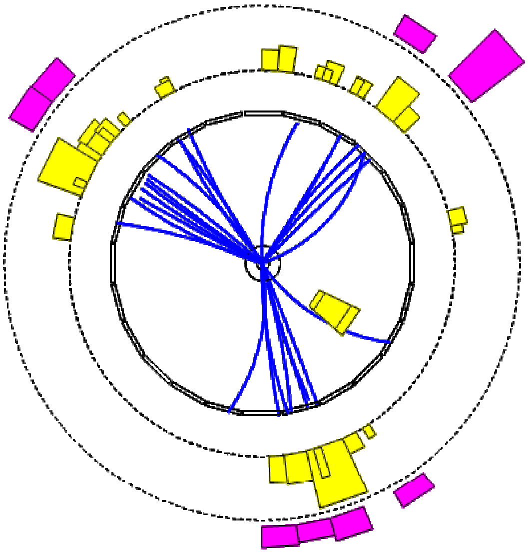
Particle Physics

Michaelmas Term 2009
Prof Mark Thomson



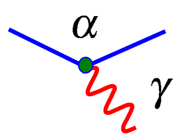
Handout 8 : Quantum Chromodynamics

Colour in QCD

★ The theory of the strong interaction, Quantum Chromodynamics (QCD), is very similar to QED but with 3 conserved "colour" charges

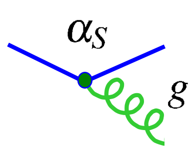
In QED:

- the electron carries one unit of charge $-e$
- the anti-electron carries one unit of anti-charge $+e$
- the force is mediated by a massless "gauge boson" – the photon



In QCD:

- quarks carry colour charge: r, g, b
- anti-quarks carry anti-charge: $\overline{r}, \overline{g}, \overline{b}$
- The force is mediated by massless gluons



★ In QCD, the strong interaction is invariant under rotations in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

i.e. the same for all three colours

⇒ **SU(3) colour symmetry**

- This is an **exact** symmetry, unlike the approximate uds flavour symmetry discussed previously.

★ Represent r, g, b SU(3) colour states by:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

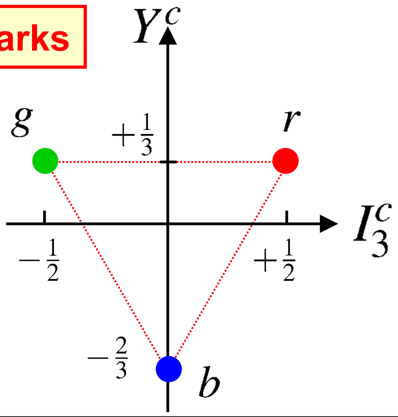
★ Colour states can be labelled by two quantum numbers:

- I_3^c colour isospin
- Y^c colour hypercharge

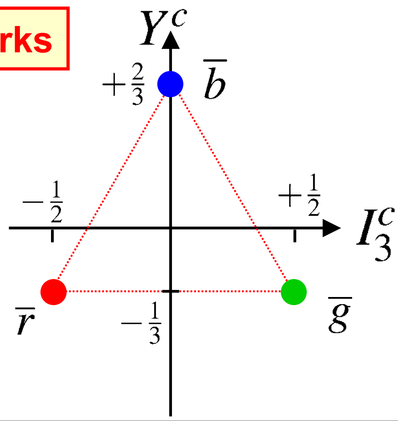
Exactly analogous to labelling u,d,s flavour states by I_3 and Y

★ Each quark (anti-quark) can have the following colour quantum numbers:

quarks



anti-quarks



Colour Confinement

★ It is believed (although not yet proven) that all observed free particles are "colourless"

- i.e. never observe a free quark (which would carry colour charge)
- consequently quarks are always found in bound states colourless hadrons

★ <u>**Colour Confinement Hypothesis:**</u>

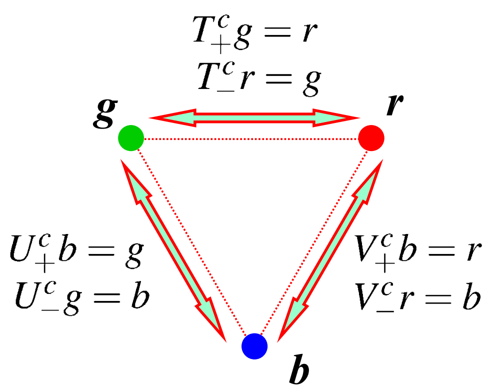
only <u>colour</u> <u>singlet</u> states can exist as free particles

★ All hadrons must be "colourless" i.e. colour singlets

★ To construct colour wave-functions for hadrons can apply results for SU(3) flavour symmetry to SU(3) colour with replacement

$$\boxed{\begin{matrix} u \longrightarrow r \\ d \longrightarrow g \\ s \longrightarrow b \end{matrix}}$$

★ just as for uds flavour symmetry can define colour ladder operators



Colour Singlets

★ It is important to understand what is meant by a singlet state

★ Consider spin states obtained from two spin 1/2 particles.

- Four spin combinations: $\uparrow\uparrow,\ \uparrow\downarrow,\ \downarrow\uparrow,\ \downarrow\downarrow$
- Gives four eigenstates of $\hat{S}^2,\ \hat{S}_z$ $\quad (2\otimes 2 = 3\oplus 1)$

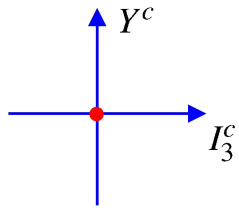
$$\left.\begin{aligned} |1,+1\rangle &= \uparrow\uparrow \\ |1,0\rangle &= \tfrac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow) \\ |1,-1\rangle &= \downarrow\downarrow \end{aligned}\right\} \boxed{\text{spin-1 triplet}} \quad \oplus \quad |0,0\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow) \quad \boxed{\text{spin-0 singlet}}$$

★ The singlet state is "spinless": it has zero angular momentum, is invariant under SU(2) spin transformations and spin ladder operators yield zero

$$S_{\pm}|0,0\rangle = 0$$

★ In the same way COLOUR SINGLETS are "colourless" combinations:

- they have zero colour quantum numbers $I_3^c = 0,\ Y^c = 0$
- invariant under SU(3) colour transformations
- ladder operators $T_{\pm},\ U_{\pm},\ V_{\pm}$ all yield zero

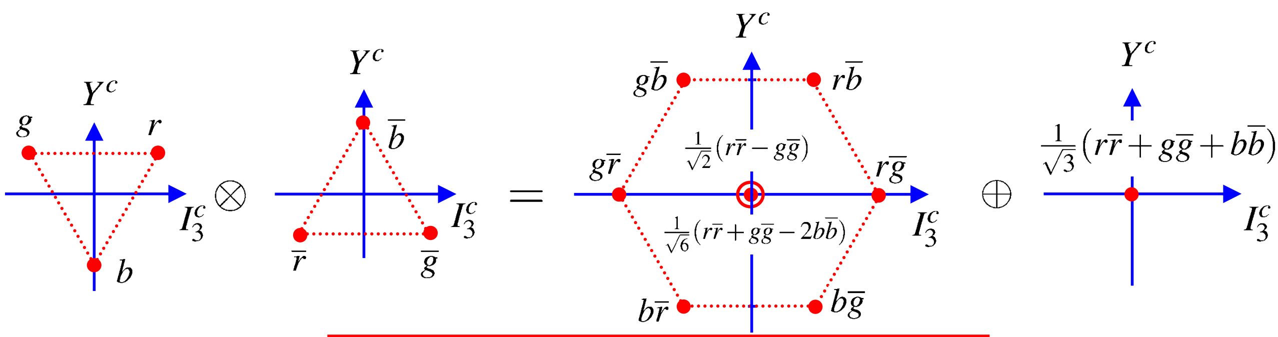


★ NOT sufficient to have $I_3^c = 0,\ Y^c = 0$: does not mean that state is a singlet

Meson Colour Wave-function

★ Consider colour wave-functions for $q\bar{q}$

★ The combination of colour with anti-colour is mathematically identical to construction of meson wave-functions with uds flavour symmetry



➡ Coloured octet and a colourless singlet

- Colour confinement implies that hadrons only exist in colour singlet states so the colour wave-function for mesons is:

$$\boxed{\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})}$$

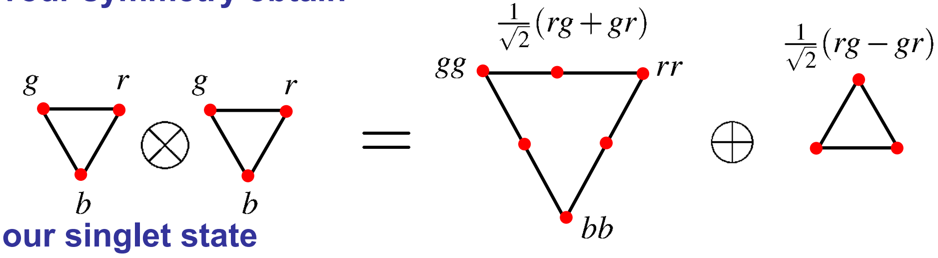
★ Can we have a $qq\bar{q}$ state ? i.e. by adding a quark to the above octet can we form a state with $Y^c = 0;\ I_3^c = 0$. The answer is clear no.

➡ $qq\bar{q}$ bound states do not exist in nature.

Baryon Colour Wave-function

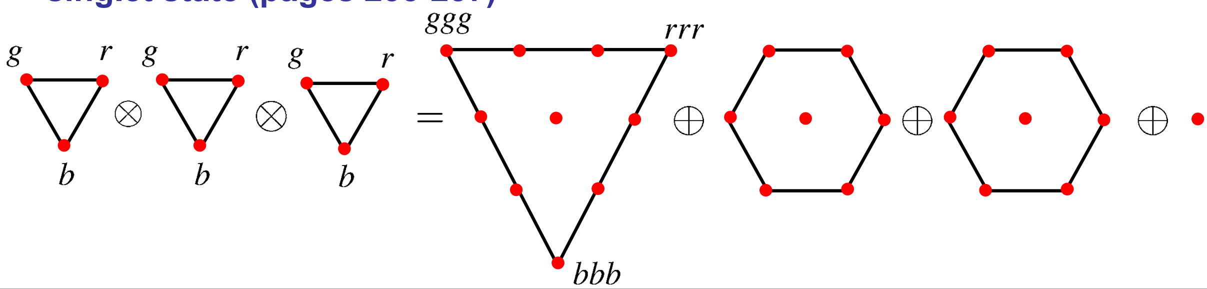
★ Do **qq** bound states exist ? This is equivalent to asking whether it possible to form a colour singlet from two colour triplets ?

- Following the discussion of construction of baryon wave-functions in SU(3) flavour symmetry obtain



- No **qq** colour singlet state
- Colour confinement ➡bound states of **qq** do not exist

★ BUT combination of three quarks (three colour triplets) gives a colour singlet state (pages 235-237)



★ The singlet colour wave-function is:

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}$$

Check this is a colour singlet…

- It has $I_3^c = 0,\ Y^c = 0$: a necessary but not sufficient condition
- Apply ladder operators, e.g. T_+ (recall $T_+ g = r$)

$$T_+\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rrb - rbr + rbr - rrb + brr - brr) = 0$$

•Similarly $T_-\psi_c^{qqq} = 0;\quad V_\pm\psi_c^{qqq} = 0;\quad U_\pm\psi_c^{qqq} = 0;$

★ Colourless singlet - therefore **qqq** bound states exist !

⇨ **Anti-symmetric colour wave-function**

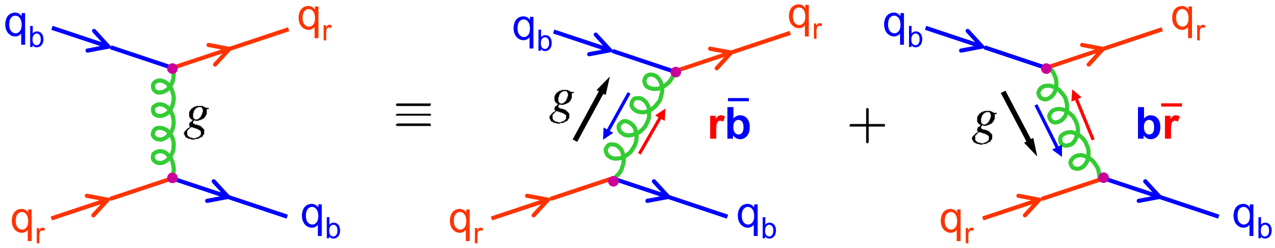
Allowed Hadrons i.e. the possible colour singlet states

- $q\bar{q},\ qqq$ **Mesons and Baryons**
- $q\bar{q}q\bar{q},\ qqqq\bar{q}$ **Exotic states, e.g. pentaquarks**

To date all confirmed hadrons are either **mesons** or **baryons**. However, some recent (but not entirely convincing) "evidence" for pentaquark states

Gluons

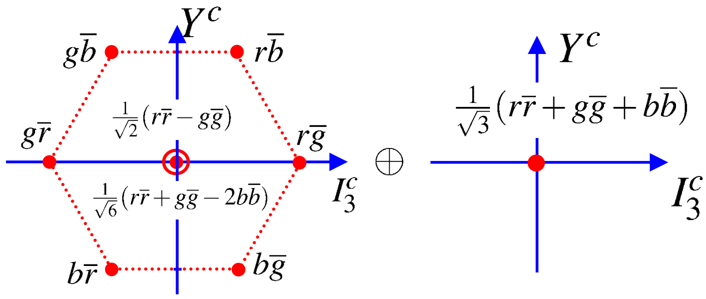
★ In QCD quarks interact by exchanging virtual massless gluons, e.g.



★ Gluons carry colour and anti-colour, e.g.

★ Gluon colour wave-functions (colour + anti-colour) are the same as those obtained for mesons (also colour + anti-colour)

⇨ OCTET + "COLOURLESS" SINGLET



★ So we might expect 9 physical gluons:

OCTET: $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

SINGLET: $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

★ **BUT**, colour confinement hypothesis:

only colour singlet states can exist as free particles ⇨ Colour singlet gluon would be unconfined. It would behave like a strongly interacting photon ➡ infinite range Strong force.

★ **Empirically**, the strong force is short range and therefore know that the physical gluons are confined. The colour singlet state does not exist in nature !

NOTE: this is not entirely ad hoc. In the context of gauge field theory (see minor option) the strong interaction arises from a fundamental SU(3) symmetry. The gluons arise from the generators of the symmetry group (the Gell-Mann λ matrices). There are 8 such matrices ➡ 8 gluons. Had nature "chosen" a U(3) symmetry, would have 9 gluons, the additional gluon would be the colour singlet state and QCD would be an unconfined long-range force.

NOTE: the "gauge symmetry" determines the exact nature of the interaction ➡ FEYNMAN RULES

Gluon-Gluon Interactions

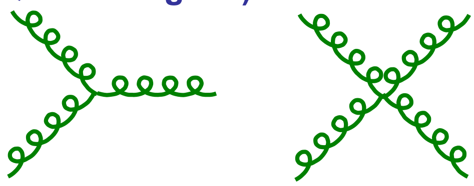
★ In QED the photon does not carry the charge of the EM interaction (photons are electrically neutral)

★ In contrast, in QCD the gluons do carry colour charge

⇒ **Gluon Self-Interactions**

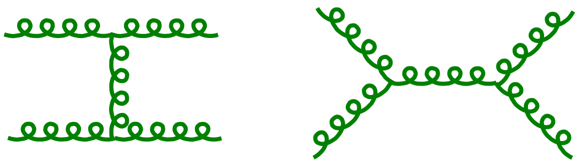
★ Two new vertices (no QED analogues)

triple-gluon vertex

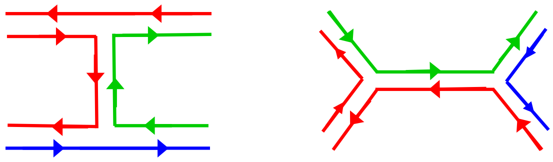


quartic-gluon vertex

★ In addition to quark-quark scattering, therefore can have gluon-gluon scattering



e.g. possible way of arranging the colour flow

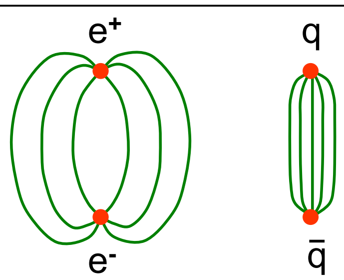


Gluon self-Interactions and Confinement

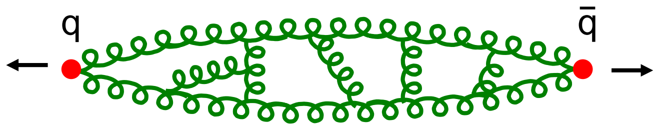
★ Gluon self-interactions are believed to give rise to colour confinement

★ Qualitative picture:

- Compare QED with QCD
- In QCD "gluon self-interactions squeeze lines of force into a flux tube"



★ What happens when try to separate two coloured objects e.g. $q\bar{q}$



- Form a flux tube of interacting gluons of approximately constant energy density $\sim 1\,\mathrm{GeV/fm}$

$$\Rightarrow \quad V(r) \sim \lambda r$$

- Require infinite energy to separate coloured objects to infinity
- Coloured quarks and gluons are always confined within colourless states
- In this way QCD provides a plausible explanation of confinement – but not yet proven (although there has been recent progress with Lattice QCD)

Hadronisation and Jets

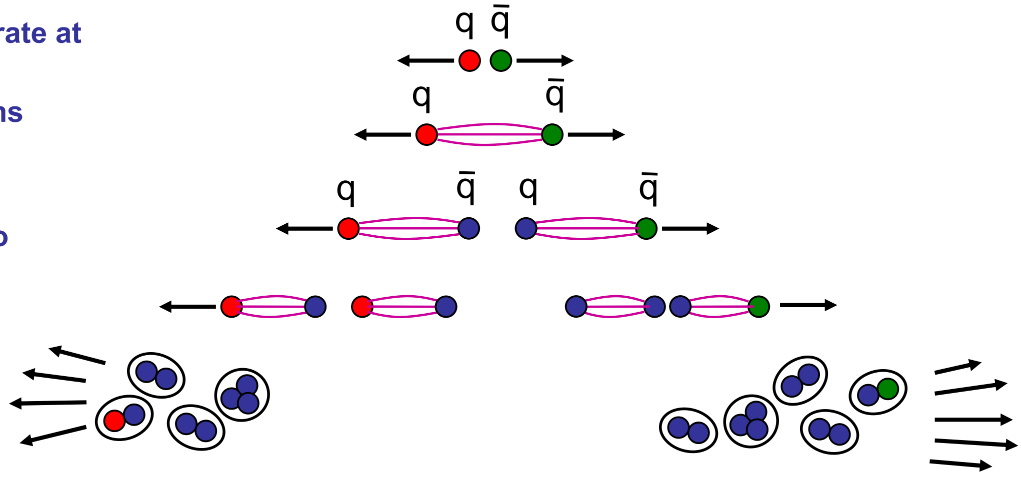
★ Consider a quark and anti-quark produced in electron positron annihilation

i) Initially Quarks separate at high velocity

ii) Colour flux tube forms between quarks

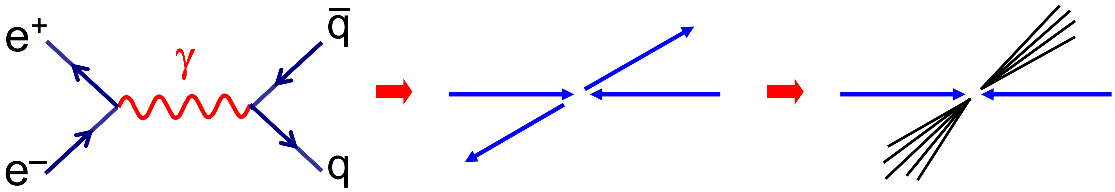
iii) Energy stored in the flux tube sufficient to produce $q\bar{q}$ pairs

iv) Process continues until quarks pair up into jets of colourless hadrons



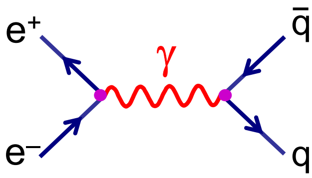
★ This process is called **hadronisation**. It is not (yet) calculable.

★ The main consequence is that at collider experiments **quarks and gluons** observed as jets of particles



QCD and Colour in e$^+$e$^-$ Collisions

★ e$^+$e$^-$ colliders are an excellent place to study QCD

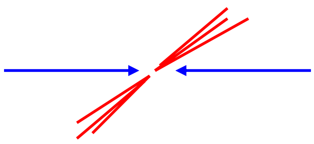


★ Well defined production of quarks
- QED process well-understood
- no need to know parton structure functions
- + experimentally very clean – no proton remnants

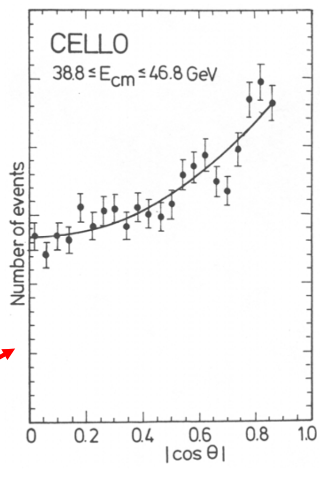
★ In handout 5 obtained expressions for the $e^+e^- \to \mu^+\mu^-$ cross-section

$$\sigma = \frac{4\pi\alpha^2}{3s} \qquad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

- In e$^+$e$^-$ collisions produce all quark flavours for which $\sqrt{s} > 2m_q$
- In general, i.e. unless producing a $q\bar{q}$ bound state, produce jets of hadrons
- Usually can't tell which jet came from the quark and came from anti-quark



★ Angular distribution of jets $\propto (1+\cos^2\theta)$



H.J.Behrend et al., Phys Lett 183B (1987) 400

➡ **Quarks are spin ½**

★ Colour is conserved and quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$

★ For a single quark flavour and single colour

$$\sigma(e^+e^- \to q_i\bar{q}_i) = \frac{4\pi\alpha^2}{3s}Q_q^2$$

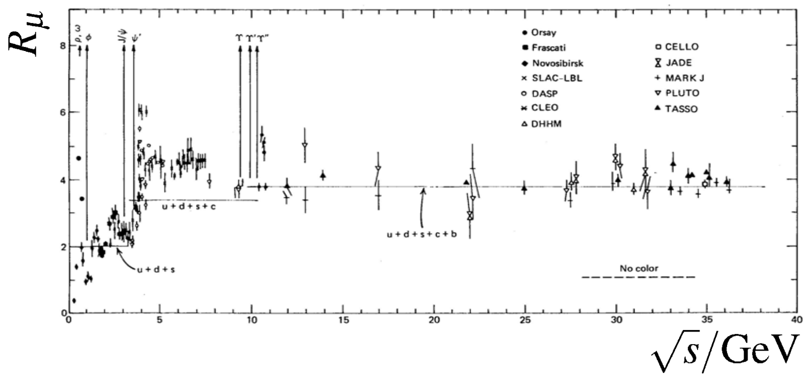
- Experimentally observe jets of hadrons:

$$\sigma(e^+e^- \to \text{hadrons}) = 3\sum_{u,d,s,..}\frac{4\pi\alpha^2}{3s}Q_q^2$$

Factor 3 comes from colours

- Usual to express as ratio compared to $\sigma(e^+e^- \to \mu^+\mu^-)$

$$R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_{u,d,s,..}Q_q^2$$



u,d,s: $R_\mu = 3\times(\frac{1}{9}+\frac{4}{9}+\frac{1}{9}) = 2$

u,d,s,c: $R_\mu = \frac{10}{3}$

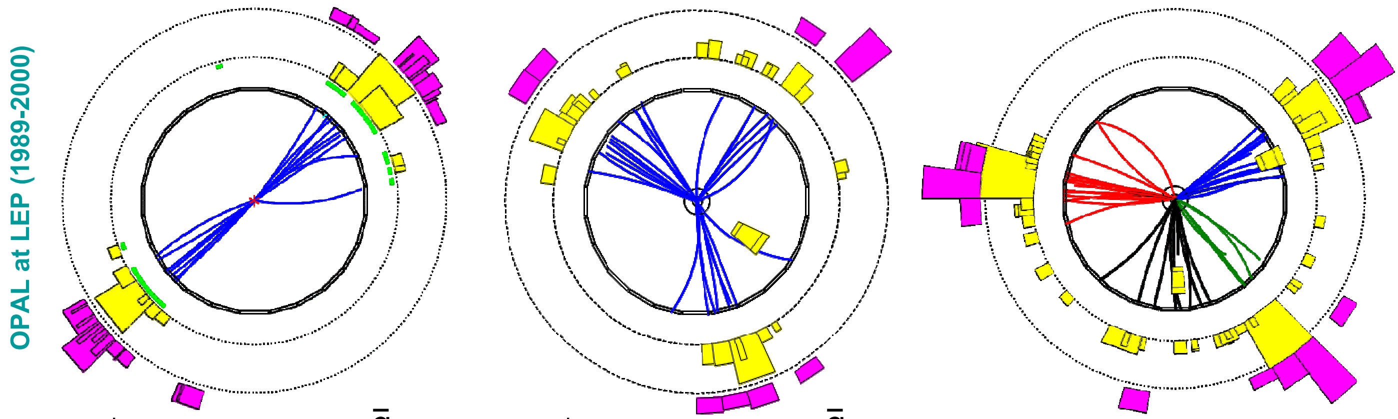
u,d,s,c,b: $R_\mu = \frac{11}{3}$

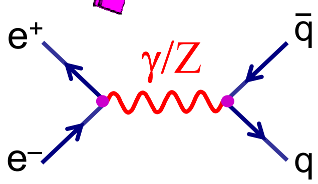
★ Data consistent with expectation with factor 3 from colour

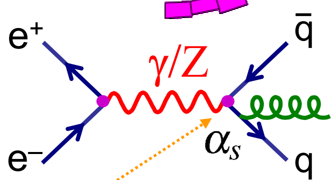
Jet production in e+e- Collisions

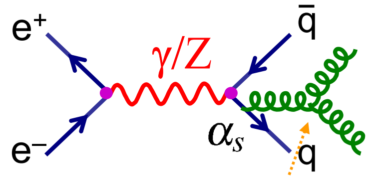
★ e^+e^- colliders are also a good place to study gluons

$e^+e^- \to q\bar{q} \to 2\text{jets}$ $\quad e^+e^- \to q\bar{q}g \to 3\text{jets}$ $\quad e^+e^- \to q\bar{q}gg \to 4\text{jets}$









Experimentally:

- Three jet rate ➡ measurement of α_s
- Angular distributions ➡ gluons are spin-1
- Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry

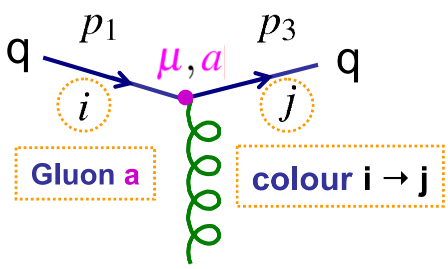
The Quark – Gluon Interaction

- Representing the colour part of the fermion wave-functions by:

$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- Particle wave-functions $u(p) \longrightarrow c_i u(p)$
- The QCD qqg vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



- Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices (justified in handout 13).
- Isolating the colour part:

$$c_j^\dagger\lambda^a c_i = c_j^\dagger \begin{pmatrix}\lambda^a_{1i}\\ \lambda^a_{2i}\\ \lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

- Hence the fundamental quark - gluon QCD interaction can be written

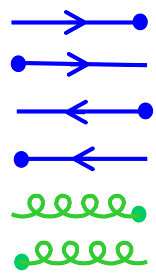
$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

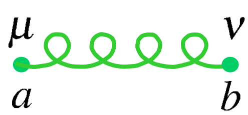
spin 1/2
- incoming quark $u(p)$
- outgoing quark $\overline{u}(p)$
- incoming anti-quark $\overline{v}(p)$
- outgoing anti-quark $v(p)$

spin 1
- incoming gluon $\epsilon^\mu(p)$
- outgoing gluon $\epsilon^\mu(p)^*$



- **Internal Lines (propagators)**

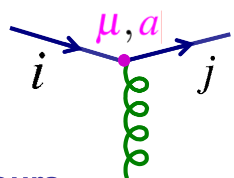
spin 1 gluon $\boxed{\dfrac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$



a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark $\boxed{-ig_s\tfrac{1}{2}\lambda^a_{ji}\gamma^\mu}$



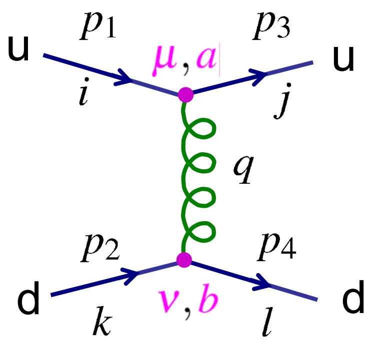
i, j = 1,2,3 are quark colours,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = **product of all factors**

Matrix Element for quark-quark scattering

★ **Consider QCD scattering of an up and a down quark**



- **The incoming and out-going quark colours are labelled by** $i, j, k, l = \{1,2,3\}$ (or $\{r,g,b\}$)
- **In terms of colour this scattering is** $$ik \rightarrow jl$$
- **The 8 different gluons are accounted for by the colour indices** $a, b = 1, 2, \ldots, 8$
- **NOTE: the δ-function in the propagator ensures** ***a*** **=** ***b*****, i.e. the gluon "emitted" at** ***a*** **is the same as that "absorbed" at** ***b***

★ **Applying the Feynman rules:**

$$-iM = \left[\overline{u}_u(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\}u_d(p_2)\right]$$

where summation over ***a*** **and** ***b*** **(and μ and ν) is implied.**

★ **Summing over** ***a*** **and** ***b*** **using the δ-function gives:**

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$

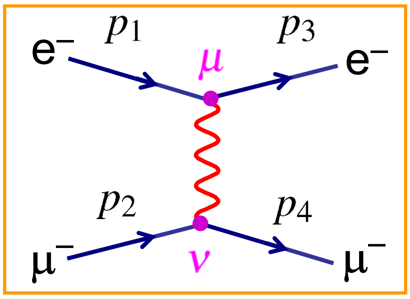
Sum over all 8 gluons (repeated indices)

QCD vs QED

QED

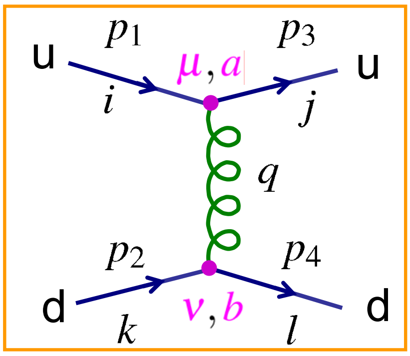
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$\boxed{M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]}$$



QCD

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$



★ **QCD Matrix Element = QED Matrix Element with:**

- $\boxed{e^2 \rightarrow g_s^2}$ **or equivalently** $\boxed{\alpha = \dfrac{e^2}{4\pi} \rightarrow \alpha_s = \dfrac{g_s^2}{4\pi}}$

+ QCD Matrix Element includes an additional "colour factor"

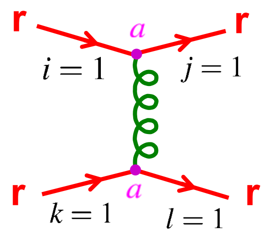
$$\boxed{C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}}$$

Evaluation of QCD Colour Factors

•QCD colour factors reflect the gluon states that are involved

$\lambda^1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\lambda^4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$	$\lambda^6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\lambda^3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$
$\lambda^2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$	$\lambda^5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}$	$\lambda^7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$	$\lambda^8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$
Gluons: $r\bar{g},\, g\bar{r}$	$r\bar{b},\, b\bar{r}$	$g\bar{b},\, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

❶ Configurations involving a single colour

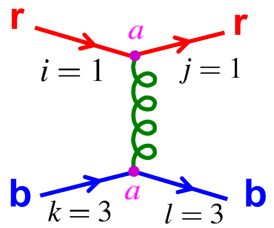


•Only matrices with non-zero entries in **11** position are involved

$$C(rr\to rr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11}+\lambda^8_{11}\lambda^8_{11})$$
$$= \frac{1}{4}\left(1+\frac{1}{3}\right) = \frac{1}{3}$$

Similarly find $\boxed{C(rr\to rr)=C(gg\to gg)=C(bb\to bb)=\frac{1}{3}}$

❷ Other configurations where quarks don't change colour e.g. $rb\to rb$

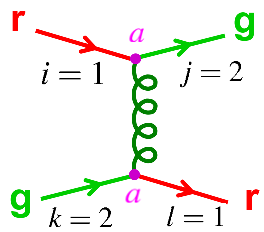


•Only matrices with non-zero entries in **11** and **33** position are involved

$$C(rb\to rb) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{11}\lambda^a_{33} = \frac{1}{4}(\lambda^8_{11}\lambda^8_{33})$$
$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}}.\frac{-2}{\sqrt{3}}\right) = -\frac{1}{6}$$

Similarly $C(rb\to rb)=C(rg\to rg)=C(gr\to gr)=C(gb\to gb)=C(br\to br)=C(bg\to bg)=-\frac{1}{6}$

❸ Configurations where quarks swap colours e.g. $rg\to gr$



•Only matrices with non-zero entries in **12** and **21** position are involved

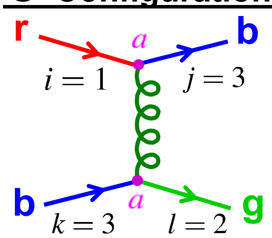
Gluons $r\bar{g},\, g\bar{r}$

$$C(rg\to gr) = \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{21}\lambda^a_{12} = \frac{1}{4}(\lambda^1_{21}\lambda^1_{12}+\lambda^2_{21}\lambda^2_{12})$$
$$= \frac{1}{4}(i(-i)+1) = \frac{1}{2}$$

$\hat{T}^{(ij)}_{+}\hat{T}^{(kl)}_{-}$

$$C(rb\to br)=C(rg\to gr)=C(gr\to rg)=C(gb\to bg)=C(br\to rb)=C(bg\to gb)=\frac{1}{2}$$

❹ Configurations involving 3 colours e.g. $rb\to bg$



•Only matrices with non-zero entries in the **13** and **32** position

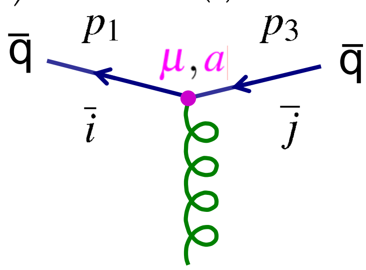
•But none of the λ matrices have non-zero entries in the **13** and **32** positions. Hence the colour factor is zero

★ colour is conserved

Colour Factors : Quarks vs Anti-Quarks

- Recall the colour part of wave-function: $r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$
- The QCD qqg vertex was written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



★Now consider the anti-quark vertex

- The QCD q̄q̄g vertex is:

$$\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$

$\boxed{\text{Note that the incoming anti-particle now enters on the LHS of the expression}}$

- For which the colour part is

$$c_i^\dagger\lambda^a c_j = c_i^\dagger\begin{pmatrix}\lambda^a_{1j}\\ \lambda^a_{2j}\\ \lambda^a_{3j}\end{pmatrix} = \lambda^a_{ij}$$

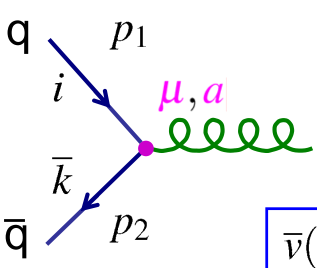
i.e indices *ij* are swapped with respect to the quark case

- Hence $\boxed{\overline{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \overline{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)}$

- c.f. the quark - gluon QCD interaction

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

★Finally we can consider the quark – anti-quark annihilation

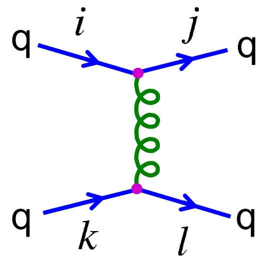


QCD vertex: $\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$

with $c_k^\dagger\lambda^a c_i = \lambda^a_{ki}$

$$\boxed{\overline{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda^a_{ki}\gamma^\mu\}u(p_1)}$$

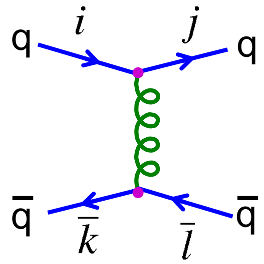
- **Consequently the colour factors for the different diagrams are:**



$$\boxed{C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}}$$

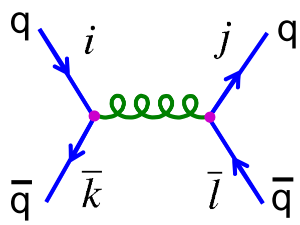
e.g.

$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



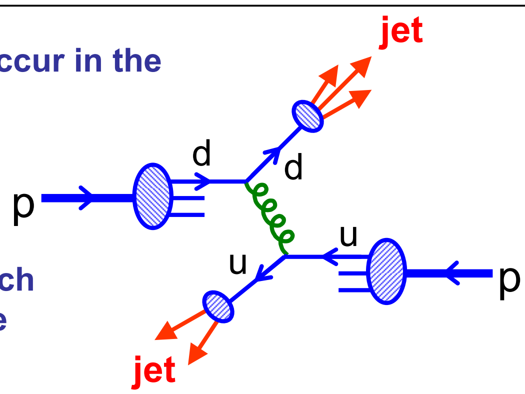
$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

Quark-Quark Scattering

- **Consider the process $u + d \to u + d$ which can occur in the high energy proton-proton scattering**
- **There are nine possible colour configurations of the colliding quarks which are all equally likely.**
- **Need to determine the average matrix element which is the sum over all possible colours divided by the number of possible initial colour states**



$$\langle |M_{fi}|^2 \rangle = \frac{1}{3}\cdot\frac{1}{3}\sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

- **The colour average matrix element contains the average colour factor**

$$\langle |C|^2 \rangle = \frac{1}{9}\sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

- **For** $qq \to qq$ $\boxed{rr\to rr,..}$ $\boxed{rb\to rb,..}$ $\boxed{rb\to br,..}$

$$\langle |C|^2 \rangle = \frac{1}{9}\left[3\times\left(\frac{1}{3}\right)^2 + 6\times\left(-\frac{1}{6}\right)^2 + 6\times\left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

•Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit (handout 6).

QED
$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1+\left(1+\frac{q^2}{s}\right)^2\right]$$

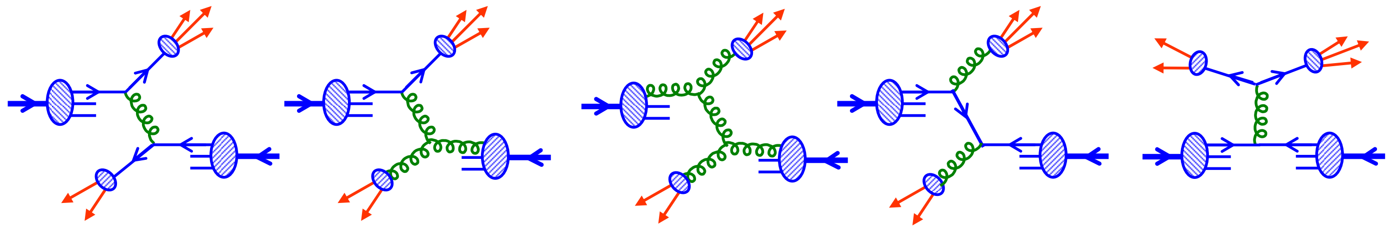
•For $ud \to ud$ in QCD replace $\alpha \to \alpha_s$ and multiply by $\langle|C|^2\rangle$

QCD
$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1+\left(1+\frac{q^2}{\hat{s}}\right)^2\right]$$

•Here $\hat{s}$ is the centre-of-mass energy of the quark-quark collision

•The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions

e.g. two jet production in proton-antiproton collisions

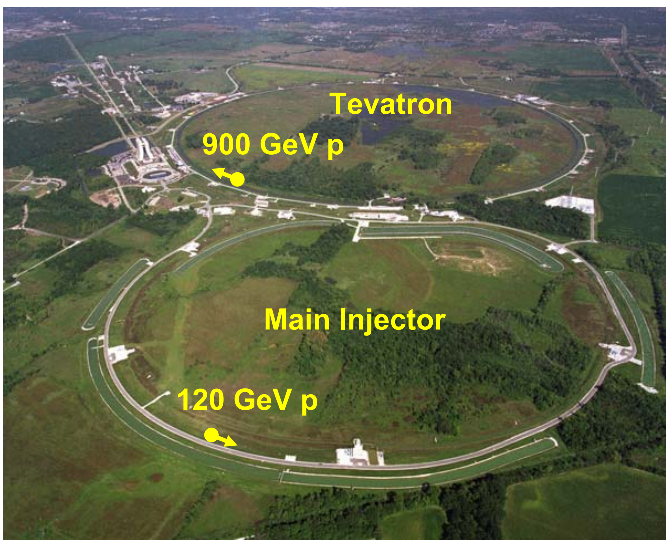


$q\bar{q} \to q\bar{q}$ $\quad qg \to qg$ $\quad gg \to gg$ $\quad q\bar{q} \to gg$ $\quad q\bar{q} \to q\bar{q}$

e.g. $p\bar{p}$ collisions at the Tevatron

★ Tevatron collider at **Fermi National Laboratory** (FNAL)
- located ~40 miles from Chigaco, US
- started operation in **1987** (will run until 2009/2010)

★ $p\bar{p}$ collisions at $\sqrt{s}$ = 1.8 TeV   c.f. 14 TeV at the LHC



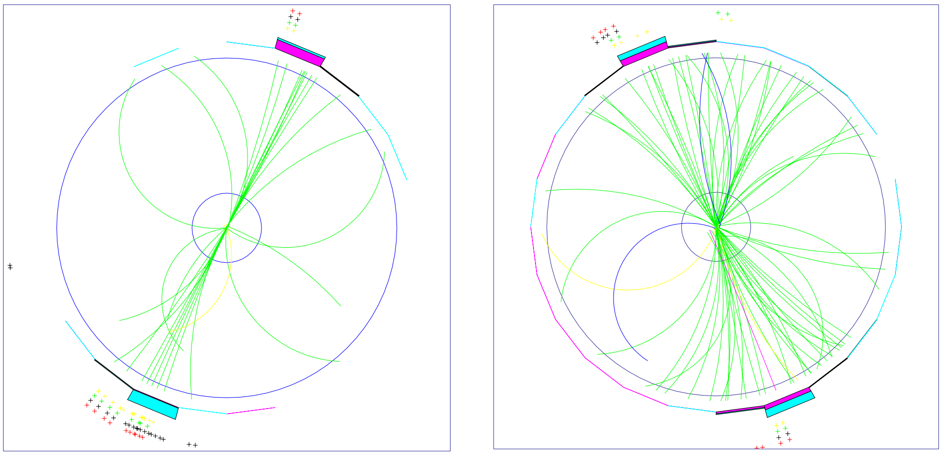
Two main accelerators:

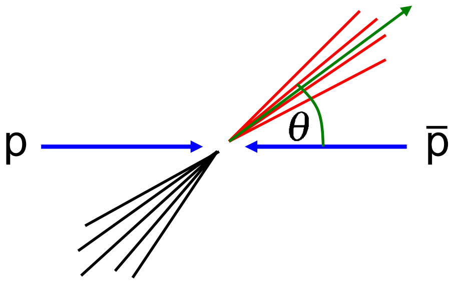
★ Main Injector
- Accelerates 8 GeV p to 120 GeV
- also $\bar{p}$ to 120 GeV
- Protons sent to Tevatron & MINOS
- $\bar{p}$ all go to Tevatron

★ Tevatron
- 4 mile circumference
- accelerates $p/\bar{p}$ from 120 GeV to 900 GeV

★ **Test QCD predictions by looking at production of pairs of high energy jets**

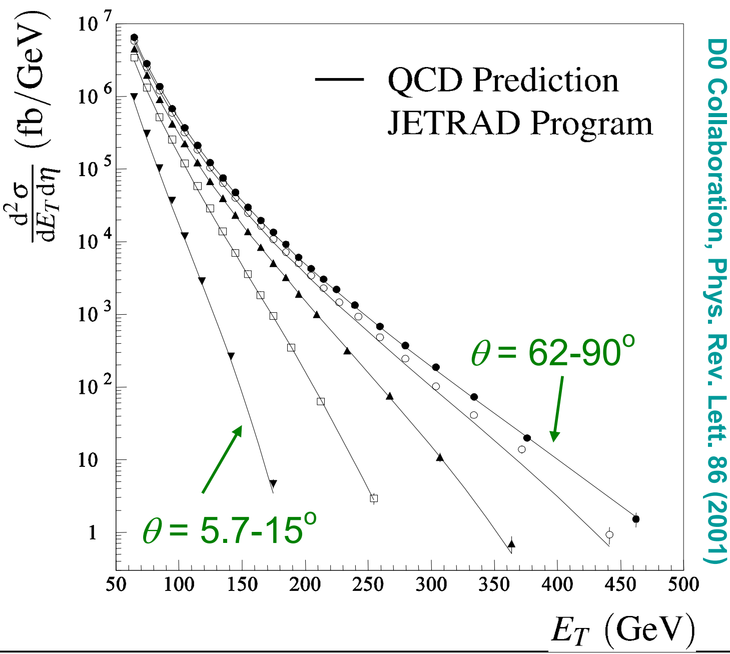
$p\bar{p} \to$ jet jet + X





★ **Measure cross-section in terms of**
- **"transverse energy"** $E_T = E_{\text{jet}} \sin\theta$
- **"pseudorapidity"** $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

…don't worry too much about the details here, what matters is that…



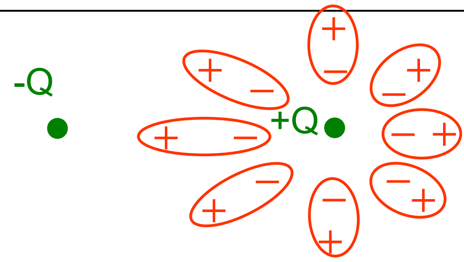
★ **QCD predictions provide an excellent description of the data**

★ **NOTE:**
- **at low E_T cross-section is dominated by low x partons i.e. gluon-gluon scattering**
- **at high E_T cross-section is dominated by high x partons i.e. quark-antiquark scattering**

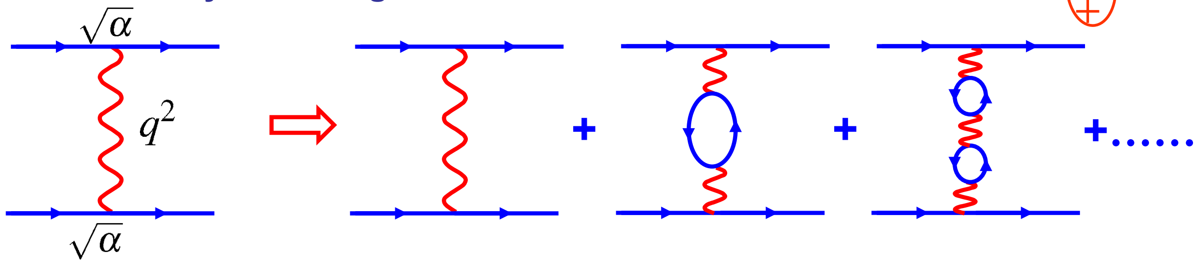
Running Coupling Constants

QED

- "bare" charge of electron screened by virtual e^+e^- pairs
- behaves like a polarizable dielectric



★ In terms of Feynman diagrams:



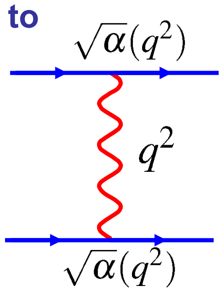
★ Some final state so add matrix element **amplitudes**: $M = M_1 + M_2 + M_3 + \ldots$

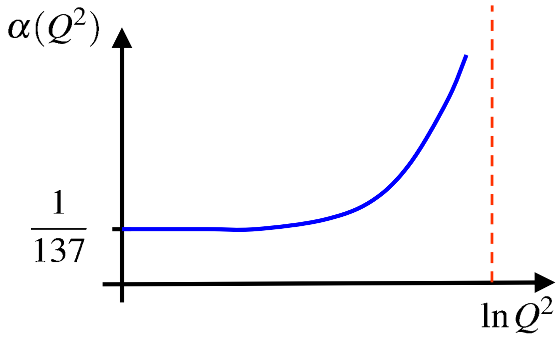
★ Giving an infinite series which can be summed and is equivalent to a single diagram with "running" coupling constant

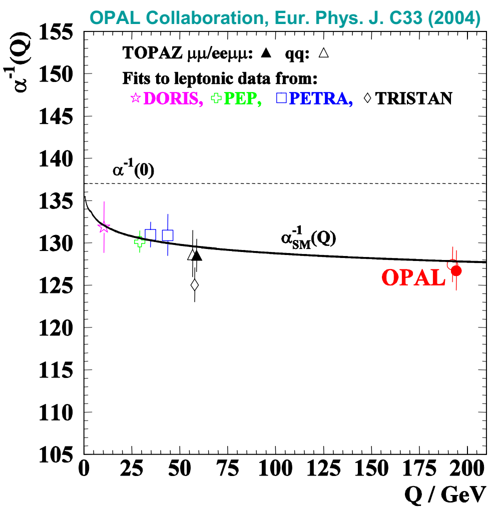
$$\alpha(Q^2) = \alpha(Q_0^2) \Big/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

Note sign

$$Q^2 \gg Q_0^2$$







★ Might worry that coupling becomes infinite at

$$\ln\left(\frac{Q^2}{Q_0^2}\right) = \frac{3\pi}{1/137}$$

i.e. at $Q \sim 10^{26}\,\mathrm{GeV}$

- But quantum gravity effects would come in way below this energy and it is highly unlikely that QED "as is" would be valid in this regime

★ In QED, running coupling **increases** very slowly

- Atomic physics: $Q^2 \sim 0$

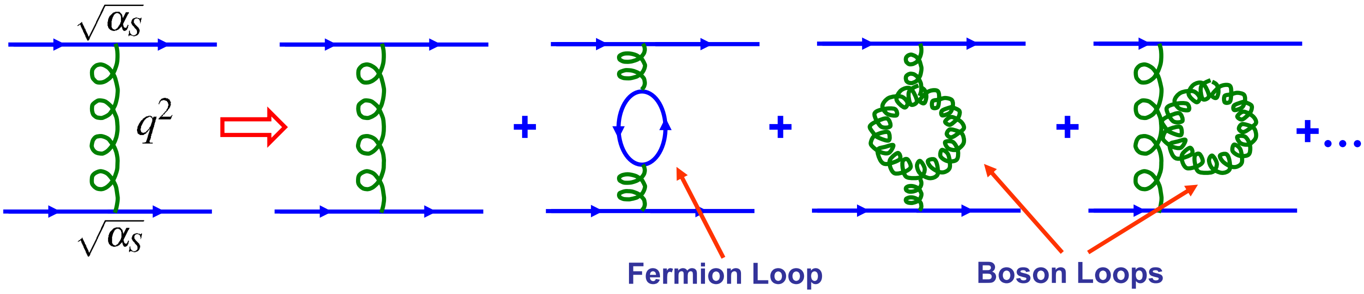
$$1/\alpha = 137.03599976(50)$$

- High energy physics:

$$1/\alpha(193\,\mathrm{GeV}) = 127.4 \pm 2.1$$

Running of α_s

QCD Similar to QED but also have gluon loops



★ Remembering **adding amplitudes**, so can get negative interference and the sum can be smaller than the original diagram alone

★ Bosonic loops "interfere negatively"

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Big/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $B = \dfrac{11N_c - 2N_f}{12\pi}$ $\begin{cases} N_c & = \text{no. of colours} \\ N_f & = \text{no. of quark flavours} \end{cases}$

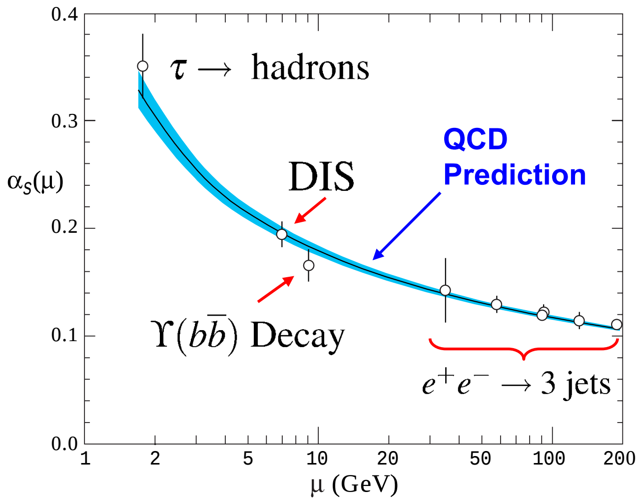
$N_c = 3;\ N_f = 6$ ➡ $B > 0$

➡ $\boxed{\alpha_S \text{ decreases with } Q^2}$

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

★ Measure α_s in many ways:
- jet rates
- DIS
- tau decays
- bottomonium decays
- +…

★ As predicted by QCD, α_s decreases with Q^2



$\mu \approx Q$

★ At low Q^2: α_s is large, e.g. at $Q^2 = 1\,\text{GeV}^2$ find $\alpha_s \sim 1$

•Can't use perturbation theory ! This is the reason why QCD calculations at low energies are so difficult, e.g. properties hadrons, hadronisation of quarks to jets,…

★ At high Q^2 : α_s is rather small, e.g. at $Q^2 = M_Z^2$ find $\alpha_s \sim 0.12$

➡ **Asymptotic Freedom**

•Can use perturbation theory and this is the reason that in DIS at high Q^2 quarks behave as if they are quasi-free (i.e. only weakly bound within hadrons)

Summary

★ Superficially QCD very similar to QED
★ But gluon self-interactions are believed to result in colour confinement
★ All hadrons are colour singlets which explains why only observe

Mesons **Baryons**

★ A low energies $\alpha_S \sim 1$

➡ Can't use perturbation theory !

Non-Perturbative regime

★ Coupling constant runs, smaller coupling at higher energy scales

$$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$$

➡ Can use perturbation theory

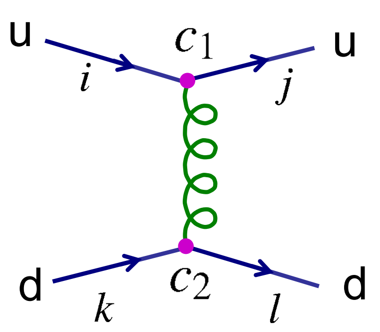
Asymptotic Freedom

★ Where calculations can be performed, QCD provides a good description of relevant experimental data

Appendix I: Alternative evaluation of colour factors

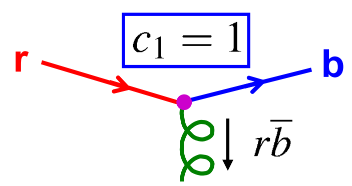
"Non-examinable" but can be used as to derive colour factors.

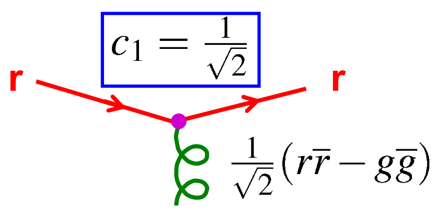
★The colour factors can be obtained (more intuitively) as follows :



•Write $C(ik \to jl) = \frac{1}{2}c_1c_2$

•Where the colour coefficients at the two vertices depend on the quark and gluon colours

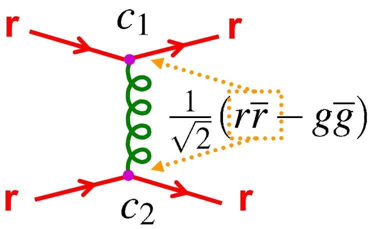


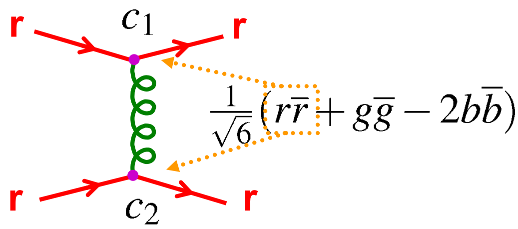


•Sum over all possible exchanged gluons conserving colour at both vertices

① Configurations involving a single colour

e.g. $rr \to rr$: two possible exchanged gluons

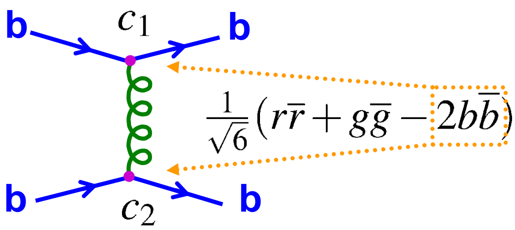




$$c_1 = c_2 = \frac{1}{\sqrt{2}} \qquad\qquad c_1 = c_2 = \frac{1}{\sqrt{6}}$$

$$C(rr \to rr) = \tfrac{1}{2}\left(\tfrac{1}{2} + \tfrac{1}{6}\right) = \tfrac{1}{3}$$

e.g. $bb \to bb$: only one possible exchanged gluon

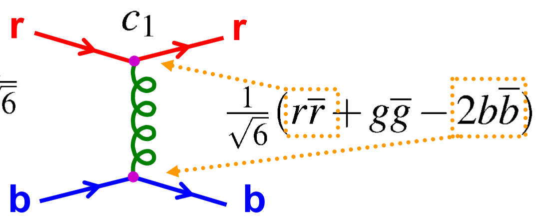


$$c_1 = c_2 = -\frac{2}{\sqrt{6}}$$

$$\Rightarrow \quad C(bb \to bb) = \tfrac{1}{2}\left(\tfrac{2}{\sqrt{6}}\tfrac{2}{\sqrt{6}}\right) = \tfrac{1}{3}$$

② Other configurations where quarks don't change colour

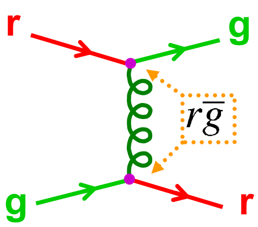
$$c_1 = \frac{1}{\sqrt{6}}$$



$$c_2 = -\frac{2}{\sqrt{6}}$$

$$C(rb \to rb) = \tfrac{1}{2}\left(-\tfrac{1}{\sqrt{6}}\tfrac{2}{\sqrt{6}}\right) = -\tfrac{1}{6}$$

③ Configurations where quarks swap colours



$$c_1 = c_2 = 1$$

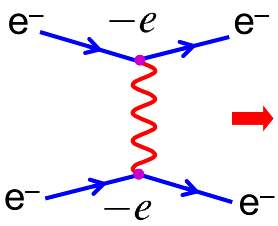
$$C(rg \to gr) = \tfrac{1}{2}$$

Appendix II: Colour Potentials

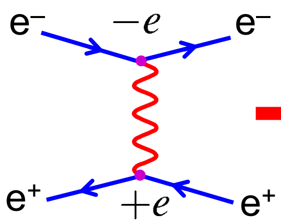
Non-examinable

- Previously argued that gluon self-interactions lead to a $+\lambda r$ long-range potential and that this is likely to explain colour confinement
- Have yet to consider the short range potential – i.e. for quarks in mesons and baryons does QCD lead to an attractive potential?
- Analogy with QED: (NOTE this is very far from a formal proof)

QED



$$\Rightarrow \quad \boxed{V(r) = +\frac{\alpha}{r}}$$

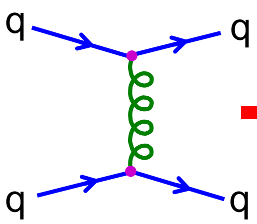


$$\Rightarrow \quad \boxed{V(r) = -\frac{\alpha}{r}}$$

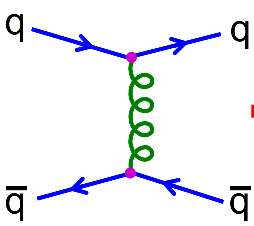
Repulsive Potential | **Static** | Attractive Potential

QCD

★ by analogy with QED expect potentials of form



$$\Rightarrow \quad \boxed{V(r) = +C\frac{\alpha_S}{r}}$$



$$\Rightarrow \quad \boxed{V(r) = -C\frac{\alpha_S}{r}}$$

★ Whether it is a attractive or repulsive potential depends on sign of colour factor

★ Consider the colour factor for a $q\bar{q}$ system in the colour singlet state:

$$\psi = \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b})$$

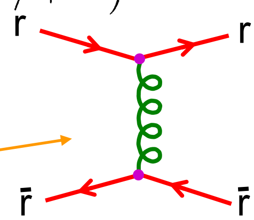
with colour potential $\langle V_{q\bar{q}}\rangle = \langle\psi|V_{\rm QCD}|\psi\rangle$

$$\Rightarrow \quad \langle V_{q\bar{q}}\rangle = \frac{1}{3}\left(\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle + \ldots + \langle r\bar{r}|V_{\rm QCD}|b\bar{b}\rangle + \ldots\right)$$

- Following the QED analogy:

$$\langle r\bar{r}|V_{\rm QCD}|r\bar{r}\rangle = -C(r\bar{r} \to r\bar{r})\frac{\alpha_S}{r}$$

which is the term arising from $r\bar{r} \to r\bar{r}$



- Have 3 terms like $r\bar{r} \to r\bar{r},\ b\bar{b} \to b\bar{b}, \ldots$ and 6 like $r\bar{r} \to g\bar{g},\ r\bar{r} \to b\bar{b}, \ldots$

$$\langle V_{q\bar{q}}\rangle = -\frac{1}{3}\frac{\alpha_S}{r}\left[3 \times C(r\bar{r} \to r\bar{r}) + 6 \times C(r\bar{r} \to g\bar{g})\right] = -\frac{1}{3}\frac{\alpha_S}{r}\left[3 \times \frac{1}{3} + 6 \times \frac{1}{2}\right]$$

$$\Rightarrow \quad \boxed{\langle V_{q\bar{q}}\rangle = -\frac{4}{3}\frac{\alpha_S}{r}}$$

NEGATIVE ➡ ATTRACTIVE

- The same calculation for a $q\bar{q}$ colour octet state, e.g. $r\bar{g}$ gives a positive repulsive potential: $C(r\bar{g} \to r\bar{g}) = -\frac{1}{6}$

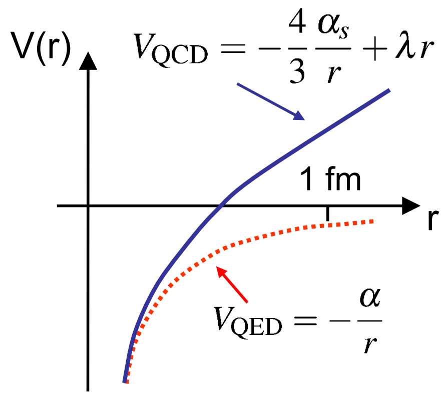
★ Whilst not a formal proof, it is comforting to see that in the colour singlet $q\bar{q}$ state the QCD potential is indeed attractive.

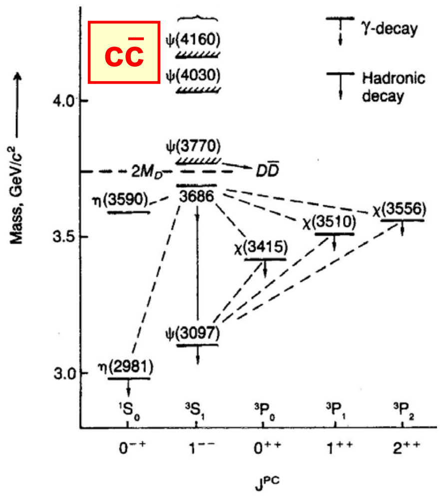
(question 15)

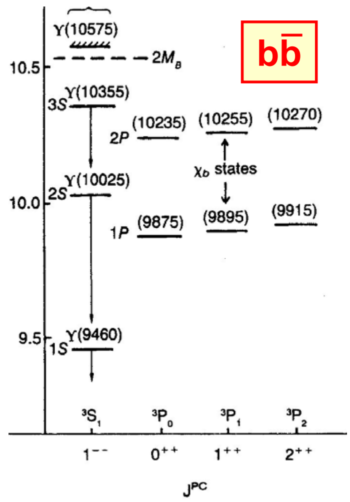
★ Combining the short-range QCD potential with the linear long-range term discussed previously:

$$\boxed{V_{\mathrm{QCD}} = -\frac{4}{3}\frac{\alpha_s}{r} + \lambda r}$$

★ This potential is found to give a good description of the observed charmonium $(c\bar{c})$ and bottomonium $(b\bar{b})$ bound states.







NOTE:

- c, b are heavy quarks
- approx. non-relativistic
- orbit close together
- probe **1/r** part of $\mathbf{V_{QCD}}$

Agreement of data with prediction provides strong evidence that V_{QCD} has the Expected form