EQUATIONS OF THE BRING-JERRARD FORM, THE GOLDEN SECTION, AND SQUARE FIBONACCI NUMBERS

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(Submitted September 1996-Final Revision February 1997)

1. INTRODUCTION

A curious problem is that of finding closed-form expressions for the positive real numbers (say x) that preserve their fractional parts when raised to the k^{th} power ($k \geq 2$, an integer). It is quite obvious that all the positive integers enjoy this property.

Since no positive number less than 1 can enjoy it, the numbers x are characterized by the fact that x^k diminished by x equals a nonnegative integer. In other words, the numbers in question are given by the *positive* roots $x_n(k)$ of the k^{th} ($k \geq 2$) degree equation

$$x^k - x = n. \tag{1.1}$$

where n is an arbitrary *nonnegative* integer. Equations like (1.1) are said to be of the Bring-Jerrard form [1, pp. 179-81]. Observe that the positive integers emerge as solutions of (1.1) when $n = a^k - a$ ($a = 1, 2, 3, \ldots$).

From this point on, the symbol $x_n(k)$ ($n = 0, 1, 2, \ldots$) will denote the n^{th} positive real number that preserves its fractional part when raised to the power k.

The case $k = 2$ has been considered in [4]. In that article k was allowed to assume negative values also, and the author proved that the golden section $\alpha = (1+\sqrt{5})/2 = x_1(-1) = x_1(2)$ is the only nonintegral number that preserves its fractional part both when one squares it and when one takes its reciprocal.

In this article we extend this study by considering the cases $k = 3, 4$, and 5. The solutions for $k = 3$ and 4 are readily found as the closed form expressions for third- and fourth-degree equations are known; we show them only for the sake of completeness. Solving the case $k = 5$ has been a bit more complicated, and is our main result. More precisely, we have established the closed-form expressions for the *only* three nonintegral numbers $x_n(5)$ for which it can be given: these numbers are $x_{15}(5)$, $x_{22440}(5)$, and $x_{2759640}(5)$. This assertion comes from the fact that the quintic of the Bring-Jerrard form $x^5 - x - r$ ($r \in \mathbb{Z}$) can be solved by radicals iff either $r = m^5 - m$, or $r = \pm 15$, ± 22440, or ± 2759640. The proof of this result involves a well-known property [2] of the Fibonacci numbers F_j.

2. THE NUMBERS $x_n(k)$ FOR $k = 2, 3$ AND 4

By using (1.1) and the well-known formulas for the solution of second-, third-, and fourth-degree equations, the following results have been established:

$$x_n(2) = (1+\sqrt{4n+1})/2 \quad (n = 0, 1, 2, \ldots) \quad (\text{see [4]}), \tag{2.1}$$

$$x_n(3) = \sqrt[3]{\frac{n}{2} - \sqrt{\left(\frac{n}{2}\right)^2 - \frac{1}{27}}} + \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 - \frac{1}{27}}} \quad (n = 0, 1, 2, \ldots), \tag{2.2}$$

and

$$x_n(4) = \frac{\sqrt{y_n} + \sqrt{-y_n + 2\sqrt{y_n^2 + 4n}}}{2} \quad (n = 0, 1, 2, \ldots), \tag{2.3}$$

where

$$y_n = \sqrt[3]{\frac{1}{2} - \sqrt{\left(\frac{4n}{3}\right)^3 + \frac{1}{4}}} + \sqrt[3]{\frac{1}{2} + \sqrt{\left(\frac{4n}{3}\right)^3 + \frac{1}{4}}}. \tag{2.4}$$

A Remark: If $n = 0$, then $x_n(2)$ and $x_n(4)$ defined by (2.1) and (2.3), respectively, clearly equal 1, as expected. Let us show that $x_0(3)$ defined by (2.2) equals 1 as well. In fact, letting $n = 0$ in (2.2) gives

$$x_0(3) = \sqrt[3]{-\sqrt{-\frac{1}{27}}} + \sqrt[3]{\sqrt{-\frac{1}{27}}} = \sqrt[6]{\frac{1}{27}}\left[\sqrt[3]{-i} + \sqrt[3]{i}\right], \tag{2.5}$$

where i is the imaginary unit. Considering the principal values of the cubic roots in (2.5) yields

$$\begin{aligned} x_0(3) &= \sqrt{\frac{1}{3}}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &= 2\sqrt{\frac{1}{3}}\cos\frac{\pi}{6} = \sqrt{\frac{1}{3}}\sqrt{3} = 1 \end{aligned}$$

as expected.

3. SOLVING $x^5 - x - r$

The quintic $q(x) = x^5 - x - r$ $(r \in \mathbb{Z})$ may be either irreducible or reducible over the rational field $\mathbb{Q}$. If it is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$ as well [9, Th. 23, p. 24]. Necessary and sufficient conditions for its decomposition are given in [8]. Since the argument leading to the complete characterization of the quintics $q(x)$ that are solvable by radicals is based essentially on properties of irreducible quintics, we settle first the irreducible case, then we complete the discussion by addressing the reducible case.

3.1 The Irreducible Case

We shall prove that, if $q(x)$ is irreducible over $\mathbb{Q}$, then it cannot be solved by radicals. To this aim, we need the following theorem by Dummit [3, p. 389] that we quote in a form specialized to our Bring-Jerrard quintic $q(x)$.

Theorem 1 (Dummit): If $q(x) = x^5 - x - r$ $(r \in \mathbb{Z})$ is irreducible, then it can be solved by radicals iff the polynomial

$$x^6 - 8x^5 + 40x^4 - 160x^3 + 400x^2 - (3125r^4 + 512)x + (9375r^4 + 256) \tag{3.1}$$

has a rational root. If this is the case, then the polynomial (3.1) factors into the product of a linear polynomial and an irreducible quintic.

Next we state our main theorem.

Theorem 2: If $q(x) = x^5 - x - r$ $(r \in \mathbb{Z})$ is irreducible, then it cannot be solved by radicals.

Proof (reductio ad absurdum): From Theorem 1, it is sufficient to prove that no integer r yields a rational root u of the monic polynomial (3.1). After observing that a rational root of a monic polynomial is necessarily an integer (from the Rational Root Theorem, e.g., see [5, p. 253]), we suppose the existence of an integer root u, thus getting a contradiction.

First, replace x by the integer u in (3.1), equate this polynomial to zero, and solve for r^4, thus obtaining the equality

$$r^4 = \frac{(u-2)^4(u^2+16)}{5^5(u-3)},$$

which can be rewritten in the form

$$u^2 + 16 = 5(u-3)\left(\frac{5r}{u-2}\right)^4. \tag{3.2}$$

Now, observe that $5[5r/(u-2)]^4$ must be an integer because $\text{g.c.d.}(u-3, u-2) = 1$. Consequently, if $u-2$ is not divisible by 5, then $r/(u-2)$ must be an integer, while, if $u-2$ is divisible by 5, then $5r/(u-2)$ must be an integer. In both cases it follows that, if u is an integer, then the quantity $v = 5r/(u-2)$ is an integer as well.

Then, from (3.2), write the quadratic equation in u,

$$u^2 - 5v^4u + 15v^4 + 16 = 0, \tag{3.3}$$

whose discriminant $25v^8 - 60v^4 - 64$ must be a perfect square (say, w^2) because u is an integer by hypothesis. Hence, v is a root of the quadratic equation in z,

$$25z^2 - 60z - w^2 - 64 = 0, \tag{3.4}$$

where $z = v^4$. Again, the discriminant $100(w^2+100)$ of (3.4) must be a perfect square (say, $100s^2$) so that w and s satisfy the diophantine equation

$$w^2 + 100 = s^2 \tag{3.5}$$

whose solutions are $(w, s) = (24, 26)$ and $(0, 10)$.

Letting $w = 24$ and 0 in (3.4) yields the roots $(z_1, z_2) = (32/5, -4)$ and $(16/5, -4/5)$, respectively. None of these roots is a fourth power, as is required by the replacement $z = v^4$ above. This contradiction comes from the fact that we supposed that u is an integer. Q.E.D.

3.2 The Reducible Case

Theorem 2 tells us that the quintics of the form $q(x)$ may be solved by radicals only if they are reducible. The solution of this case has been given by Rabinowitz in his nice paper [8]. In fact, after showing that, if $r = m^5 - m$ $(m \in \mathbb{Z})$, then

$$x^5 - x - (m^5 - m) = (x-m)(x^4 + mx^3 + m^2x^2 + m^3x + m^4 - 1), \tag{3.6}$$

this author proves the following.

Theorem 3 (Rabinowitz): If $r \neq m^5 - m$, then $q(x)$ is reducible iff

$$r^2 = \begin{cases} F_{2j-1}^2 F_{2j}^2 F_{2j+2}, \\ F_{2j}^2 F_{2j+1}^2 F_{2j-2}. \end{cases} \tag{3.7}$$

Since the only square Fibonacci numbers with even subscript are $F_0 = 0$, $F_2 = 1$, and $F_{12} = 144$ (e.g., see [2]), the nonzero values of r (note that $r = 0$ has the form $m^5 - m$ with $m = -1$, 0, or 1) that satisfy (3.7) are given by

$$r = \begin{cases} \pm F_4 F_5 \sqrt{F_2} = \pm 15, \\ \pm F_9 F_{10} \sqrt{F_{12}} = \pm 22440, \\ \pm F_{14} F_{15} \sqrt{F_{12}} = \pm 2759640. \end{cases} \tag{3.8}$$

4. THE NUMBERS $x_n(5)$ THAT HAVE A CLOSED-FORM EXPRESSION

First, from (3.6) and (1.1), it is immediately seen that

$$x_{a^5-a}(5) = a \quad (a = 1, 2, 3, \ldots). \tag{4.1}$$

Then one can readily ascertain that the decompositions of the polynomials $q(x)$ having the *positive* values of r given by (3.8) are

$$x^5 - x - 15 = (x^2 - x + 3)(x^3 + x^2 - 2x - 5),$$
$$x^5 - x - 22440 = (x^2 + 12x + 55)(x^3 - 12x^2 + 89x - 408),$$
$$x^5 - x - 2759640 = (x^2 - 12x + 377)(x^3 + 12x^2 - 233x - 7320).$$

The real positive roots of the above polynomials give the solution of our problem. Namely, we get

$$x_{15}(5) = -\frac{1}{3} + \sqrt[3]{\frac{115}{54} + \frac{\sqrt{1317}}{18}} + \sqrt[3]{\frac{115}{54} - \frac{\sqrt{1317}}{18}}, \tag{4.2}$$

$$x_{22440}(5) = 4 + \sqrt[3]{90 + \frac{\sqrt{862863}}{9}} - \sqrt[3]{-90 + \frac{\sqrt{862863}}{9}}, \tag{4.3}$$

and

$$x_{2759640}(5) = -4 + \sqrt[3]{3130 + \frac{\sqrt{726984777}}{9}} + \sqrt[3]{3130 - \frac{\sqrt{726984777}}{9}}. \tag{4.4}$$

5. CONCLUDING COMMENTS

For solving the problem of finding all numbers $x_n(5)$ that have a closed-form expression, we have characterized all the quintics of the Bring-Jerrard form $x^5 - x - r$ over $\mathbb{Z}$ that are solvable by radicals. This result is not trivial because there are examples of irreducible polynomials of degree five over $\mathbb{Q}$ that can either be solved by radicals or not; e.g., $x^5 + 15x + 12$ can be solved [3], whereas $x^5 - 6x + 3$ cannot [10, p. 147].

Formal solutions applicable to unsolvable quintics were sought by using elliptic functions [6]; in particular, that given by Hermite is based on the Bring-Jerrard form [7].

Let us conclude our paper by posing ourselves the following question.

Question: Do there exist nonintegral numbers $x_n(k)$ with $k \geq 6$, that can be expressed by radicals?

ACKNOWLEDGMENTS

The contribution of the first author has been financially supported by CNR (Italian Council for National Research), whereas the contribution of the second author has been given within the framework of an agreement between the Italian PT Administration (Istituto Superiore PT) and the Fondazione Ugo Bordoni.

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AMS Classification Numbers: 12D05, 11B39, 14H52

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