3 = H(X_3, Y_5), U_5 = H(X_5, Y_1). \end{array} \tag{7}$$

Figure 8 shows the generated points. We can verify the following facts:

1. the following three polyhedra

$$\begin{array}{l} X_1X_2Z_2U_1Z_1U_2Y_2Y_1,\ Y_3Z_3X_3U_3Y_4U_4X_4Z_4, \\ Z_6U_5Y_5Y_6X_6U_6Z_5X_5 \end{array}$$

   are regular cubes of edge length equals 1, all isometric to the $3D$ cube $[0, 1] \times [0, 1] \times [0, 1]$;
2. the following three polyhedra

$$\begin{array}{ll} U_1Z_2Y_2Y_1\text{-}Y_3Z_3U_4U_4, & U_3Z_4Y_4Y_3\text{-}Y_5Z_5U_6Y_6, \\ U_5Z_6Y_6Y_5\text{–}Y_1Z_1U_2Y_2 & \end{array}$$

   are right cylinders, all isometric to $[0, 1] \times [0, 1] \times (0, a)$;
3. the set of the following 24 points

$$\begin{array}{lll} X_1, X_2, Y_2, U_1, & Y_1, U_2, Z_2, Z_1; & Z_3, Y_3, X_3, U_3, \\ Z_4, U_4, X_4, Y_4; & Y_6, U_5, Z_5, Z_6, & X_6, U_6, Y_5, X_5 \end{array}$$

   is isometric to $\{\alpha, \beta, \cdots, \omega\} \subset \mathit{\Delta}(a)$, up to appropriate permutation;
4. and finally, the cylinders

$$\begin{array}{l} X_1X_2U_2Z_1\text{-}U_3X_3X_4Z_4,\ X_3X_4U_4Z_3\text{-}U_5X_5X_6Z_6, \\ X_5X_6U_6Z_5\text{-}U_1X_1X_2Z_2 \end{array}$$

   are mutually perpendicular, all isometric to $[0, 1] \times [0, 1] \times [0, a + 2]$. There union $\hat{F}(\mathit{\Delta}(a))$ forms a Penrose triangle (actually, tribar) in $\mathbb{R}^n$.

□

In the rest of this section we prove that the 12 extremal points of the Penrose triangle $\mathit{\Delta}(a)$ can be isometrically embedded in $\mathbb{R}^n$ for $n = 5$, and $n = 5$ is the least dimension for embedding the Penrose triangle into Euclidean space. For this, we need consider the metric on the extremals of $\mathit{\Delta}(a)$. Let

$$X_{12} := \{\alpha, \beta, \theta, \eta; \lambda, o, \iota, \nu; \omega, \phi, \tau, \upsilon\}$$

be the set of the 12 extremals of $\mathit{\Delta}(a)$. As $\mathit{\Delta}(a)$ can be isometrically immersed (projected) in the three dimensional Euclidean space $\mathbb{R}^n$, in a piecewise way, as depicted in Figure 6 and Figure 7, we can define the distance $d(x, y)$ between any two extremal points $x, y \in \mathit{\Delta}(a)$ by

$$d(x, y) := \max_{I \in \mathit{\Pi}} D(I(x), I(y)), \tag{8}$$

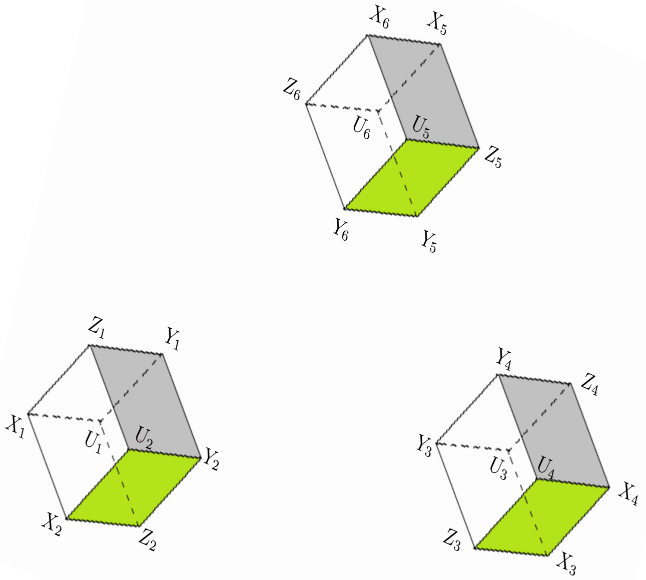


Fig. 8. $X_1, X_2, \cdots, X_6; Y_1, Y_2, \cdots, Y_6$ are the images of F, an isometric map from the 12 extremals of the Penrose triangle $\Delta(a)$ to $\mathbb{R}^n$, and three cubes are generated from $X_i, Y_i (1 \leq i \leq 6)$ and functions G, H.

here Π is the set of all piecewise isometric immersion of $\Delta(a)$ in $\mathbb{R}^3$, and $D(X,Y)$ is the usual distance between two points X, Y in the three dimensional Euclidean space. It is clear that (X_{12}, d) is a metric space, i.e.,

$$\begin{aligned} &d(x,y) \geq 0, \text{ and } d(x,y) = 0 \text{ if and only if } x = y, \\ &d(x,y) = d(y,x), \\ &d(x,y) + d(y,z) \geq d(x,z), \end{aligned} \tag{9}$$

hold for all $x, y, z \in X_{12}$. Since Π contains all projection from all possible $3D$ components of $\Delta(a)$ to $\mathbb{R}^3$, we have the following inequality:

$$d(x,y) \geq \max_{P \in \Pi(x,y)} D(P(x), P(y)) \tag{10}$$

here $\Pi(x,y)$ is the set of all mapping that projects a $3D$ polyhedral component of $\Delta(a)$ which contains x, y to $\mathbb{R}^3$.

Taking an example for $x = \alpha, y = \beta$, it is easy to see that any projection $P : \Delta(a) \to \mathbb{R}^3$ we have $D(P(\alpha), P(\beta)) \leq \alpha\beta = 1$, and there is also a projection that project the component $A_1 \subset \Delta(a)$ to

$$P(\alpha) = (0,0,0), P(\beta) = (1,0,0),$$

thus $d(\alpha, \beta) \geq D(P(\alpha), P(\beta)) = 1$. For points $x = \alpha, y = o$, we see that $\alpha, o \in C_1 = A_1 \cup B_3 \cup A_2$, and a point initially at position $\alpha \in \Delta(a)$ can move to position $o \in \Delta(a)$ as follows: first along $\boldsymbol{u} = \alpha\beta$ to the position $\beta \in \Delta(a)$, then turn 90° on the place $\alpha\beta\gamma$, continue to move along the line $\boldsymbol{v} = \beta\gamma\kappa\lambda$ to the position $\lambda \in \Delta(a)$, then turn 90° on the plane $\kappa\lambda o$, and move to position $o \in \Delta(a)$ finally. Apply the following Pythogoras Theorem we can computer the distance between α and o is $\sqrt{a^2 + 4a + 6}$.

Theorem 2. *(Pythagoras of Samos, c.570-495 BCE) Assume that a point X started to move along the direction $\boldsymbol{u}$ for a straight distance a, then move along the direction $\boldsymbol{v}$ for a straight distance b, and so on, and move along the direction $\boldsymbol{w}$ for a distance c, and finally arrived the point Y. Assume that $\boldsymbol{u}, \boldsymbol{v}, \cdots, \boldsymbol{w}$ are pair-wisely perpendicular to each other, Then the straight length between points X and Y is $\sqrt{a^2 + b^2 + \cdots + c^2}$.*

The 12×12 distance matrix for (X_{12}, d) can be expressed as a block matrix as follows:

$$M_d = \begin{pmatrix} M_1 & M_2 & M_2^T \\ M_2^T & M_1 & M_2 \\ M_2 & M_2^T & M_1 \end{pmatrix}, \tag{11}$$

where

$$M_1 = \begin{array}{c} \\ \alpha \\ \beta \\ \theta \\ \eta \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \theta & \eta \end{array} \\ \begin{pmatrix} 0 & 1 & \sqrt{2} & \sqrt{3} \\ 1 & 0 & \sqrt{3} & \sqrt{2} \\ \sqrt{2} & \sqrt{3} & 0 & 1 \\ \sqrt{3} & \sqrt{2} & 1 & 0 \end{pmatrix} \end{array}, \tag{12}$$

is the 4×4 distance matrix on point set $\{\alpha, \beta, \theta, \eta\}$,

$$M_2 = \begin{array}{c} \\ \alpha \\ \beta \\ \theta \\ \eta \end{array} \begin{array}{c} \begin{array}{cccc} \lambda & o & \iota & \nu \end{array} \\ \begin{pmatrix} a_{2,1} & a_{2,2} & a_{1,0} & a_{1,1} \\ a_{2,0} & a_{2,1} & a_{1,1} & a_{2,2} \\ a_{1,2} & a_{1,1} & a_{0,1} & a_{0,0} \\ a_{1,1} & a_{1,0} & a_{0,2} & a_{0,1} \end{pmatrix} \end{array}, \tag{13}$$

and

$$a_{i,j} := \sqrt{(a+i)^2 + j} \; (i, j = 0, 1, 2)$$

for shorter.

Isometric embeddability in the Euclidean space has been well understood since the classical works of Menger, von Neumann, Schoenberg, and others (see, e.g., [10–13]). Given a set of finite points $X = \{p_0, p_1, \cdots, p_N\}$, and a metric $d : X \times X \to \mathbb{R}_{\geq 0}$, the problem of isometric embedding (X, d) in the Euclidean space $\mathbb{R}^n$ can be characterized by the Cayley-Menger determinant of X (and its subset). For (X, d), let $d_{i,j} = d(p_i, p_j)$ for $i, j = 0, 1, \cdots, N$. The *Cayley-Menger* *determinant* is defined by

$$D(X) := \det \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & d_{0,1}^2 & d_{0,2}^2 & \cdots & d_{0,N}^2 \\ 1 & d_{1,0}^2 & 0 & d_{1,2}^2 & \cdots & d_{1,N}^2 \\ 1 & d_{2,0}^2 & d_{2,1}^2 & 0 & \cdots & d_{2,N}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d_{N,0}^2 & d_{N,1}^2 & d_{N,2}^2 & \cdots & 0 \end{pmatrix},$$

Following result can be found in [11].

Theorem 3. *(Blumenthal, [11]) A finite metric space (X, d) is Euclidean with dimension n if and only if there are $p_0, p_1, \cdots, p_n \in X$ such that*

$$\begin{aligned} &\text{(i) } (-1)^{j+1} D(p_0, \ldots, p_j) > 0 \textit{ for } 1 \leq j \leq n, \textit{ and} \\ &\text{(ii) } D(p_0, \cdots, p_n, x) = D(p_0, \cdots, p_n, y) \\ &\quad = D(p_0, \cdots, p_n, x, y) = 0 \textit{for all } x, y \in X. \end{aligned} \tag{14}$$

Applying the above theorem, we can search maximal subset $p_0, p_1, \cdots, p_n$ of X_{12} that satisfies the conditions (i) and (ii) of the Theorem 3. As $\#X_{12} = 12$ and X_{12} has only $2^{12} = 4096$ subsets, it is very easy to use any symbolic computation software to find all subsets of X_{12} satisfying (i) and (ii). The searching result is that there are altogether 64 different subsets satisfying the required conditions, and each of them contains 6 points. From the symbolic computation result we can prove the isometric embeddability of X_{12} in the five dimensional Euclidean space $\mathbb{R}^5$.

Theorem 4. *The point set X_{12} with distance matrix M_d given by (11) can be isometrically embedded into $\mathbb{R}^n$ for $n \geq 5$, and $n = 5$ is the least dimension for the isometric embedding.*

Proof. For saving place here we prove this theorem by a constructive method. Consider points $\alpha, \beta, \lambda, o, \omega, \phi \in X_{12}$. Then for $D(\alpha, \beta)$, $-D(\alpha, \beta, \lambda)$ and $D(\alpha, \beta, \lambda, o)$ we have:

$$+\det \begin{array}{c} \begin{array}{ccc} & \alpha & \beta \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{array}{c} \\ \alpha \\ \beta \end{array} \end{array} = 2 > 0, \tag{15}$$

$$\begin{aligned} -\det &\begin{array}{c} \begin{array}{cccc} & \alpha & \beta & \lambda \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & (a+2)^2+1 \\ 1 & 1 & 0 & (a+2)^2 \\ 1 & (a+2)^2+1 & (a+2)^2 & 0 \end{pmatrix} \begin{array}{c} \\ \alpha \\ \beta \\ \lambda \end{array} \end{array} \\ &= 4(a+2)^2 > 0, \end{aligned} \tag{16}$$

$$\begin{aligned}&+\det\begin{pmatrix}0 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & a^2+1 & a^2+2\\ 1 & 1 & 0 & a^2 & a^2+1\\ 1 & a^2+1 & a^2 & 0 & 1\\ 1 & a^2+2 & a^2+1 & 1 & 0\end{pmatrix}\\ &= 8(a+2)^2 > 0,\end{aligned} \tag{17}$$

for $-D(\alpha,\beta,\lambda,o,\omega)$ and $D(\alpha,\beta,\lambda,o,\omega,\phi)$, using notation

$$a_{2,0} = a+2, a_{2,1} = \sqrt{(a+2)^2+1}, a_{2,2} = \sqrt{(a+2)^2+2}.$$

for better print quality, we have

$$\begin{aligned}&-\det\begin{pmatrix}0 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & a_{2,1}^2 & a_{2,2}^2 & a_{2,1}^2\\ 1 & 1 & 0 & a_{2,0}^2 & a_{2,1}^2 & a_{2,2}^2\\ 1 & a_{2,1}^2 & a_{2,0}^2 & 0 & 1 & a_{2,1}^2\\ 1 & a_{2,2}^2 & a_{2,1}^2 & 1 & 0 & a_{2,0}^2\\ 1 & a_{2,1}^2 & a_{2,2}^2 & a_{2,1}^2 & a_{2,0}^2 & 0\end{pmatrix}\\ &= 4\ (a+3)\,(a+1)\left(3\,a^2+12\,a+13\right) > 0,\end{aligned} \tag{18}$$

$$\begin{aligned}&+\det\begin{pmatrix}0 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & a_{2,1}^2 & a_{2,2}^2 & a_{2,1}^2 & a_{2,0}^2\\ 1 & 1 & 0 & a_{2,0}^2 & a_{2,1}^2 & a_{2,2}^2 & a_{2,1}^2\\ 1 & a_{2,1}^2 & a_{2,0}^2 & 0 & 1 & a_{2,1}^2 & a_{2,2}^2\\ 1 & a_{2,2}^2 & a_{2,1}^2 & 1 & 0 & a_{2,0}^2 & a_{2,1}^2\\ 1 & a_{2,1}^2 & a_{2,2}^2 & a_{2,1}^2 & a_{2,0}^2 & 0 & 1\\ 1 & a_{2,0}^2 & a_{2,1}^2 & a_{2,2}^2 & a_{2,1}^2 & 1 & 0\end{pmatrix}\\ &= 24\ (a+3)^2\,(a+1)^2 > 0.\end{aligned} \tag{19}$$

With symbolic computation software it is easy to verify that

$$D(\alpha,\beta,\lambda,o,\omega,\phi,x) = 0, \quad D(\alpha,\beta,\lambda,o,\omega,\phi,x,y) = 0, \tag{20}$$

hold for all $x, y \in \{\eta,\theta,\iota,\nu,\tau,\upsilon\}$. Indeed, according to symmetry, we need only to check this for $x \in \{\eta,\theta\}, y \in \{\iota,\nu\}$. Combine (15) to (20) and Theorem 3, we proved that X_{12} can be isometrically embedded into $\mathbb{R}^5$. This implies also that the Penrose triangle can not be isometrically embedded into $\mathbb{R}^4$ as people generally believed that. □

Combining Theorem 1 and Theorem 4, we proved that the Penrose triangle can be embedded into $\mathbb{R}^n$, but not $\mathbb{R}^4$.

Actually, we have used `Maple` software running on a notebook computer with Intel(R) Core(TM) i7 CPU and 8GB proved that there are 64 different selections of $p_0, p_1, \cdots, p_5$ satisfying the conditions (i) and (ii) of Theorem 3, and the 64 subsets can be represented as members of the Cartesian product

$$\{\alpha, \theta\} \times \{\beta, \eta\} \times \{\lambda, \iota\} \times \{o, \nu\} \times \{\omega, \tau\} \times \{\phi, \upsilon\}.$$

Note that both

$$\text{extremal}(\mathfrak{P}) = \{\alpha, \beta, \lambda, o, \omega, \phi\}$$

and

$$\text{extremal}(\mathfrak{Q}) = \{\theta, \eta, \iota, \nu, \tau, \upsilon\}$$

are members of the product set, which implies that $\mathfrak{P}$ and $\mathfrak{Q}$ can be viewed as two simpleces in $\mathbb{R}^5$, therefore, and, the Penrose triangle $\mathit{\Delta}(a)$ as a three dimensional topological manifold, is indeed a polyhedral belt contained in the shell $\mathfrak{P} \setminus \mathfrak{Q}$.

3 Solving embedding equations using symbolic computation

Theorem 4 confirms the existence of an isometric embedding of (the 12 extremal points on) the Penrose triangle in the space $\mathbb{R}^n$. One may use the general method given by Blumenthal [11] or Lu Yang and Jingzhong Zhang [14] to create an explicit representation of the embedding, i.e., the concrete coordinates $(x_i, y_i, z_i, u_i, v_i) \in \mathbb{R}^n$ $(i = 1, 2, \cdots, 12)$ as the images of

$$X_{12} = \{\alpha, \beta, \eta, \theta, \iota, \lambda, \nu, o, \tau, \upsilon, \varphi, \omega\}$$

in $\mathbb{R}^5$ under the isometric embedding. Due to the symmetry of the Penrose triangle, we can also find a solution of the embedding equations using symbolic computation. We will show this in this section. Suppose that the Penrose triangle has been realized in the five dimensional Euclidean space, then the projection of the configuration in the $\mathbb{R}^n$ along any direction (an oriented straight line) into the three dimensional space would be the three cubes (that is a rigid movement of $[0,1] \times [0,1] \times [0,1] \subset \mathbb{R}^4$) connected by three bars (that is isometric to the cylinder $[0,1] \times [0,1] \times [0,a]$. As depicted in the Figure 7, we may assume the cubes are

$$\mathit{\Delta} := \alpha\beta\gamma\delta\epsilon\varepsilon\zeta\eta, \quad \mathit{\Pi} := \iota\kappa\lambda\mu\nu\xi\, o\pi, \quad \mathit{\Sigma} := \rho\upsilon\tau\sigma\phi\chi\psi\omega.$$

With an unitary orthogonal transforms, we may change that the vertex δ on the first cube $\mathit{\Delta}$ is lie on the origin $(0,0,0,0,0)$, and the coordinates of other vertices are

$$\begin{array}{lll} \alpha = (0,1,0,0,0), & \beta = (1,1,0,0,0), & \gamma = (0,1,0,0,0), \\ \varepsilon = (0,1,1,0,0), & \zeta = (1,1,1,0,0), & \\ \eta = (0,1,1,0,0), & \theta = (0,0,1,0,0). & \end{array} \tag{21}$$

As shown in Figure 9, the original design of Reutersvärd, the cube $\mathit{\Pi}$ can move along a straight line ℓ_s (in the appropriate space, here, we assume it is a line in the $\mathbb{R}^5$) to $\mathit{\Delta}$, so that $\mathit{\Pi}$ coincides $\mathit{\Delta}$ with

$$\begin{aligned} &\rho \to \alpha, \quad \upsilon \to \beta, \quad \tau \to \gamma, \quad \sigma \to \delta, \\ &\phi \to \varepsilon, \quad \chi \to \zeta, \quad \psi \to \eta, \quad \omega \to \theta, \end{aligned} \tag{22}$$

after movement, and the other cube $\mathit{\Pi}$ can be moved to $\mathit{\Delta}$ along a line ℓ_p so that

$$\begin{aligned} &\iota \to \alpha, \quad \kappa \to \beta, \quad \lambda \to \gamma, \quad \mu \to \delta, \\ &\nu \to \varepsilon, \quad \xi \to \zeta, \quad o \to \eta, \quad \pi \to \theta. \end{aligned} \tag{23}$$

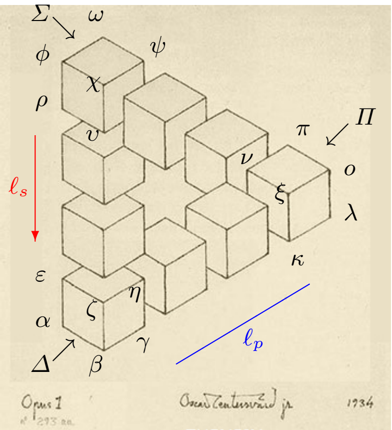


Fig. 9. ℓ_s, ℓ_p are two lines in $\mathbb{R}^5$ so that the cube $\mathit{\Sigma}$ can be moved to coincide $\mathit{\Delta}$ in parallel to ℓ_s, and the cube can be move to $\mathit{\Delta}$ in parallel to ℓ_p. Some vertices of the cubes $\mathit{\Delta}, \mathit{\Pi}, \mathit{\Sigma}$ are not marked in the picture. When viewing $\mathit{\Delta}, \mathit{\Pi}, \mathit{\Sigma}$ as necker cubes, take ζ, ξ, χ to the front-most positions.

Without loss of generality, we may assume the line ℓ_s and the line ℓ_p line in the plane

$$\{(x, y, z, u, v) | x = 0, y = 0, z = 0\} \subset \mathbb{R}^5,$$

and the coordinates of vertices $\delta \in \mathit{\Delta}, \mu \in \mathit{\Sigma}, \sigma \in \mathit{\Pi}$ are

$$\delta = (0, 0, 0, u_1, v_1), \mu = (0, 0, 0, u_2, v_2), \sigma = (0, 0, 0, u_3, v_3), \tag{24}$$

respectively. We can take $u_1 = 0, v_1 = 0$ as in (21), here we use this form just for symmetry. Therefore, the coordinates of all other 21 vertices of $\mathit{\Delta}, \mathit{\Pi}, \mathit{\Sigma}$ can be determined by (21) and (24). In particular, we have

$$\begin{aligned} &\alpha = (1, 0, 0, u_1, v_1), \beta = (1, 1, 0, u_1, v_1), \lambda = (0, 1, 0, u_2, v_2), \\ &o = (0, 1, 1, u_2, v_2), \omega = (0, 0, 1, u_3, v_3), \phi = (1, 0, 1, u_3, v_3). \end{aligned} \tag{25}$$

Using the coordinates to compute the distance $\alpha\beta, \alpha\lambda, \alpha o, \alpha\omega, \alpha\phi$, we establish the following equation system:

$$\begin{array}{l} 1+(u_1-u_2)^2+(v_1-v_2)^2-(a+2)^2=0, \\ 1+(u_1-u_3)^2+(v_1-v_3)^2-(a+2)^2=0, \\ 1+(u_2-u_3)^2+(v_2-v_3)^2-(a+2)^2=0, \\ u_1=0, v_1=0. \end{array} \tag{26}$$

Solving this equation system we obtained

$$u_2=u_3=\frac{\sqrt{3}}{2}\sqrt{a^2+4a+3}, v_2=-v_3=\frac{1}{2}\sqrt{a^2+4a+3},$$

and hence, the coordinates of the 24 vertices of the cube Δ, Π, Σ. For saving space, here we omit the detail data.

4 Epilogue

As we have proved, the Penrose triangle has an isometric embedding in lowest dimension Euclidean space $\mathbb{R}^5$, as a subset of $\mathfrak{P} \setminus \mathfrak{Q}$, the difference set of two simplexes, where $\mathfrak{P}$ is formed by $\alpha, \beta, \lambda, o, \omega, \phi$, and $\mathfrak{Q}$ formed by $\theta, \eta, \iota, \nu, \tau, \upsilon$. Figure 10 gives an intuitive explanation to this fact.

It is clear that $\dim(\mathfrak{P} \setminus \mathfrak{Q}) = 5$, otherwise $\Delta(a)$ could be isometrically embedded into lower dimensional space. Therefore, viewing it from the $\mathbb{R}^5$, the Penrose triangle is a bounded set of co-dimension 2, locally flat, with 0 genus, like a non-planar closed space curve in $\mathbb{R}^3$.

To conclude the paper, we indicate that the way we have used to imbed the Penrose triangle into the space $\mathbb{R}^n$ can be applied to construct a *flat* isometric embedding of the Möbius band into $\mathbb{R}^4$. Namely, let

$$\begin{array}{l} A=(0,0,a,a), B=(1,0,a,a), C=(0,a,0,a), \\ D=(1,a,0,a), E=(0,a,a,0), F=(1,a,a,0), \end{array} \tag{27}$$

then, the rectangles $ABCD, CDEF, EFAB$ in the 4-dim space form a Möbius band in $\mathbb{R}^4$ so that every interior point of the rectangles has a flat neighborhood, as shown in Figure 11. See [15] for more works on isometric embeddings and immersions of Möbius bands. We wonder if the similar method can be applied to construct an isometric embedding of the impossible cuble (Figure 2(left) into $\mathbb{R}^n$ so that the embedded cube produces a weird view from three-dimensional space.

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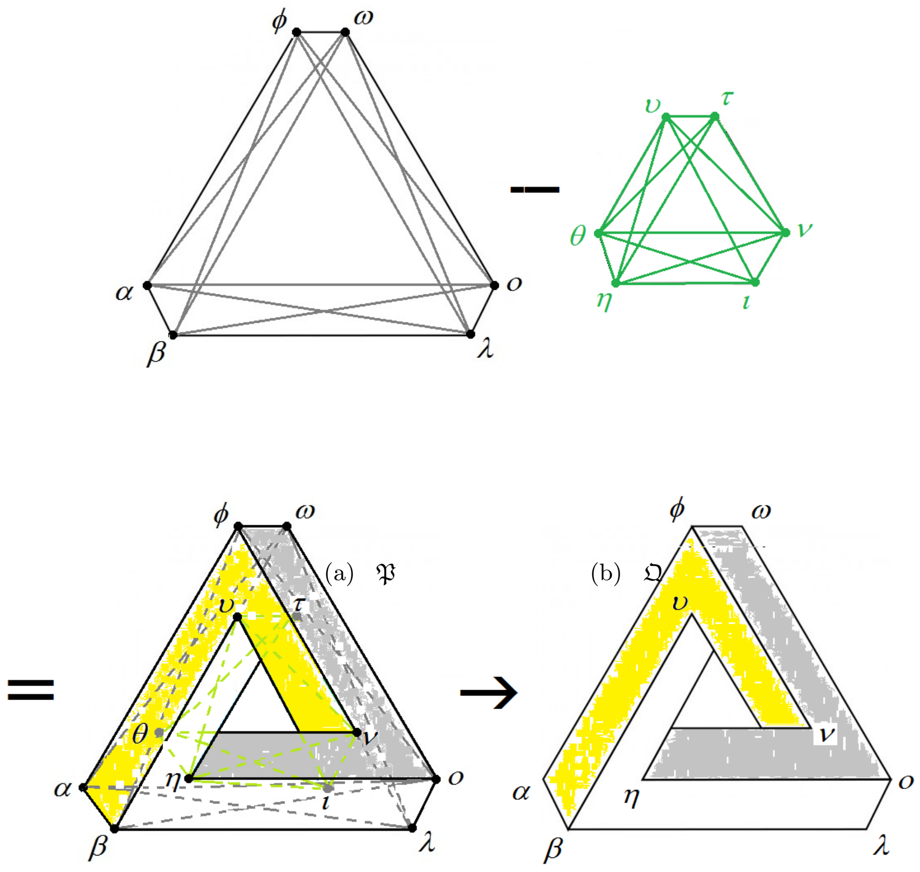

(a) $\mathfrak{P}$

(b) $\mathfrak{Q}$

(c) $\mathfrak{P} \setminus \mathfrak{Q}$

(d) a Penrose triangle

Fig. 10. (a): $\mathfrak{P}$: the convex hull of the Penrose triangle, also a simplex formed by $\alpha, \beta, \lambda, o, \omega, \phi$ in the space $\mathbb{R}^5$; (b): $\mathfrak{Q}$, the simplex formed by points $\theta, \eta, \iota, \nu, \tau, \upsilon$ in $\mathbb{R}^5$; (c): $\Delta(a) \subset \mathfrak{P} \setminus \mathfrak{Q}$; (d): the piecewise isometric immersion of the Penrose triangle in the three dimensional space as we see.

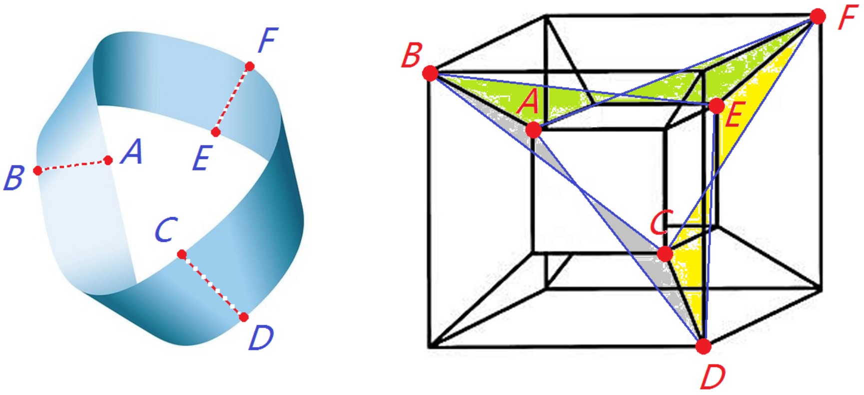

Fig. 11. (*left*): the Möbius band in the space $\mathbb{R}^3$; (*right*): an isometric embedding of the Möbius band in $\mathbb{R}^4$.