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HOW SHOULD RELATIVE CHANGES BE MEASURED?

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ABSTRACT

Various indicators of relative change (or difference) are considered. It is shown that the log-change $H_{10}(y/x) = \log_e(y/x)$ is the only symmetric, additive and normed indicator of relative change. It is proposed that the values of the log-change in per cent, 100 $\log_e(y/x)$ be denoted by the symbol %̲, the log-percentage. It is hoped that the symmetric and additive log-percentages (%̲) will gradually replace the ordinary asymmetric and nonadditive percentages(%).

1. Introduction

Communication of information about economic and other phenomena is often blurred by various difficulties connected with the measurement of changes or differences. In the following we consider difficulties associated with indicators of relative differences of a variable (e.g., price, volume, value, height, length, etc.) measured on a ratio scale. We do not discuss at any length the reasons why relative rather than absolute differences are to be preffered for many purposes. The advantages of relative differences are connected with the fact that they are pure numbers and independent of the units of measurement. Therefore, relative differences are also directly comparable for, e.g., commodities or variables having different units of measurement. Relative changes play a crucial role in, e.g., index number calculations and in the measurement of productivity, and they may have important advantages over levels of variables in estimating economic relationships, see e.g. Törnqvist (1936), Y. Vartia (1976), Diewert (1978), Christensen and Jorgenson (1970), Fisher (1966), P. Vartia (1974, p. 33-47).

Many of the questions dealt with have earlier been discussed in the literature. However, since the present state of affairs is clearly unsatisfactory in many respects, we want to present a definite proposal as to how relative

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changes should be measured and how these measures and indicators should be named. Giving things suggestive names is often helpful. Initially, some examples of the difficulties will be given.

Example 1: Many people have been troubled by the fact that relative differences are ordinarily calculated in an asymmetric way. Consider, for instance, two commodities A and B with prices Mk 200 and Mk 250 respectively. Their relative difference, calculated in the ordinary way, unfortunately depends, on which of them is used as a point of comparison: B is 25 % more expensive than A, but A costs only 20 % less than B.

Example 2: In connection with exchange rate changes a 25 per cent rise in the value of foreign currencies corresponds to a 20 per cent decrease in the value of domestic currency. These two measures for one single change are often mixed up and - in a country like Finland, which has already devalued its currency ten times since the Second World War - a lot of mistakes and quarrels would have been avoided had only a single measure of relative change, the "true exchange rate change percentage", been available. Similarly confusion may be caused by the fact that the premium of the Finnmark against the dollar is not equal (in absolute value) to the discount of the dollar against the mark in the forward markets.

Example 3: With most indicators of relative change, successive relative changes are not additive; this applies e.g., to the calculation fo compound interest. Thus, for instance two successive increases of 5 % correspond to an increase of 10.25 %, not of 10 %. Also, the calculation of annual interest becomes dependent on the number of times per year that interest is added to the principal. This causes astonishment among the depositing public and only few cashier girls are able to explain why this is the case. In Finland, banks have even used the number of times per year interest is accumulated as a means of competition.

Example 4: For most indicators of relative change, the additive identity between changes in value, volume and price does not hold. Thus, e.g., an equal, 10 % increase in both volume and price does not lead to a 20 % but to a 21 % rise in value. The common approximation procedure of adding up relative changes in price and volume to obtain the relative change in value is useful only for small changes.

Example 5: In calculating arc-elasticities from discrete observations, the absolute differences in, say, the volume and price of a commodity can be related to several means of the two price and volume situations. Economists have suggested several ways to overcome this difficulty, see P. Vartia (1977).

2. Indicators of relative changes or differences

These difficulties are well known and the list could well be continued. Can troubles of this kind be avoided? How should relative differences between the values x and y of a ratio-scale variable be measured? An indicator of the relative difference between x and y (or the change from x to y) is defined here as a real-valued function C(x,y), defined for all positive x and y, $C: \mathbb{R}^2_+ \to \mathbb{R}$, which has the following properties, see Y. Vartia (1976, p. 9-25).

1. $C(x,y) = 0$ iff $x = y$
2. $C(x,y) > 0$ iff $y > x$
   $C(x,y) < 0$ iff $y < x$
3. C is a continuous and increasing function of y when x is fixed
4. $\forall a: a > 0 \Rightarrow C(ax,ay) = C(x,y)$

The last-mentioned property requires that the values of an indicator of relative difference must be independent of the unit of measurement; i.e., relative differences in height should be the same irrespective of whether height is expressed in centimetres or inches, and relative changes in price should be the same irrespective of whether they are expressed in marks or dollars. Setting $a = 1/x$ in 4. it is shown that every indicator $C(x,y)$ of relative difference is a function of the ratio y/x alone, i.e., there exists a function $H: \mathbb{R}_+ \to \mathbb{R}$ such that $C(x,y) = H(y/x) = C(y/x,1)$. The properties 1.-4. for H are

1'. $H(y/x) = 0$ iff $y/x = 1$
2'. $H(y/x) > 0$ iff $y/x > 1$
    $H(y/x) < 0$ iff $y/x < 1$
3'. H is a continuous increasing function of its argument y/x
4'. $H(ay/ax) = H(y/x)$

The following functions $H(y/x)$, i.a., are indicators of relative difference:

1. $H_1(y/x) = (y-x)/x \qquad = (y/x) - 1$
2. $H_2(y/x) = (y-x)/y \qquad = 1 - (x/y)$
3. $H_3(y/x) = (y-x)/\frac{1}{2}(x+y) \qquad = ((y/x)-1)/\frac{1}{2}(1+(y/x))$
4. $H_4(y/x) = (y-x)/\sqrt{xy} \qquad = ((y/x)-1)/\sqrt{y/x}$
5. $H_5(y/x) = (y-x)/[\frac{1}{2}(x^{-1}+y^{-1})]^{-1} \qquad = ((y/x)-1)(1+(x/y))/2$
6. $H_6(y/x) = (y-x)/[\frac{1}{2}(x^k+y^k)]^{1/k} \qquad = ((y/x)-1)/[\frac{1}{2}(1+(y/x)^k)]^{1/k}$
7. $H_7(y/x) = (y-x)/\min(x,y) \qquad = ((y/x)-1)/\min(1,y/x)$
8. $H_8(y/x) = (y-x)/\max(x,y) \qquad = ((y/x)-1)/\max(1,y/x)$
9. $H_9(y/x) = (y-x)/K(x,y) \qquad = ((y/x)-1)/K(1,y/x)$
10. $H_{10}(y/x) \qquad = \log_e(y/x)$.

Here $K(y,y)$ is any <u>mean</u> of x and y for which $H_9(y/x)$ is increasing in y/x. Thus, when the absolute difference $y - x$ is compared to such a mean $K(x,y)$, we arrive at various indicators of relative difference. <u>A mean</u> $K(x,y)$ of two numbers x and y in a subset A of $\mathbb{R}^2$ is a real-valued function $K: A \to \mathbb{R}$ which satisfies, for all (x,y) in A, the following:

A. Mean property: $\min(x,y) \leq K(x,y) \leq \max(x,y)$

B. Continuity: K is continuous

C. Homogeneity: $\forall a: a > 0 \Rightarrow K(ax,ay) = aK(x,y)$

D. Symmetry: $K(x,y) = K(y,x)$

Note that if $H(y/x)$ is an indicator of relative change, then $\bar{H}(y/x) = cH(y/x)$ (c a positive constant) is one too. Therefore, any of the $H_i(y/x)$ above could be multiplied by a positive constant. Still others can easily be invented; e.g., linear combinations (with positive coefficients) of any set of indicators of relative change are also such indicators. For example, $H_{11} = \frac{1}{3}H_3 + \frac{2}{3}H_4$ may be used to approximate H_{10} for small changes.

It is clear that the indicators H_3 - H_8 are all special cases of H_9; where K(x,y) has been chosen so as to be respectively the arithmetic mean, geometric mean, harmonic mean, moment mean of order k, the minimum or the maximum. However, it is not generally known that $H_{10}(y/x) = \log_e(y/x)$, too, can be interpreted as the ratio of the absolute difference (y - x) to a mean L(x,y). The definition of this mean L(x,y) of the positive numbers x and y

$$(1) \qquad L(x,y) \begin{cases} = (y-x)/\log_e(y/x) & \text{for } x \neq y \\ = x & \text{for } x = y \end{cases}$$

was presented already by Törnqvist (1935, p. 35), who mentioned that $\sqrt{xy} < L(x,y) < (x+y)/2$ for $x \neq y$. Vartia (1974, 1976) showed that L(x,y) is a mean with properties A -D above. It has been also shown that for unequal x and y the logarithmic mean satisfies

$$(2) \qquad \sqrt{xy} < T(x,y) < L(x,y) < S(x,y) < \tfrac{1}{2}(x+y),$$

where $T(x,y) = \sqrt[3]{xy\frac{1}{2}(x+y)}$ is the Theil (1973) mean and $S(x,y) = \frac{1}{3}(2\sqrt{xy} + \frac{1}{2}(x+y))$ is the Sato (1974) mean. For x = y all the means are equal. Diewert (1978, p. 900) gives an even tighter upper bound for L(x,y), L(x,y) < D(x,y) < S(x,y), where $D(x,y) = (\frac{1}{2}(x^{\frac{1}{3}} + y^{\frac{1}{3}}))^3$, while Lau (1979) denotes L(x,y) by V(x,y) referring to it as the Vartia mean. Mustonen (1976) has generalized the logarithmic mean for n arguments.

By (1) we have, for all positive x and y, an important representation for the log-difference

$$(3) \qquad \log_e(y/x) = (y - x)/L(x,y),$$

indicating that the log-difference is literally a relative difference with respect to the logarithmic mean. In practice, the easiest way of calculating log-differences is to use the formula $\log_e(y/x)$ directly, since the logarithmic function is nowadays a standard option in most pocket calculators. The representation (3) is often useful in theoretical considerations involving log-differences, e.g., in the context of index numbers; see Sato (1976), Y. Vartia (1976, 1976b); or elasticities, see P. Vartia (1977).

A comparison of the two indicators of relative change, the customary $H_1(y/x) = (y - x)/x$ and $H_{10}(y/x) = \log_e(y/x)$, is presented in Table 1 and Figure 1, where also the terminology suggested in chapter 3 is applied.

Several of the problems related to various indicators of relative change are due to some of these indicators' being asymmetric. We define here the indicator of relative change H(y/x) to be symmetric iff

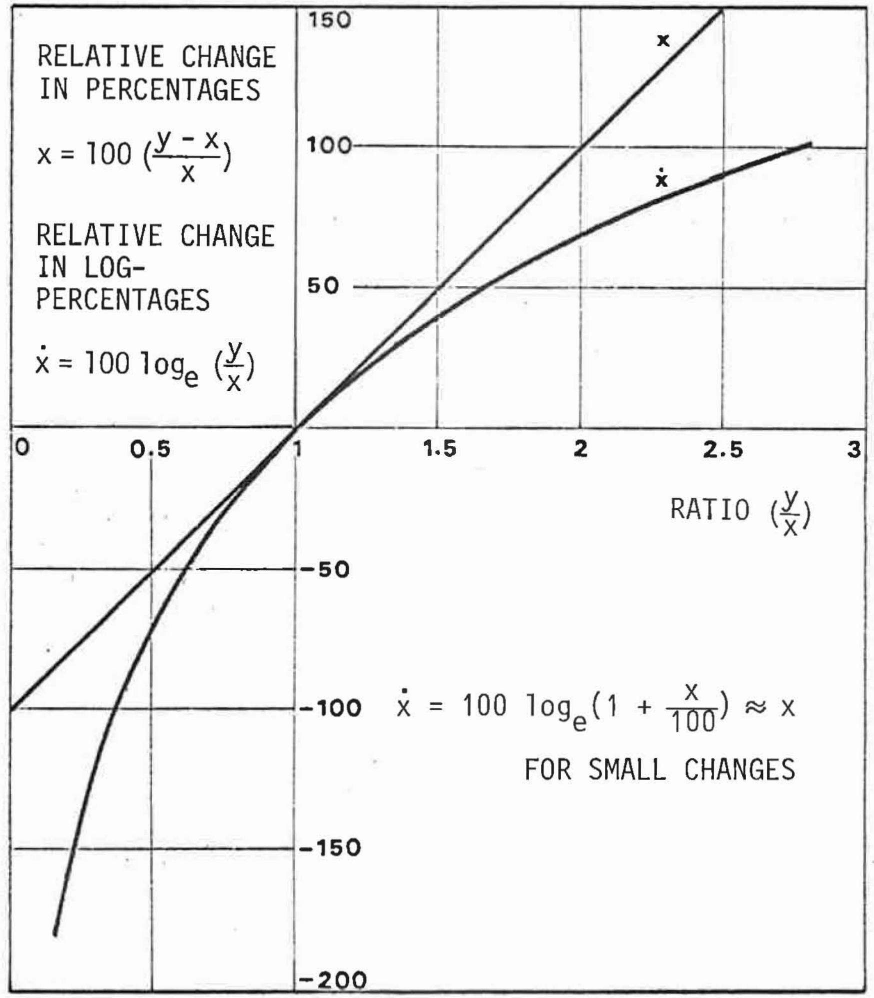
$$(4) \qquad H(x/y) = - H(y/x).$$

Thus, the indicators H_1 and presented above are not symmetric whereas the others are. The problems in examples 1 and 2 arise just from the customarily used indicator's of relative change, $H_1(x/y) = (y - x)/x$,

Table 1. Comparison of percentages and log-percentages as indicators of relative change

y/x	$100\frac{y-x}{x}$	$100\log_e(\frac{y}{x})$
0	-100	$-\infty$
.01	- 99	-460.517
.1	- 90	-230.259
.2	- 80	-160.944
.5	- 50	- 69.315
.9	- 10	- 10.536
.95	- 5	- 5.129
.99	- 1	- 1.005
1.0	0	0.000
1.01	1	0.995
1.05	5	4.879
1.1	10	9.531
1.5	50	40.547
2.0	100	69.315
3	200	109.861
4	300	138.629
5	400	160.944
10	900	230.259

Figure 1. Comparison of the ordinary percentage change $100\,H_1(y/x) = 100(y-x)/x$ and the log-percentage change $100\,H_{10}(y/x) = 100\log_e(y/x)$ as indicators of relative change



being asymmetric. Correspondingly, the numerical value of the relative difference between two numbers x and y depends on which of the two is used as the point of comparison. This asymmetry is obviously an annoying property, and symmetric indicators should be preferred to asymmetric ones. Indicators of the form $H(y/x) = (y - x)/K(x,y)$, including H_7 and H_8 above, are symmetric and often suggested just because of this property; see Samuelson (1969) and Lipsey (1975), who are in favour of H_3; and Rao and Miller (1971, p. 17), who in fact propose the use of H_7.

Let us consider the two-stage change $x \to y \to z$ and examine how the relative change $x \to z$ can be expressed in terms of the changes $x \to y$ and $y \to z$. We define the indicator of the relative change $H(y/x)$ as additive iff

(5) $$H(\frac{z}{x}) = H(\frac{y}{x}) + H(\frac{z}{y}).$$

Setting $z = x$, we immediately see that an additive indicator of relative change must be symmetric. But we may even infer the functional form of an additive indicator. Denoting $y/x = p$, $z/y = q$ and $z/x = r$, we have $r = pq$; and $H\colon \mathbb{R}_+ \to \mathbb{R}$ is additive iff $H(pq) = H(p) + H(q)$ for all positive p and q. The only solution H of this functional Cauchy-type equation (which is continuous at least at one point) is $H(p) = c\log_e(p)$, $c \in \mathbb{R}$; see Eichhorn (1978, p. 13). Therefore, the only additive indicators of relative change are positive multiplies of H_{10}; i.e., of the form $H(y/x) = cH_{10}(y/x) = c\log_e(y/x)$, $c > 0$. The use of additive indicators will solve the problem posed in example 3 above. It is also easy to see that, for additive indicators, the additive identity between volume, price and value changes (problem 4) holds, i.e.

(6) $$H(\frac{p_1q_1}{p_0q_0}) = H(\frac{p_1}{p_0}) + H(\frac{q_1}{q_0}) .$$

As mentioned above, each of the indicators H_1 - H_{10} can be multiplied by a positive constant to obtain a new indicator of relative change. To exclude the indicators that do not behave approximately as $H_1(y/x) = (y - x)/x$ when $y/x \approx 1$, we further require that a useful indicator has to be a normed one. An indicator H(s) of relative change is normed iff

(7) $$\lim_{s\to 1} \frac{H(s)}{H_1(s)} = \lim_{s\to 1} (\frac{H(s) - H(1)}{s - 1}) = H'(1) = 1 .$$

Thus, the above indicators H_1 - H_{10} are all normed, but their positive multiplies are not. Of non-normed indicators, only some multiplies of H_{10} have an established position in scientific usage. For instance, in acoustics and electronics a non-normed indicator of relative difference $dB = 10\log_{10}(P_1/P_0)$ $= 4.3429\log_e(P_1/P_0) = 4.3429H_{10}(P_1/P_0)$ is used to measure relative differences (e.g. damping factors) in terms of decibels between two intensities or powers[1)] P_1 and P_0 . Thus doubling an acoustic power makes an increase of 3 decibels, which is hardly noticable.

Also, in information theory a non-normed indicator of relative difference is in use, namely, an indicator of the information of an event measured in bits, $inf(A) = \log_2(1/P(A)) = 1.4427 \log_e(1/P(A))$. It actually compares the relative difference between the probability of A and that of the certain event E, because $inf(A) = 1.4427H_{10}(P(E)/P(A))$. On the other hand, the cont-measure of information, $cont(A) = 1 - P(A)$, see Hintikka (1968), is a normed indicator of the relative difference between P(A) and P(E) = 1, because $cont(A) = (P(E) - P(A))/P(E) = H_2(P(E)/P(A))$.

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1) When the power P of an electrical current is proportional to the square of its current I (or voltage U) we have $10\log_{10}(P_1/P_0) = 10\log_{10}(I_1^2/I_0^2)$ $= 20\log_{10}(I_1/I_0)$. This explains why the multiplier 20 instead of 10 is conventionally used for currents or voltages to express their relative changes in terms of decibels.

Other multiples $cH_{10}(y/x) = c\log_e(y/x)$ of H_{10} are also widely used in various brances of science: the DIN-scale in photography, Richter's scale for measuring the energy of earth-quakes and the phone-scale in acoustics provide examples of such logarithmic 'scales'. The constants c are chosen in these scales in such a way that the figures of the scale are of convenient magnitude. None of these logarithmic indicators are in use in economics, where relative changes are usually small.

To conclude, the indicator

$$(8) \qquad H_{10}(y/x) = \log_e(y/x) = \frac{y - x}{L(x,y)}$$

is the only indicator of relative difference that is both additive and normed. It is also symmetric, of course, and a special case of indicators of the form $H_9(y/x) = (y - x)/K(x,y)$.

If we require, as is natural, that the indicators used in economics for scientific purposes should be symmetric, additive and normed, we are left only with the indicator $H_{10}(y/x) = \log_e(y/x)$. Törnqvist (1935, p. 36) already stated that "there are no good reasons for giving numbers in the form of percentages, because natural logarithms of the indices are at least for scientists far more interesting." We suggest that this indicator should be used more extensively to express relative changes or differences in economic variables.

3. Log-percentages (%) instead of ordinary percentages (%)

It is common practice to shortly call the logarithmic change $\log_e(x_1/x_0) = \log_e x_1 - \log_e x_0 = \Delta\log_e x$ the log-change, see e.g. Theil (1967). We further propose that $\log_e(y/x) = \log_e(y) - \log_e(x)$ be called, accordingly, the log-difference between y and x. Theil obviously derived the prefix log from logarithm, which comes from the Greek words logos (meaning ratio, sense, nature) and arithmos (meaning number). Thus log-differences are literally logarithmic differences. However, log-differences could also be described as "logical" or "natural" relative differences, because the words logarithmic and logical are both derived from the same word logos. Note also that particularly the logarithmus naturalis ($\log_e = \ln$) should be used to calculate log-differences.

It is customary to indicate small relative differences (or changes) in percentage (or per thousand) form, i.e., to multiply them by 100 (1000). This shifting of the decimal point serves mainly practical purposes. Thus a relative increase in a price from \$ 80 to \$ 90 is usually said to be 12.5 % (or 125 $^{o}/oo$) instead of 0.125. The lack of a comparable helpful terminology and symbols in connection with log-changes causes unnecessary difficulties for a researcher using these. Although log-changes are often similarly multiplied by 100 to arrive at more comfortable figures in tables, see Theil (1975, p. 182), the decimal point is usually shifted back to its original position when these figures are referred to, or the figures are awkwardly referred to as "log-changes multiplied by 100". To speak concretely, the log-change from 80 to 90 is $\log_e(1.125) \approx 0.118$ and its 100-multiple is 11.8, corresponding to an increase of 12.5 in "ordinary percentage" terms.

We propose that the log-changes (or differences) multiplied by 100, 100 $\log_e(y/x)$, be called log-percentages and denoted by the symbol %. E.g., the relative change from 80 to 90 would thus be 11.8 % (read 11.8 "logical per cent" or shortly "log-per-cent"). Nothing else, other than the symbol % is needed to distinquish this from the corresponding increase of 12.5 in terms of "ordinary percentages" (12.5 %). In order to also make a distinction on a verbal level between the "percentage change" $100(\frac{y-x}{x})$ and 100 $\log_e(\frac{y}{x})$ the latter could be termed the "log-percentage change". Then log-percentages (%) would be units of the log-percentage change in the same way as ordinary percentages (%) are units of the ordinary percentage change. As 100(y - x)/x is the ordinary relative change in per cent, we may similarly say that 100$\log_e(y/x)$ is the log-change in per cent. Literally, 100$\log_e(y/x)$ = 100(y - x)/L(x,y) tells how many per cent the absolute change (y - x) is of the logarithmic mean L(x,y). These symbols and terminology have already been used succesfully in some research reports in Finland.

In order to demonstrate how easily these symbols and concepts can be used in practice we finally present two tables where changes are expressed both in terms of ordinary and log-percentages respectively.

Table 2. Finland's trade balance in 1978-79

	1978		Change 78 → 79, %			1979	
	Value share, %	Value mill.mk	Price	Volume	Value	Value mill.mk	Value share, %
Exports of goods	52.1	35 206	12.3	9.9	23.4	43 430	49.7
Imports of goods	47.9	32 338	15.3	18.1	36.2	44 045	50.3
Foreign trade	100.0	67 544	13.7	13.9	29.5	87 475	100.0
Surplus	4.3	2 868	.	.	.	-615	-0.7

Here the dot (·) denotes that the corresponding figure cannot appear because of logical reasons. A similar table where log-percentages have been substituted for ordinary percentages is as follows.

Table 3. Finland's trade balance in 1978-79 (changes are expressed in terms of log-percentages)

	1978		Change 78 → 79, %			1979	
	Value share, %	Value mill.mk	Price	Volume	Value	Value mill.mk	Value share, %
Exports of goods	52.1	35 206	11.6	9.4	21.0	43 430	49.7
Imports of goods	47.9	32 338	14.2	16.7	30.9	44 045	50.3
Foreign trade	100.0	67 544	12.9	13.0	25.9	87 475	100.0
Surplus	4.3	2 868	.	.	.	-615	-0.7

For instance, the changes in the prices and the volume of exports from 1978 to 1979 are 11.6 % (= 100 $\log_e(1.123)$) and 9.4 %, respectively. In consequence, the change in export value in terms of log-percentages is 11.6 + 9.4 = 21.0 %. The value shares are expressed, of course, in terms of ordinary percentages in both tables.

4. Conclusions

We hope that the use of the log-change $\log_e(y/x)$ and the log-percentage change $100\log_e(y/x)$, together with the terminology and symbols presented above, will find its way first to research reports and econometric work and gradually also to ordinary statistical and economic publications to replace the ordinary asymmetric and nonadditive percentage change.

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