Partial Metric Spaces

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1. INTRODUCTION. This is an article about a generalization of a landmark construction introduced in 1906 by the French mathematician Maurice René Fréchet [**2**].

Definition 1. A **metric space** is a pair $(X, d : X \times X \to \mathbb{R})$ such that

M0: $0 \leq d(x, y)$ (nonnegativity),
M1: if $x = y$ then $d(x, y) = 0$ (equality implies indistancy),
M2: if $d(x, y) = 0$ then $x = y$ (indistancy implies equality),
M3: $d(x, y) = d(y, x)$ (symmetry), and
M4: $d(x, z) \leq d(x, y) + d(y, z)$ (triangularity).

As with many mathematical concepts these axioms are chosen to ensure that two things are equal if and only if some property expressible in terms of the concept holds. For a metric space, $x = y$ if and only if $d(x, y) = 0$. Thus as there is the *equality* relation $x = y$ in a metric space, so there is what we call an *indistancy* relation $d(x, y) = 0$. Axioms M1 and M2 work together to identify *equality* with *indistancy*. That is x and y are *equal* if and only if x and y have *no distance between them*. Such identification may seem to be so fundamental that to suggest otherwise would serve no purpose. However, there is a longstanding precedent for relaxing the axioms which ensure this identification. The relation defined by $x \equiv y$ if and only if $d(x, y) = 0$ is an equivalence, which can be useful, as in the construction of the classical l^p-spaces. In this construction, spaces are considered in which axiom M2 is dropped while the others hold, giving a *pseudometric space*. This article retains M2 but drops M1, introducing the possibility of equality without indistancy, and leading to the study of self-distances $d(x, x)$ which may not be zero. Originally motivated by the experience of computer science, as discussed below, we show how a mathematics of nonzero self-distance for metric spaces has been established, and is now leading to interesting research into the foundations of topology.

The approach of this article is to retrace the steps of a standard introduction to metric and topological spaces [**10**], seeing why and how it can be generalized to accommodate nonzero self-distance. The article then concludes with a discussion of research directions. Proofs of results presented here consist of straightforward reasoning about distances or topology, and as such are left as informative exercises for the reader. For more material and publications please visit [**7**], the authors' web site `partialmetric.org`

2. NONZERO SELF-DISTANCE. Let us begin with an example of a metric space, and why nonzero self-distance is worth considering. Let S^ω be the set of all infinite sequences $x = \langle x_0, x_1, \ldots \rangle$ over a set S. For all such sequences x and y let $d_S(x, y) = 2^{-k}$, where k is the largest number (possibly ∞) such that $x_i = y_i$ for each $i < k$. Thus $d_S(x, y)$ is defined to be 1 over 2 to the power of the length of the longest initial sequence common to both x and y. It can be shown that (S^ω, d_S) is a metric space.

How might computer scientists view this metric space? To be interested in an infinite sequence x they would want to know how to compute it, that is, how to write

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a computer program to print out (on either a screen or paper) the values x_0, then x_1, then x_2, and so on. As x is an infinite sequence, its values cannot be printed out in any finite amount of time, and so computer scientists are interested in how the sequence x is formed from its *parts*, the finite sequences $\langle\ \rangle$, $\langle x_0 \rangle$, $\langle x_0, x_1 \rangle$, $\langle x_0, x_1, x_2 \rangle$, and so on. After each value x_k is printed, the finite sequence $\langle x_0, \ldots, x_k \rangle$ represents that part of the infinite sequence produced so far. Each finite sequence is thus thought of in computer science as being a *partially computed* version of the infinite sequence x, which is *totally computed*. Suppose now that the above definition of d_S is extended to S^*, the set of all finite sequences over S. Then axioms M0, M2, M3, and M4 still hold. However, if x is a finite sequence then $d_S(x, x) = 2^{-k}$ for some number $k < \infty$, which is not 0, since $x_j = x_j$ can only hold if x_j is defined. Thus axiom M1 (equality implies indistancy) does not hold for finite sequences. This raises an intriguing contrast between 20th century mathematics, of which the theory of metric spaces is our working example, and the contemporary experience of computer science. The truth of the statement $x = x$ is surely unchallenged in mathematics, while in computer science its truth can only be asserted to the extent to which x is computed. This article will show that rather than collapsing, the theory of metric spaces is actually expanded and enriched by the generalization of dropping the requirement for equality to imply indistancy.

3. PARTIAL METRIC SPACES. Nonzero self-distance is thus motivated by experience from computer science, and seen to be plausible for the example of finite and infinite sequences. The question we now ask is whether nonzero self-distance can be introduced to any metric space. That is, is there a generalization of the metric space axioms M0-M4 to introduce nonzero self-distance such that familiar metric and topological properties are retained? The following is suggested.

Definition 2. A *partial metric space* is a pair $(X, p : X \times X \to \mathbb{R})$ such that

P0: $0 \leq p(x, x) \leq p(x, y)$ (nonnegativity and small self-distances),

P2: if $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (indistancy implies equality),

P3: $p(x, y) = p(y, x)$ (symmetry), and

P4: $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ (triangularity) [**9**].

Why these axioms and not others? We are not seeking an alternative, but an extension to the theory of metric spaces. If $d_p(x, y)$ is defined to be $2p(x, y) - p(x, x) - p(y, y)$ then from the axioms P0, P2, P3, and P4 for p, it can be shown that M0, M2, M3, and M4 respectively hold for d_p. In particular, note how $p(y, y)$ is included in P4 in order to ensure that M4 will hold for d_p. Thus as d_p also satisfies M1, (X, d_p) is a metric space. Each partial metric space thus gives rise to a metric space with the additional notion of nonzero self-distance introduced. Also, a partial metric space is a generalization of a metric space; indeed, if an axiom P1: $p(x, x) = 0$ is imposed, then the above axioms reduce to their metric counterparts. Thus, a metric space can be defined to be a partial metric space in which each self-distance is zero.

Why should axiom P2 deserve the title of *indistancy implies equality*? It can be argued that this axiom reduces to M2 for (X, d_p), but there should be a justification in terms of (X, p). Let us define *indistancy* for (X, p) to be $p(x, y) = 0$. Then if $p(x, y) = 0$ it can be shown by P0 and P3 that $p(x, x) = p(x, y) = p(y, y)$, and hence $x = y$ by P2. It is a recurring theme of this article to find as many ways as possible in which partial metric spaces may be said to extend metric spaces. This is to apply as much as possible the existing theory of metric spaces to partial metric spaces, but also to see how the notion of *nonzero self-distance* can influence our understanding of metric spaces.

Let us now consider three examples of partial metric spaces, beginning with the sequences studied in the previous section. $(S^* \cup S^\omega, d_S)$ is a partial metric space, where the finite sequences are precisely those having nonzero self-distance, and the infinite sequences are precisely those having zero self-distance. For a second example note a very familiar function that just happens to be a partial metric. Let $\max(a, b)$ be the maximum of any two nonnegative real numbers a and b; then max is a partial metric over $\mathbb{R}^+ = [0, \infty)$.

For a third example, let $\mathcal{I}$ be the collection of nonempty closed bounded intervals in $\mathbb{R}$: $\mathcal{I} = \{[a, b] : a \leq b\}$. For $[a, b], [c, d] \in \mathcal{I}$ let $p([a, b], [c, d]) = \max(b, d) - \min(a, c)$. Then it can be shown that p is a partial metric over $\mathcal{I}$, and the self-distance of $[a, b]$ is the length $b - a$. This is related to the real line as follows: $|a - b| = p([a, a], [b, b])$, and so by mapping each a in $\mathbb{R}$ to $[a, a]$ we embed the usual metric structure of $\mathbb{R}$ into that of the partial metric structure of intervals.

And so partial metric spaces demonstrate that although zero self-distance has always been taken for granted in the theory of metric spaces, it is not necessary in order to establish a mathematics of distance. What partial metric spaces do is to introduce a symmetric metric-style treatment of the nonsymmetric relation *is part of,* which, as explained in this article, is fundamental in computer science. This relation is a partial order:

Definition 3. A *partial order on* X is a binary relation $\sqsubseteq$ on X such that

$x \sqsubseteq x$ (reflexivity),
if $x \sqsubseteq y$ and $y \sqsubseteq x$ then $x = y$ (antisymmetry), and
if $x \sqsubseteq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$ (transitivity).

A *partially ordered set* (or *poset*) is a pair $(X, \sqsubseteq)$ such that $\sqsubseteq$ is a partial order on X.

For each partial metric space (X, p) let $\sqsubseteq_p$ be the binary relation over X such that $x \sqsubseteq_p y$ (to be read, x *is part of* y) if and only if $p(x, x) = p(x, y)$. Then it can be shown that $(X, \sqsubseteq_p)$ is a *poset*.

Let us now see the poset for each of our earlier partial metric spaces. For sequences, $x \sqsubseteq_{d_S} y$ if and only if there exists some $k \leq \infty$ such that the length of x is k, and for each $i < k,\ x_i = y_i$. In other words, $x \sqsubseteq_{d_S} y$ if and only if x is an initial part of y. For example, suppose we wrote a computer program to print out all the prime numbers. Then the printing out of each prime number is described by the chain

$$\langle\,\rangle \sqsubseteq_{d_S} \langle 2 \rangle \sqsubseteq_{d_S} \langle 2, 3 \rangle \sqsubseteq_{d_S} \langle 2, 3, 5 \rangle \sqsubseteq_{d_S} \ldots,$$

whose least upper bound is the infinite sequence $\langle 2, 3, 5, \ldots \rangle$ of all prime numbers.

For the partial metric $\max(a, b)$ over the nonnegative reals, $\sqsubseteq_{max}$ is the usual $\geq$ ordering. For intervals, $[a, b] \sqsubseteq_p [c, d]$ if and only if $[c, d]$ is a subset of $[a, b]$.

Thus the notion of a partial metric extends that of a metric by introducing nonzero self-distance, which can then be used to define the relation *is part of*, which, for example, can be applied to model the output from a computer program.

4. THE CONTRACTION FIXED POINT THEOREM.

We now consider how a familiar theorem from the theory of metric spaces can be carried over to partial metric spaces. *Complete spaces*, *Cauchy sequences,* and the *contraction fixed point theorem* are all well known in the theory of metric spaces, and can be generalized to partial metric spaces as follows. The next definition generalizes the metric space notion of *Cauchy sequence* to partial metric spaces.

Definition 4. A sequence $x = (x_n)$ of points in a partial metric space (X, p) is *Cauchy* if there exists $a \geq 0$ such that for each $\epsilon > 0$ there exists k such that for all $n, m > k$, $|p(x_n, x_m) - a| < \epsilon$.

In other words, x is Cauchy if the numbers $p(x_n, x_m)$ converge to some a as n and m approach infinity, that is, if $\lim_{n,m\to\infty} p(x_n, x_m) = a$. Note that then $\lim_{n\to\infty} p(x_n, x_n) = a$, and so if (X, p) is a metric space then $a = 0$.

Definition 5. A sequence $x = (x_n)$ of points in a partial metric space (X, p) *converges* to y in X if

$$\lim_{n\to\infty} p(x_n, y) = \lim_{n\to\infty} p(x_n, x_n) = p(y, y).$$

Thus if a sequence converges to a point then the *self-distances* converge to the self-distance of that point.

Definition 6. A partial metric space (X, p) is *complete* if every Cauchy sequence converges.

Definition 7. For each partial metric space (X, p), a *contraction* is a function $f : X \to X$ for which there exists a $c \in [0, 1)$ such that for all x, y in X, $p(f(x), f(y)) \leq c \cdot p(x, y)$.

Theorem 1 (Matthews [8]). *For each contraction f over a complete partial metric space (X, p) there exists a unique x in X such that $x = f(x)$. Also, $p(x, x) = 0$.*

Thus the contraction fixed point theorem is extended to partial metric spaces. This highlights an additional feature: the fixed point has self-distance 0, which, although trivial in metric spaces, can be useful for reasoning about posets found in computer science. In the context of computer science where a computable function can also be proved to be a contraction, the partial metric extension of the contraction fixed point theorem can be used to prove that the unique fixed point, which is the program's output, will be totally computed [**8, 11**].

5. EQUIVALENTS FOR PARTIAL METRIC SPACES. Partial metric spaces arose from the need to develop a version of the contraction fixed point theorem which would work for partially computed sequences as well as totally computed ones. Since then much research has been aimed at extrapolating away from computer science in order to develop a mathematics of posets for metric spaces. To discover more about the properties of partial metric spaces we now look at equivalent formulations.

Definition 8. A *weighted metric space* is a triple $(X, d, |\cdot| : X \to \mathbb{R})$ such that (X, d) is a metric space and

$0 \leq |x|$ for each x in X, and
$|x| - |y| \leq d(x, y)$ for all x and y in X [**9**].

Thus a weighted metric space is a metric space with a nonnegative real number assigned to each point as a *weight*. Let $(X, d, |\cdot|)$ be a weighted metric space, and let

$$p(x, y) = \frac{d(x, y) + |x| + |y|}{2}.$$

Then (X, p) is a partial metric space, and $p(x, x) = |x|$. Conversely, if (X, p) is a partial metric space, then $(X, d_p, |\cdot|)$, where (as before) $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ and $|x| = p(x, x)$, is a weighted metric space. It can be seen that from either space we can move to the other and back again. In a weighted metric space the ordering can be defined by $x \sqsubseteq_p y$ if $|x| = d(x, y) + |y|$. Note that any metric space can be trivially weighted by defining $|x| = 0$ for each x. Thus a partial metric space combines the metric notion of *distance, weight*, and *poset* in a single formalism.

Now we consider another variant of the metric space concept that bears a close relationship to partial metric spaces.

Definition 9. A *based metric space* is a triple (X, d, ϕ) such that (X, d) is a metric space and ϕ is any member of X.

That is, a based metric space is a metric space with an arbitrarily chosen *base point*. This can be turned into an equivalent weighted metric space as follows. Let $|x| = d(x, \phi)$; by the triangle inequality and symmetry, $|x| \leq d(x, y) + |y|$, so $|x| - |y| \leq d(x, y)$ and therefore $(X, d, |\cdot| : X \to \mathbb{R})$ is a weighted metric space, and can be turned into an equivalent partial metric space as previously discussed. Then for each $x \in X,\ p(x, x) = d(x, \phi)$, so ϕ is the largest member of the associated poset $(X, \sqsubseteq_p)$. Conversely, if there happens to exist a largest member ϕ in X for $\sqsubseteq_p$, then (X, d_p, ϕ) is a based metric space from which (X, p) can be recovered as above.

Consider the following example of a based metric space. Suppose we wish to design an interactive computer game consisting of a Euclidean space, and players who move around in the space. The space itself could be modeled by a metric space (X, d), and each player's position at any time in the space by a base point. One of many challenges for the game's programmers would be to ensure that at all times each player's view of space is consistent with the space itself, both of which are displayed as one upon the computer's screen. Then the movement of each player through space could be modeled by a sequence of the form $((X, d, \phi_n))$, to which can be associated a sequence of posets $((X, \sqsubseteq_n))$ to describe that player's changing view of space.

Each of our earlier examples of partial metric spaces does not have a unique largest member. However, those posets used in computer science, such as our example of finite and infinite sequences, usually do have a unique least member, this being the place where any computation begins. An equivalence between any based bounded metric space and a partial metric space having a unique least member can be defined as follows. Suppose that (X, d, ϕ) is a based metric space such that the metric is bounded by some value, say a. Let

$$p(x, y) = a \ + \ \frac{d(x, y) - d(x, \phi) - d(y, \phi)}{2}.$$

Then it can be shown that (X, p) is a partial metric space having ϕ as a unique least member, and $(X, d_p) = (X, d)$.

Having now introduced weighted metric spaces and based metric spaces, we introduce a third equivalent formulation for partial metric spaces. A connection between posets and metric spaces existed long before partial metric spaces were introduced.

Definition 10. A *quasimetric space* is a pair $(X, q : X \times X \to \mathbb{R})$ such that

Q0: $0 \leq q(x, y)$ (nonnegativity),
Q1: if $x = y$ then $q(x, y) = 0$ (equality implies indistancy),
Q2: if $q(x, y) = q(y, x) = 0$ then $x = y$ (indistancy implies equality), and
Q4: $q(x, z) \leq q(x, y) + q(y, z)$ (triangularity).

As quasimetrics are not in general symmetric, we revise our definition of *indistancy* to be $q(x, y) = q(y, x) = 0$. Thus in quasimetric spaces *equality* is identified with *indistancy*. A metric space (X, d) can be formed by defining $d(x, y) = q(x, y) + q(y, x)$. For any quasimetric q, a partial order $\sqsubseteq_q$ is described by $x \sqsubseteq_q y \iff q(x, y) = 0$. Also, any partial order $\sqsubseteq$ is $\sqsubseteq_q$ for some quasimetric q; in fact,

$$q(x, y) = \begin{cases} 0 & x \sqsubseteq y \\ 1 & x \not\sqsubseteq y \end{cases}$$

is such a quasimetric. This connection between posets and quasimetric spaces can be related to partial metric spaces as follows.

Definition 11. A *weighted quasimetric space* is a triple $(X, q, |\cdot| : X \to \mathbb{R})$ such that (X, q) is a quasimetric space and

$0 \leq |x|$ for each x in X, and
$|x| + q(x, y) = |y| + q(y, x)$ for all x and y in X.

If we define $p(x, y) = |x| + q(x, y)$ then (X, p) is a partial metric space. Conversely, if (X, p) is a partial metric space then $(X, q_p, |\cdot|_p)$, where $q_p(x, y) = p(x, y) - p(x, x)$ and $|x|_p = p(x, x)$, is a weighted quasimetric space. With these definitions, for any partial metric, $\sqsubseteq_p = \sqsubseteq_{q_p}$. Not every quasimetric space has a weight function $|\cdot|$ (see [**9**]).

6. PARTIAL METRIC TOPOLOGY.

A first course in metric spaces would usually progress into a discussion of their topological properties [**10**]. For example, it would show that the notion of *convergent sequence* in a metric space can be expressed in terms of topology. This section shows how a course in partial metric spaces would progress into a discussion of their topological properties. Recall the following definitions in topology.

Definition 12. A *topological space* is a pair (X, τ) so that τ is a set of subsets of X, the empty set is in τ, X is in τ, τ is closed under finite intersections, and τ is closed under arbitrary unions. Each member of τ is termed an *open set*.

Topologies are often determined by special open sets:

Definition 13. A *basis* for a topological space (X, τ) is a subset β of τ so that whenever $x \in T$, T open, there is a $B \in \beta$ such that $x \in B \subseteq T$. The sets in β are called *basic* open sets.

In particular, the open balls in a metric space give rise to a topology called the *metric* topology. This is easily generalized to quasimetric and partial metric spaces as follows:

Definition 14. Given a quasimetric space (X, q), $x \in X$, and $\epsilon > 0$,

$$B^q_\epsilon(x) = \{y : q(x, y) < \epsilon\}$$

is the *open ball* with center x and radius ϵ.
For a partial metric p, we abbreviate $B^{q_p}_\epsilon(x)$ to $B^p_\epsilon(x)$.

Certainly by the above definitions, for a partial metric p,

$$B^p_\epsilon(x) = \{y : p(x, y) < p(x, x) + \epsilon\}.$$

The usual proof that the open balls in a metric space form a basis for a topology carries over, essentially unchanged, to any quasimetric space. This topology is denoted τ_q (and τ_{q_p} is abbreviated to τ_p). In particular, when (X, p) is a metric space then this is the usual open ball topology.

But there is an essential difference due to the lack of symmetry: one should wonder why we did not define $B^q_\epsilon(x) = \{y : q(y, x) < \epsilon\}$ (rather than $q(x, y) < \epsilon$), and obtain our topology from this basis instead. It turns out we must take into account both of the topologies just mentioned, and a third. We do this by considering, for any quasimetric q, its *dual* q^* and its *symmetrization* q^S, defined by $q^*(x, y) = q(y, x)$ and $q^S = q + q^*$. Then it is easily seen that q^* is a quasimetric and q^S is a metric (and the topology mentioned at the beginning of this paragraph is τ_{q^*}). Of course $q = q^*$ if and only if q is a metric, and in this case the topologies τ_q, τ_{q^*}, and τ_{q^S} are identical.

For a partial metric p, $(q_p)^*(x, y) = p(x, y) - p(y, y)$. We abbreviate q_p to p, $(q_p)^*$ to p^*, and $(q_p)^S$ to p^S (or d_p) in notations such as $B^{q_p}_\epsilon(x)$ and $\tau_{(q_p)^*}$. To discuss this array of topologies, we need:

Definition 15. A *bitopological space* is a triple (X, τ, τ^*) such that τ and τ^* are topologies.

Bitopological spaces were first introduced in [**3**]. A thorough discussion is in [**5**]. These spaces naturally occur when there is a lack of symmetry to be considered.

Each bitopological space (X, τ, τ^*) gives rise to a third topology important in the study of these spaces. It is $\tau^S = \tau \vee \tau^*$, the join of the topologies τ and τ^*, that is, the smallest topology which contains both of them. The topology τ^S often has symmetric properties, and is called the *symmetrization* topology. For example, it is easy to check that

$$B^q_{\epsilon/2}(x) \cap B^{q^*}_{\epsilon/2}(x) \subseteq B^{q^S}_\epsilon(x) \subseteq B^q_\epsilon(x) \cap B^{q^*}_\epsilon(x).$$

As a result, $\tau_q \vee \tau_{q^*}$ is τ_{q^S}, a metric topology.

As an example, for the partial metric space $(\mathbb{R}^+, max)$ discussed earlier, $B^{max}_\epsilon(x) = \{y : \max(x, y) < x + \epsilon\} = [0, x + \epsilon)$, so $\tau_{max} = \{[0, t) : 0 \leq t \leq \infty\}$, $B^{max^*}_\epsilon(x) = \{y : \max(x, y) < y + \epsilon\} = (x - \epsilon, \infty)$, so $\tau_{max^*} = \{(t, \infty) : 0 \leq t \leq \infty\} \cup \{[0, \infty)\}$, and thus τ_{max^S} is the usual real topology on this set.

The relation *x is part of y* is also understood topologically. We have seen it captured in terms of distance by $p(x, x) = p(x, y)$, or $q_p(x, y) = 0$, which leads to a poset formulation $x \sqsubseteq_p y$. Topologically, this is expressed by noting that $x \sqsubseteq_p y$ if and only if y is in each $B_\epsilon(x)$, which in turn holds if and only if $x \in \mathrm{cl}(\{y\})$. Indeed, for any topology τ, the relation $\sqsubseteq_\tau$, defined by $x \sqsubseteq_\tau y \iff x \in \mathrm{cl}(\{y\})$, is called its *specialization order*. This relation is always reflexive and transitive.

As a result of P2, (X, τ_p) is a T_0 *space*: $x \neq y$ if and only if there is an open set containing exactly one of x and y; equivalently, $x \in \text{cl}(\{y\})$ and $y \in \text{cl}(\{x\})$ only when $x = y$. Put differently, a topological space (X, τ) is T_0 if and only if its specialization order, $\sqsubseteq_\tau$, is a partial order.

In contrast, each metric topology τ is *Hausdorff*; that is, $x \neq y$ if and only if there are disjoint open sets O and O' such that x is in O and y is in O'. Note that in Hausdorff spaces, if $x \neq y$ then there is an open set O' such that $y \in O'$ and $x \notin O'$, so $y \not\sqsubseteq_\tau x$. Therefore $\sqsubseteq_\tau$ is equality, thus a symmetric relation.

So key properties of partial metric spaces and the reality they represent are implicit in their bitopological spaces, just as key properties of metric spaces are abstracted into topological spaces. Finally, we consider the idea of determining the end product of a computation (such as an infinite string, or a real number) as the result of a limit of its parts.

Definition 16. For a topological space (X, τ) a sequence $x = (x_n)$ of points in X *converges* to a point y in X if for each open set O containing y there exists k such that for each $n > k$, x_n is in O.

That is, (x_n) converges to y if it is eventually in any open set containing y. It is easy to check that given a basis β, (x_n) converges to y if and only if it is eventually in any basic open set containing y. Thus in particular, for a quasimetric space (X, q), $x_n \to y$ in τ_q if and only if, for each $\epsilon > 0$, eventually $q(y, x_n) < \epsilon$.

Our definition of partial metric convergence of a sequence (x_n) to a point y is that $\lim_{n\to\infty} p(x_n, y) = \lim_{n\to\infty} p(x_n, x_n) = p(y, y)$. This is equivalent to saying that $\lim_{n\to\infty} q_p(y, x_n) = 0 = \lim_{n\to\infty} q_p^*(y, x_n)$. This in turn happens if and only if for each $\epsilon > 0$, eventually $q_p(y, x_n) < \epsilon$ and eventually $q_p^*(y, x_n) < \epsilon$, that is, if and only if $x_n \to y$ with respect to both τ_p and τ_{p^*}, that is, if and only if $x_n \to y$ with respect to τ_{d_p}.

Thus in the case of the nonnegative reals with the partial metric max, our definition of partial metric convergence reduces to the usual real convergence. On the other hand, by the above, $\lim_{n\to\infty} q_{max}(y, x_n) = 0$ if and only if for each $\epsilon > 0$ eventually $x_n < y + \epsilon$, and that holds if and only if $y \geq \limsup(x_n)$. Similarly, $\lim_{n\to\infty} (q_{max})^*(y, x_n) = 0$ if and only if $y \leq \liminf(x_n)$.

Therefore, partial metric convergence in $(\mathbb{R}^+, max)$ is the usual real convergence, while its two parts are closely related to lim sup and lim inf. In general, a sequence converges with respect to p in the partial metric sense if and only if it converges with respect to d_p, which in turn holds if and only if it converges with respect to τ_{d_p}.

In the central motivating case of sequences over a set S, $x_n \to x$, where $x = (s_1, s_2, ...)$ and each x_n are in S^ω, if and only if for each positive integer k, the initial segment $(s_1, s_2, ..., s_k)$ is an initial segment of x_n for sufficiently large n. In the other key example of nonempty closed bounded intervals in $\mathbb{R}$, for any real number a, $\{a\} = [a, a] = \lim_{n\to\infty}[b_n, c_n]$ if and only if for each k, $[b_n, c_n] \subseteq [a - 1/k, a + 1/k]$ for sufficiently large n.

It is easy to check that in a weighted metric space $(X, d, |\cdot|)$, a sequence (x_n) converges to y in the associated partial metric if and only if the distances $d(x_n, y)$ converge to 0 and the weights $|x_n|$ converge to $|y|$.

Thus given a metric space (X, d) there is just one topology, but from a partial metric space (X, p), three related topologies can be identified (they are all equal if p is a metric). Also, a key partial order is represented by $(X, \sqsubseteq_p)$ ($\sqsubseteq_p$ is $=$ if p is a metric).

7. CONCLUSION. We have shown above how partial metrics model the sort of asymmetric convergence implicit in computing an object, as metric spaces model the more traditional symmetric spaces of analysis. Similarities, such as the involvement of topology, and dissimilarities, such as the need for an order and for bitopology in this newer case, have also been pointed out.

We close with a discussion of some possible uses of partial metric spaces. The partial metrics originally studied and discussed above are valued in $\mathbb{R}$, and this imposes countability issues that are irrelevant for our purposes (for partial metrics valued in $\mathbb{R}$ and $x \in X$, $\{B_{1/n}(x) : n = 1, 2, 3, ...\}$ is a countable base for the neighborhoods of x in τ_p, and similar issues arise for τ_{p^*} and τ_{d_p}). To avoid this, we allow our partial metrics and quasimetrics below to be valued elsewhere, such as in sets of the form $[0, \infty]^I$.

But other problems must be overcome. We get our topology by saying that a set T is open when for each $x \in T$ there is some $r > 0$ such that $\{y : q_p(x, y) < r\} \subseteq T$; equivalently, when for each $x \in T$ there is some $r > 0$ such that $N_r(x) = \{y : q_p(x, y) \leq r\} \subseteq T$. It turns out that in some of the spaces we want to study, there are pairs $r, s > 0$ such that $\inf\{r, s\} = 0$. Four properties of the set $G = (0, \infty)$ of positive reals are centrally important in the use of metrics:

(a) if $r \in G$ and $r \leq s$ then $s \in G$,

(b) if $r, s \in G$ then for some $t \in G$, $t \leq r$ and $t \leq s$,

(c) if $r \in G$ then for some $t \in G$, $t + t \leq r$,

(d) for each a, b, if $a \leq b + r$ for each $r \in G$, then $a \leq b$.

As a result, we define a *set of positives, G,* to have these properties which we often use. Then $\tau_{q,G}$ is defined to be the topology whose open sets are those $T \subseteq X$ such that for each $x \in T$, $N_r(x) \subseteq T$ for some $r \in G$. Details are given in an earlier MONTHLY article [**4**], and in [**6**]. Note for example that if I has at least two elements, then $\{r > 0 : r \in [0, \infty]^I\}$ fails to satisfy (b) above, so it is not a set of positives. A particularly useful set of positives in $[0, \infty]^I$ is $\{r \in (0, \infty]^I : \{i : r(i) \neq \infty\}$ is finite$\}$.

Now we give some examples to show variety in the kinds of concepts that can be modeled by such partial metrics.

Partial metrics were designed to discuss computer programs, and our first example comes from this area. A type of poset $(X, \sqsubseteq)$ termed a *domain* has been defined to model computation. We now give part of this definition; much more can be learned in [**1**]:

A *directed complete partially ordered set (dcpo)* is a poset, $(P, \leq)$, in which each directed subset S has a supremum $\bigvee S$ (recall that a set S is directed by an order $\leq$ if for each $r, s \in S$ there is a $t \in S$ such that $r \leq t$ and $s \leq t$). For any poset $(P, \leq)$, the *way-below relation* $\ll$ is defined by $b \ll a$ if whenever $a \leq \bigvee D$, D directed, then $b \leq d$ for some $d \in D$. A dcpo is *continuous* if for each $a \in P$, $\{b : b \ll a\}$ is a directed set and $a = \bigvee\{b : b \ll a\}$.

The above axioms are best understood by considering the elements of P as sets of accumulated knowledge, and interpreting $a \leq b$ to mean that the knowledge represented by b implies all the knowledge represented by a. Then $(P, \leq)$ is a dcpo if, whenever a directed set of knowledge is accumulated, then there is an element which represents this knowledge. The example of finite and infinite sequences (discussed in Sections 2 and 3) is a continuous poset; in it, the directed union of a set of sequences is the sequence which has precisely the information held by the elements of the set. The example of the closed bounded intervals (also in Section 3) is also a continuous poset; here the knowledge that a point is in each of a collection of such intervals is given by the fact that it is in their intersection, which is also a closed bounded interval. These, like all continuous posets, are natural examples of spaces in which information is gathered.

For sequences, $b \ll a$ if and only if b is a finite (initial) subsequence of a, and a is clearly the supremum of $\{b : b \ll a\}$, so this example (which abstracts the Turing machine) is a continuous dcpo. For the closed bounded intervals, it can be seen that $[u, v] \ll [x, y]$ if and only if $[x, y] \subseteq (u, v)$, and therefore that $\{[u, v] : [u, v] \ll [x, y]\}$ is directed by $\supseteq$ and $[x, y]$ is its supremum, so this poset is also a continuous dcpo.

Given a poset, its *Scott topology,* σ, is the one whose closed sets are the lower sets which contain the suprema of their directed subsets. That is, a set C is Scott closed if whenever $x \leq y \in C$ then $x \in C$, and whenever $D \subseteq C$ is directed then $\bigvee D \in C$ (assuming $\bigvee D$ exists, as it must for a dcpo).

The Scott topology is seen to be appropriate by thinking of $\leq$ as the "knowledge order", with $x \geq y$ meaning that x implies y. Then it is natural to think that a set is closed if it contains all objects implied by each of its elements, and whenever it contains increasing amounts of knowledge, it contains an object that implies all this knowledge. For each $x \in P$, the smallest closed set containing x is $\{y : y \leq x\}$; thus the poset order is the specialization order of the Scott topology, and so this topology can only arise from a metric if $\leq$ is equality.

Due to the lack of symmetry embodied in $\leq$, it is useful to consider a second topology, and the one most often used is the *lower topology,* ω, whose closed sets are generated by the sets of the form $\{y : y \geq x\}$ for $x \in P$.

In [**6**] it is shown that for each continuous dcpo, there is a partial metric into a power of the unit interval, $[0, 1]^I$, such that τ_p is the Scott topology and τ_{p^*} is the lower topology. Thus the poset order is the specialization order, so in particular $(P, \leq) = (P, \leq_p)$.

But many other bitopological spaces can be so represented (to be precise, the ones that so arise are the pairwise Tychonoff spaces; see [**6**]). It is unclear whether a reasonable characterization of continuous dcpo's can be found in terms of partial metrics.

More traditional examples are found by looking at topologies on $\mathbb{R}^X$, the real valued functions on a set X. The best known of these is the topology of uniform convergence, given by the metric $d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in X\}$. The partial metric $p_\infty(f, g) = \sup\{\max(f(x), g(x)) : x \in X\}$ gives rise to this topology, since $d_{p_\infty}(f, g) = \|f - g\|_\infty$, the sup norm distance between f and g. By earlier discussion, this splits the topology of uniform convergence into two subtopologies: τ_{p_∞}, its lower open sets, and $\tau_{(p_\infty)^*}$, its upper open sets.

In fact, each topology on each set X arises from a *pseudo-partial metric*: a function $p : X \times X \to H$, where H is some abelian lattice ordered group, satisfying all the partial metric axioms except P2 (if $p(x, x) = p(x, y) = p(y, y)$ then $x = y$), together with a set of positives $G \subseteq H$; the T_0 topologies are those for which P2 also holds. This is shown using the discussions in [**6**] and [**4**] and viewing $[0, 1]^I$ as a subset of the lattice ordered abelian group $\mathbb{R}^I$.

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