The Haskell School of Music

— From Signals to Symphonies —

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Version 2.4 (February 22, 2012)

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Cover image: *Euterpe*, the Greek Muse of Music
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Preface

In the year 2000 I wrote a book called *The Haskell School of Expression – Learning Functional Programming through Multimedia* [Hud00]. In that book I used graphics, animation, music, and robotics as a way to motivate learning how to program, and specifically how to learn *functional programming* using Haskell, a purely functional programming language. Haskell [P+03] is quite a bit different from conventional imperative or object-oriented languages such as C, C++, Java, C#, and so on. It takes a different mind-set to program in such a language, and appeals to the mathematically inclined and to those who seek purity and elegance in their programs. Although Haskell was designed over twenty years ago, it has only recently begun to catch on in a significant way, not just because of its purity and elegance, but because with it you can solve real-world problems quickly and efficiently, and with great economy of code.

I have also had a long, informal, yet passionate interest in music, being an amateur jazz pianist and having played in several bands over the years. About fifteen years ago, in an effort to combine work with play, I and my students wrote a Haskell library called *Haskore* for expressing high-level computer music concepts in a purely functional way [HMGW96, Hud96, Hud03]. Indeed, three of the chapters in *The Haskell School of Expression* summarize the basic ideas of this work. Soon after that, with the help of another student, Matt Zamec, I designed a Haskell library called *HasSound* that was, essentially, a Haskell interface to *csound* [Ver86] for doing sound synthesis and instrument design.

Thus, when I recently became responsible for the Music Track in the new *Computing and the Arts* major at Yale, and became responsible for teaching not one, but two computer music courses in the new curriculum, it was natural to base the course material on Haskell. This current book is a rewrite of *The Haskell School of Expression* with a focus on computer music, based on, and greatly improving upon, the ideas in Haskore and HasSound.

The new Haskell library that incorporates all of this is called *Euterpea*.

Haskell was named after the logician Haskell B. Curry who, along with Alonzo Church, helped establish the theoretical foundations of functional programming in the 1940's, when digital computers were mostly just a gleam in researchers' eyes. A curious historical fact is that Haskell Curry's father, Samuel Silas Curry, helped found and direct a school in Boston called the *School of Expression*. (This school eventually evolved into what is now *Curry College*.) Since pure functional programming is centered around the notion of an *expression*, I thought that *The Haskell School of Expression* would be a good title for my first book. And it was thus quite natural to choose *The Haskell School of Music* for my second!

How To Read This Book

As mentioned earlier, there is a certain mind-set, a certain viewpoint of the world, and a certain approach to problem solving that collectively work best when programming in Haskell (this is true for any programming paradigm). If you teach only Haskell language details to a C programmer, he or she is likely to write ugly, incomprehensible functional programs. But if you teach how to think differently, how to see problems in a different light, functional solutions will come easily, and elegant Haskell programs will result. As Samuel Silas Curry once said:

> All expression comes *from within outward*, from the center to the surface, from a hidden source to outward manifestation. The study of expression as a natural process brings you into contact with cause and makes you feel the source of reality.

What is especially beautiful about this quote is that music is also a form of expression, although Curry was more likely talking about writing and speech. In addition, as has been noted by many, music has many ties to mathematics. So for me, combining the elegant mathematical nature of Haskell with that of music is as natural as singing a nursery tune.

Using a high-level language to express musical ideas is, of course, not new. But Haskell is unique in its insistence on purity (no side effects), and this alone makes it particularly suitable for expressing musical ideas. By focusing on *what* a musical entity is rather than on *how* to create it, we allow musical ideas to take their natural form as Haskell expressions. Haskell's many abstraction mechanisms allow us to write computer music programs

that are elegant, concise, yet powerful. We will consistently attempt to let the music express itself as naturally as possible, without encoding it in terms of irrelevant language details.

Of course, my ultimate goal is not just to teach computer music concepts. Along the way you will also learn Haskell. There is no limit to what one might wish to do with computer music, and therefore the better you are at programming, the more success you will have. This is why I think that many languages designed specifically for computer music—although fun to work with, easy to use, and cute in concept—face the danger of being too limited in expressiveness.

You do not need to know much, if any, music theory to read this book, and you do not need to play an instrument. Of course, the more you know about music, the more you will be able to apply the concepts learned in this text in musically creative ways.

My general approach to introducing computer music concepts is to first provide an intuitive explanation, then a mathematically rigorous definition, and finally fully executable Haskell code. In the process I introduce Haskell features as they are needed, rather than all at once. I believe that this interleaving of concepts and applications makes the material easier to digest.

Another characteristic of my approach is that I do not hide any details—I want Euterpea to be as transparent as possible! There are no magical built-in operations, no special computer music commands or values. This works out well for several reasons. First, there is in fact nothing ugly or difficult to hide—so why hide anything at all? Second, by reading the code, you will better and more quickly understand Haskell. Finally, by stepping through the design process with me, you may decide that you prefer a different approach—there is, after all, no One True Way to express computer music ideas. I expect that this process will position you well to write rich, creative musical applications on your own.

I encourage the seasoned programmer having experience only with conventional imperative and/or object-oriented languages to read this text with an open mind. Many things will be different, and will likely feel awkward. There will be a tendency to rely on old habits when writing new programs, and to ignore suggestions about how to approach things differently. If you can manage to resist those tendencies I am confident that you will have an enjoyable learning experience. Those who succeed in this process often find that many ideas about functional programming can be applied to imperative and object-oriented languages as well, and that their imperative coding

style changes for the better.

I also ask the experienced programmer to be patient while in the earlier chapters I explain things like "syntax," "operator precedence," etc., since it is my goal that this text should be readable by someone having only modest prior programming experience. With patience the more advanced ideas will appear soon enough.

If you are a novice programmer, I suggest taking your time with the book; work through the exercises, and don't rush things. If, however, you don't fully grasp an idea, feel free to move on, but try to re-read difficult material at a later time when you have seen more examples of the concepts in action. For the most part this is a "show by example" textbook, and you should try to execute as many of the programs in this text as you can, as well as every program that you write. Learn-by-doing is the corollary to show-by-example.

Finally, I note that some section titles are prefaced with the parenthetical phrase, "[**Advanced**]". These sections may be skipped upon first reading, especially if the focus is on learning computer music concepts, as opposed to programming concepts.

Haskell Implementations

There are several good implementations of Haskell, all available free on the Internet through the Haskell users' website at `http://haskell.org`. One that I especially recommend is *GHC*, an easy-to-use and easy-to-install Haskell compiler and interpreter (see `http://haskell.org/ghc`). GHC runs on a variety of platforms, including PC's (Windows 7, XP, and Vista), various flavors of Unix (Linux, FreeBSD, etc.), and Mac OS X. The preferred way to install GHC is through the *Haskell Platform* (`http://hackage.haskell.org/platform/`). Any text editor can be used to create source files, but I prefer to use emacs (see `http://www.gnu.org/software/emacs`), along with its Haskell mode (see `http://projects.haskell.org/haskellmode-emacs/`). The entire Euterpea library is available on the community Haskell server, including all of the source code from this textbook. Instructions on how to install Euterpea can be found at `http://haskell.cs.yale.edu`. Feel free to email me at `mailto:paul.hudak@yale.edu` with any comments, suggestions, or questions.

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Tom Makucevich, a talented computer music practitioner and composer in New Haven, was the original motivator, and first user, of Haskore, which preceded Euterpea. Watching him toil endlessly with low-level csound programs was simply too much for me to bear! Several undergraduate students at Yale contributed to the original design and implementation of Haskore. I would like to thank in particular the contributions of Syam Gadde and Bo Whong, who co-authored the original paper on Haskore. Additionally, Matt Zamec helped me greatly in the creation of HasSound.

I wish to thank my more recent graduate students, in particular Hai (Paul) Liu, Eric Cheng, Donya Quick, and Daniel Winograd-Cort for their help in writing much of the code that constitutes the current Euterpea library. In addition, many students in my computer music classes at Yale provided valuable feedback through earlier drafts of the manuscript.

Finally, I wish to thank my wife, Cathy Van Dyke, my best friend and ardent supporter, whose love, patience, and understanding have helped me get through some bad times, and enjoy the good.

Happy Haskell Music Making!

Paul Hudak
New Haven
January 2012

Chapter 1

Overview of Computer Music, Euterpea, and Haskell

Computers are everywhere. And so is music! Although some might think of the two as being at best distant relatives, in fact they share many deep properties. Music comes from the soul, and is inspired by the heart, yet it has the mathematical rigor of computers. Computers have mathematical rigor of course, yet the most creative ideas in mathematics and computer science come from the soul, just like music. Both disciplines demand both left-brain and right-brain skills. It always surprises me how many computer scientists and mathematicians have a serious interest in music. It seems that those with a strong affinity or acuity in one of these disciplines is often strong in the other as well.

It is quite natural then to consider how the two might interact. In fact there is a long history of interactions between music and mathematics, dating back to the Greeks' construction of musical scales based on arithmetic relationships, and including many classical composers use of mathematical structures, the formal harmonic analysis of music, and many modern music composition techniques. Advanced music theory uses ideas from diverse branches of mathematics such as number theory, abstract algebra, topology, category theory, calculus, and so on.

There is also a long history of efforts to combine computers and music. Most consumer electronics today are digital, as are most forms of audio processing and recording. But in addition, digital musical instruments provide new modes of expression, notation software and sequencers have become standard tools for the working musician, and those with the most computer

science savvy use computers to explore new modes of composition, transformation, performance, and analysis.

This textbook explores the fundamentals of computer music using a language-centric approach. In particular, the functional programming language *Haskell* is used to express all of the computer music concepts. Thus a by-product of learning computer music concepts will be learning how to program in Haskell. The core musical ideas are collected into a Haskell library called *Euterpea*. The name "Euterpea" is derived from "Euterpe," who was one of the nine Greek muses, or goddesses of the arts, specifically the muse of music. A hypothetical picture of Euterpe graces the cover of this textbook.

1.1 The Note vs. Signal Dichotomy

The field of computer music has grown astronomically over the past several decades, and the material can be structured and organized along several dimensions. A dimension that proves particulary useful with respect to a programming language is one that separates *high-level* musical concerns from *low-level* musical concerns. Since a "high-level" programming language—namely Haskell—is used to program at both of these musical levels, to avoid confusion the terms *note level* and *signal level* will be used in the musical dimension.

At the *note level*, a *note* (i.e. pitch and duration) is the lowest musical entity that is considered, and everything else is built up from there. At this level, in addition to conventional representations of music, one can study interesting aspects of so-called *algorithmic composition*, including the use of fractals, grammar-based systems, stochastic processes, and so on. From this basis one can also study the harmonic and rhythmic *analysis* of music, although that is not currently an emphasis in this textbook. Haskell facilitates programming at this level through its powerful data abstraction facilities, higher-order functions, and declarative semantics.

In contrast, at the *signal level* the focus is on the actual sound generated in a computer music application, and thus a *signal* is the lowest entity that is considered. Sound is concretely represented in a digital computer by a discrete sampling of the continuous audio signal, at a high enough rate that human ears cannot distinguish the discrete from the continuous, usually 44,100 samples per second (the standard sampling rate used for CDs, mp3 files, and so on). But in Euterpea, these details are hidden: signals are

treated abstractly as continuous quantities. This greatly eases the burden of programming with sequences of discrete values. At the signal level, one can study sound synthesis techniques (to simulate the sound of a conventional instrument, say, or something completely artificial), audio processing (e.g. determining the frequency spectrum of a signal), and special effects (reverb, panning, distortion, and so on).

Suppose for a moment that one is playing music using a metronome set at 96, which corresponds to 96 beats per minute. That means that one beat takes $60/96 = 0.625$ seconds. At a stereo sampling rate of 44,100 samples per second, that in turn translates into $2 \times 0.625 \times 44,100 = 55{,}125$ samples, and each sample typically occupies several bytes of computer memory. This is typical of the minimum memory requirements of a computation at the signal level. In contrast, at the note level, one only needs some kind of operator or data structure that says "play this note," which requires a total of only a small handful of bytes. This dramatic difference highlights one of the key computational differences between programming at the note level versus the signal level.

Of course, many computer music applications involve both the note level *and* the signal level, and indeed there needs to be a mechanism to mediate between the two. Although such mediation can take many forms, it is for the most part straightforward. Which is another reason why the distinction between the note level and the signal level is so natural.

This textbook begins with a treatment of the note level (Chapters 1-16) and follows with a treatment of the signal level (Chapters 17-20). If the reader is interested only in the signal level, one could skip Chapters 8-16.

1.2 Basic Principles of Programming

Programming, in its broadest sense, is *problem solving*. It begins by recognizing problems that can and should be solved using a digital computer. Thus the first step in programming is answering the question, "What problem am I trying to solve?"

Once the problem is understood, a solution must be found. This may not be easy, of course, and in fact one may discover several solutions, so a way to measure success is needed. There are various dimensions in which to do this, including correctness ("Will I get the right answer?") and efficiency ("Will it run fast enough, or use too much memory?"). But the distinction of which solution is better is not always clear, since the number of dimensions

can be large, and programs will often excel in one dimension and do poorly in others. For example, there may be one solution that is fastest, one that uses the least amount of memory, and one that is easiest to understand. Deciding which to choose can be difficult, and is one of the more interesting challenges in programming.

The last measure of success mentioned above—clarity of a program—is somewhat elusive: difficult to quantify and measure. Nevertheless, in large software systems clarity is an especially important goal, since such systems are worked on by many people over long periods of time, and evolve considerably as they mature. Having easy-to-understand code makes it much easier to modify.

In the area of computer music, there is another reason why clarity is important: namely, that the code often represents the author's thought process, musical intent, and artistic choices. A conventional musical score does not say much about what the composer thought as she wrote the music, but a program often does. So when you write your programs, write them for others to see, and aim for elegance and beauty, just like the musical result that you desire.

Programming is itself a creative process. Sometimes programming solutions (or artistic creations) come to mind all at once, with little effort. More often, however, they are discovered only after lots of hard work! One may write a program, modify it, throw it away and start over, give up, start again, and so on. It's important to realize that such hard work and reworking of programs is the norm, and in fact you are encouraged to get into the habit of doing so. Don't always be satisfied with your first solution, and always be prepared to go back and change or even throw away those parts of your program that you're not happy with.

1.3 Computation by Calculation

It's helpful when learning a new programming language to have a good grasp of how programs in that language are executed—in other words, an understanding of what a program *means*. The execution of Haskell programs is perhaps best understood as *computation by calculation*. Programs in Haskell can be viewed as *functions* whose input is that of the problem being solved, and whose output is the desired result—and the behavior of functions can be effectively understood as computation by calculation.

An example involving numbers might help to demonstrate these ideas.

Numbers are used in many applications, and computer music is no exception. For example, integers might be used to represent pitch, and floating-point numbers might be used to perform calculations involving frequency or amplitude.

Suppose onne wishes to perform an arithmetic calculation such as $3 \times (9 + 5)$. In Haskell this would be written as $3 * (9 + 5)$, since most standard computer keyboards and text editors do not recognize the special symbol $\times$. The result can be calculated as follows:

$$\begin{aligned} & 3 * (9 + 5) \\ \Rightarrow\ & 3 * 14 \\ \Rightarrow\ & 42 \end{aligned}$$

It turns out that this is not the only way to compute the result, as evidenced by this alternative calculation:[1]

$$\begin{aligned} & 3 * (9 + 5) \\ \Rightarrow\ & 3 * 9 + 3 * 5 \\ \Rightarrow\ & 27 + 3 * 5 \\ \Rightarrow\ & 27 + 15 \\ \Rightarrow\ & 42 \end{aligned}$$

Even though this calculation takes two extra steps, it at least gives the same, correct answer. Indeed, an important property of each and every program written in Haskell is that it will always yield the same answer when given the same inputs, regardless of the order chosen to perform the calculations.[2] This is precisely the mathematical definition of a *function*: for the same inputs, it always yields the same output.

On the other hand, the first calculation above required fewer steps than the second, and thus it is said to be more *efficient*. Efficiency in both space (amount of memory used) and time (number of steps executed) is important when searching for solutions to problems. Of course, if the computation returns the wrong answer, efficiency is a moot point. In general it is best to search first for an elegant (and correct!) solution to a problem, and later refine it for better performance. This strategy is sometimes summarized as, "Get it right first!"

The above calculations are fairly trivial, but much more sophisticated computations will be introduced soon enough. For starters—and to intro-

[1] This assumes that multiplication distributes over addition in the number system being used, a point that will be returned to later in the text.

[2] This is true as long as a non-terminating sequence of calculations is not chosen, another issue that will be addressed later.

duce the idea of a Haskell function—the arithmetic operations performed in the previous example can be *generalized* by defining a function to perform them for any numbers x, y, and z:

$$simple\ x\ y\ z = x * (y + z)$$

This equation defines *simple* as a function of three *arguments*, x, y, and z. In mathematical notation this definition might be written differently, such as one of the following:

$$simple(x, y, z) = x \times (y + z)$$
$$simple(x, y, z) = x \cdot (y + z)$$
$$simple(x, y, z) = x(y + z)$$

In any case, it should be clear that "*simple* 3 9 5" is the same as "$3*(9+5)$." In fact the proper way to calculate the result is:

$$\begin{aligned} &simple\ 3\ 9\ 5 \\ &\Rightarrow 3 * (9 + 5) \\ &\Rightarrow 3 * 14 \\ &\Rightarrow 42 \end{aligned}$$

The first step in this calculation is an example of *unfolding* a function definition: 3 is substituted for x, 9 for y, and 5 for z on the right-hand side of the definition of *simple*. This is an entirely mechanical process, not unlike what the computer actually does to execute the program.

simple 3 9 5 is said to *evaluate* to 42. To express the fact that an expression e evaluates (via zero, one, or possibly many more steps) to the value v, one writes $e \Longrightarrow v$ (this arrow is longer than that used earlier). So one can say directly, for example, that $simple\ 3\ 9\ 5 \Longrightarrow 42$, which should be read "*simple* 3 9 5 evaluates to 42."

With *simple* now suitably defined, one can repeat the sequence of arithmetic calculations as often as one likes, using different values for the arguments to *simple*. For example, $simple\ 4\ 3\ 2 \Longrightarrow 20$.

One can also use calculation to *prove properties* about programs. For example, it should be clear that for any a, b, and c, $simple\ a\ b\ c$ should yield the same result as $simple\ a\ c\ b$. For a proof of this, one calculates *symbolically*; that is, using the symbols a, b, and c rather than concrete numbers such as 3, 5, and 9:

$$\begin{aligned} &simple\ a\ b\ c \\ &\Rightarrow a * (b + c) \\ &\Rightarrow a * (c + b) \\ &\Rightarrow simple\ a\ c\ b \end{aligned}$$

Note that the same notation is used for these symbolic steps as for concrete ones. In particular, the arrow in the notation reflects the direction of formal reasoning, and nothing more. In general, if $e_1 \Rightarrow e_2$, then it's also true that $e_2 \Rightarrow e_1$.

These symbolic steps are also referred to as as "calculations," even though the computer will not typically perform them when executing a program (although it might perform them *before* a program is run if it thinks that it might make the program run faster). The second step in the calculation above relies on the commutativity of addition (namely that, for any numbers x and y, $x + y = y + x$). The third step is the reverse of an unfold step, and is appropriately called a *fold* calculation. It would be particularly strange if a computer performed this step while executing a program, since it does not seem to be headed toward a final answer. But for proving properties about programs, such "backward reasoning" is quite important.

When one wishes to make the justification for each step clearer, whether symbolic or concrete, a calculation can be annotated with more detail, as in:

$$
\begin{aligned}
&\mathit{simple}\ a\ b\ c \\
&\Rightarrow \{\, \mathit{unfold} \,\} \\
&a * (b + c) \\
&\Rightarrow \{\, \mathit{commutativity} \,\} \\
&a * (c + b) \\
&\Rightarrow \{\, \mathit{fold} \,\} \\
&\mathit{simple}\ a\ c\ b
\end{aligned}
$$

In most cases, however, this will not be necessary.

Proving properties of programs is another theme that will be repeated often in this text. Computer music applications often have some kind of a mathematical basis, and that mathematics must be reflected somewhere in your program. But how do you know that you got it right? Proof by calculation is one way to connect the problem specification with the program solution.

More broadly speaking, as the world begins to rely more and more on computers to accomplish not just ordinary tasks such as writing term papers, sending email, and social networking, but also life-critical tasks such as controlling medical procedures and guiding spacecraft, then the correctness of programs gains in importance. Proving complex properties of large, complex programs is not easy—and rarely if ever done in practice—but that should not deter you from proving simpler properties of the whole system,

or complex properties of parts of the system, since such proofs may uncover errors, and if not, at least give you confidence in your effort.

If you are someone who is already an experienced programmer, the idea of computing *everything* by calculation may seem odd at best, and naïve at worst. How does one write to a file, play a sound, draw a picture, or respond to mouse-clicks? If you are wondering about these things, it is hoped that you have patience reading the early chapters, and that you find delight in reading the later chapters where the full power of this approach begins to shine.

In many ways this first chapter is the most difficult, since it contains the highest density of new concepts. If the reader has trouble with some of the concepts in this overview chapter, keep in mind that most of them will be revisited in later chapters. And don't hesitate to return to this chapter later to re-read difficult sections; they will likely be much easier to grasp at that time.

Details: In the remainder of this textbook the need will often arise to explain some aspect of Haskell in more detail, without distracting too much from the primary line of discourse. In those circumstances the explanations will be offset in a box such as this one, proceeded with the word "Details."

Exercise 1.1 Write out all of the steps in the calculation of the value of

simple (*simple* 2 3 4) 5 6

Exercise 1.2 Prove by calculation that *simple* $(a - b)\ a\ b \Longrightarrow a^2 - b^2$.

1.4 Expressions and Values

In Haskell, the entities on which calculations are performed are called *expressions*, and the entities that result from a calculation—i.e. "the answers"—are called *values*. It is helpful to think of a value just as an expression on which no more calculation can be carried out—every value is an expression, but not the other way around.

Examples of expressions include *atomic* (meaning, indivisible) values such as the integer 42 and the character `'a'`, which are examples of two *primitive* atomic values. The next chapter introduces examples of *user-defined* atomic values, such as the musical note names C, Cs, Df, etc., which in music notation are written C, C♯, D♭, etc. (In music theory, note names are called *pitch classes*.)

In addition, there are *structured* expressions (i.e., made from smaller pieces) such as the *list* of pitches $[C, Cs, Df]$, the character/number *pair* $(\texttt{'b'}, 4)$ (lists and pairs are different in a subtle way, to be described later), and the string `"Euterpea"`. Each of these structured expressions is also a value, since by themselves there is no further calculation that can be carried out. As another example, $1 + 2$ is an expression, and one step of calculation yields the expression 3, which is a value, since no more calculations can be performed. As a final example, as was expained earlier, the expression *simple* 3 9 5 evaluates to the value 42.

Sometimes, however, an expression has only a never-ending sequence of calculations. For example, if x is defined as:

$$x = x + 1$$

then here is what happens when trying to calculate the value of x:

$$\begin{aligned} & x \\ & \Rightarrow x + 1 \\ & \Rightarrow (x + 1) + 1 \\ & \Rightarrow ((x + 1) + 1) + 1 \\ & \Rightarrow (((x + 1) + 1) + 1) + 1 \\ & \ldots \end{aligned}$$

Similarly, if a function f is defined as:

$$f\ x = f\ (x - 1)$$

then an expression such as $f\ 42$ runs into a similar problem:

$$\begin{aligned} & f\ 42 \\ & \Rightarrow f\ 41 \\ & \Rightarrow f\ 40 \\ & \Rightarrow f\ 39 \\ & \ldots \end{aligned}$$

Both of these clearly result in a never-ending sequence of calculations. Such expressions are said to not terminate, or *diverge*. In such cases the symbol $\perp$, pronounced "bottom," is used to denote the value of the expression. This means that every diverging computation in Haskell denotes the same

$\perp$ value,[3] reflecting the fact that, from an observer's point of view, there is nothing to distinguish one diverging computation from another.

1.5 Types

Every expression (and therefore every value) also has an associated *type*. One can think of types as sets of expressions (or values), in which members of the same set have much in common. Examples include the primitive atomic types *Integer* (the set of all integers) and *Char* (the set of all characters), the user-defined atomic type *PitchClass* (the set of all pitch classes, i.e. note names), as well as the structured types $[Integer]$ and $[PitchClass]$ (the sets of all lists of integers and lists of pitch classes, respectively), and *String* (the set of all Haskell strings).

The association of an expression or value with its type is very important, and there is a special way of expressing it in Haskell. Using the examples of values and types above:

Cs	$:: PitchClass$
42	$:: Integer$
`'a'`	$:: Char$
`"Euterpea"`	$:: String$
$[C, Cs, Df]$	$:: [PitchClass]$
`('b'`$, 4)$	$:: (Char, Integer)$

Each association of an expression with its type is called a *type signature*.

Details: Note that the names of specific types are capitalized, such as *Integer* and *Char*, but the names of values are not, such as *simple* and x. This is not just a convention: it is required when programming in Haskell. In addition, the case of the other characters matters, too. For example, *test*, *teSt*, and *tEST* are all distinct names for values, as are *Test*, *TeST*, and *TEST* for types.

[3] Technically, each type has its own version of $\perp$.

Details: Literal characters are written enclosed in single forward quotes (apostrophes), as in `'a'`, `'A'`, `'b'`, `','`, `'!'`, `' '` (a space), and so on. (There are some exceptions, however; see the Haskell Report for details.) Strings are written enclosed in double quote characters, as in `"Euterpea"` above. The connection between characters and strings will be explained in a later chapter.

The "::" should be read "has type," as in "42 has type *Integer*." Note that square braces are used both to construct a list value (the left-hand side of (::) above), and to describe its type (the right-hand side above). Analogously, the round braces used for pairs are used in the same way. But also note that all of the elements in a list, however long, must have the same type, whereas the elements of a pair can have different types.

Haskell's *type system* ensures that Haskell programs are *well-typed*; that is, that the programmer has not mismatched types in some way. For example, it does not make much sense to add together two characters, so the expression `'a'+'b'` is *ill-typed*. The best news is that Haskell's type system will tell you if your program is well-typed *before you run it*. This is a big advantage, since most programming errors are manifested as type errors.

1.6 Function Types and Type Signatures

What should the type of a function be? It seems that it should at least convey the fact that a function takes values of one type—T_1, say—as input, and returns values of (possibly) some other type—T_2, say—as output. In Haskell this is written $T_1 \rightarrow T_2$, and such a function is said to "map values of type T_1 to values of type T_2." If there is more than one argument, the notation is extended with more arrows. For example, if the intent is that the function *simple* defined in the previous section has type $Integer \rightarrow Integer \rightarrow Integer \rightarrow Integer$, one can include a type signature with the definition of *simple*:

$$\begin{aligned} &simple && :: Integer \rightarrow Integer \rightarrow Integer \rightarrow Integer \\ &simple\ x\ y\ z && = x * (y + z) \end{aligned}$$

Details: When writing Haskell programs using a typical text editor, there typically will not be nice fonts and arrows as in $Integer \rightarrow Integer$. Rather, you will have to type `Integer -> Integer`.

Haskell's type system also ensures that user-supplied type signatures such as this one are correct. Actually, Haskell's type system is powerful enough to allow one to avoid writing any type signatures at all, in which case the type system is said to *infer* the correct types.[4] Nevertheless, judicious placement of type signatures, as was done for *simple*, is a good habit, since type signatures are an effective form of documentation and help bring programming errors to light. In fact, it is a good habit to first write down the type of each function you are planning to define, as a first approximation to its full specification—a way to grasp its overall functionality before delving into its details.

The normal use of a function is referred to as *function application*. For example, *simple* 3 9 5 is the application of the function *simple* to the arguments 3, 9, and 5. Some functions, such as (+), are applied using what is known as *infix syntax*; that is, the function is written between the two arguments rather than in front of them (compare $x + y$ to $f\ x\ y$).

Details: Infix functions are often called *operators*, and are distinguished by the fact that they do not contain any numbers or letters of the alphabet. Thus ^! and $*\#$: are infix operators, whereas $thisIsAFunction$ and $f9g$ are not (but are still valid names for functions or other values). The only exception to this is that the symbol ' is considered to be alphanumeric; thus f' and $one's$ are valid names, but not operators.

In Haskell, when referring to an infix operator as a value, it is enclosed in parentheses, such as when declaring its type, as in:

$$(+) :: Integer \to Integer \to Integer$$

Also, when trying to understand an expression such as $f\ \ x + g\ \ y$, there is a simple rule to remember: function application *always* has "higher precedence" than operator application, so that $f\ x + g\ y$ is the same as $(f\ x) + (g\ y)$.

Despite all of these syntactic differences, however, operators are still just functions.

Exercise 1.3 Identify the well-typed expressions in the following, and, for each, give its proper type:

[4] There are a few exceptions to this rule, and in the case of *simple* the inferred type is actually a bit more general than that written above. Both of these points will be returned to later.

$[(2,3),(4,5)]$
$[Cs, 42]$
$(Df, -42)$
simple `'a' 'b' 'c'`
(*simple* 1 2 3, *simple*)
`["hello","world"]`

1.7 Abstraction, Abstraction, Abstraction

The title of this section is the answer to the question: "What are the three most important ideas in programming?" Webster defines the verb "abstract" as follows:

> **abstract**, *vt* (1) remove, separate (2) to consider apart from application to a particular instance.

In programming this is done when a repeating pattern of some sort occurs, and one wishes to "separate" that pattern from the "particular instances" in which it appears. In this textbook this process is called the *abstraction principle*. The following sections introduce several different kinds of abstraction, using examples involving both simple numbers and arithmetic (things everyone should be familiar with) as well as musical examples (that are specific to Euterpea).

1.7.1 Naming

One of the most basic ideas in programming—for that matter, in every day life—is to *name* things. For example, one may wish to give a name to the value of π, since it is inconvenient to retype (or remember) the value of π beyond a small number of digits. In mathematics the greek letter π in fact *is* the name for this value, but unfortunately one doesn't have the luxury of using greek letters on standard computer keyboards and text editors. So in Haskell one writes:

$pi :: Double$
$pi = 3.141592653589793$

to associate the name *pi* with the number 3.141592653589793. The type signature in the first line declares *pi* to be a double-precision *floating-point*

number, which mathematically and in Haskell is distinct from an integer.[5] Now the name *pi* can be used in expressions whenever it is in scope; it is an abstract representation, if you will, of the number 3.141592653589793. Furthermore, if there is ever a need to change a named value (which hopefully won't ever happen for *pi*, but could certainly happen for other values), one would only have to change it in one place, instead of in the possibly large number of places where it is used.

For a simple musical example, note first that in music theory, a *pitch* consists of a *pitch class* and an *octave*. For example, in Euterpea one simply writes $(A, 4)$ to represent the pitch class A in the fourth octave. This particular note is called "concert A" (because it is often used as the note to which an orchestra tunes its instruments) or "A440" (because its frequency is 440 cycles per second). Because this particular pitch is so common, it may be desirable to give it a name, which is easily done in Haskell, as was done above for π:

$$
\begin{aligned}
&concertA, a440 :: (PitchClass, Octave)\\
&concertA = (A, 4) \quad \text{-- concert A}\\
&a440 \quad\;\; = (A, 4) \quad \text{-- A440}
\end{aligned}
$$

Details: This example demonstrates the use of program *comments*. Any text to the right of "--" till the end of the line is considered to be a programmer comment, and is effectively ignored. Haskell also permits *nested* comments that have the form {-this is a comment -} and can appear anywhere in a program, including across multiple lines.

This example demonstrates the (perhaps obvious) fact that several different names can be given to the same value—just as your brother John might have the nickname "Moose." Also note that the name *concertA* requires more typing than $(A, 4)$; nevertheless, it has more mnemonic value, and, if mistyped, will more likely result in a syntax error. For example, if you type "*concrtA*" by mistake, you will likely get an error saying, "Undefined variable," whereas if you type "$(A, 5)$" you will not.

[5] We will have more to say about floating-point numbers later.

Details: This example also demonstrates that two names having the same type can be combined into the same type signature, separated by a comma. Note finally, as a reminder, that these are names of values, and thus they both begin with a lowercase letter.

Consider now a problem whose solution requires writing some larger expression more than once. For example:

$x :: Float$
$x = f\ (pi * r ** 2) + g\ (pi * r ** 2)$

Details: $(**)$ is Haskell's floating-point exponentiation operator. Thus $pi * r ** 2$ is analogous to πr^2 in mathematics. $(**)$ has higher precedence than $(*)$ and the other binary arithmetic operators in Haskell.

Note in the definition of x that the expression $pi * r ** 2$ (presumably representing the area of a circle whose radius is r) is repeated—it has two instances—and thus, applying the abstraction principle, it can be separated from these instances. From the previous examples, doing this is straightforward—it's called *naming*—so one might choose to rewrite the single equation above as two:

$area = pi * r ** 2$
$x \quad = f\ area + g\ area$

If, however, the definition of *area* is not intended for use elsewhere in the program, then it is advantageous to "hide" it within the definition of x. This will avoid cluttering up the namespace, and prevents *area* from clashing with some other value named *area*. To achieve this, one could simply use a **let** expression:

$x = \mathbf{let}\ area = pi * r ** 2$
$\quad \mathbf{in}\ f\ area + g\ area$

A **let** expression restricts the *visibility* of the names that it creates to the internal workings of the **let** expression itself. For example, if one writes:

$area = 42$
$x \quad = \mathbf{let}\ area = pi * r ** 2$
$\quad\quad \mathbf{in}\ f\ area + g\ area$

then there is no conflict of names—the "outer" *area* is completely different from the "inner" one enclosed in the **let** expression. Think of the inner *area* as analogous to the first name of someone in your household. If your brother's name is "John" he will not be confused with John Thompson who lives down the street when you say, "John spilled the milk."

So you can see that naming—using either top-level equations or equations within a **let** expression—is an example of the abstraction principle in action.

Details: An equation such as $c = 42$ is called a *binding*. A simple rule to remember when programming in Haskell is never to give more than one binding for the same name in a context where the names can be confused, whether at the top level of your program or nestled within a **let** expression. For example, this is not allowed:

$$a = 42$$
$$a = 43$$

nor is this:

$$a = 42$$
$$b = 43$$
$$a = 44$$

1.7.2 Functional Abstraction

The design of functions such as *simple* can be viewed as the abstraction principle in action. To see this using the example above involving the area of a circle, suppose the original program looked like this:

$$x :: Float$$
$$x = f\ (pi * r_1 ** 2) + g\ (pi * r_2 ** 2)$$

Note that there are now two areas involved—one of a circle whose radius is r_1, the other r_2. Now the expressions in parentheses have a *repeating pattern of operations*. In discerning the nature of a repeating pattern it's sometimes helpful to first identify those things that are *not* repeating, i.e. those things that are *changing*. In the case above, it is the radius that is changing. A repeating pattern of operations can be abstracted as a *function* that takes the changing values as arguments. Using the function name *areaF* (for "area function") one can write:

$$x = \textbf{let } areaF\ r = pi * r ** 2$$
$$\quad \textbf{in } f\ (areaF\ r_1) + g\ (areaF\ r_2)$$

This is a simple generalization of the previous example, where the function now takes the "variable quantity"—in this case the radius—as an argument. A very simple proof by calculation, in which *areaF* is unfolded where it is used, can be given to demonstrate that this program is equivalent to the old.

This application of the abstraction principle is called *functional abstraction*, since a sequence of operations is abstracted as a *function* such as *areaF*.

For a musical example, a few more concepts from Euterpea are first introduced, concepts that are addressed more formally in the next chapter:

1. Recall that in music theory a *note* is a pitch combined with a *duration*. Duration is measured in beats, and in Euterpea has type *Dur*. A note whose duration is one beat is called a whole note; one with duration 1/2 is called a half note; and so on. A note in Euterpea is the smallest entity, besides a rest, that is actually a performable piece of music, and its type is *Music Pitch* (other variations of this type will be introduced in later chapters).

2. In Euterpea there are functions:

   $$note :: Dur \to Pitch \to Music\ Pitch$$
   $$rest :: Dur \to Music\ Pitch$$

   such that *note d p* is a note whose duration is d and pitch is p, and *rest d* is a rest with duration d. For example, *note* $(1/4)$ $(A, 4)$ is a quarter note concert A.

3. In Euterpea the following infix operators combine smaller *Music* values into larger ones:

   $$(:+:) :: Music\ Pitch \to Music\ Pitch \to Music\ Pitch$$
   $$(:=:) :: Music\ Pitch \to Music\ Pitch \to Music\ Pitch$$

   Intuitively:

   - m_1 :+: m_2 is the music value that represents the playing of m_1 followed by m_2.
   - m_1 :=: m_2 is the music value that represents the playing of m_1 and m_2 simultaneously.

4. Finally, Eutperepa has a function $trans :: Int \to Pitch \to Pitch$ such that $trans\ i\ p$ is a pitch that is i semitones (half steps, or steps on a piano) higher than p.

Now for the example. Consider the simple melody:

$$note\ qn\ p_1 :+: note\ qn\ p_2 :+: note\ qn\ p_3$$

where qn is a quarter note:

$$qn = 1/4$$

Suppose one wishes to harmonize each note with a note played a minor third lower. In music theory, a minor third corresponds to three semitones, and thus the harmonized melody can be written as:

$$\begin{aligned} mel = (&note\ qn\ p_1 :=: note\ qn\ (trans\ (-3)\ p_1)) \ :+: \\ &(note\ qn\ p_2 :=: note\ qn\ (trans\ (-3)\ p_2)) \ :+: \\ &(note\ qn\ p_3 :=: note\ qn\ (trans\ (-3)\ p_3)) \end{aligned}$$

Note as in the previous example a repeating pattern of operations—namely, the operations that harmonize a single note with a note three semitones below it. As before, to abstract a sequence of operations such as this, a function can be defined that takes the "variable quantities"—in this case the pitch—as arguments. One could take this one step further, however, by noting that in some other context one might wish to vary the duration. Recognizing this is to anticipate the need for abstraction. Calling this function *hNote* (for "harmonize note") one can then write:

$$\begin{aligned} &hNote \quad :: Dur \to Pitch \to Music\ Pitch \\ &hNote\ d\ p = note\ d\ p :=: note\ d\ (trans\ (-3)\ p) \end{aligned}$$

There are three instances of the pattern in *mel*, each of which can be replaced with an application of *hNote*. This leads to:

$$\begin{aligned} &mel :: Music\ Pitch \\ &mel = hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hNote\ qn\ p_3 \end{aligned}$$

Again using the idea of unfolding described earlier in this chapter, it is easy to prove that this definition is equivalent to the previous one.

As with *areaF*, this use of *hNote* is an example of functional abstraction. In a sense, functional abstraction can be seen as a generalization of naming. That is, $area\ r_1$ is just a name for $pi * r_1 ** 2$, $hNote\ d\ p_1$ is just a name for $note\ d\ p_1 :=: note\ d\ (trans\ (-3)\ p_1)$, and so on. Stated another way, named quantities such as *area*, *pi*, *concertA*, and *a440* defined earlier can be thought of as functions with no arguments.

Of course, the definition of *hNote* could also be hidden within *mel* using a **let** expression as was done in the previous example:

$$
\begin{aligned}
&mel :: Music\ Pitch \\
&mel = \textbf{let}\ hNote\ d\ p = note\ d\ p :=: note\ d\ (trans\ (-3)\ p) \\
&\qquad \textbf{in}\ hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hNote\ qn\ p_3
\end{aligned}
$$

1.7.3 Data Abstraction

The value of *mel* is the sequential composition of three harmonized notes. But what if in another situation one must compose together five harmonized notes, or in other situations even more? In situations where the number of values is uncertain, it is useful to represent them in a *data structure*. For the example at hand, a good choice of data structure is a *list*, briefly introduced earlier, that can have any length. The use of a data structure motivated by the abstraction principle is one form of *data abstraction*.

Imagine now an entire list of pitches, whose length isn't known at the time the program is written. What now? It seems that a function is needed to convert a list of pitches into a sequential composition of harmonized notes. Before defining such a function, however, there is a bit more to say about lists.

Earlier the example $[C, Cs, Df]$ was given, a list of pitch classes whose type is thus $[PitchClass]$. A list with *no* elements is—not surprisingly—written $[\,]$, and is called the *empty list*.

To add a single element x to the front of a list xs, one writes $x : xs$ in Haskell. (Note the naming convention used here; xs is the plural of x, and should be read that way.) For example, $C : [Cs, Df]$ is the same as $[C, Cs, Df]$. In fact, this list is equivalent to $C : (Cs : (Df : [\,]))$, which can also be written $C : Cs : Df : [\,]$ since the infix operator (:) is right associative.

Details: In mathematics one rarely worries about whether the notation $a+b+c$ stands for $(a+b)+c$ (in which case $+$ would be "left associative") or $a+(b+c)$ (in which case $+$ would "right associative"). This is because in situations where the parentheses are left out it's usually the case that the operator is *mathematically* associative, meaning that it doesn't matter which interpretation is chosen. If the interpretation *does* matter, mathematicians will include parentheses to make it clear. Furthermore, in mathematics there is an implicit assumption that some operators have higher *precedence* than others; for example, $2 \times a + b$ is interpreted as $(2 \times a) + b$, not $2 \times (a + b)$.

In many programming languages, including Haskell, each operator is defined to have a particular precedence level and to be left associative, right associative, or to have no associativity at all. For arithmetic operators, mathematical convention is usually followed; for example, $2 * a + b$ is interpreted as $(2 * a) + b$ in Haskell. The predefined list-forming operator $(:)$ is defined to be right associative. Just as in mathematics, this associativity can be overridden by using parentheses: thus $(a : b) : c$ is a valid Haskell expression (assuming that it is well-typed; it must be a list of lists), and is very different from $a : b : c$. A way to specify the precedence and associativity of user-defined operators will be discussed in a later chapter.

Returning now to the problem of defining a function (call it *hList*) to turn a list of pitches into a sequential composition of harmonized notes, one should first express what its type should be:

$$hList :: Dur \to [Pitch] \to Music\ Pitch$$

To define its proper behavior, it is helpful to consider, one by one, all possible cases that could arise on the input. First off, the list could be empty, in which case the sequential composition should be a *Music Pitch* value that has zero duration. So:

$$hList\ d\ [\,] = rest\ 0$$

The other possibility is that the list *isn't* empty—i.e. it contains at least one element, say p, followed by the rest of the elements, say ps. In this case the result should be the harmonization of p followed by the sequential composition of the harmonization of ps. Thus:

$$hList\ d\ (p : ps) = hNote\ d\ p :+: hList\ d\ ps$$

Note that this part of the definition of *hList* is *recursive*—it refers to itself! But the original problem—the harmonization of $p : ps$—has been reduced to the harmonization of p (previously captured in the function *hNote*) and the harmonization of ps (a slightly smaller problem than the original one).

Combining these two equations with the type signature yields the complete definition of the function *hList*:

$$
\begin{aligned}
&hList :: Dur \to [Pitch] \to Music\ Pitch \\
&hList\ d\ [\,] \quad\ = rest\ 0 \\
&hList\ d\ (p : ps) = hNote\ d\ p :+: hList\ d\ ps
\end{aligned}
$$

Recursion is a powerful technique that will be used many times in this textbook. It is also an example of a general problem-solving technique where a large problem is broken down into several smaller but similar problems; solving these smaller problems one-by-one leads to a solution to the larger problem.

Details: Although intuitive, this example highlights an important aspect of Haskell: *pattern matching*. The left-hand sides of the equations contain *patterns* such as $[\,]$ and $x : xs$. When a function is applied, these patterns are *matched* against the argument values in a fairly intuitive way ($[\,]$ only matches the empty list, and $p : ps$ will successfully match any list with at least one element, while naming the first element p and the rest of the list ps). If the match succeeds, the right-hand side is evaluated and returned as the result of the application. If it fails, the next equation is tried, and if all equations fail, an error results. All of the equations that define a particular function must appear together, one after the other.

Defining functions by pattern matching is quite common in Haskell, and you should eventually become familiar with the various kinds of patterns that are allowed; see Appendix D for a concise summary.

Given this definition of *hList* the definition of *mel* can be rewritten as:

$$mel = hList\ qn\ [p_1, p_2, p_3]$$

One can prove that this definition is equivalent to the old via calculation:

$$
\begin{aligned}
&mel = hList\ qn\ [p_1, p_2, p_3] \\
&\Rightarrow hList\ qn\ (p_1 : p_2 : p_3 : [\,]) \\
&\Rightarrow hNote\ qn\ p_1 :+: hList\ qn\ (p_2 : p_3 : [\,]) \\
&\Rightarrow hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hList\ qn\ (p_3 : [\,]) \\
&\Rightarrow hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hNote\ qn\ p_3 :+: hList\ qn\ [\,] \\
&\Rightarrow hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hNote\ qn\ p_3 :+: rest\ 0
\end{aligned}
$$

The first step above is not really a calculation, but rather a rewriting of the

list syntax. The remaining calculations each represent an unfolding of *hList*.

Lists are perhaps the most commonly used data structure in Haskell, and there is a rich library of functions that operate on them. In subsequent chapters lists will be used in a variety of interesting computer music applications.

Exercise 1.4 Modify the definitions of *hNote* and *hList* so that they each take an extra argument that specifies the interval of harmonization (rather than being fixed at -3). Rewrite the definition of *mel* to take these changes into account.

1.8 Haskell Equality vs. Euterpean Equality

The astute reader will have objected to the proof just completed, arguing that the original version of *mel*:

$$hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hNote\ qn\ p_3$$

is not the same as the terminus of the above proof:

$$hNote\ qn\ p_1 :+: hNote\ qn\ p_2 :+: hNote\ qn\ p_3 :+: rest\ 0$$

Indeed, that reader would be right! As Haskell values, these expressions are *not* equal, and if you printed each of them you would get different results. So what happened? Did proof by calculation fail?

No, proof by calcultation did not fail, since, as just pointed out, as Haskell values these two expressions are not the same, and proof by calculation is based on the equality of Haskell values. The problem is that a "deeper" notion of equivalence is needed, one based on the notion of *musical* equality. Adding a rest of zero duration to the beginning or end of any piece of music should not change what one *hears*, and therefore it seems that the above two expressions are *musically* equivalent. But it is unreasonable to expect Haskell to figure this out for the programmer!

As an analogy, consider the use of an ordered list to represent a set (which is unordered). The Haskell values $[x_1, x_2]$ and $[x_2, x_1]$ are not equal, yet in a program that "interprets" them as sets, they *are* equal.

The way this problem is approached in Euterpea is to formally define a notion of *musical interpretation*, from which the notion *musical equivalence* is defined. This leads to a kind of "algebra of music" that includes, among others, the following axiom:

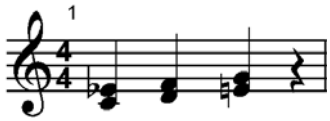


Figure 1.1: Polyphonic vs. Contrapuntal Interpretation

$$m :+: rest\ 0 \equiv m$$

The operator ($\equiv$) should be read, "is musically equivalent to." With this axiom it is easy to see that the original two expressions above *are* in fact musically equivalent.

For a more extreme example of this idea, and to entice the reader to learn more about musical equivalence in later chapters, note that *mel*, given pitches $p_1 = Ef$, $p_2 = F$, $p_3 = G$, and duration $d = 1/4$, generates the harmonized melody shown in Figure 1.1. One can write this concretely in Euterpea as:

$$\begin{aligned} mel_1 = (&note\ (1/4)\ (Ef, 4) :=: note\ (1/4)\ (C, 4))\ :+: \\ &(note\ (1/4)\ (F,\ 4) :=: note\ (1/4)\ (D, 4))\ :+: \\ &(note\ (1/4)\ (G,\ 4) :=: note\ (1/4)\ (E, 4)) \end{aligned}$$

The definition of mel_1 can then be seen as a *polyphonic* interpretation of the musical phrase in Figure 1.1, where each pair of notes is seen as a harmonic unit. In contrast, a *contrapuntal* interpretation sees two independent *lines* of notes, in this case the line ⟨E♭,F,G⟩ and the line ⟨C,D,E⟩. In Euterpea one can write this as:

$$\begin{aligned} mel_2 = (&note\ (1/4)\ (Ef, 4) :+: note\ (1/4)\ (F, 4) :+: note\ (1/4)\ (G, 4)) \\ &:=: \\ &(note\ (1/4)\ (C,\ 4) :+: note\ (1/4)\ (D, 4) :+: note\ (1/4)\ (E, 4)) \end{aligned}$$

mel_1 and mel_2 are clearly not equal as Haskell values. Yet if they are played, they will *sound* the same—they are, in the sense described earlier, *musically* equivalent. But proving these two phrases musically equivalent will require far more than a simple axiom involving *rest* 0. In fact this can be done in an elegant way, using the algebra of music developed in Chapter 11.

1.9 Code Reuse and Modularity

There doesn't seem to be much repetition in the last definition of *hList*, so perhaps the end of the abstraction process has been reached. In fact, it's worth considering how much progress has been made. The original

definition:

$$\begin{aligned} mel = (&note\ qn\ p_1 :=: note\ qn\ (trans\ (-3)\ p_1)) \ :+: \\ &(note\ qn\ p_2 :=: note\ qn\ (trans\ (-3)\ p_2)) \ :+: \\ &(note\ qn\ p_3 :=: note\ qn\ (trans\ (-3)\ p_3)) \end{aligned}$$

was replaced with:

$$mel = hList\ qn\ [p_1, p_2, p_3]$$

But additionally, definitions for the auxiliary functions *hNote* and *hList* were introduced:

$$\begin{aligned} &hNote \quad :: Dur \to Pitch \to Music\ Pitch \\ &hNote\ d\ p = note\ d\ p :=: note\ d\ (trans\ (-3)\ p) \\ &hList \quad :: Dur \to [Pitch] \to Music\ Pitch \\ &hList\ d\ [\,] \quad = rest\ 0 \\ &hList\ d\ (p : ps) = hNote\ d\ p :+: hList\ d\ ps \end{aligned}$$

In terms of code size, the final program is actually larger than the original! So has the program improved in any way?

Things have certainly gotten better from the standpoint of "removing repeating patterns," and one could argue that the resulting program therefore is easier to understand. But there is more. Now that auxiliary functions such as *hNote* and *hList* have been defined, one can *reuse* them in other contexts. Being able to reuse code is also called *modularity*, since the reused components are like little modules, or bricks, that can form the foundation of many applications.[6] In a later chapter, techniques will be introduced—most notably, *higher-order functions* and *polymorphism*—for improving the modularity of this example even more, and substantially increasing the ability to reuse code.

1.10 [Advanced] Programming with Numbers

In computer music programming, it is often necessary to program with numbers. For example, it is often convenient to represent pitch on a simple absolute scale using integer values. And when computing with analog signals that represent a particular sound wave, it is necessary to use floating point numbers as an approximation to the reals. So it is a good idea to understand precisely how numbers are represented inside a computer, and within a particular language such as Haskell.

[6] "Code reuse" and "modularity" are important software engineering principles.

In mathematics there are many different kinds of number systems. For example, there are integers, natural numbers (i.e. non-negative integers), real numbers, rational numbers, and complex numbers. These number systems possess many useful properties, such as the fact that multiplication and addition are commutative, and that multiplication distributes over addition. You have undoubtedly learned many of these properties in your studies, and have used them often in algebra, geometry, trigonometry, physics, and so on.

Unfortunately, each of these number systems places great demands on computer systems. In particular, a number can in general require an *arbitrary amount of memory* to represent it. Clearly, for example, an irrational number such as π cannot be represented exactly; the best one can do is approximate it, or possibly write a program that computes it to whatever (finite) precision is needed in a given application. But even integers (and therefore rational numbers) present problems, since any given integer can be arbitrarily large.

Most programming languages do not deal with these problems very well. In fact, most programming languages do not have exact forms of many of these number systems. Haskell does slightly better than most, in that it has exact forms of integers (the type *Integer*) as well as rational numbers (the type *Rational*, defined in the Ratio Library). But in Haskell and most other languages there is no exact form of real numbers, for example, which are instead approximated by *floating-point numbers* with either single-word precision (*Float* in Haskell) or double-word precision (*Double*). What's worse, the behavior of arithmetic operations on floating-point numbers can vary somewhat depending on what kind of computer is being used, although hardware standardization in recent years has reduced the degree of this problem.

The bottom line is that, as simple as they may seem, great care must be taken when programming with numbers. Many computer errors, some quite serious and renowned, were rooted in numerical incongruities. The field of mathematics known as *numerical analysis* is concerned precisely with these problems, and programming with floating-point numbers in sophisticated applications often requires a good understanding of numerical analysis to devise proper algorithms and write correct programs.

As a simple example of this problem, consider the distributive law, expressed here as a calculation in Haskell, and used earlier in this chapter in calculations involving the function *simple*:

$$a * (b + c) \Rightarrow a * b + a * c$$

For most floating-point numbers, this law is perfectly valid. For example, in the GHC implementation of Haskell, the expressions $pi * (3 + 4) :: Float$ and $pi * 3 + pi * 4 :: Float$ both yield the same result: 21.99115. But funny things can happen when the magnitude of $b + c$ differs significantly from the magnitude of either b or c. For example, the following two calculations are from GHC:

$$5 * (-0.123456 + 0.123457) \quad :: Float \Rightarrow 4.991889\text{e} - 6$$
$$5 * (-0.123456) + 5 * (0.123457) :: Float \Rightarrow 5.00679\text{e} - 6$$

Although the error here is small, its very existence is worrisome, and in certain situations it could be disastrous. The precise behavior of floating-point numbers will not be discussed further in this textbook. Just remember that they are *approximations* to the real numbers. If real-number accuracy is important to your application, further study of the nature of floating-point numbers is probably warranted.

On the other hand, the distributive law (and many others) is valid in Haskell for the exact data types *Integer* and *Ratio Integer* (i.e. rationals). However, another problem arises: although the representation of an *Integer* in Haskell is not normally something to be concerned about, it should be clear that the representation must be allowed to grow to an arbitrary size. For example, Haskell has no problem with the following number:

$$veryBigNumber :: Integer$$
$$veryBigNumber = 432083457203485932198765123721340 59$$

and such numbers can be added, multiplied, etc. without any loss of accuracy. However, such numbers cannot fit into a single word of computer memory, most of which are limited to 32 bits. Worse, since the computer system does not know ahead of time exactly how many words will be required, it must devise a dynamic scheme to allow just the right number of words to be used in each case. The overhead of implementing this idea unfortunately causes programs to run slower.

For this reason, Haskell (and most other languages) provides another integer data type called *Int* that has maximum and minimum values that depend on the word-size of the particular computer being used. In other words, every value of type *Int* fits into one word of memory, and the primitive machine instructions for binary numbers can be used to manipulate them efficiently.[7] Unfortunately, this means that *overflow* or *underflow* errors

[7] The Haskell Report requires that every implementation support *Int*s at least in the range -2^{29} to $2^{29} - 1$, inclusive. The GHC implementation running on a Pentium processor, for example, supports the range -2^{31} to $2^{31} - 1$.

could occur when an *Int* value exceeds either the maximum or minimum values. However, most implementations of Haskell (as well as most other languages) do not tell you when this happens. For example, in GHC, the following *Int* value:

$i :: Int$
$i = 1234567890$

works just fine, but if you multiply it by two, GHC returns the value -1825831516! This is because twice i exceeds the maximum allowed value, so the resulting bits become nonsensical,[8] and are interpreted in this case as a negative number of the given magnitude.

This is alarming! Indeed, why should anyone ever use *Int* when *Integer* is available? The answer, as implied earlier, is efficiency, but clearly care should be taken when making this choice. If you are indexing into a list, for example, and you are confident that you are not performing index calculations that might result in the above kind of error, then *Int* should work just fine, since a list longer than 2^{31} will not fit into memory anyway! But if you are calculating the number of microseconds in some large time interval, or counting the number of people living on earth, then *Integer* would most likely be a better choice. Choose your number data types wisely!

In this textbook the numeric data types *Integer*, *Int*, *Float*, *Double*, *Rational*, and *Complex* will be used for a variety of different applications; for a discussion of the other number types, consult the Haskell Report. As these data types are used, there will be little discussion about their properties—this is not, after all, a book on numerical analysis—but a warning will be cast whenever reasoning about, for example, floating-point numbers, in a way that might not be technically sound.

[8] Actually, these bits are perfectly sensible in the following way: the 32-bit binary representation of i is 01001001100101100000001011010010, and twice that is 10010011001011000000010110100100. But the latter number is seen as negative because the 32nd bit (the highest-order bit on the CPU on which this was run) is a one, which means it is a negative number in "twos-complement" representation. The twos-complement of this number is in turn 01101100110100111111101001011100, whose decimal representation is 1825831516.

Chapter 2

Simple Music

module *Euterpea.Music.Note.Music* **where**
infixr 5:+:, :=:

The previous chapters introduced some of the fundamental ideas of functional programming in Haskell. Also introduced were several of Euterpea's functions and operators, such as *note*, *rest*, (:+:), (:=:), and *trans*. This chapter will reveal the actual definitions of these functions and operators, thus exposing Euterpea's underlying structure and overall design at the note level. In addition, a number of other musical ideas will be developed, and in the process more Haskell features will be introduced as well.

2.1 Preliminaries

Sometimes it is useful to use a built-in Haskell data type to directly represent some concept of interest. For example, one may wish to use *Int* to represent *octaves*, where by convention octave 4 corresponds to the octave containing middle C on the piano. One can express this in Haskell using a *type synonym*:

type *Octave* = *Int*

A type synonym does not create a new data type—it just gives a new name to an existing type. Type synonyms can be defined not just for atomic types such as *Int*, but also for structured types such as pairs. For example, as discussed in the last chapter, in music theory a pitch is defined as a pair, consisting of a *pitch class* and an *octave*. Assuming the existence of a data type called *PitchClass* (which will be returned to shortly), one can write the following type synonym:

type *Pitch* = (*PitchClass*, *Octave*)

For example, concert A (i.e. A440) corresponds to the pitch $(A, 4)$, and the lowest and highest notes on a piano correspond to $(A, 0)$ and $(C, 8)$, respectively.

Another important musical concept is *duration*. Rather than use either integers or floating-point numbers, Euterpea uses *rational* numbers to denote duration:

type *Dur* = *Rational*

Rational is the data type of rational numbers expressed as ratios of *Integer*s in Haskell. The choice of *Rational* is somewhat subjective, but is justified by three observations: (1) many durations are expressed as ratios in music theory (5:4 rhythm, quarter notes, dotted notes, and so on), (2) *Rational* numbers are exact (unlike floating point numbers), which is important in many computer music applications, and (3) irrational durations are rarely needed.

Rational numbers in Haskell are printed by GHC in the form $n \,\%\, d$, where n is the numerator, and d is the denominator. Even a whole number, say the number 42, will print as $42 \,\%\, 1$ if it is a *Rational* number. To create a *Rational* number in a program, however, once it is given the proper type, one can use the normal division operator, as in the following definition of a quarter note:

```
qn :: Dur
qn = 1/4   -- quarter note
```

So far so good. But what about *PitchClass*? One might try to use integers to represent pitch classes as well, but this is not very elegant—ideally one would like to write something that looks more like the conventional pitch class names C, C♯, D♭, D, etc. The solution is to use an *algebraic data type* in Haskell:

```
data PitchClass = Cff | Cf | C | Dff | Cs | Df | Css | D | Eff | Ds
                | Ef | Fff | Dss | E | Es | Ff | F | Gff | Ess | Fs
                | Gf | Fss | G | Aff | Gs | Af | Gss | A | Bff | As
                | Bf | Ass | B | Bs | Bss
```

Details: All of the names to the right of the equal sign in a **data** declaration are called *constructors*, and must be capitalized. In this way they are syntactically distinguished from ordinary values. This distinction is useful since only constructors can be used in the pattern matching that is part of a function definition, as will be described shortly.

The *PitchClass* data type declaration essentially enumerates 35 pitch class names (five for each of the note names A through G). Note that both double-sharps and double-flats are included, resulting in many enharmonics (i.e., two notes that "sound the same," such as G♯ and A♭).

Keep in mind that *PitchClass* is a completely new, user-defined data type that is not equal to any other. This is what distinguishes a **data** declaration from a **type** declaration. As another example of the use of a **data** declaration to define a simple enumerated type, Haskell's Boolean data type, called *Bool*, is predefined in Haskell simply as:

data *Bool* = *False* | *True*

2.2 Notes, Music, and Polymorphism

One can of course define other data types for other purposes. For example, one will want to define the notion of a *note* and a *rest*. Both of these can be thought of as "primitive" musical values, and thus as a first attempt one might write:

data *Primitive* = *Note Dur Pitch*
| *Rest Dur*

For example, *Note qn a440* would be concert A played as a quarter note, and *Rest* 1 is a whole-note rest.

This definition is not completely satisfactory, however, because one may wish to attach other information to a note, such as its loudness, or some other annotation or articulation. Furthermore, the pitch itself may actually be a percussive sound, having no true pitch at all. To resolve this, Euterpea uses an important concept in Haskell, namely *polymorphism*—the ability to parameterize, or abstract, over types (*poly* means *many* and *morphism* refers to the structure, or *form*, of objects). *Primitive* can be redefined as a *polymorphic data type* as follows.

Instead of fixing the type of the pitch of a note, it is left unspecified

through the use of a *type variable*:

```
data Primitive a = Note Dur a
                 | Rest Dur
```

Note that the type variable a is used as an argument to *Primitive*, and then used in the body of the declaration—just like a variable in a function. This version of *Primitive* is more general than the previous version—indeed, note that *Primitive Pitch* is the same as (or, technically, is *isomorphic to*) the previous version of *Primitive*. But additionally, *Primitive* is now more flexible than the previous version, since, for example, one can add loudness by pairing loudness with pitch, as in *Primitive* (*Pitch*, *Loudness*). Other concrete instances of this idea will be introduced later.

Another way to interpret this data declaration is to say that for any type a, this declaration declares the types of its constructors to be:

```
Note :: Dur → a → Primitive a
Rest :: Dur →     Primitive a
```

Even though *Note* and *Rest* are called data constructors, they are still functions, and they have a type. Since they both have type variables in their type signatures, they are examples of *polymorphic functions*.

Note that one can think of polymorphism as applying the abstraction principle at the type level—indeed it is often called *type abstraction*. Many more examples of both polymorphic functions and polymorphic data types will be explored in detail in Chapter 3.

So far Euterpea's primitive notes and rests have been introuced—but how does one combine many notes and rests into a larger composition? To achieve this, Euterpea defines another polymorphic data type, perhaps the most important data type used in this textbook, which defines the fundamental structure of a note-level musical entity:

```
data Music a =
    Prim (Primitive a)          -- primitive value
  | Music a :+: Music a         -- sequential composition
  | Music a :=: Music a         -- parallel composition
  | Modify Control (Music a)    -- modifier
```

Following the reasoning above, the types of these constructors are:

```
Prim   :: Primitive a           → Music a
(:+:)  :: Music a → Music a → Music a
(:=:)  :: Music a → Music a → Music a
Modify :: Control → Music a → Music a
```

These four constructors then are also polymorphic functions.

Details: Note the use of the *infix constructors* (:+:) and (:=:). Infix constructors are just like infix operators in Haskell, but they must begin with a colon. This syntactic distinction makes it clear when one is pattern matching, and is analogous to the distinction between ordinary names (which must begin with a lower-case character) and constructor names (which must begin with an upper-case character).

The observant reader will also recall that at the very beginning of this chapter—corresponding to the module containing all the code in this chapter—the following line appeared:

infixr 5:+:, :=:

This is called a *fixity declaration*. The "*r*" after the word "**infix**" means that the specified operators—in this case (:+:) and (:=:)—are to have *right* associativity, and the "5" specifies their *precedence level* (these operators will bind more tightly than an operator with a lower precedence).

The *Music* data type declaration essentially says that a value of type *Music a* has one of four possible forms:

- *Prim p*, where *p* is a primitive value of type *Primitive a*, for some type *a*. For example:

  $$a440m :: Music\ Pitch$$
  $$a440m = Prim\ (Note\ qn\ a440)$$

  is the musical value corresponding to a quarter-note rendition of concert A.

- m_1 :+: m_2 is the *sequential composition* of m_1 and m_2; i.e. m_1 and m_2 are played in sequence.

- m_1 :=: m_2 is the *parallel composition* of m_1 and m_2; i.e. m_1 and m_2 are played simultaneously. The duration of the result is the duration of the longer of m_1 and m_2.

  (Recall that these last two operators were introduced in the last chapter. You can see now that they are actually constructors of an algebraic data type.)

- *Modify cntrl m* is an "annotated" version of *m* in which the control parameter *cntrl* specifies some way in which *m* is to be modified.

Details: Note that *Music a* is defined in terms of *Music a*, and thus the data type is said to be *recursive* (analogous to a recursive function). It is also often called an *inductive* data type, since it is, in essence, an inductive definition of an infinite number of values, each of which can be arbitrarily complex.

It is convenient to represent these musical ideas as a recursive datatype because it allows one to not only *construct* musical values, but also take them apart, analyze their structure, print them in a structure-preserving way, transform them, interpret them for performance purposes, and so on. Many examples of these kinds of processes will be seen in this textbook.

The *Control* data type is used by the *Modify* consructor to allow one to annotate a *Music* value with a *tempo change*, a *transposition*, a *phrase attribute*, a *player name*, or an *instrument*. This data type is unimportant at the moment, but for completeness here is its full definition:

```
data Control =
    Tempo      Rational              -- scale the tempo
  | Transpose  AbsPitch              -- transposition
  | Instrument InstrumentName        -- instrument label
  | Phrase     [PhraseAttribute]     -- phrase attributes
  | Player     PlayerName            -- player label

type PlayerName = String
```

AbsPitch ("absolute pitch," defined in Section 2.4) is just a type synonym for *Int*. Instrument names are borrowed from the General MIDI standard, and are captured as an algebraic data type in Figure 2.1. Phrase attributes and the concept of a "player" are closely related, but a full explanation is deferred until Chapter 8.

2.3 Convenient Auxiliary Functions

For convenience, and in anticipation of their frequent use, a number of functions are defined in Euterpea to make it easier to write certain kinds of musical values. For starters:

```
note       :: Dur -> a -> Music a
note d p   = Prim (Note d p)
rest       :: Dur -> Music a
```

```
data InstrumentName =
    AcousticGrandPiano | BrightAcousticPiano | ElectricGrandPiano
  | HonkyTonkPiano | RhodesPiano | ChorusedPiano
  | Harpsichord | Clavinet | Celesta
  | Glockenspiel | MusicBox | Vibraphone
  | Marimba | Xylophone | TubularBells
  | Dulcimer | HammondOrgan | PercussiveOrgan
  | RockOrgan | ChurchOrgan | ReedOrgan
  | Accordion | Harmonica | TangoAccordion
  | AcousticGuitarNylon | AcousticGuitarSteel | ElectricGuitarJazz
  | ElectricGuitarClean | ElectricGuitarMuted | OverdrivenGuitar
  | DistortionGuitar | GuitarHarmonics | AcousticBass
  | ElectricBassFingered | ElectricBassPicked | FretlessBass
  | SlapBass1 | SlapBass2 | SynthBass1
  | SynthBass2 | Violin | Viola
  | Cello | Contrabass | TremoloStrings
  | PizzicatoStrings | OrchestralHarp | Timpani
  | StringEnsemble1 | StringEnsemble2 | SynthStrings1
  | SynthStrings2 | ChoirAahs | VoiceOohs
  | SynthVoice | OrchestraHit | Trumpet
  | Trombone | Tuba | MutedTrumpet
  | FrenchHorn | BrassSection | SynthBrass1
  | SynthBrass2 | SopranoSax | AltoSax
  | TenorSax | BaritoneSax | Oboe
  | Bassoon | EnglishHorn | Clarinet
  | Piccolo | Flute | Recorder
  | PanFlute | BlownBottle | Shakuhachi
  | Whistle | Ocarina | Lead1Square
  | Lead2Sawtooth | Lead3Calliope | Lead4Chiff
  | Lead5Charang | Lead6Voice | Lead7Fifths
  | Lead8BassLead | Pad1NewAge | Pad2Warm
  | Pad3Polysynth | Pad4Choir | Pad5Bowed
  | Pad6Metallic | Pad7Halo | Pad8Sweep
  | FX1Train | FX2Soundtrack | FX3Crystal
  | FX4Atmosphere | FX5Brightness | FX6Goblins
  | FX7Echoes | FX8SciFi | Sitar
  | Banjo | Shamisen | Koto
  | Kalimba | Bagpipe | Fiddle
  | Shanai | TinkleBell | Agogo
  | SteelDrums | Woodblock | TaikoDrum
  | MelodicDrum | SynthDrum | ReverseCymbal
  | GuitarFretNoise | BreathNoise | Seashore
  | BirdTweet | TelephoneRing | Helicopter
  | Applause | Gunshot | Percussion
  | Custom String
```

Figure 2.1: General MIDI Instrument Names

$$
\begin{array}{ll}
rest\ d & = Prim\ (Rest\ d) \\
tempo & :: Dur \to Music\ a \to Music\ a \\
tempo\ r\ m & = Modify\ (Tempo\ r)\ m \\
transpose & :: AbsPitch \to Music\ a \to Music\ a \\
transpose\ i\ m & = Modify\ (Transpose\ i)\ m \\
instrument & :: InstrumentName \to Music\ a \to Music\ a \\
instrument\ i\ m & = Modify\ (Instrument\ i)\ m \\
phrase & :: [PhraseAttribute] \to Music\ a \to Music\ a \\
phrase\ pa\ m & = Modify\ (Phrase\ pa)\ m \\
player & :: PlayerName \to Music\ a \to Music\ a \\
player\ pn\ m & = Modify\ (Player\ pn)\ m
\end{array}
$$

Note that each of these functions is poymorphic, a trait inherited from the data types that it uses. Also recall that the first two of these functions were used in an example in the last chapter.

One can also create simple names for familiar notes, durations, and rests, as shown in Figures 2.2 and 2.3. Despite the large number of them, these names are sufficiently "unusual" that name clashes are unlikely.

Details: Figures 2.2 and 2.3 demonstrate that at the top level of a program, more than one equation can be placed on one line, as long as they are separated by a semicolon. This allows one to save vertical space on the page, and is useful whenever each line is relatively short. The semicolon is not needed at the end of a single equation, or at the end of the last equation on a line. This convenient feature is part of Haskell's *layout* rule, and will be explained in more detail later.

More than one equation can also be placed on one line in a **let** expression, as demonstrated below:

$$
\begin{array}{l}
\textbf{let}\ x = 1; y = 2 \\
\textbf{in}\ x + y
\end{array}
$$

2.3.1 A Simple Example

As a simple example, suppose one wishes to generate a ii-V-I chord progression in a particular key. In music theory, such a chord progression begins with a minor chord on the second degree of a major scale, followed by a major chord on the fifth degree, and ending in a major chord on the first

```
cff, cf, c, cs, css, dff, df, d, ds, dss, eff, ef, e, es, ess, fff, ff, f,
  fs, fss, gff, gf, g, gs, gss, aff, af, a, as, ass, bff, bf, b, bs, bss ::
    Octave → Dur → Music Pitch

cff  o d = note d (Cff, o); cf   o d = note d (Cf,  o)
c    o d = note d (C,   o); cs   o d = note d (Cs,  o)
css  o d = note d (Css, o); dff  o d = note d (Dff, o)
df   o d = note d (Df,  o); d    o d = note d (D,   o)
ds   o d = note d (Ds,  o); dss  o d = note d (Dss, o)
eff  o d = note d (Eff, o); ef   o d = note d (Ef,  o)
e    o d = note d (E,   o); es   o d = note d (Es,  o)
ess  o d = note d (Ess, o); fff  o d = note d (Fff, o)
ff   o d = note d (Ff,  o); f    o d = note d (F,   o)
fs   o d = note d (Fs,  o); fss  o d = note d (Fss, o)
gff  o d = note d (Gff, o); gf   o d = note d (Gf,  o)
g    o d = note d (G,   o); gs   o d = note d (Gs,  o)
gss  o d = note d (Gss, o); aff  o d = note d (Aff, o)
af   o d = note d (Af,  o); a    o d = note d (A,   o)
as   o d = note d (As,  o); ass  o d = note d (Ass, o)
bff  o d = note d (Bff, o); bf   o d = note d (Bf,  o)
b    o d = note d (B,   o); bs   o d = note d (Bs,  o)
bss  o d = note d (Bss, o)
```

Figure 2.2: Convenient Note Names

```
bn, wn, hn, qn, en, sn, tn, sfn, dwn, dhn,
  dqn, den, dsn, dtn, ddhn, ddqn, dden :: Dur

bnr, wnr, hnr, qnr, enr, snr, tnr, dwnr, dhnr,
  dqnr, denr, dsnr, dtnr, ddhnr, ddqnr, ddenr :: Music Pitch

bn   = 2;     bnr   = rest bn     -- brevis rest
wn   = 1;     wnr   = rest wn     -- whole note rest
hn   = 1/2;   hnr   = rest hn     -- half note rest
qn   = 1/4;   qnr   = rest qn     -- quarter note rest
en   = 1/8;   enr   = rest en     -- eighth note rest
sn   = 1/16;  snr   = rest sn     -- sixteenth note rest
tn   = 1/32;  tnr   = rest tn     -- thirty-second note rest
sfn  = 1/64;  sfnr  = rest sfn    -- sixty-fourth note rest

dwn  = 3/2;   dwnr  = rest dwn    -- dotted whole note rest
dhn  = 3/4;   dhnr  = rest dhn    -- dotted half note rest
dqn  = 3/8;   dqnr  = rest dqn    -- dotted quarter note rest
den  = 3/16;  denr  = rest den    -- dotted eighth note rest
dsn  = 3/32;  dsnr  = rest dsn    -- dotted sixteenth note rest
dtn  = 3/64;  dtnr  = rest dtn    -- dotted thirty-second note rest

ddhn = 7/8;   ddhnr = rest ddhn   -- double-dotted half note rest
ddqn = 7/16;  ddqnr = rest ddqn   -- double-dotted quarter note rest
dden = 7/32;  ddenr = rest dden   -- double-dotted eighth note rest
```

Figure 2.3: Convenient Duration and Rest Names

degree. One can write this in Euterpea, using triads in the key of C major, as follows:

```
t251 :: Music Pitch
t251 = let dMinor = d 4 wn :=: f 4 wn :=: a 4 wn
           gMajor = g 4 wn :=: b 4 wn :=: d 5 wn
           cMajor = c 4 bn :=: e 4 bn :=: g 4 bn
       in dMinor :+: gMajor :+: cMajor
```

Details: Note that more than one equation is allowed in a **let** expression, just like at the top level of a program. The first characters of each equation, however, must line up vertically, and if an equation takes more than one line then the subsequent lines must be to the right of the first characters. For example, this is legal:

```
let a = aLongName
        + anEvenLongerName
    b = 56
in ...
```

but neither of these are:

```
let  a = aLongName
   + anEvenLongerName
     b = 56
in ...
```

```
let a = aLongName
        + anEvenLongerName
  b = 56
in ...
```

(The second line in the first example is too far to the left, as is the third line in the second example.)

Details: Although this rule, called the *layout rule*, may seem a bit *ad hoc*, it avoids having to use special syntax to denote the end of one equation and the beginning of the next (such as a semicolon), thus enhancing readability. In practice, use of layout is rather intuitive. Just remember two things:

First, the first character following **let** (and a few other keywords that will be introduced later) is what determines the starting column for the set of equations being written. Thus one can begin the equations on the same line as the keyword, the next line, or whatever.

Second, be sure that the starting column is further to the right than the starting column associated with any immediately surrounding **let** clause (otherwise it would be ambiguous). The "termination" of an equation happens when something appears at or to the left of the starting column associated with that equation.

In order to play this simple example, one can use Euterpea's *play* function and simply type:

play t251

at the GHCi command line. Default instruments and tempos are used to convert *t251* into MIDI and then play the result through your computer's standard sound card.

Exercise 2.1 The above example is fairly concrete, in that, for one, it is rooted in C major, and furthermore it has a fixed tempo. Define a function $twoFiveOne :: Pitch \to Dur \to Music\ Pitch$ such that $twoFiveOne\ p\ d$ constructs a ii-V-I chord progression starting on the pitch p (which is assumed to be the second degree of the major scale on which the progression is being constructed), where the duration of the first two chords is each d, and the duration of the last chord is $2 * d$.

To verify your code, prove by calculation that $twoFiveOne\ (D, 4)\ wn = t251$.

2.4 Absolute Pitches

Treating pitches simply as integers is useful in many settings, so Euterpea uses a type synonym to define the concept of an "absolute pitch:"

type $AbsPitch = Int$

The absolute pitch of a (relative) pitch can be defined mathematically as 12 times the octave, plus the index of the pitch class. One can express this in Haskell as follows:

$$\begin{aligned} &absPitch && :: Pitch \to AbsPitch \\ &absPitch\ (pc, oct) && = 12 * oct + pcToInt\ pc \end{aligned}$$

Details: Note the use of pattern matching to match the argument of *absPitch* to a pair.

pcToInt is a function that converts a particular pitch class to an index, easily but tediously expressed as shown in Figure 2.4. But there is a subtlety: according to music theory convention, pitches are assigned integers in the range 0 to 11, i.e. modulo 12, starting on pitch class C. In other words, the index of C is 0, C♭ is 11, and B♯ is 0. However, that would mean the absolute pitch of $(C, 4)$, say, would be 48, whereas $(Cf, 4)$ would be 59. Somehow the latter does not seem right—47 would be a more logical choice. Therefore the definition in Figure 2.4 is written in such a way that the wrap-round does not happen, i.e. numbers outside the range 0 to 11 are used. With this definition, *absPitch* $(Cf, 4)$ yields 47, as desired.

$$
\begin{aligned}
&pcToInt :: PitchClass \to Int \\
&pcToInt\ Cff = -2; pcToInt\ Dff = 0; pcToInt\ Eff = 2 \\
&pcToInt\ Cf\ = -1; pcToInt\ Df\ = 1; pcToInt\ Ef\ = 3 \\
&pcToInt\ C\ = 0;\ \ pcToInt\ D\ = 2; pcToInt\ E\ = 4 \\
&pcToInt\ Cs\ = 1;\ \ pcToInt\ Ds\ = 3; pcToInt\ Es\ = 5 \\
&pcToInt\ Css = 2;\ \ pcToInt\ Dss = 4; pcToInt\ Ess = 6 \\
&pcToInt\ Fff = 3;\ \ pcToInt\ Gff = 5; pcToInt\ Aff = 7 \\
&pcToInt\ Ff\ = 4;\ \ pcToInt\ Gf\ = 6; pcToInt\ Af\ = 8 \\
&pcToInt\ F\ = 5;\ \ pcToInt\ G\ = 7; pcToInt\ A\ = 9 \\
&pcToInt\ Fs\ = 6;\ \ pcToInt\ Gs\ = 8; pcToInt\ As\ = 10 \\
&pcToInt\ Fss = 7;\ \ pcToInt\ Gss = 9; pcToInt\ Ass = 11 \\
&pcToInt\ Bff = 9 \\
&pcToInt\ Bf\ = 10 \\
&pcToInt\ B\ = 11 \\
&pcToInt\ Bs\ = 12 \\
&pcToInt\ Bss = 13
\end{aligned}
$$

Figure 2.4: Converting Pitch Classes to Integers

Details: The repetition of "*pcToInt*" above can be avoided by using a Haskell **case** expression, resulting in a more compact definition:

$$
\begin{aligned}
&pcToInt\ \ \ :: PitchClass \to Int \\
&pcToInt\ pc = \textbf{case}\ pc\ \textbf{of} \\
&\quad Cff \to -2; Cf \to -1; C \to 0;\ \ Cs \to 1;\ \ Css \to 2; \\
&\quad Dff \to 0;\ \ \ Df \to 1;\ \ \ D \to 2;\ \ Ds \to 3;\ \ Dss \to 4; \\
&\quad Eff \to 2;\ \ \ Ef \to 3;\ \ \ E \to 4;\ \ Es \to 5;\ \ Ess \to 6; \\
&\quad Fff \to 3;\ \ \ Ff \to 4;\ \ \ F \to 5;\ \ Fs \to 6;\ \ Fss \to 7; \\
&\quad Gff \to 5;\ \ \ Gf \to 6;\ \ \ G \to 7;\ \ Gs \to 8;\ \ Gss \to 9; \\
&\quad Aff \to 7;\ \ \ Af \to 8;\ \ \ A \to 9;\ \ As \to 10; Ass \to 11; \\
&\quad Bff \to 9;\ \ \ Bf \to 10;\ B \to 11; Bs \to 12; Bss \to 13
\end{aligned}
$$

As you can see, a **case** expression allows multiple pattern-matches on an expression without using equations. Note that layout applies to the body of a case expression, and can be overriden as before using a semicolon. (As in a function type signature, the right-pointing arrow in a **case** expression must be typed as "->" on your computer keyboard.)

The body of a **case** expression observes layout just as a **let** expression, including the fact that semicolons can be used, as above, to place more than one pattern match on the same line.

Converting an absolute pitch to a pitch is a bit more tricky, because of enharmonic equivalences. For example, the absolute pitch 15 might correspond to either $(Ds, 1)$ or $(Ef, 1)$. Euterpea takes the approach of always returning a sharp in such ambiguous cases:

$$
\begin{aligned}
&\mathit{pitch} \quad :: \mathit{AbsPitch} \to \mathit{Pitch} \\
&\mathit{pitch}\ \mathit{ap} = \\
&\quad \textbf{let}\ (\mathit{oct}, n) = \mathit{divMod}\ \mathit{ap}\ 12 \\
&\quad \textbf{in}\ ([\, C, Cs, D, Ds, E, F, Fs, G, Gs, A, As, B\,] \mathbin{!!}\ n, \mathit{oct})
\end{aligned}
$$

Details: (!!) is Haskell's zero-based list-indexing function; list !! n returns the $(n+1)$th element in list. $\mathit{divMod}\ x\ n$ returns a pair (q, r), where q is the integer quotient of x divided by n, and r is the value of x modulo n.

Given *pitch* and *absPitch*, it is now easy to define a function *trans* that transposes pitches:

$$
\begin{aligned}
&\mathit{trans} \quad :: \mathit{Int} \to \mathit{Pitch} \to \mathit{Pitch} \\
&\mathit{trans}\ i\ p = \mathit{pitch}\ (\mathit{absPitch}\ p + i)
\end{aligned}
$$

With this definition, all of the operators and functions introduced in the previous chapter have been covered.

Exercise 2.2 Show that $\mathit{abspitch}\ (\mathit{pitch}\ \mathit{ap}) = \mathit{ap}$, and, up to enharmonic equivalences, $\mathit{pitch}\ (\mathit{abspitch}\ p) = p$.

Exercise 2.3 Show that $\mathit{trans}\ i\ (\mathit{trans}\ j\ p) = \mathit{trans}\ (i + j)\ p$.

Chapter 3

Polymorphic and Higher-Order Functions

Several examples of polymorphic data types were introduced in the last couple of chapters. In this chapter the focus is on *polymorphic functions*, which are most commonly functions defined over polymorphic data types.

The already familiar *list* is the most common example of a polymorphic data type, and it will be studied in depth in this chapter. Although lists have no direct musical connection, they are perhaps the most commonly used data type in Haskell, and have many applications in computer music programming. But in addition the *Music* data type is polymorphic, and several new functions that operate on it polymorphiccally will also be defined,

(A more detailed discussion of predefined polymorphic functions that operate on lists can be found in Appendix A.)

This chapter also introduces *higher-order functions*, which are functions that take one or more functions as arguments or return a function as a result (functions can also be placed in data structures). Higher-order functions permit the elegant and concise expression of many musical concepts. Together with polymorphism, higher-order functions substantially increase the programmer's expressive power and ability to reuse code.

Both of these new ideas naturally follow the foundations that have already been established.

3.1 Polymorphic Types

In previous chapters, examples of lists containing several different kinds of elements—integers, characters, pitch classes, and so on—were introduced, and one can well imagine situations requiring lists of other element types. Sometimes, however, it isn't necessary to be so particular about the type of the elements. For example, suppose one wishes to define a function *length* that determines the number of elements in a list. It doesn't really matter whether the list contains integers, pitch classes, or even other lists—one can imagine computing the length in exactly the same way in each case. The obvious definition is:

$$\begin{aligned} &\mathit{length}\ [\,] &&= 0 \\ &\mathit{length}\ (x : \mathit{xs}) &&= 1 + \mathit{length}\ \mathit{xs} \end{aligned}$$

This recursive definition is self-explanatory. One can read the equations as saying: "The length of the empty list is 0, and the length of a list whose first element is x and remainder is xs is 1 plus the length of xs."

But what should the type of *length* be? Intuitively, one would like to say that, for *any* type a, the type of *length* is $[a] \to \mathit{Integer}$. In mathematics one might write:

$$\mathit{length} :: (\forall\ a)\ [a] \to \mathit{Integer}$$

But in Haskell this is written simply as:

$$\mathit{length} :: [a] \to \mathit{Integer}$$

In other words, the universal quantification of the type variable a is implicit.

Details: Generic names for types, such as a above, are called *type variables*, and are uncapitalized to distinguish them from concrete types such as $\mathit{Integer}$.

So *length* can be applied to a list containing elements of *any* type. For example:

$$\begin{aligned} &\mathit{length}\ [1,2,3] &&\Longrightarrow 3 \\ &\mathit{length}\ [C, \mathit{Cs}, \mathit{Df}] &&\Longrightarrow 3 \\ &\mathit{length}\ [[1],[],[2,3,4]] &&\Longrightarrow 3 \end{aligned}$$

Note that the type of the argument to *length* in the last example is $[[\mathit{Integer}]]$; that is, a list of lists of integers.

Here are two other examples of polymorphic list functions, which happen

to be predefined in Haskell:

$$
\begin{aligned}
&\mathit{head} && :: [a] \to a \\
&\mathit{head}\ (x\colon \_) && = x \\
&\mathit{tail} && :: [a] \to [a] \\
&\mathit{tail}\ (\_ : \mathit{xs}) && = \mathit{xs}
\end{aligned}
$$

Details: The _ on the left-hand side of these equations is called a *wildcard* pattern. It matches any value, and binds no variables. It is useful as a way of documenting the fact that one does not care about the value in that part of the pattern. Note that one could (perhaps should) have used a wildcard in place of the variable x in the definition of *length*.

These two functions take the "head" and "tail," respectively, of any non-empty list. For example:

$$
\begin{aligned}
&\mathit{head}\ [1,2,3] && \Rightarrow 1 \\
&\mathit{head}\ [C, \mathit{Cs}, \mathit{Df}] && \Rightarrow C \\
&\mathit{tail}\ [1,2,3] && \Rightarrow [2,3] \\
&\mathit{tail}\ [C, \mathit{Cs}, \mathit{Df}] && \Rightarrow [\mathit{Cs}, \mathit{Df}]
\end{aligned}
$$

Note that, for any non-empty list xs, *head* and *tail* obey the following law:

$$\mathit{head}\ \mathit{xs} : \mathit{tail}\ \mathit{xs} = \mathit{xs}$$

Functions such as *length*, *head*, and *tail* are said to be *polymorphic*. Polymorphic functions arise naturally when defining functions on lists and other polymorphic data types, including the *Music* data type defined in the last chapter.

3.2 Abstraction Over Recursive Definitions

Given a list of pitches, suppose one wishes to convert each pitch into an absolute pitch. One might write a function:

$$
\begin{aligned}
&\mathit{toAbsPitches} && :: [\mathit{Pitch}] \to [\mathit{AbsPitch}] \\
&\mathit{toAbsPitches}\ [\,] && = [\,] \\
&\mathit{toAbsPitches}\ (p : \mathit{ps}) && = \mathit{absPitch}\ p : \mathit{toAbsPitches}\ \mathit{ps}
\end{aligned}
$$

One might also want to convert a list of absolute pitches to a list of pitches:

$$
\begin{aligned}
&\mathit{toPitches} &&:: [\mathit{AbsPitch}] \to [\mathit{Pitch}] \\
&\mathit{toPitches}\ [\,] &&= [\,] \\
&\mathit{toPitches}\ (a : as) &&= \mathit{pitch}\ a : \mathit{toPitches}\ as
\end{aligned}
$$

These two functions are different, but share something in common: there is a repeating pattern of operations. But the pattern is not quite like any of the examples studied earlier, and therefore it is unclear how to apply the abstraction principle. What distinguishes this situation is that there is a repeating pattern of *recursion*.

In discerning the nature of a repeating pattern, recall that it's sometimes helpful to first identify those things that *are not* repeating—i.e. those things that are *changing*—since these will be the sources of *parameterization*: those values that must be passed as arguments to the abstracted function. In the case above, these changing values are the functions *absPitch* and *pitch*; consider them instances of a new name, f. Rewriting either of the above functions as a new function—call it *map*—that takes an extra argument f, yields:

$$
\begin{aligned}
&\mathit{map}\ f\ [\,] &&= [\,] \\
&\mathit{map}\ f\ (x : xs) &&= f\ x : \mathit{map}\ f\ xs
\end{aligned}
$$

This recursive pattern of operations is so common that *map* is predefined in Haskell (and is why the name *map* was chosen in the first place).

With *map*, one can now redefine *toAbsPitches* and *toPitches* as:

$$
\begin{aligned}
&\mathit{toAbsPitches} &&:: [\mathit{Pitch}] \to [\mathit{AbsPitch}] \\
&\mathit{toAbsPitches}\ ps &&= \mathit{map}\ \mathit{absPitch}\ ps \\
&\mathit{toPitches} &&:: [\mathit{AbsPitch}] \to [\mathit{Pitch}] \\
&\mathit{toPitches}\ as &&= \mathit{map}\ \mathit{pitch}\ as
\end{aligned}
$$

Note that these definitions are non-recursive; the common pattern of recursion has been abstracted away and isolated in the definition of *map*. They are also very succinct; so much so, that it seems unnecessary to create new names for these functions at all! One of the powers of higher-order functions is that they permit concise yet easy-to-understand definitions such as this, and you will see many similar examples throughout the remainder of the text.

A proof that the new versions of these two functions are equivalent to the old ones can be done via calculation, but requires a proof technique called *induction*, because of the recursive nature of the original function definitions. Inductive proofs are discussed in detail, including for these two examples, in Chapter 10.

3.2.1 Map is Polymorphic

What should the type of *map* be? Looking first at its use in *toAbsPitches*, note that it takes the function $absPitch :: Pitch \rightarrow AbsPitch$ as its first argument and a list of *Pitch*s as its second argument, returning a list of *AbsPitch*s as its result. So its type must be:

$$map :: (Pitch \rightarrow AbsPitch) \rightarrow [Pitch] \rightarrow [AbsPitch]$$

Yet a similar analysis of its use in *toPitches* reveals that *map*'s type should be:

$$map :: (AbsPitch \rightarrow Pitch) \rightarrow [AbsPitch] \rightarrow [Pitch]$$

This apparent anomaly can be resolved by noting that *map*, like *length*, *head*, and *tail*, does not really care what its list element types are, *as long as its functional argument can be applied to them.* Indeed, *map* is *polymorphic*, and its most general type is:

$$map :: (a \rightarrow b) \rightarrow [a] \rightarrow [b]$$

This can be read: "*map* is a function that takes a function from any type a to any type b, and a list of a's, and returns a list of b's." The correspondence between the two a's and between the two b's is important: a function that converts *Int*'s to *Char*'s, for example, cannot be mapped over a list of *Char*'s. It is easy to see that in the case of *toAbsPitches*, a is instantiated as *Pitch* and b as *AbsPitch*, whereas in *toPitches*, a and b are instantiated as *AbsPitch* and *Pitch*, respectively.

Note, by the way, that the above reasoning can be viewed as the abstraction principle at work at the type level.

Details: In Chapter 1 it was mentioned that every expression in Haskell has an associated type. But with polymorphism, one might wonder if there is just one type for every expression. For example, *map* could have any of these types:

$(a \to b) \to [a] \to [b]$
$(Integer \to b) \to [Integer] \to [b]$
$(a \to Float) \to [a] \to [Float]$
$(Char \to Char) \to [Char] \to [Char]$

and so on, depending on how it will be used. However, notice that the first of these types is in some fundamental sense more general than the other three. In fact, every expression in Haskell has a unique type known as its *principal type*: the least general type that captures all valid uses of the expression. The first type above is the principal type of *map*, since it captures all valid uses of *map*, yet is less general than, for example, the type $a \to b \to c$. As another example, the principal type of *head* is $[a] \to a$; the types $[b] \to a$, $b \to a$, or even a are too general, whereas something like $[Integer] \to Integer$ is too specific. (The existence of unique principal types is the hallmark feature of the *Hindley-Milner type system* [Hin69, Mil78] that forms the basis of the type systems of Haskell, ML [MTH90] and several other functional languages [Hud89].)

3.2.2 Using map

For a musical example involving the use of *map*, consider the task of generating a six-note whole-tone scale starting at a given pitch:[1]

$$\begin{aligned} &wts \quad :: Pitch \to [Music\ Pitch] \\ &wts\ p = \textbf{let}\ f\ ap = note\ qn\ (pitch\ (absPitch\ p + ap)) \\ &\qquad\quad \textbf{in}\ map\ f\ [0,2,4,6,8] \end{aligned}$$

For example:

$$\begin{aligned} &wts\ a440 \\ &\Longrightarrow [note\ qn\ (A,4), \quad note\ qn\ (B,4), note\ qn\ (C\#,4), \\ &\qquad note\ qn\ (D\#,4), note\ qn\ (F,4), note\ qn\ (G,4)] \end{aligned}$$

[**To do:** Add exercises involving map.]

[1] A whole-tone scale is a sequence of six ascending notes, with each adjacent pair of notes separated by two semitones, i.e. a whole note.

3.3 Append

Consider now the problem of *concatenating* or *appending* two lists together; that is, creating a third list that consists of all of the elements from the first list followed by all of the elements of the second. Once again the type of list elements does not matter, so one can define this as a polymorphic infix operator $(+\!\!+)$:

$$(+\!\!+) :: [a] \to [a] \to [a]$$

For example, here are two uses of $(+\!\!+)$ on different types:

$$[1,2,3] +\!\!+ [4,5,6] \quad \Longrightarrow [1,2,3,4,5,6]$$
$$[C,E,G] +\!\!+ [D,F,A] \Longrightarrow [C,E,G,D,F,A]$$

As usual, one can approach this problem by considering the various possibilities that could arise as input. But in the case of $(+\!\!+)$ there are *two* inputs—so which should be considered first? In general this is not an easy question to answer, so one could just try the first list first: it could be empty, or non-empty. If it is empty the answer is easy:

$$[\,] +\!\!+ ys = ys$$

and if it is not empty the answer is also straightforward:

$$(x : xs) +\!\!+ ys = x : (xs +\!\!+ ys)$$

Note the recursive use of $(+\!\!+)$. The full definition is thus:

$$\begin{aligned}(+\!\!+) &:: [a] \to [a] \to [a]\\ [\,] &+\!\!+ ys = ys\\ (x : xs) &+\!\!+ ys = x : (xs +\!\!+ ys)\end{aligned}$$

Details: Note that an infix operator can be defined just as any other function, including pattern-matching, except that on the left-hand-side it is written using its infix syntax.

Also be aware that this textbook takes liberty in typesetting by displaying the append operator as $+\!\!+$. When you type your code, however, you will need to write ++. Recall that infix operators in Haskell must not contain any numbers or letters of the alphabet, and also must not begin with a colon (because those are reserved to be infix constructors).

If one were to have considered instead the second list first, then the first equation would still be easy:

$$xs +\!\!+ [\,] = xs$$

but the second is not so obvious:

$$xs +\!\!+ (y : ys) = ??$$

So it seems that the right choice was made to begin with.

Like *map*, the concatenation operator $(+\!\!+)$ is used so often that it is predefined in Haskell.

3.3.1 [Advanced] The Efficiency and Fixity of Append

In Chapter 10 the following simple property about $(+\!\!+)$ will be proved:

$$(xs +\!\!+ ys) +\!\!+ zs = xs +\!\!+ (ys +\!\!+ zs)$$

That is, $(+\!\!+)$ is *associative*.

But what about the efficiency of the left-hand and right-hand sides of this equation? It is easy to see via calculation that appending two lists together takes a number of steps proportional to the length of the first list (indeed the second list is not evaluated at all). For example:

$$\begin{aligned}
&[1,2,3] +\!\!+ xs \\
&\Rightarrow 1 : ([2,3] +\!\!+ xs) \\
&\Rightarrow 1 : 2 : ([3] +\!\!+ xs) \\
&\Rightarrow 1 : 2 : 3 : ([\,] +\!\!+ xs) \\
&\Rightarrow 1 : 2 : 3 : xs
\end{aligned}$$

Therefore the evaluation of $xs +\!\!+ (ys +\!\!+ zs)$ takes a number of steps proportional to the length of xs plus the length of ys. But what about $(xs +\!\!+ ys) +\!\!+ zs$? The leftmost append will take a number of steps proportional to the length of xs, but then the rightmost append will require a number of steps proportional to the length of xs plus the length of ys, for a total cost of:

$$2 * \mathit{length}\ xs + \mathit{length}\ ys$$

Thus $xs +\!\!+ (ys +\!\!+ zs)$ is more efficient than $(xs +\!\!+ ys) +\!\!+ zs$. This is why the Standard Prelude defines the fixity of $(+\!\!+)$ as:

infixr 5 $+\!\!+$

In other words, if you just write $xs +\!\!+ ys +\!\!+ zs$, you will get the most efficient association, namely the right association $xs +\!\!+ (ys +\!\!+ zs)$. In the next section a more dramatic example of this property will be introduced.

3.4 Fold

Suppose one wishes to take a list of notes (each of type *Music a*) and convert them into a *line*, or *melody*. One can define a recursive function to do this:

$$\begin{aligned} &\mathit{line} && :: [\mathit{Music}\ a] \to \mathit{Music}\ a \\ &\mathit{line}\ [\,] && = \mathit{rest}\ 0 \\ &\mathit{line}\ (m : ms) && = m :+: \mathit{line}\ ms \end{aligned}$$

Note that this function is polymorphic—the first example so far, in fact, of a polymorphic function involving the *Music* data type.

One might also wish to have a function *chord* that operates in an analogous way, but using (:=:) instead of (:+:):

$$\begin{aligned} &\mathit{chord} && :: [\mathit{Music}\ a] \to \mathit{Music}\ a \\ &\mathit{chord}\ [\,] && = \mathit{rest}\ 0 \\ &\mathit{chord}\ (m : ms) && = m :=: \mathit{chord}\ ms \end{aligned}$$

This function is also polymorphic.

[**To do:** Consider alternative to *maxPitch*, since it is more properly defined with *fold1*.]

In a completely different context one might wish to compute the highest pitch in a list of pitches, which one might capture in the following way:

$$\begin{aligned} &\mathit{maxPitch} && :: [\mathit{Pitch}] \to \mathit{Pitch} \\ &\mathit{maxPitch}\ [\,] && = \mathit{pitch}\ 0 \\ &\mathit{maxPitch}\ (p : ps) && = p\ !!!\ \mathit{maxPitch}\ ps \end{aligned}$$

where (!!!) is defined as:

$$p_1\ !!!\ p_2 = \textbf{if}\ \mathit{absPitch}\ p_1 > \mathit{absPitch}\ p_2\ \textbf{then}\ p_1\ \textbf{else}\ p_2$$

Details: An expression **if** *pred* **then** *cons* **else** *alt* is called a *conditional expression*. If *pred* (called the *predicate*) is true, then *cons* (called the *consequence*) is the result; if *pred* is false, then *alt* (called the *alternative*) is the result.

Once again this is a situation where several definitions share something in common: a repeating recursive pattern. Using the process used earlier to discover *map*, one first identifies those things that are changing. There are two situations: the *rest* 0 and *pitch* 0 values (for which the generic name *init*, for "initial value," will be used), and the (:+:), (:=:), and (!!!) operators (for which the generic name *op*, for "operator," will be used). Now rewriting

any of the above three functions as a new function—call it *fold*—that takes extra arguments *op* and *init*, one arrives at:[2]

$$\begin{aligned} &fold\ op\ init\ [\,] &&= init \\ &fold\ op\ init\ (x : xs) &&= x\ `op`\ fold\ op\ init\ xs \end{aligned}$$

Details: Any normal binary function name can be used as an infix operator by enclosing it in backquotes; $x\ `f`\ y$ is equivalent to $f\ x\ y$. Using infix application here for *op* better reflects the structure of the repeating pattern that is being abstracted, but could also have been written $op\ x\ (fold\ op\ init\ xs)$.

With this definition of *fold* one can now rewrite the definitions of *line*, *chord*, and *maxPitch* as:

$$\begin{aligned} &line, chord :: [Music\ a] \to Music\ a \\ &line\ \ ms = fold\ (:+:)\ (rest\ 0)\ ms \\ &chord\ ms = fold\ (:=:)\ (rest\ 0)\ ms \end{aligned}$$

$$\begin{aligned} &maxPitch\ \ :: [Pitch] \to Pitch \\ &maxPitch\ ps = fold\ (!!!)\ (pitch\ 0)\ ps \end{aligned}$$

Details: Just as one can turn a function into an operator by enclosing it in backquotes, one can turn an operator into a function by enclosing it in parentheses. This is required in order to pass an operator as a value to another function, as in the examples above. (If one wrote $fold\ !!!\ 0\ ps$ instead of $fold\ (!!!)\ 0\ ps$ it would look like one were trying to compare *fold* to $0\ ps$, which is nonsensical and ill-typed.)

In Chapter 10, induction is used to prove that these new definitions are equivalent to the old ones.

fold, like *map*, is a highly useful—reusable—function, as will be seen through several other examples later in the text. Indeed, it too is polymorphic, for note that it does not depend on the type of the list elements. Its most general type—somewhat trickier than that for *map*—is:

[2] The use of the name "*fold*" for this function is historical, and has nothing to do with the use of "fold" and "unfold" in Chapter 1 to describe steps in a calculation.

$$fold :: (a \to b \to b) \to b \to [a] \to b$$

This allows one to use *fold* whenever one needs to "collapse" a list of elements using a binary (i.e. two-argument) operator.

As a final example, recall the definition of *hList* from Chapter 1:

$$\begin{aligned} hList &:: Dur \to [Pitch] \to Music\ Pitch \\ hList\ d\ [\,] &= rest\ 0 \\ hList\ d\ (p : ps) &= hNote\ d\ p :+: hList\ d\ ps \end{aligned}$$

A little thought should convince the reader that this can be rewritten as:

$$\begin{aligned} hList\ d\ ps = &\textbf{let}\ f\ p = hNote\ d\ p \\ &\textbf{in}\ line\ (map\ f\ ps) \end{aligned}$$

One could argue that this is more modular, since it avoids explicit recursion, and is instead built up from smaller building blocks, namely *line* (which uses *fold*) and *map*.

3.4.1 Haskell's Folds

Haskell actually defines two versions of *fold* in the Standard Prelude. The first is called *foldr* ("fold-from-the-right") whose definition is the same as that of *fold* given earlier:

$$\begin{aligned} foldr &:: (a \to b \to b) \to b \to [a] \to b \\ foldr\ op\ init\ [\,] &= init \\ foldr\ op\ init\ (x : xs) &= x\ `op`\ foldr\ op\ init\ xs \end{aligned}$$

A good way to think about *foldr* is that it replaces all occurrences of the list operator (:) with its first argument (a function), and replaces [ ] with its second argument. In other words:

$$\begin{aligned} &foldr\ op\ init\ (x_1 : x_2 : ... : xn : [\,]) \\ &\Longrightarrow x_1\ `op`\ (x_2\ `op`\ (...(xn\ `op`\ init)...)) \end{aligned}$$

This might help in better understanding the type of *foldr*, and also explains its name: the list is "folded from the right." Stated another way, for any list *xs*, the following always holds:[3]

$$foldr\ (:)\ [\,]\ xs \Longrightarrow xs$$

Haskell's second version of *fold* is called *foldl*:

$$\begin{aligned} foldl &:: (b \to a \to b) \to b \to [a] \to b \\ foldl\ op\ init\ [\,] &= init \\ foldl\ op\ init\ (x : xs) &= foldl\ op\ (init\ `op`\ x)\ xs \end{aligned}$$

[3] This will be formally proved in Chapter 10.

A good way to think about *foldl* is to imagine "folding the list from the left:"

$$\begin{aligned}&\mathit{foldl}\ \mathit{op}\ \mathit{init}\ (x_1 : x_2 : ... : xn : [\,])\\ &\Longrightarrow (...((\mathit{init}\ `\mathit{op}`\ x_1)\ `\mathit{op}`\ x_2)...)\ `\mathit{op}`\ xn\end{aligned}$$

3.4.2 [Advanced] Why Two Folds?

Note that if one had used *foldl* instead of *foldr* in the definitions given earlier then not much would change; *foldr* and *foldl* would give the same result. Indeed, judging from their types, it looks like the only difference between *foldr* and *foldl* is that the operator takes its arguments in a different order.

So why does Haskell define two versions of *fold*? It turns out that there are situations where using one is more efficient, and possibly "more defined," than the other (that is, the function terminates on more values of its input domain) .

Probably the simplest example of this is a generalization of the associativity of $(+\!\!+)$ discussed in the last section. Suppose that one wishes to collapse a list of lists into one list. The Standard Prelude defines the polymorphic function *concat* for this purpose:

$$\begin{aligned}&\mathit{concat} \quad :: [[a]] \to [a]\\ &\mathit{concat}\ \mathit{xss} = \mathit{foldr}\ (+\!\!+)\ [\,]\ \mathit{xss}\end{aligned}$$

For example:

$$\begin{aligned}&\mathit{concat}\ [[1],[3,4],[\,],[5,6]]\\ &\Longrightarrow [1,3,4,5,6]\end{aligned}$$

More importantly, from the earlier discussion it should be clear that this property holds:

$$\begin{aligned}&\mathit{concat}\ [xs_1, xs_2, ..., xsn]\\ &\Rightarrow\ \ \mathit{foldr}\ (+\!\!+)\ [\,]\ [xs_1, xs_2, ..., xsn]\\ &\Longrightarrow xs_1 +\!\!+ (xs_2 +\!\!+ (...(xn +\!\!+ [\,]))...)\end{aligned}$$

The total cost of this computation is proportional to the sum of the lengths of all of the lists. If each list has the same length *len*, and there are n lists, then this cost is $(n-1) * len$.

On the other hand, if one had defined *concat* this way:

$$\mathit{slowConcat}\ \mathit{xss} = \mathit{foldl}\ (+\!\!+)\ [\,]\ \mathit{xss}$$

then:

$$\mathit{slowConcat}\ [xs_1, xs_2, ..., xsn]$$

$$\begin{aligned} &\Rightarrow \quad \mathit{foldl}\ (+\!\!+)\ [\,]\ [\mathit{xs}_1, \mathit{xs}_2, ..., \mathit{xsn}] \\ &\Longrightarrow (...(([\,] +\!\!+ x_1) +\!\!+ x_2)...) +\!\!+ \mathit{xn} \end{aligned}$$

If each list has the same length *len*, then the cost of this computation will be:

$$\begin{aligned} &\mathit{len} + (\mathit{len} + \mathit{len}) + (\mathit{len} + \mathit{len} + \mathit{len}) + ... + (n-1) * \mathit{len} \\ &= n * (n-1) * \mathit{len}/2 \end{aligned}$$

which is considerably worse than $(n-1) * \mathit{len}$. Thus the choice of *foldr* in the definition of *concat* is quite important.

Similar examples can be given to demonstrate that *foldl* is sometimes more efficient than *foldr*. On the other hand, in many cases the choice does not matter at all (consider, for example, (+)). The moral of all this is that care must be taken in the choice between *foldr* and *foldl* if efficiency is a concern.

3.4.3 Fold for Non-empty Lists

In certain contexts it may be understood that the functions *line* and *chord* should not be applied to an empty list. For such situations the Standard Prelude provides functions *foldr1* and *foldl1*, which return an error if applied to an empty list. And thus one may desire to define versions of *line* and *chord* that adopt this behavior:

$$\begin{aligned} &\mathit{line1}, \mathit{chord1} :: [\mathit{Music}\ a] \to \mathit{Music}\ a \\ &\mathit{line1}\ \mathit{ms} \quad\ = \mathit{foldr1}\ (:+:)\ \mathit{ms} \\ &\mathit{chord1}\ \mathit{ms} \quad = \mathit{foldr1}\ (:=:)\ \mathit{ms} \end{aligned}$$

Note that *foldr1* and *foldl1* do not take an *init* argument.

In the case of *maxPitch* one could go a step further and say that the previous definition is in fact flawed, for who is to say what the maximum pitch of an empty list is? The choice of 0 was indeed arbitrary, and in a way it is nonsensical—how can 0 be the maximum if it is not even in the list? In such situations one might wish to define only one function, and to have that function return an error when presented with an empty list. For consistency with *line* and *chord*, however, that function is defined here with a new name:

$$\begin{aligned} &\mathit{maxPitch1} \quad\ :: [\mathit{Pitch}] \to \mathit{Pitch} \\ &\mathit{maxPitch1}\ \mathit{ps} = \mathit{foldr1}\ (!!!)\ \mathit{ps} \end{aligned}$$

3.5 [Advanced] A Final Example: Reverse

As a final example of a useful list function, consider the problem of *reversing* a list, which will be captured in a function called *reverse*. This could be useful, for example, when constructing the *retrograde* of a musical passage, i.e. the music as if it were played backwards. For example, *reverse* $[C, Cs, Df]$ is $[Df, Cs, C]$.

Thus *reverse* takes a single list argument, whose possibilities are the normal ones for a list: it is either empty, or it is not. And thus:

$$\begin{aligned} &\mathit{reverse} && :: [a] \to [a] \\ &\mathit{reverse}\ [\,] && = [\,] \\ &\mathit{reverse}\ (x : xs) && = \mathit{reverse}\ xs +\!\!+ [x] \end{aligned}$$

This, in fact, is a perfectly good definition for *reverse*—it is certainly clear—except for one small problem: it is terribly inefficient! To see why, first recall that the number of steps needed to compute $xs +\!\!+ ys$ is proportional to the length of xs. Now suppose that the list argument to *reverse* has length n. The recursive call to *reverse* will return a list of length $n-1$, which is the first argument to $(+\!\!+)$. Thus the cost to reverse a list of length of n will be proportional to $n-1$ plus the cost to reverse a list of length $n-1$. So the total cost is proportional to $(n-1)+(n-2)+\cdots+1 = n(n-1)/2$, which in turn is proportional to the square of n.

Can one do better than this? In fact, yes.

There is another algorithm for reversing a list, which can be described intuitively as follows: take the first element, and put it at the front of an empty auxiliary list; then take the next element and add it to the front of the auxiliary list (thus the auxiliary list now consists of the first two elements in the original list, but in reverse order); then do this again and again until the end of the original list is reached. At that point the auxiliary list will be the reverse of the original one.

This algorithm can be expressed recursively, but the auxiliary list implies that one needs a function that takes *two* arguments—the original list and the auxiliary one—yet *reverse* only takes one. This can be solved by creating an auxiliary function *rev*:

$$\begin{aligned} \mathit{reverse}\ xs = \mathbf{let}\ &\mathit{rev}\ \mathit{acc}\ [\,] && = \mathit{acc} \\ &\mathit{rev}\ \mathit{acc}\ (x : xs) && = \mathit{rev}\ (x : \mathit{acc})\ xs \\ \mathbf{in}\ &\mathit{rev}\ [\,]\ xs \end{aligned}$$

The auxiliary list is the first argument to *rev*, and is called *acc* since it

behaves as an "accumulator" of the intermediate results. Note how it is returned as the final result once the end of the original list is reached.

A little thought should convince the reader that this function does not have the quadratic (n^2) behavior of the first algorithm, and indeed can be shown to execute a number of steps that is directly proportional to the length of the list, which one can hardly expect to improve upon.

But now, compare the definition of *rev* with the definition of *foldl*:

$$\begin{aligned} &\mathit{foldl}\ \mathit{op}\ \mathit{init}\ [\,] &&= \mathit{init} \\ &\mathit{foldl}\ \mathit{op}\ \mathit{init}\ (x : \mathit{xs}) &&= \mathit{foldl}\ \mathit{op}\ (\mathit{init}\ `\mathit{op}`\ x)\ \mathit{xs} \end{aligned}$$

They are somewhat similar. In fact, suppose one were to slightly rewrite *rev*, yielding:

$$\begin{aligned} &\mathit{rev}\ \mathit{op}\ \mathit{acc}\ [\,] &&= \mathit{acc} \\ &\mathit{rev}\ \mathit{op}\ \mathit{acc}\ (x : \mathit{xs}) &&= \mathit{rev}\ \mathit{op}\ (\mathit{acc}\ `\mathit{op}`\ x)\ \mathit{xs} \end{aligned}$$

Now *rev* looks strongly like *foldl*, and the question becomes whether or not there is a function that can be substituted for *op* that would make the latter definition of *rev* equivalent to the former one. Indeed there is:

$$\mathit{revOp}\ a\ b = b : a$$

For note that:

$$\begin{aligned} &\mathit{acc}\ `\mathit{revOp}`\ x \\ &\Rightarrow \mathit{revOp}\ \mathit{acc}\ x \\ &\Rightarrow x : \mathit{acc} \end{aligned}$$

So *reverse* can be rewritten as:

$$\begin{aligned} \mathit{reverse}\ \mathit{xs} = \textbf{let}\ &\mathit{rev}\ \mathit{op}\ \mathit{acc}\ [\,] &&= \mathit{acc} \\ &\mathit{rev}\ \mathit{op}\ \mathit{acc}\ (x : \mathit{xs}) &&= \mathit{rev}\ \mathit{op}\ (\mathit{acc}\ `\mathit{op}`\ x)\ \mathit{xs} \\ \textbf{in}\ &\mathit{rev}\ \mathit{revOp}\ [\,]\ \mathit{xs} \end{aligned}$$

which is the same as:

$$\mathit{reverse}\ \mathit{xs} = \mathit{foldl}\ \mathit{revOp}\ [\,]\ \mathit{xs}$$

If all of this seems like magic, well, you are starting to see the beauty of functional programming!

3.6 Currying

One can improve further upon some of the definitions given in this chapter using a technique called *currying simplification*. To understand this idea, first look closer at the notation used to write function applications, such as

simple x y z. Although, as noted earlier, this is similar to the mathematical notation $simple(x, y, z)$, in fact there is an important difference, namely that *simple x y z* is actually shorthand for $(((simple\ x)\ y)\ z)$. In other words, function application is *left associative*, taking one argument at a time.

Now look at the expression $(((simple\ x)\ y)\ z)$ a bit closer: there is an application of *simple* to x, the result of which is applied to y; so $(simple\ x)$ must be a function! The result of this application, $((simple\ x)\ y)$, is then applied to z, so $((simple\ x)\ y)$ must also be a function!

Since each of these intermediate applications yields a function, it seems perfectly reasonable to define a function such as:

$$multSumByFive = simple\ 5$$

What is *simple* 5? From the above argument it is clear that it must be a function. And from the definition of *simple* in Section 1 one might guess that this function takes two arguments, and returns 5 times their sum. Indeed, one can *calculate* this result as follows:

$$\begin{aligned} &multSumByFive\ a\ b \\ &\Rightarrow (simple\ 5)\ a\ b \\ &\Rightarrow simple\ 5\ a\ b \\ &\Rightarrow 5 * (a + b) \end{aligned}$$

The intermediate step with parentheses is included just for clarity. This method of applying functions to one argument at a time, yielding intermediate functions along the way, is called *currying*, after the logician Haskell B. Curry who popularized the idea.[4] It is helpful to look at the types of the intermediate functions as arguments are applied:

$$\begin{aligned} &simple && :: Float \to Float \to Float \to Float \\ &simple\ 5 && :: Float \to Float \to Float \\ &simple\ 5\ a && :: Float \to Float \\ &simple\ 5\ a\ b && :: Float \end{aligned}$$

For a musical example of this idea, recall the function $note :: Dur \to Pitch \to Music\ Pitch$. So *note qn* is a function that, given a pitch, yields a quarter note rendition of that pitch. A common use of this idea is simplifying something like:

$$note\ qn\ p_1\ :+:\ note\ qn\ p_2\ :+:\ ...\ :+:\ note\ qn\ pn$$

to:

[4] It was actually Schönfinkel who first called attention to this idea [Sch24], but the word "schönfinkelling" is rather a mouthful!

$$line\ (map\ (note\ qn)\ [p_1, p_2, ..., pn])$$

Indeed, this idea is used extentively in the larger example in the next chapter.

3.6.1 Currying Simplification

One can also use currying to improve some of the previous function definitions as follows. Suppose that the expressions $f\ x$ and $g\ x$ are the same, for all values of x. Then it seems clear that the functions f and g are equivalent.[5] So, if one wishes to define f in terms of g, instead of writing:

$$f\ x = g\ x$$

one could instead simply write:

$$f = g$$

One can Apply this reasoning to the definitions of *line* and *chord* from Section 3.4:

$$\begin{aligned} line\ ms &= fold\ (:+:)\ (rest\ 0)\ ms \\ chord\ ms &= fold\ (:=:)\ (rest\ 0)\ ms \end{aligned}$$

Since function application is left associative, one can rewrite these as:

$$\begin{aligned} line\ ms &= (fold\ (:+:)\ (rest\ 0))\ ms \\ chord\ ms &= (fold\ (:=:)\ (rest\ 0))\ ms \end{aligned}$$

But now applying the same reasoning here as was used for f and g above means that one can write these simply as:

$$\begin{aligned} line &= fold\ (:+:)\ (rest\ 0) \\ chord &= fold\ (:=:)\ (rest\ 0) \end{aligned}$$

Similarly, the definitions of *toAbsPitches* and *toPitches* from Section 3.2:

$$\begin{aligned} toAbsPitches\ ps &= map\ absPitch\ ps \\ toPitches\ \ \ as &= map\ pitch\ as \end{aligned}$$

can be rewritten as:

$$\begin{aligned} toAbsPitches &= map\ absPitch \\ toPitches\ \ \ &= map\ pitch \end{aligned}$$

Furthermore, the definition *hList*, most recently defined as:

$$\begin{aligned} hList\ d\ ps = &\textbf{ let } f\ p = hNote\ d\ p \\ &\textbf{ in } line\ (map\ f\ ps) \end{aligned}$$

can be rewritten as:

[5] In mathematics, one would say that the two functions are *extensionally* equivalent.

$$hList\ d\ ps = \textbf{let}\ f = hNote\ d \\ \textbf{in}\ line\ (map\ f\ ps)$$

and since the definition of f is now so simple, one might as well in-line it:

$$hList\ d\ ps = line\ (map\ (hNote\ d)\ ps)$$

This kind of simplification will be referred to as "currying simplification" or just "currying."[6]

Details: Some care should be taken when using this simplification idea. In particular, note that an equation such as $f\ x = g\ x\ y\ x$ cannot be simplified to $f = g\ x\ y$, since then the remaining x on the right-hand side would become undefined!

3.6.2 [Advanced] Simplification of *reverse*

Here is a more interesting example, in which currying simplification is used three times. Recall from Section 3.5 the definition of *reverse* using *foldl*:

$$reverse\ xs = \textbf{let}\ revOp\ acc\ x = x : acc \\ \textbf{in}\ foldl\ revOp\ [\,]\ xs$$

Using the polymorphic function *flip* which is defined in the Standard Prelude as:

$$flip \qquad :: (a \to b \to c) \to (b \to a \to c) \\ flip\ f\ x\ y = f\ y\ x$$

it should be clear that *revOp* can be rewritten as:

$$revOp\ acc\ x = flip\ (:)\ acc\ x$$

But now currying simplification can be used twice to reveal that:

$$revOp = flip\ (:)$$

This, along with a third use of currying, allows one to rewrite the definition of *reverse* simply as:

$$reverse = foldl\ (flip\ (:))\ [\,]$$

This is in fact the way *reverse* is defined in the Standard Prelude.

Exercise 3.1 Show that $flip\ (flip\ f)$ is the same as f.

[6] In the Lambda Calculus this is called "eta contraction."

Exercise 3.2 What is the type of *ys* in:

$$xs = [1, 2, 3] :: [Integer]$$
$$ys = map\ (+)\ xs$$

Exercise 3.3 Define a function *applyEach* that, given a list of functions, applies each to some given value. For example:

$$applyEach\ [simple\ 2\ 2, (+3)]\ 5 \Longrightarrow [14, 8]$$

where *simple* is as defined in Chapter 1.

Exercise 3.4 Define a function *applyAll* that, given a list of functions $[f_1, f_2, ..., fn]$ and a value v, returns the result $f_1\ (f_2\ (...(fn\ v)...))$. For example:

$$applyAll\ [simple\ 2\ 2, (+3)]\ 5 \Longrightarrow 20$$

Exercise 3.5 Recall the discussion about the efficiency of $(+\!\!+)$ and *concat* in Chapter 3. Which of the following functions is more efficient, and why?

$$appendr, appendl :: [[a]] \to [a]$$
$$appendr = foldr\ (flip\ (+\!\!+))\ [\,]$$
$$appendl = foldl\ (flip\ (+\!\!+))\ [\,]$$

3.7 Errors

The last section suggested the idea of "returning an error" when the argument to *foldr1* is the empty list. As you might imagine, there are other situations where an error result is also warranted.

There are many ways to deal with such situations, depending on the application, but sometimes one wishes to literally stop the program, signalling to the user that some kind of an *error* has occurred. In Haskell this is done with the Standard Prelude function $error :: String \to a$. Note that *error* is polymorphic, meaning that it can be used with any data type. The value of the expression *error s* is $\perp$, the completely undefined, or "bottom" value. As an example of its use, here is the definition of *foldr1* from the Standard Prelude:

$$foldr1 :: (a \to a \to a) \to [a] \to a$$
$$foldr1\ f\ [x] = x$$

$$\begin{aligned} &\mathit{foldr1}\ f\ (x : xs) = f\ x\ (\mathit{foldr1}\ f\ xs) \\ &\mathit{foldr1}\ f\ [\,] \quad\ = \mathit{error}\ \texttt{"Prelude.foldr1: empty list"} \end{aligned}$$

Thus if the anomalous situation arises, the program will terminate immediately, and the string `"Prelude.foldr1: empty list"` will be printed.

Exercise 3.6 Rewrite the definition of *length* non-recursively.

Exercise 3.7 Define a function that behaves as each of the following:

a) Doubles each number in a list. For example:

$$\mathit{doubleEach}\ [1, 2, 3] \Longrightarrow [2, 4, 6]$$

b) Pairs each element in a list with that number and one plus that number. For example:

$$\mathit{pairAndOne}\ [1, 2, 3] \Longrightarrow [(1, 2), (2, 3), (3, 4)]$$

c) Adds together each pair of numbers in a list. For example:

$$\mathit{addEachPair}\ [(1, 2), (3, 4), (5, 6)] \Longrightarrow [3, 7, 11]$$

d) Adds "pointwise" the elements of a list of pairs. For example:

$$\mathit{addPairsPointwise}\ [(1, 2), (3, 4), (5, 6)] \Longrightarrow (9, 12)$$

In the next two exercises, give both recursive and (if possible) non-recursive definitions, and be sure to include type signatures.

Exercise 3.8 Define a function *maxAbsPitch* that determines the maximum absolute pitch of a list of absolute pitches. Define *minAbsPitch* analogously. Both functions should return an error if applied to the empty list.

Exercise 3.9 Define a function $\mathit{chrom} :: \mathit{Pitch} \to \mathit{Pitch} \to \mathit{Music\ Pitch}$ such that *chrom* p_1 p_2 is a chromatic scale of quarter-notes whose first pitch is p_1 and last pitch is p_2. If $p_1 > p_2$, the scale should be descending, otherwise it should be ascending. If $p_1 == p_2$, then the scale should contain just one note. (A chromatic scale is one whose successive pitches are separated by one absolute pitch (i.e. one semitone)).

Exercise 3.10 Abstractly, a scale can be described by the intervals between successive notes. For example, the 7-note major scale can be defined as the

sequence of 6 intervals $[2, 2, 1, 2, 2, 2]$, and the 12-note chromatic scale by the 11 intervals $[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$. Define a function $mkScale :: Pitch \rightarrow [Int] \rightarrow Music\ Pitch$ such that $mkScale\ p\ ints$ is the scale beginning at pitch p and having the intervallic structure $ints$.

Exercise 3.11 Define an enumerated data type that captures each of the standard major scale modes: Ionian, Dorian, Phrygian, Lydian, Mixolydian, Aeolian, and Locrian. Then define a function *genScale* that, given one of these contructors, generates a scale in the intervalic form described in Exercise 3.10.

Exercise 3.12 Write the melody of "Frère Jacques" (or, "Are You Sleeping") in Euterpea. Try to make it as succinct as possible. Then, using functions already defined, generate a traditional four-part round, i.e. four identical voices, each delayed successively by two measures. Use a different instrument to realize each voice.

Exercise 3.13 Freddie the Frog wants to communicate privately with his girlfriend Francine by *encrypting* messages sent to her. Frog brains are not that large, so they agree on this simple strategy: each character in the text shall be converted to the character "one greater" than it, based on the representation described below (with wrap-around from 255 to 0). Define functions *encrypt* and *decrypt* that will allow Freddie and Francine to communicate using this strategy.

Characters are often represented inside a computer as some kind of an integer; in the case of Haskell, a 16-bit unicode representation is used. However, the standard keyboard is adequately represented by a standard byte (eight bits), and thus one only needs to consider the first 256 codes in the unicode representation. For this exercise, you will want to use two Haskell functions, *toEnum* and *fromEnum*. The first will convert an integer into a character, the second will convert a character into an integer.

Chapter 4

A Musical Interlude

At this point enough detail about Haskell and Euterpea has been covered that it is worth developing a small but full application or two. In this chapter an existing composition will be transcribed into Euterpea, thus exemplifying how to express conventional musical ideas in Euterpea. Then a simple form of algorithmic composition will be presented, where it will become apparent that more exotic things can easily be expressed as well.

But before tackling either of these, Haskell's *modules* will be described in more detail.

4.1 Modules

Haskell programs are partitioned into *modules* that capture common types, functions, etc. that naturally comprise an application. The first part of a module is called the module *header*, and in it one declares what the name of the module is, and what other modules it might import. For this chapter the module's name is *Interlude*, into which the module *Euterpea* is imported:

module *Euterpea.Examples.Interlude* **where**
import *Euterpea*

Details: Module names must always be capitalized (just like type names).

If one wishes to use this module in another module M, it may be imported into M, just as was done above in importing *Euterpea* into *Interlude*:

module M **where**
import *Interlude*

This will make available in M all of the names of functions, types, and so on that were defined at the top-level of *Interlude*.

But this isn't always what the programmer would like. Another purpose of a module is to manage the overall *name space* of an application. Modules allow one to structure an application in such a way that only the functionality intended for the end user is visible—everything else needed to implement the system is effectively hidden. In the case of *Interlude*, there are only a few names whose visibillity is desirable: *childSong6*, ..., and .... This can be achieved by writing the module header as follows:

module *Interlude* (*childSong6*, ...) **where**
import *Euterpea*

This set of visible names is sometimes called the *export list* of the module. And if type signatures are included, as in:

module *Interlude* (*childSong6* :: *Music Pitch*) **where**
import *Euterpea*

the list of names is sometimes called the *interface* to the module. If the list is omitted, as was done initially, then *all* names defined at the top-level of the module are exported.

There are many other rules concerning the import and export of names to and from modules. Rather than introduce them all at once, they will be introduced as needed in future chapter.

4.2 Transcribing an Existing Score

Figure 4.1 shows the first 28 bars of Chick Corea's *Child Song No. 6*, written for electric piano. Analyzing this tune explores several basic issues that arise in the transcription of an existing score into Euterpea, including repeating phrases, grace notes, triplets, tempo, and specifying an instrument. To begin, however, a couple of auxiliary functions are defined to make the job easier.

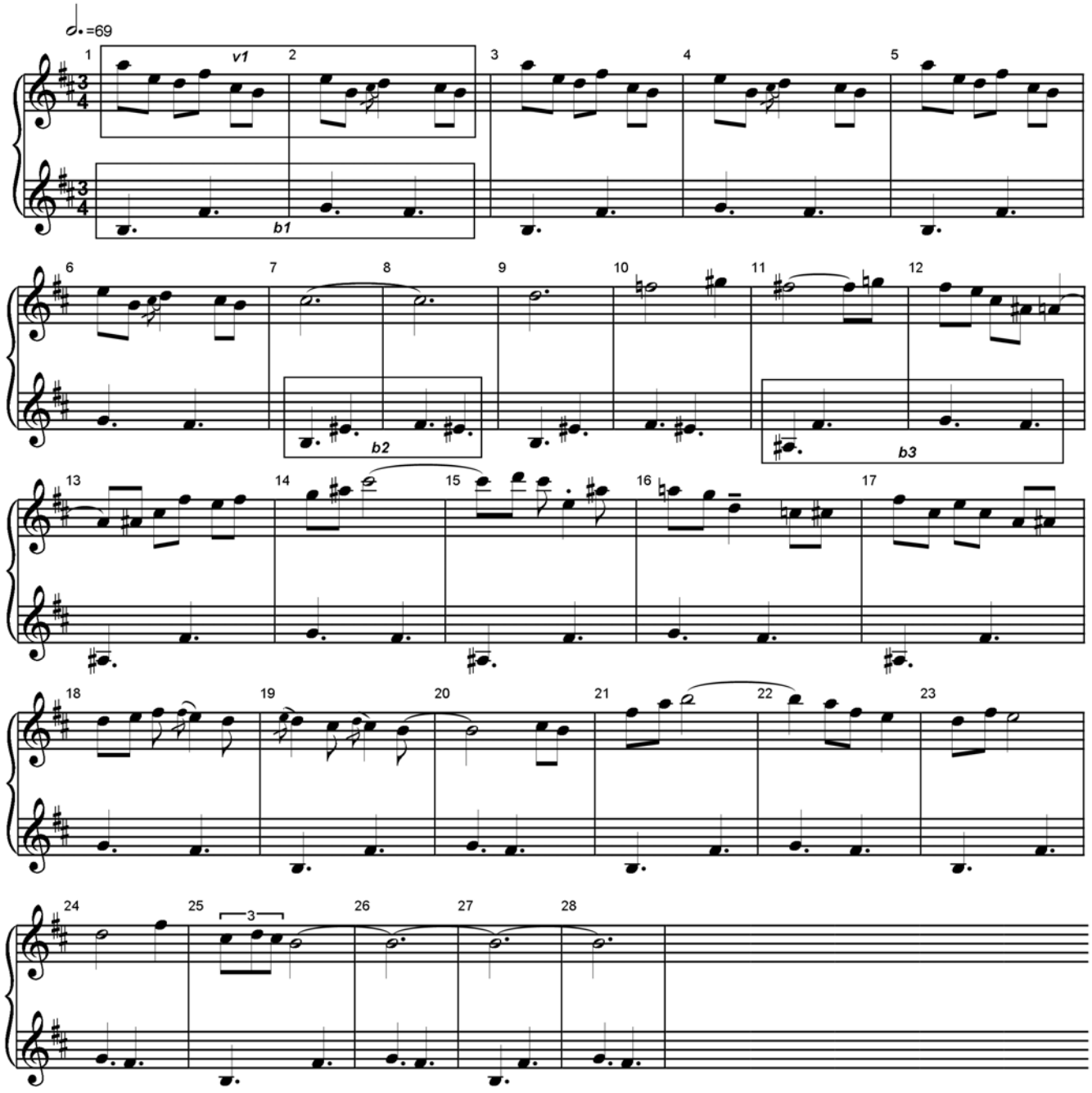


Figure 4.1: Excerpt from Chick Corea's *Child Song No. 6*

4.2.1 Auxiliary Functions

For starters, note that there are several repeating patterns of notes in this composition, each enclosed in a rectangle in Figure 4.1. In fact, the bass line consists *entirely* of three repeating phrases. In anticipation of this, a function can be defined that repeats a phrase a particular number of times:

$$
\begin{aligned}
&\mathit{timesM} && :: \mathit{Int} \to \mathit{Music}\ a \to \mathit{Music}\ a \\
&\mathit{timesM}\ 0\ m && = \mathit{rest}\ 0 \\
&\mathit{timesM}\ n\ m && = m :+: \mathit{timesM}\ (n-1)\ m
\end{aligned}
$$

Details: Note that pattern-matching can be used on numbers. As mentioned earlier, when there is more than one equation that defines a function, the first equation is tried first. If it fails, the second equation is tried, and so on. In the case above, if the first argument to *timesM* is not 0, the first equation will fail. The second equation is then tried, which always succeeds.

So, for example, $\mathit{timesM}\ 3\ b_1$ will repeat the baseline b_1 (to be defined shortly) three times.

To motivate the second auxiliary function, note in Figure 4.1 that there are many melodic lines that consist of a sequence of consecutive notes having the same duration (for example eighth notes in the melody, and dotted quarter notes in the bass). To avoid having to write each of these durations explicitly, a function that specifies them just once is defined. To do this, recall that $a\ 4\ qn$ is a concert A quarter note. Then note that, because of currying, $a\ 4$ is a function that can be applied to any duration—i.e. its type is $\mathit{Dur} \to \mathit{Music}\ a$. In other words, it's a note whose duration hasn't been specified yet.

With this thought in mind, one can return to the original problem and define a function that takes a duration and a *list* of notes with the aforementioned type, returning a *Music* value with the duration attached to each note appropriately. In Haskell:

$$
\begin{aligned}
&\mathit{addDur} && :: \mathit{Dur} \to [\mathit{Dur} \to \mathit{Music}\ a] \to \mathit{Music}\ a \\
&\mathit{addDur}\ d\ ns && = \mathbf{let}\ f\ n = n\ d \\
&&& \quad \mathbf{in}\ \mathit{line}\ (\mathit{map}\ f\ ns)
\end{aligned}
$$

Finally, a function to add a grace note to a note is defined. Grace notes can approach the principal note from above or below; sometimes starting a

half-step away, and sometimes a whole step; and having a rhythmic interpretation that is to a large extent up to the performer. In the case of the six uses of grace notes in *Child Song No. 6*, the assumption will be that the grace note begins on the downbeat of the principal note, and thus its duration will subtract from that of the principal note. It will also be assumed that the grace note duration is 1/8 of that of the principal note. Thus the goal is to define a function:

$$graceNote :: Int \to Music\ Pitch \to Music\ Pitch$$

such that $graceNote\ n\ (note\ d\ p)$ is a *Music* value consisting of two notes, the first being the grace note whose duration is $d/8$ and whose pitch is n semitones higher (or lower if n is negative) than p, and the second being the principal note at pitch p but now with duration $7\ d/8$. In Haskell:

```
graceNote n (Prim (Note d p)) =
  note (d/8) (trans n p) :+: note (7 * d/8) p
graceNote n _                 =
  error "Can only add a grace note to a note."
```

Note that pattern-matching is performed against the nested constructors of *Prim* and *Note*—one cannot match against the application of a function such as *note*. Also note the error message—programs are not expected to ever apply *graceNote* to something other than a single note.

(In Chapter 6 a slightly more general form of *graceNote* will be defined.)

The only special cases that will not be handled using auxiliary functions are the single staccato on note four of bar fifteen, and the single portamento on note three of bar sixteen. These situations will be addressed differently in a later chapter.

4.2.2 Bass Line

With these auxilary functions now defined, the base line in Figure 4.1 can be defined by first noting the three repeating phrases (enclosed in rectangular boxes), which can be captured as follows:

$$b_1 = addDur\ dqn\ [b\ 3,\ \ fs\ 4,\ g\ 4,\ fs\ 4]$$
$$b_2 = addDur\ dqn\ [b\ 3,\ \ es\ 4, fs\ 4,\ es\ 4]$$
$$b_3 = addDur\ dqn\ [as\ 3, fs\ 4,\ g\ 4,\ fs\ 4]$$

Using *timesM* it is then easy to define the entire 28 bars of the base line:

```
bassLine = timesM 3 b1 :+: timesM 2 b2 :+:
           timesM 4 b3 :+: timesM 5 b1
```

4.2.3 Main Voice

The upper voice of this composition is a bit more tedious to define, but is still straightforward. At the highest level, it consists of the phrase v_1 in the first two bars (in the rectangular box) repeated three times, followed by the remaining melody, which will be named v_2:

$$mainVoice = timesM\ 3\ v_1 :+: v_2$$

The repeating phrase v_1 is defined by:

$$\begin{aligned} v_1 &= v_{1a} :+: graceNote\ (-1)\ (d\ 5\ qn) :+: v_{1b} \quad \text{-- bars 1-2} \\ v_{1a} &= addDur\ en\ [a\ 5, e\ 5, d\ 5, fs\ 5, cs\ 5, b\ 4, e\ 5, b\ 4] \\ v_{1b} &= addDur\ en\ [cs\ 5, b\ 4] \end{aligned}$$

Note the treatment of the grace note.

The remainder of the main voice, v_2, is defined in seven pieces:

$$v_2 = v_{2a} :+: v_{2b} :+: v_{2c} :+: v_{2d} :+: v_{2e} :+: v_{2f} :+: v_{2g}$$

with each of the pieces defined in Figure 4.2. Note that:

- The phrases are divided so as to (for the most part) line up with bar lines, for convenience. But it may be that this is not the best way to organize the music—for example, one could argue that the last two notes in bar 20 form a pick-up to the phrase that follows, and thus more logically fall with that following phrase. The organization of the Euterpea code in this way is at the discretion of the composer.
- The stacatto is treated by playing the qurater note as an eighth note; the portamento is ignored. As mentioned earlier, these ornamentations will be addressed differently in a later chapter.
- The triplet of eighth notes in bar 25 is addressed by scaling the tempo by a factor of $3/2$.

4.2.4 Putting It All Together

In the Preface to *Children's Songs – 20 Pieces for Keyboard* [Cor94], Chick Corea notes that, "Songs 1 through 15 were composed for the Fender Rhodes." Therefore the MIDI instrument *RhodesPiano* is a logical choice for the transcription of his composition. Furthermore, note that a dotted half-note is specified to have a metronome value of 69. By default, the *play* function in Euterpea uses a tempo equivalent to a quarter note having a

```
v2a = line [cs 5 (dhn + dhn), d 5 dhn,
            f 5 hn, gs 5 qn, fs 5 (hn + en), g 5 en]   -- bars 7-11
v2b = addDur en [fs 5, e 5, cs 5, as 4] :+: a 4 dqn :+:
      addDur en [as 4, cs 5, fs 5, e 5, fs 5]          -- bars 12-13
v2c = line [g 5 en, as 5 en, cs 6 (hn + en), d 6 en, cs 6 en] :+:
      e 5 en :+: enr :+:
      line [as 5 en, a 5 en, g 5 en, d 5 qn, c 5 en, cs 5 en]
                                                       -- bars 14-16
v2d = addDur en [fs 5, cs 5, e 5, cs 5,
                 a 4, as 4, d 5, e 5, fs 5]            -- bars 17-18.5
v2e = line [graceNote 2 (e 5 qn), d 5 en, graceNote 2 (d 5 qn), cs 5 en,
            graceNote 1 (cs 5 qn), b 4 (en + hn), cs 5 en, b 4 en]
                                                       -- bars 18.5-20
v2f = line [fs 5 en, a 5 en, b 5 (hn + qn), a 5 en, fs 5 en, e 5 qn,
            d 5 en, fs 5 en, e 5 hn, d 5 hn, fs 5 qn]  -- bars 21-23
v2g = tempo (3/2) (line [cs 5 en, d 5 en, cs 5 en]) :+:
      b 4 (3 * dhn + hn)                               -- bars 24-28
```

Figure 4.2: Bars 7-28

metronome value of 120. Therefore the tempo should be scaled by a factor of $(dhn\,/\,qn) * (69/120)$.

These two observations lead to the final definition of the transcription of *Children's Song No. 6* into Euterpea:

```
childSong6 :: Music Pitch
childSong6 = let t = (dhn / qn) * (69/120)
             in instrument RhodesPiano
                          (tempo t (bassLine :=: mainVoice))
```

The intent is that this is the only value that will be of interest to users of this module, and thus *childSong6* is the only name exported from this section of the module, as discussed in Section 4.1.

This example can be played through the command *play childSong6*.

Exercise 4.1 Find a simple piece of music written by your favorite composer, and transcribe it into Euterpea. In doing so, look for repeating patterns, transposed phrases, etc. and reflect this in your code, thus revealing deeper structural aspects of the music than that found in common practice notation.

4.3 Simple Algorithmic Composition

TBD

Chapter 5

Syntactic Magic

This chapter introduces several more of Haskell's syntactic devices that faciliate writing concise and intuitive programs. These devices will be used frequently in the remainder of the text.

5.1 Sections

The use of currying was introduced in Chapter 3 as a way to simplify programs. This is a syntactic device that relies on the way that normal functions are applied, and how those applications are parsed.

With a bit more syntax, one can also curry applications of infix operators such as $(+)$. This syntax is called a *section*, and the idea is that, in an expression such as $(x + y)$, one can omit either the x or the y, and the result (with the parentheses still intact) is a function of that missing argument. If *both* variables are omitted, it is a function of *two* arguments. In other words, the expressions $(x+)$, $(+y)$ and $(+)$ are equivalent, respectively, to the functions:

$$\begin{aligned} f_1\ y &= x + y \\ f_2\ x &= x + y \\ f_3\ x\ y &= x + y \end{aligned}$$

For example, suppose one wishes to remove all absolute pitches greater than 99 from a list, perhaps because everything above that value is assumed to be unplayable. There is a pre-defined function in Haskell that can help to achieve this:

$$\mathit{filter} :: (a \to \mathit{Bool}) \to [a] \to [a]$$

filter p *xs* returns a list for which each element x satisfies the predicate p; i.e. $p\ x$ is *True*.

Using *filter*, one can then write:

$$\begin{aligned}&\mathit{playable} &&:: [\mathit{AbsPitch}] \to [\mathit{AbsPitch}]\\ &\mathit{playable}\ \mathit{xs} &&= \textbf{let}\ \mathit{test}\ \mathit{ap} = \mathit{ap} < 100\\ &&&\quad\ \textbf{in}\ \mathit{filter}\ \mathit{test}\ \mathit{xs}\end{aligned}$$

But using a section, one can write this more succinctly as:

$$\begin{aligned}&\mathit{playable} &&:: [\mathit{AbsPitch}] \to [\mathit{AbsPitch}]\\ &\mathit{playable}\ \mathit{xs} &&= \mathit{filter}\ (<100)\ \mathit{xs}\end{aligned}$$

which can be further simplified using currying:

$$\begin{aligned}&\mathit{playable} &&:: [\mathit{AbsPitch}] \to [\mathit{AbsPitch}]\\ &\mathit{playable} &&= \mathit{filter}\ (<100)\end{aligned}$$

This is an extremely concise definition. As you gain experience with higher-order functions you will not only be able to start writing definitions such as this directly, but you will also start *thinking* in "higher-order" terms. Many more examples of this kind of reasoning will appear throughout the text.

Exercise 5.1 Define a function *twice* that, given a function f, returns a function that applies f twice to its argument. For example:

$$(\mathit{twice}\ (+1))\ 2 \Rightarrow 4$$

What is the principal type of *twice*? Describe what *twice twice* does, and give an example of its use. Also consider the functions *twice twice twice* and *twice* (*twice twice*)?

Exercise 5.2 Generalize *twice* defined in the previous exercise by defining a function *power* that takes a function f and an integer n, and returns a function that applies the function f to its argument n times. For example:

$$\mathit{power}\ (+2)\ 5\ 1 \Longrightarrow 11$$

Use *power* in a musical context to define something useful.

5.2 Anonymous Functions

Another way to define a function in Haskell is in some sense the most fundamental: it is called an *anonymous function*, or *lambda expression* (since the concept is drawn directly from Church's lambda calculus [Chu41]). The idea is that functions are values, just like numbers and characters and strings, and therefore there should be a way to create them without having to give them a name. As a simple example, an anonymous function that increments its numeric argument by one can be written $\lambda x \rightarrow x + 1$. Anonymous functions are most useful in situations where you don't wish to name them, which is why they are called "anonymous." Anonymity is a property also shared by sections, but sections can only be derived from an existing infix operator.

Details: The typesetting used in this textbook prints an actual Greek lambda character, but in writing $\lambda x \rightarrow x + 1$ in your programs you will have to type "`\x -> x+1`" instead.

As another example, to raise the pitch of every element in a list of pitches *ps* by an octave, one could write:

$$map\ (\lambda p \rightarrow pitch\ (absPitch\ p + 12))\ ps$$

An even better example is an anonymous function that pattern-matches its argument, as in the following, which doubles the duration of every note in a list of notes *ns*:

$$map\ (\lambda(Note\ d\ p) \rightarrow Note\ (2 * d)\ p)\ ns$$

Details: Anonymous functions can only perform one match against an argument. That is, you cannot stack together several anonymous functions to define one function, as you can with equations.

Anonymous functions are considered most fundamental because definitions such as that for *simple* given in Chapter 1:

$$simple\ x\ y\ z = x * (y + z)$$

can be written instead as:

$$simple = \lambda x\ y\ z \rightarrow x * (y + z)$$

Details: $\lambda x\ y\ z \rightarrow exp$ is shorthand for $\lambda x \rightarrow \lambda y \rightarrow \lambda z \rightarrow exp$.

One can also use anonymous functions to explain precisely the behavior of sections. In particular, note that:

$$(x+) \Rightarrow \lambda y \rightarrow x + y$$
$$(+y) \Rightarrow \lambda x \rightarrow x + y$$
$$(+) \ \Rightarrow \lambda x\ y \rightarrow x + y$$

Exercise 5.3 Suppose one defines a function *fix* as:

$$fix\ f = f\ (fix\ f)$$

What is the principal type of *fix*? (This is tricky!) Suppose further that one has a recursive function:

$$\begin{aligned} &remainder \quad :: Integer \rightarrow Integer \rightarrow Integer \\ &remainder\ a\ b = \textbf{if}\ a < b\ \textbf{then}\ a \\ &\qquad\qquad\qquad\ \textbf{else}\ remainder\ (a - b)\ b \end{aligned}$$

Rewrite this function using *fix* so that it is not recursive. (Also tricky!) Do you think that this process can be applied to *any* recursive function?

5.3 List Comprehensions

Haskell has a convenient and intuitive way to define a list in such a way that it resembles the definition of a *set* in mathematics. For example, recall in the last chapter the definition of the function *addDur*:

$$\begin{aligned} &addDur \quad :: Dur \rightarrow [Dur \rightarrow Music\ a] \rightarrow Music\ a \\ &addDur\ d\ ns = \textbf{let}\ f\ n = n\ d \\ &\qquad\qquad\qquad\ \textbf{in}\ line\ (map\ f\ ns) \end{aligned}$$

Here *ns* is a list of notes, each of which does not have a duration yet assigned to it. If one thinks of this as a set, one might be led to write the following solution in mathematical notation:

$$\{n\ d \mid n \in ns\}$$

which can be read, "the set of all notes n d such that n is an element of ns." Indeed, using a Haskell *list comprehension* one can write almost exactly the same thing:

$$[n\ d \mid n \leftarrow ns]$$

The difference, of course, is that the above expression generates an (ordered) list in Haskell, not an (unordered) set in mathematics.

List comprehensions allow one to rewrite the definition of *addDur* much more succinctly and elegantly:

$$\begin{aligned} &addDur \quad :: Dur \rightarrow [Dur \rightarrow Music\ a] \rightarrow Music\ a \\ &addDur\ d\ ns = line\ [n\ d \mid n \leftarrow ns] \end{aligned}$$

Details: Liberty is again taken in type-setting by using the symbol $\leftarrow$ to mean "is an element of." When writing your programs, you will have to type "<-" instead.

The expression $[exp \mid x \leftarrow xs]$ is actually shorthand for the expression $map\ (\lambda x \rightarrow exp)\ xs$. The form $x \leftarrow xs$ is called a *generator*, and in general more than one is allowed, as in:

$$[(x, y) \mid x \leftarrow [0, 1, 2], y \leftarrow [\texttt{'a'}, \texttt{'b'}]]$$

which evaluates to the list:

$$[(0, \texttt{'a'}), (0, \texttt{'b'}), (1, \texttt{'a'}), (1, \texttt{'b'}), (2, \texttt{'a'}), (2, \texttt{'b'})]$$

The order here is important; that is, note that the left-most generator changes least quickly.

It is also possible to *filter* values as they are generated; for example, one can modify the above example to eliminate the odd integers in the first list:

$$[(x, y) \mid x \leftarrow [0, 1, 2], even\ x, y \leftarrow [\texttt{'a'}, \texttt{'b'}]]$$

where $even\ n$ returns $True$ if n is even. This example evaluates to:

$$[(0, \texttt{'a'}), (0, \texttt{'b'}), (2, \texttt{'a'}), (2, \texttt{'b'})]$$

Details: When reasoning about list comprehensions (e.g. when doing proof by calculation), one can use the following syntactic translation into pure functions:

$$
\begin{array}{ll}
[e \mid \mathit{True}] & = [e] \\
[e \mid q] & = [e \mid q, \mathit{True}] \\
[e \mid b, qs] & = \textbf{if } b \textbf{ then } [e \mid qs] \textbf{ else } [\,] \\
[e \mid p \leftarrow xs, qs] & = \textbf{let } ok\ p = [e \mid qs] \\
 & \qquad\quad ok\ \_ = [\,] \\
 & \quad \textbf{in } \mathit{concatMap}\ ok\ xs \\
[e \mid \textbf{let } \mathit{decls}, qs] & = \textbf{let } \mathit{decls} \textbf{ in } [e \mid qs]
\end{array}
$$

where q is a single qualifier, qs is a sequence of qualifiers, b is a Boolean, p is a pattern, and decls is a sequence of variable bindings (a feature of list comprehensions not explained earlier).

5.3.1 Arithmetic Sequences

Another convenient syntax for lists whose elements can be enumerated is called an *arithmetic sequence*. For example, the arithmetic sequence $[1\,..\,10]$ is equivalent to the list:

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$$

There are actually four different versions of arithmetic sequences, some of which generate *infinite* lists (whose use will be discussed in a later chapter). In the following, let $a = n' - n$:

$$
\begin{array}{ll}
[n\,..] & \text{-- infinite list } n,\ n+1,\ n+2,\ ... \\
[n, n'\,..] & \text{-- infinite list } n,\ n+a,\ n+2*a,\ ... \\
[n\,..\,m] & \text{-- finite list } n,\ n+1,\ n+2,\ ...,\ m \\
[n, n'\,..\,m] & \text{-- finite list } n,\ n+a,\ n+2*a,\ ...,\ m
\end{array}
$$

Arithmetic sequences are discussed in greater detail in Appendix B.

Exercise 5.4 Using list comprehensions, define a function:

$$\mathit{apPairs} :: [\mathit{AbsPitch}] \rightarrow [\mathit{AbsPitch}] \rightarrow [(\mathit{AbsPitch}, \mathit{AbsPitch})]$$

such that $\mathit{apPairs}\ \mathit{aps}_1\ \mathit{aps}_2$ is a list of all combinations of the absolute pitches in aps_1 and aps_2. Furthermore, for each pair $(\mathit{ap}_1, \mathit{ap}_2)$ in the result, the absolute value of $\mathit{ap}_1 - \mathit{ap}_2$ must be greater than two and less than eight.

Finally, write a function to turn the result of *apPairs* into a *Music Pitch* value by playing each pair of pitches in parallel, and stringing them all together sequentially. Try varying the rhythm by, for example, using an

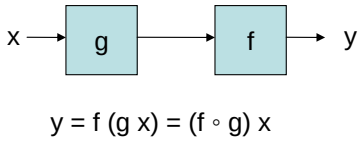


Figure 5.1: Gluing Two Functions Together

eighth note when the first absolute pitch is odd, and a sixteenth note when it is even, or some other criterion.

Test your functions by using arithemtic sequences to generate the two lists of arguments given to *apPairs*.

5.4 Function Composition

An example of polymorphism that has nothing to do with data structures arises from the desire to take two functions f and g and "glue them together," yielding another function h that first applies g to its argument, and then applies f to that result. This is called function *composition* (just as in mathematics), and Haskell pre-defines a simple infix operator $(\circ)$ to achieve it, as follows:

$$(\circ) \quad :: (b \to c) \to (a \to b) \to a \to c$$
$$(f \circ g)\ x = f\ (g\ x)$$

Details: The symbol for function composition is typeset in this textbook as $\circ$, which is consistent with mathematical convention. When writing your programs, however, you will have to use a period, as in "`f . g`".

Note the type of the operator $(\circ)$; it is completely polymorphic. Note also that the result of the first function to be applied—some type b—must be the same as the type of the argument to the second function to be applied. Pictorially, if one thinks of a function as a black box that takes input at one end and returns some output at the other, function composition is like connecting two boxes together, end to end, as shown in Figure 5.1.

The ability to compose functions using $(\circ)$ is quite handy. For example, recall the last version of *hList*:

$$hList\ d\ ps = line\ (map\ (hNote\ d)\ ps)$$

One can do two simplifications here. First, rewrite the right-hand side using function composition:

$$hList\ d\ ps = (line \circ map\ (hNote\ d))\ ps$$

Then, use currying simplification:

$$hList\ d = line \circ map\ (hNote\ d)$$

5.5 Higher-Order Thinking

It's worth taking a deep breath here and contemplating what has been done with *hList*, which has gone through quite a few transformations. Here is the original definition given in Chapter 1:

$$\begin{aligned} hList\ d\ [\,] &= rest\ 0 \\ hList\ d\ (p : ps) &= hNote\ d\ p :+: hList\ d\ ps \end{aligned}$$

Compare this to the definition above. You may be distressed to think that you have to go through all of these transformations just to write a relatively simple function! There are two points to make about this: First, you don't have to make *any* of these transformations if you don't want to. All of these versions of *hList* are correct, and they all run about equally fast. They are explained here for pedagogical purposes, so that you understand the full power of Haskell. Second, with practice, you will find that you can write the concise higher-order versions of many functions straight away, without going through all of the steps presented here.

As mentioned earlier, one thing that helps is to start *thinking* in "higher-order" terms. To facilitate this way of thinking one can write type signatures that reflect more closely their higher-order nature. For example, recall these type signatures for *map*, *filter*, and $(\circ)$:

$$\begin{aligned} map &:: (a \to b) \to [a] \to [b] \\ filter &:: (a \to Bool) \to [a] \to [a] \\ (\circ) &:: (b \to c) \to (a \to b) \to a \to c \end{aligned}$$

Also recall that the arrow in function types is right associative. Therefore, another completely equivalent way to write the above type signatures is:

$$\begin{aligned} map &:: (a \to b) \to ([a] \to [b]) \\ filter &:: (a \to Bool) \to ([a] \to [a]) \\ (\circ) &:: (b \to c) \to (a \to b) \to (a \to c) \end{aligned}$$

Although equivalent, these versions emphasize the fact that each of these

functions returns a function as its result. *map* essentially "lifts" a function on elements to a function on lists of elements. *filter* converts a predicate into a function on lists. And $(\circ)$ returns a function that is the composition of its two functional arguments.

So for example, using higher-order thinking, *map* $(+12)$ is a function that transposes a list of absolute pitches by one octave. *filter* (<100) is a function that removes all absolute pitches greater than or equal to 100 (as discussed earlier). And therefore *map* $(+12) \circ$ *filter* (<100) first does the filtering, and then does the transposition. All very consise and very natural using higher-order thinking.

In the remainder of this textbook definitions such as this will be written directly, using a small set of rich polymorphic functions such as *foldl*, *map*, *filter*, $(\circ)$, and a few other functions drawn from the Standard Prelude and other standard libraries.

5.6 Infix Function Application

Haskell predefines an infix operator to apply a function to a value:

$$f \$ x = f\ x$$

At first this doesn't seem very useful—after all, why wouldn't one simply write $f\ x$ instead of $f \$ x$?

But in fact this operator has a very useful purpose: eliminating parentheses! In the Standard Prelude, (\$) right associative, and to have the lowest precedence level, via the fixity declaration:

infixr 0 \$

Therefore, note that $f\ (g\ x)$ is the same as $f \$ g\ x$ (remember that normal function application always has higher precedence than infix operator application), and $f\ (x + 1)$ is the same as $f \$ x + 1$. This "trick" is especially useful when there is a sequence of nested, parenthesized expresssions. For example, recall the following definition from the last chapter:

$$\begin{aligned} \mathit{childSong6} = {} & \textbf{let}\ t = (\mathit{dhn} / \mathit{qn}) * (69/120) \\ & \textbf{in}\ \mathit{instrument}\ \mathit{RhodesPiano} \\ & \qquad (\mathit{tempo}\ t\ (\mathit{bassLine} :=: \mathit{mainVoice})) \end{aligned}$$

One can write the last few lines a bit more clearly as follows:

$$\begin{aligned} \mathit{childSong6} = {} & \textbf{let}\ t = (\mathit{dhn} / \mathit{qn}) * (69/120) \\ & \textbf{in}\ \mathit{instrument}\ \mathit{RhodesPiano} \$ \end{aligned}$$

$$\begin{array}{ll} tempo\ t & \$ \\ bassLine :=: mainVoice & \end{array}$$

Or, on a single line, instead of:

$$instrument\ RhodesPiano\ (tempo\ t\ (bassLine :=: mainVoice))$$

one can write:

$$instrument\ RhodesPiano\ \$\ tempo\ t\ \$\ bassLine :=: mainVoice$$

Exercise 5.5 The last definition of *hList* still has an argument d on the left-hand side, and one occurence of d on the right-hand side. Is there some way to eliminate it using currying simplification? (Hint: the answer is yes, but the solution is a bit perverse, and is not recommended as a way to write your code!)

Exercise 5.6 Use *line*, *map* and (\$) to give a concise definition of *addDur*.

Exercise 5.7 Rewrite this example:

$$map\ (\lambda x \rightarrow (x + 1)/2)\ xs$$

using a composition of sections.

Exercise 5.8 Consider the expression:

$$map\ f\ (map\ g\ xs)$$

Rewrite this using function composition and a single call to *map*. Then rewrite the earlier example:

$$map\ (\lambda x \rightarrow (x + 1)/2)\ xs$$

as a "map of a map" (i.e. using two maps).

Exercise 5.9 Go back to any exercises prior to this chapter, and simplify your solutions using ideas learned here.

Exercise 5.10 Using higher-order functions introduced in this chapter, fill in the two missing functions, f_1 and f_2, in the evaluation below so that it is valid:

$$f_1\ (f_2\ (*)\ [1, 2, 3, 4])\ 5 \Rightarrow [5, 10, 15, 20]$$

Chapter 6

More Music

```
module Euterpea.Music.Note.MoreMusic where
import Euterpea.Music.Note.Music
```

This chapter explores a number of simple musical ideas, and contributes to a growing collection of Euterpea functions for expressing those ideas.

6.1 Delay and Repeat

One can delay the start of a music value simply by inserting a rest in front of it, which can be packaged in a function as follows:

```
delayM       :: Dur → Music a → Music a
delayM d m = rest d :+: m
```

With *delayM* it is easy to write canon-like structures such as $m :=: delayM\ d\ m$, a song written in rounds (see Exercise 3.12), and so on.

Recall from Chapter 4 the function *timesM* that repeats a musical phrase a certain number of times:

```
timesM       :: Int → Music a → Music a
timesM 0 m  = rest 0
timesM n m = m :+: timesM (n − 1) m
```

More interestingly, Haskell's non-strict semantics allows one to define *infinite* musical values. For example, a musical value may be repeated *ad nauseam* using this simple function:

```
repeatM     :: Music a → Music a
```

```
repeatM m = m :+: repeatM m
```

Thus, for example, an infinite ostinato can be expressed in this way, and then used in different contexts that automatically extract only the portion that is actually needed. Functions that create such contexts will be described shortly.

6.2 Inversion and Retrograde

The notions of inversion, retrograde, retrograde inversion, etc. as used in twelve-tone theory are also easily captured in Euterpea. These terms are usually applied only to "lines" of notes, i.e. a melody (in twelve-tone theory it is called a "row"). The *retrograde* of a line is simply its reverse—i.e. the notes played in the reverse order. The *inversion* of a line is with respect to a given pitch (by convention usually the first pitch), where the intervals between successive pitches are inverted, i.e. negated. If the absolute pitch of the first note is ap, then each pitch p is converted into an absolute pitch $ap - (absPitch\ p - ap)$, in other words $2 * ap - absPitch\ p$.

[**To do:** Put in an example.]

To do all this in Haskell, a transformation from a line created by *line* to a list is defined:

```
lineToList                  :: Music a → [Music a]
lineToList (Prim (Rest 0))  = []
lineToList (n :+: ns)       = n : lineToList ns
lineToList _                =
  error "lineToList: argument not created by function line"
```

Using this function it is then straightforward to define *invert*:

```
invert :: Music Pitch → Music Pitch
invert m =
  let l@(Prim (Note _ r) : _) = lineToList m
      inv (Prim (Note d p)) =
                  note d (pitch (2 * absPitch r - absPitch p))
      inv (Prim (Rest d))   = rest d
  in line (map inv l)
```

Details: The pattern $l@(Prim\ (Note\ \_\ r) : \_)$ is called an *as pattern*. It behaves just like the pattern $Prim\ (Note\ \_\ r) : \_$ but additionally binds l to the value of a successful match to that pattern. l can then be used wherever it is in scope, such as in the last line of the function definition.

With *lineToList* and *invert* it is then easy to define the remaining functions via composition:

$$\begin{aligned} &retro, retroInvert, invertRetro :: Music\ Pitch \to Music\ Pitch \\ &retro \quad\quad = line \circ reverse \circ lineToList \\ &retroInvert \ = retro \circ invert \\ &invertRetro = invert \circ retro \end{aligned}$$

Exercise 6.1 Show that $retro \circ retro$, $invert \circ invert$, and $retroInvert \circ invertRetro$ are the identity on values created by *line*.

Exercise 6.2 Define a function $properRow :: Music\ Pitch \to Bool$ that determines whether or not its argument is a "proper" twelve-tone row, meaning that: (a) it must have exactly twelve notes, and (b) each unique pitch class is used exactly once (regardless of the octave). Enharmonically equivalent pitch classes are *not* considered unique. You may assume that the *Music Pitch* value is generated by the function *line*, but note that rests are allowed.

Exercise 6.3 Define a function $retroPitches :: Music\ Pitch \to Music\ Pitch$ that reverses the pitches in a line, but maintains the durations in the same order from beginning to end. For example:

$$\begin{aligned} &retroPitches\ (line\ [c\ 4\ en, d\ 4\ qn]) \\ &\Longrightarrow (line\ [d\ 4\ en, c\ 4\ qn]) \end{aligned}$$

6.3 Polyrhythms

For some rhythmical ideas, first note that if m is a line of three eighth notes, then $tempo\ (3/2)\ m$ is a *triplet* of eighth notes (recall that this idea was used in Chapter 4). In fact *tempo* can be used to create quite complex rhythmical patterns. For example, consider the "nested polyrhythms" shown in Figure

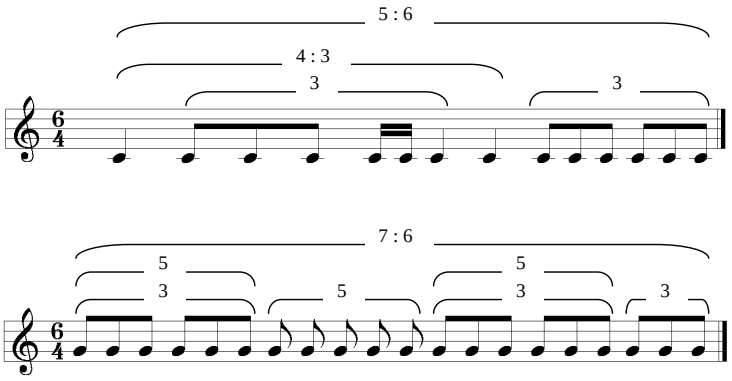


Figure 6.1: Nested Polyrhythms (top: pr_1; bottom: pr_2)

6.1. They can be expressed naturally in Euterpea as follows (note the use of a **let** clause in pr_2 to capture recurring phrases):

```
pr1, pr2 :: Pitch → Music Pitch
pr1 p = tempo (5/6)
        (tempo (4/3) (mkLn 1 p qn :+:
                      tempo (3/2) (mkLn 3 p en :+:
                                   mkLn 2 p sn :+:
                                   mkLn 1 p qn) :+:
                      mkLn 1 p qn) :+:
         tempo (3/2) (mkLn 6 p en))

pr2 p =
  let m1 = tempo (5/4) (tempo (3/2) m2 :+: m2)
      m2 = mkLn 3 p en
  in tempo (7/6) (m1 :+:
                  tempo (5/4) (mkLn 5 p en) :+:
                  m1 :+:
                  tempo (3/2) m2)
mkLn n p d = line $ take n $ repeat $ note d p
```

Details: *take* n *lst* is the first n elements of the list *lst*. For example:

$take\ 3\ [C, Cs, Df, D, Ds] \Longrightarrow [C, Cs, Df]$

repeat x is the infinite list of the same value x. For example:

$take\ 3\ (repeat\ 42) \Longrightarrow [42, 42, 42]$

To play polyrhythms pr_1 and pr_2 in parallel using middle C and middle G, respectively, one can write:

$$pr_{12} :: Music\ Pitch$$
$$pr_{12} = pr_1\ (C, 4) :=: pr_2\ (G, 4)$$

6.4 Symbolic Meter Changes

One can implement the notion of "symbolic meter changes" of the form "oldnote = newnote" (quarter note = dotted eighth, for example) by defining an infix function:

$$(=:=) \qquad :: Dur \to Dur \to Music\ a \to Music\ a$$
$$old =:= new = tempo\ (new\,/\,old)$$

Of course, using the new function is not much shorter than using *tempo* directly, but it may have mnemonic value.

6.5 Computing Duration

It is often desirable to compute the *duration*, in whole notes, of a musical value; one can do so as follows:

$$\begin{aligned}
&dur && :: Music\ a \to Dur\\
&dur\ (Prim\ (Note\ d\ _)) && = d\\
&dur\ (Prim\ (Rest\ d)) && = d\\
&dur\ (m_1 :+: m_2) && = dur\ m_1 + dur\ m_2\\
&dur\ (m_1 :=: m_2) && = dur\ m_1\ `max`\ dur\ m_2\\
&dur\ (Modify\ (Tempo\ r)\ m) && = dur\ m\,/\,r\\
&dur\ (Modify\ _\ m) && = dur\ m
\end{aligned}$$

The duration of a primitive value is obvious. The duration of $m_1 :+: m_2$ is the sum of the two, and the duration of $m_1 :=: m_2$ is the maximum of the two. The only tricky case is the duration of a music value that is modified by the

Tempo atttribute—in this case the duration must be scaled appropriately.

Note that the duration of a music value that is conceptually infinite in duration will be $\bot$, since *dur* will not terminate. (Similary, taking the length of an infinite list is $\bot$.) For example:

$$
\begin{aligned}
& dur\ (repeatM\ (a\ 4\ qn)) \\
& \Rightarrow dur\ (a\ 4\ qn :+: repeatM\ (a\ 4\ qn)) \\
& \Rightarrow dur\ (a\ 4\ qn) + dur\ (repeatM\ (a\ 4\ qn)) \\
& \Rightarrow qn + dur\ (repeatM\ (a\ 4\ qn)) \\
& \Rightarrow qn + qn + dur\ (repeatM\ (a\ 4\ qn)) \\
& \Rightarrow ... \\
& \Rightarrow \bot
\end{aligned}
$$

6.6 Super-retrograde

Using *dur* one can define a function *revM* that reverses any *Music* value whose duration is finite (and is thus considerably more useful than *retro* defined earlier):

$$
\begin{aligned}
& revM && :: Music\ a \to Music\ a \\
& revM\ n@(Prim\ \_) && = n \\
& revM\ (Modify\ c\ m) && = Modify\ c\ (revM\ m) \\
& revM\ (m_1 :+: m_2) && = revM\ m_2 :+: revM\ m_1 \\
& revM\ (m_1 :=: m_2) && = \\
& \quad \textbf{let}\ d_1 = dur\ m_1 \\
& \qquad\ d_2 = dur\ m_2 \\
& \quad \textbf{in if}\ d_1 > d_2\ \textbf{then}\ revM\ m_1 :=: (rest\ (d_1 - d_2) :+: revM\ m_2) \\
& \qquad\qquad\qquad \textbf{else}\ (rest\ (d_2 - d_1) :+: revM\ m_1) :=: revM\ m_2
\end{aligned}
$$

The first three cases are easy, but the last case is a bit tricky. The parallel constructor (:=:) implicitly begins each of its music values at the same time. But if one is shorter than the other, then, when reversed, a *rest* must be inserted before the shorter one, to account for the difference.

Note that *revM* of a *Music* value whose duration is infinite is $\bot$. (Similary, reversing an infinite list is $\bot$.)

6.7 Truncating Parallel Composition

Note that the duration of $m_1 :=: m_2$ is the maximum of the durations of m_1 and m_2 (and thus if one is infinite, so is the result). Sometimes one would rather have the result be of duration equal to the shorter of the two. This is not as easy as it sounds, since it may require truncating the longer one in the middle of a note (or notes).

The goal is to define a "truncating parallel composition" operator $(/=:)$, but first an auxiliary function *cut* will be defined such that $cut\ d\ m$ is the musical value m "cut short" to have at most duration d:

$$
\begin{aligned}
&cut :: Dur \to Music\ a \to Music\ a \\
&cut\ d\ m \mid d \leqslant 0 &&= rest\ 0 \\
&cut\ d\ (Prim\ (Note\ oldD\ p)) &&= note\ (min\ oldD\ d)\ p \\
&cut\ d\ (Prim\ (Rest\ oldD)) &&= rest\ (min\ oldD\ d) \\
&cut\ d\ (m_1 :=: m_2) &&= cut\ d\ m_1 :=: cut\ d\ m_2 \\
&cut\ d\ (m_1 :+: m_2) &&= \mathbf{let}\ m'_1 = cut\ d\ m_1 \\
&&&\qquad\ m'_2 = cut\ (d - dur\ m'_1)\ m_2 \\
&&&\quad \mathbf{in}\ m'_1 :+: m'_2 \\
&cut\ d\ (Modify\ (Tempo\ r)\ m) &&= tempo\ r\ (cut\ (d * r)\ m) \\
&cut\ d\ (Modify\ c\ m) &&= Modify\ c\ (cut\ d\ m)
\end{aligned}
$$

Note that *cut* is equipped to handle a *Music* value of infinite duration.

With *cut*, the definition of $(/=:)$ is now straightforward:

$$
\begin{aligned}
&(/=:) \quad :: Music\ a \to Music\ a \to Music\ a \\
&m_1\ /=:\ m_2 = cut\ (min\ (dur\ m_1)\ (dur\ m_2))\ (m_1 :=: m_2)
\end{aligned}
$$

Unfortunately, whereas *cut* can handle infinite-duration music values, $(/=:)$ cannot. This is because $(/=:)$ computes the duration of both of its arguments, but if m has infinite duration, then $dur\ m \Rightarrow \bot$. If, in a particular context, you know that only one of the two arguments is infinite, and you know which one (say m_1), it is always possible to do the following:

$$cut\ (dur\ m_2)\ m_1 :=: m_2$$

Exercise 6.4 Define a version of $(/=:)$ that shortens correctly when either one or the other of its arguments is infinite in duration. Assume that it is not known ahead of time which one is infinite.

Exercise 6.5 Define a version of $(/=:)$ that shortens correctly when either one or the other *or both* of its arguments are infinite in duration. When

they are both infinite, an infinite-duration result is returned. (This is much harder than the previous exercise.)

6.8 Trills

A *trill* is an ornament that alternates rapidly between two (usually adjacent) pitches. Two versions of a trill function will be defined, both of which take the starting note and an interval for the trill note as arguments (the interval is usually one or two, but can actually be anything). One version will additionally have an argument that specifies how long each trill note should be, whereas the other will have an argument that specifies how many trills should occur. In both cases the total duration will be the same as the duration of the original note.

Here is the first trill function:

```
trill :: Int → Dur → Music Pitch → Music Pitch
trill i sDur (Prim (Note tDur p)) =
  if sDur ⩾ tDur then note tDur p
                 else note sDur p :+:
                      trill (negate i) sDur
                            (note (tDur − sDur) (trans i p))
trill i d (Modify (Tempo r) m) = tempo r (trill i (d ∗ r) m)
trill i d (Modify c m)         = Modify c (trill i d m)
trill _ _ _                    =
  error "trill: input must be a single note."
```

Using this function it is simple to define a version that starts on the trill note rather than the start note:

```
trill′ :: Int → Dur → Music Pitch → Music Pitch
trill′ i sDur m = trill (negate i) sDur (transpose i m)
```

The second way to define a trill is in terms of the number of subdivided notes to be included in the trill. One can use the first trill function to define this new one:

```
trilln :: Int → Int → Music Pitch → Music Pitch
trilln i nTimes m = trill i (dur m / fromIntegral nTimes) m
```

This, too, can be made to start on the other note.

```
trilln′ :: Int → Int → Music Pitch → Music Pitch
trilln′ i nTimes m = trilln (negate i) nTimes (transpose i m)
```

```
ssfMel :: Music Pitch
ssfMel = line (l1 +++ l2 +++ l3 +++ l4)
  where l1 = [trilln 2 5 (bf 6 en), ef 7 en, ef 6 en, ef 7 en]
        l2 = [bf 6 sn, c 7 sn, bf 6 sn, g 6 sn, ef 6 en, bf 5 en]
        l3 = [ef 6 sn, f 6 sn, g 6 sn, af 6 sn, bf 6 en, ef 7 en]
        l4 = [trill 2 tn (bf 6 qn), bf 6 sn, denr]
starsAndStripes :: Music Pitch
starsAndStripes = instrument Flute ssfMel
```

Figure 6.2: Trills in *Stars and Stripes Forever*

Finally, a *roll* can be implemented as a trill whose interval is zero. This feature is particularly useful for percussion.

```
roll  :: Dur → Music Pitch → Music Pitch
rolln :: Int → Music Pitch → Music Pitch

roll dur m = trill 0 dur m
rolln nTimes m = trilln 0 nTimes m
```

Figure 6.2 shows a nice use of the trill functions in encoding the opening lines of John Philip Sousa's *Stars and Stripes Forever*.

6.9 Grace Notes

Recall from Chapter 4 the function *graceNote* to generate grace notes. A more general version is defined below, which takes a *Rational* argument that specifies that fraction of the principal note's duration to be used for the grace note's duration:

```
grace :: Int → Rational → Music Pitch → Music Pitch
grace n r (Prim (Note d p)) =
  note (r * d) (trans n p) :+: note ((1 - r) * d) p
grace n r _                 =
  error "grace: can only add a grace note to a note"
```

Thus *grace* n r (*note* d p) is a *Music* value consisting of two notes, the first being the grace note whose duration is $r * d$ and whose pitch is n semitones higher (or lower if n is negative) than p, and the second being the principal note at pitch p but now with duration $(1 - r) * d$.

Note that *grace* places the downbeat of the grace note at the point written for the principal note. Sometimes the interpretation of a grace note

is such that the downbeat of the principal note is to be unchanged. In that case, the grace note reduces the duration of the *previous* note. One can define a function *grace2* that takes two notes as arguments, and places the grace note appropriately:

$$
\begin{aligned}
&\mathit{grace2} :: \mathit{Int} \to \mathit{Rational} \to \\
&\qquad \mathit{Music\ Pitch} \to \mathit{Music\ Pitch} \to \mathit{Music\ Pitch} \\
&\mathit{grace2}\ n\ r\ (\mathit{Prim}\ (\mathit{Note}\ d_1\ p_1))\ (\mathit{Prim}\ (\mathit{Note}\ d_2\ p_2)) = \\
&\quad \mathit{note}\ (d_1 - r * d_2)\ p_1 :+: \mathit{note}\ (r * d_2)\ (\mathit{trans}\ n\ p_2) :+: \mathit{note}\ d_2\ p_2 \\
&\mathit{grace2}\ \_\ \_\ \_\ \_ = \\
&\quad \mathit{error}\ \texttt{"grace2: can only add a grace note to a note"}
\end{aligned}
$$

6.10 Percussion

Percussion is a difficult notion to represent in the abstract. On one hand, a percussion instrument is just another instrument, so why should it be treated differently? On the other hand, even common practice notation treats it specially, although it has much in common with non-percussive notation. The MIDI standard is equally ambiguous about the treatment of percussion: on one hand, percussion sounds are chosen by specifying an octave and pitch, just like any other instrument; on the other hand these pitches have no tonal meaning whatsoever: they are just a convenient way to select from a large number of percussion sounds. Indeed, part of the General MIDI Standard is a set of names for commonly used percussion sounds.

Since MIDI is such a popular platform, it is worth defining some handy functions for using the General MIDI Standard. In Figure 6.3 a data type is defined that borrows its constructor names from the General MIDI standard. The comments reflecting the "MIDI Key" numbers will be explained later, but basically a MIDI Key is the equivalent of an absolute pitch in Euterpea terminology. So all that remains to be done is a way to convert these percussion sound names into a *Music* value; i.e. a *Note*:

$$
\begin{aligned}
&\mathit{perc} :: \mathit{PercussionSound} \to \mathit{Dur} \to \mathit{Music\ Pitch} \\
&\mathit{perc}\ ps\ dur = \mathit{note}\ dur\ (\mathit{pitch}\ (\mathit{fromEnum}\ ps + 35))
\end{aligned}
$$

```
data PercussionSound =
    AcousticBassDrum   -- MIDI Key 35
  | BassDrum1          -- MIDI Key 36
  | SideStick          -- ...
  | AcousticSnare | HandClap      | ElectricSnare | LowFloorTom
  | ClosedHiHat   | HighFloorTom | PedalHiHat    | LowTom
  | OpenHiHat     | LowMidTom    | HiMidTom      | CrashCymbal1
  | HighTom       | RideCymbal1  | ChineseCymbal | RideBell
  | Tambourine    | SplashCymbal | Cowbell       | CrashCymbal2
  | Vibraslap     | RideCymbal2  | HiBongo       | LowBongo
  | MuteHiConga   | OpenHiConga  | LowConga      | HighTimbale
  | LowTimbale    | HighAgogo    | LowAgogo      | Cabasa
  | Maracas       | ShortWhistle | LongWhistle   | ShortGuiro
  | LongGuiro     | Claves       | HiWoodBlock   | LowWoodBlock
  | MuteCuica     | OpenCuica    | MuteTriangle
  | OpenTriangle          -- MIDI Key 82
  deriving (Show, Eq, Ord, Enum)
```

Figure 6.3: General MIDI Percussion Names

Details: *fromEnum* is an operator in the *Enum* class, which is all about enumerations, and will be discussed in more detail in Chapter 7. A data type that is a member of this class can be *enumerated*—i.e. the elements of the data type can be listed in order. *fromEnum* maps each value to its index in this enumeration. Thus *fromEnum AcousticBassDrum* is 0, *fromEnum BassDrum1* is 1, and so on.

Recall the *InstrumentName* data type from Chapter 2. If a *Music* value returned from *perc* is played using, say, the *AcousticGrandPiano* instrument, then you will hear an acounstic grand piano sound at the appropriate pitch. But if you specify the *Percussion* instrument, then you will here the percussion sound that was specified as an argument to *perc*.

For example, here are eight bars of a simple rock or "funk groove" that uses *perc* and *roll*:

```
funkGroove
  = let p1 = perc LowTom        qn
        p2 = perc AcousticSnare en
    in tempo 3 $ instrument Percussion $ cut 8 $ repeatM
       ((p1 :+: qnr :+: p2 :+: qnr :+: p2 :+:
```

$$p_1 :+: p_1 :+: qnr :+: p_2 :+: enr)$$
$$:=: roll\ en\ (perc\ ClosedHiHat\ 2))$$

Exercise 6.6 Write a program that generates all of the General MIDI percussion sounds, playing through each of them one at a time.

Exercise 6.7 Find a drum beat that you like, and express it in Euterpea. Then use *repeatM*, *cut*, and (:=:) to add a simple melody to it.

6.11 A Map for Music

Recall from Chapter 3 the definition of *map*:

$$\begin{aligned} &map && :: (a \to b) \to [a] \to [b] \\ &map\ f\ [\,] && = [\,] \\ &map\ f\ (x : xs) && = f\ x : map\ f\ xs \end{aligned}$$

This function is defined on the list data type. Is there something analogous for *Music*? I.e. a function:[1]

$$mMap :: (a \to b) \to Music\ a \to Music\ b$$

Such a function is indeed straightforward to define, but it helps to first define a map-like function for the *Primitive* type:

$$\begin{aligned} &pMap && :: (a \to b) \to Primitive\ a \to Primitive\ b \\ &pMap\ f\ (Note\ d\ x) && = Note\ d\ (f\ x) \\ &pMap\ f\ (Rest\ d) && = Rest\ d \end{aligned}$$

With *pMap* in hand one can now define *mMap*:

$$\begin{aligned} &mMap && :: (a \to b) \to Music\ a \to Music\ b \\ &mMap\ f\ (Prim\ p) && = Prim\ (pMap\ f\ p) \\ &mMap\ f\ (m_1 :+: m_2) && = mMap\ f\ m_1 :+: mMap\ f\ m_2 \\ &mMap\ f\ (m_1 :=: m_2) && = mMap\ f\ m_1 :=: mMap\ f\ m_2 \\ &mMap\ f\ (Modify\ c\ m) && = Modify\ c\ (mMap\ f\ m) \end{aligned}$$

Just as $map\ f\ xs$ for lists replaces each polymorphic element x in xs with $f\ x$, $mMap\ f\ m$ for *Music* replaces each polymorphic element p in m with $f\ p$.

[1] The name *mapM* would perhaps have been a better choice here, to be consistent with previous names. However, *mapM* is a predefined function in Haskell, and thus *mMap* is used instead. Similarly, Haskell's *Monad* library defines a function *foldM*, and thus in the next section the name *mFold* is used instead.

As an example of how *mMap* can be used, suppose that one introduces a *Volume* type for a note:

```
type Volume = Int
```

The goal is to convert a value of type *Music Pitch* into a value of type *Music* (*Pitch*, *Volume*)—that is, to pair each pitch with a volume attribute. One can define a function to do so as follows:

```
addVolume  :: Volume → Music Pitch → Music (Pitch, Volume)
addVolume v = mMap (λp → (p, v))
```

Exercise 6.8 Using *mMap*, define a function:

```
scaleVolume :: Rational → Music (Pitch, Volume)
                        → Music (Pitch, Volume)
```

such that *scaleVolume s m* scales the volume of each note in *m* by a factor of *s*.

6.12 A Fold for Music

One can also define a fold-like operator for *Music*. But whereas the list data type has only two constructors (the nullary constructor [] and the binary constructor (:)), *Music* has *four* constructors. Thus the following function takes four arguments in addition to the *Music* value it is transforming, instead of two:

```
mFold :: (Primitive a → b) → (b → b → b) → (b → b → b) →
         (Control → b → b) → Music a → b
mFold f (+:) (=:) g m =
  let rec = mFold f (+:) (=:) g
  in case m of
    Prim p      → f p
    m1 :+: m2   → rec m1 +: rec m2
    m1 :=: m2   → rec m1 =: rec m2
    Modify c m → g c (rec m)
```

This somewhat unwieldy function basically takes apart a *Music* value and puts it back together with different constructors. Indeed, note that:

```
mFold Prim (:+:) (:=:) Modify m == m
```

Although intuitive, proving this property requires induction, a proof technique discussed in Chapter 10.

To see how *mFold* might be used, note first of all that it is more general than *mMap*—indeed, *mMap* can be defined in terms of *mFold* like this:

```
mMap   :: (a → b) → Music a → Music b
mMap f = mFold g (:+:) (:=:) Modify where
  g (Note d x) = note d (f x)
  g (Rest d)   = rest d
```

More interestingly, one can use *mFold* to more succinctly define functions such as *dur* from Section 6.5:

```
dur :: Music a → Dur
dur = mFold getDur (+) max modDur where
  getDur (Note d _)     = d
  getDur (Rest d)       = d
  modDur (Tempo r) d = d/r
  modDur _ d            = d
```

Exercise 6.9 Redefine *revM* from Section 6.6 using *mFold*.

Exercise 6.10 Define a function *insideOut* that inverts the role of serial and parallel composition in a *Music* value. Using *insideOut*, see if you can (a) find a non-trivial value $m :: Music\ Pitch$ such that $m == insideOut\ m$ and (b) find a value $m :: Music\ Pitch$ such that:

```
m :+: insideOut m :+: m
```

sounds interesting. (You are free to define what "sounds interesting" means.)

6.13 Crazy Recursion

With all the functions and data types that have been defined, and the power of recursion and higher-order functions well understood, one can start to do some wild and crazy things. Here is just one such idea.

The goal is to Define a function to recursively apply transformations f (to elements in a sequence) and g (to accumulated phrases) some specified number of times:

```
rep :: (Music a → Music a) → (Music a → Music a) → Int
       → Music a → Music a
rep f g 0 m = rest 0
rep f g n m = m :=: g (rep f g (n - 1) (f m))
```

With this simple function one can create some interesting phrases of music with very little code. For example, *rep* can be used three times, nested together, to create a "cascade" of sounds:

$$\begin{aligned} run &= rep\ (transpose\ 5)\ (delayM\ tn)\ 8\ (c\ 4\ tn) \\ cascade &= rep\ (transpose\ 4)\ (delayM\ en)\ 8\ run \\ cascades &= rep\ id\ (delayM\ sn)\ 2\ cascade \end{aligned}$$

One can then make the cascade run up, and then down:

$$final = cascades :+: revM\ cascades$$

What happens if the f and g arguments are reversed?

$$\begin{aligned} run' &= rep\ (delayM\ tn)\ (transpose\ 5)\ 8\ (c\ 4\ tn) \\ cascade' &= rep\ (delayM\ en)\ (transpose\ 4)\ 8\ run' \\ cascades' &= rep\ (delayM\ sn)\ id\ 2\ cascade' \\ final' &= cascades' :+: revM\ cascades' \end{aligned}$$

Exercise 6.11 Consider this sequence of 8 numbers:

$$s_1 = [1, 5, 3, 6, 5, 0, 1, 1]$$

One might interpret this as a sequence of pitches, i.e. a melody. Another way to represent this sequence is as a sequence of 7 intervals:

$$s_2 = [4, -2, 3, -1, -5, 1, 0]$$

Together with the starting pitch (i.e. 1), this sequence of intervals can be used to reconstruct the original melody. But, with a suitable transposition to eliminate negative numbers, it can also be viewed as another melody. Indeed, one can repeat the process: s_2 can be represented by this sequence of 6 intervals:

$$s_3 = [-6, 5, -4, -4, 6, -1]$$

Together with the starting number (i.e. 4), s_3 can be used to reconstruct s_2. Continuing the process:

$$\begin{aligned} s_4 &= [11, -9, 0, 10, -7] \\ s_5 &= [-20, 9, 10, -17] \\ s_6 &= [29, 1, -27] \\ s_7 &= [-28, -28] \\ s_8 &= [0] \end{aligned}$$

Now, if one takes the first element of each of these sequences to form this 8-number sequence:

$$ic = [0, -28, 29, -20, 11, -6, 4, 1]$$

then it alone can be used to re-create the original 8-number sequence in its entirety. Of course, it can also be used as the original melody was used, and one could derive another 8-note sequence from it—and so on. The list *ic* will be referred to as the "interval closure" of the original list s_1.

Your job is to:

a) Define a function *toIntervals* that takes a list of n numbers, and generates a list of n lists, such that the i^{th} list is the sequence si as defined above.

b) Define a function *getHeads* that takes a list of n lists and returns a list of n numbers such that the i^{th} element is the head of the i^{th} list.

c) Compose the above two functions in a suitable way to define a function *intervalClosure* that takes an n-element list and returns its interval closure.

d) Define a function *intervalClosures* that takes an n-element list and returns an infinite sequence of interval closures.

e) Now for the open-ended part of this exercise: Interpret the outputs of any of the functions above to create some "interesting" music.

Exercise 6.12 Do something wild and crazy with Euterpea.

Chapter 7

Qualified Types and Type Classes

This chapter introduces the notions of *qualified types* and *type classes*. These concepts can be viewed as a refinement of the notion of polymorphism, and increase the ability to write modular programs.

7.1 Motivation

A polymorphic type such as $(a \to a)$ can be viewed as shorthand for $\forall(a)\, a \to a$, which can be read "*for all* types a, functions mapping elements of type a to elements of type a." Note the emphasis on "*for all*."

In practice, however, there are times when one would prefer to limit a polymorphic type to a smaller number of possibilities. A good example is a function such as (+). It's probably not a good idea to limit (+) to a *single* (that is, *monomorphic*) type such as $Integer \to Integer \to Integer$, since there are other kinds of numbers—such as rational and floating-point numbers—that one would like to perform addition on as well. Nor is it a good idea to have a different addition function for each number type, since that would require giving each a different name, such as *addInteger*, *addRational*, *addFloat*, etc. And, unfortunately, giving (+) a type such as $a \to a \to a$ will not work, since this would imply that one could add things other than numbers, such as characters, pitch classes, lists, tuples, functions, and any type that you might define on your own!

Haskell provides a solution to this problem through the use of *qualified*

types. Conceptually, one can think of a qualified type just as a polymorphic type, except that in place of "*for all* types a" it will be possible to say "for all types *a that are members of the type class* C," where the type class C can be thought of as a set of types. For example, suppose there is a type class *Num* with members *Integer*, *Rational*, and *Float*. Then an accurate type for $(+)$ would be $\forall(a{\in}Num)a \to a \to a$. But in Haskell, instead of writing $\forall(a \in Num)\cdots$, the notation $Num\ a \Rightarrow\cdots$ is used. So the proper type signature for $(+)$ is:

$$(+) :: Num\ a \Rightarrow a \to a \to a$$

which should be read: "for all types a that are members of the type class *Num*, $(+)$ has type $a \to a \to a$." Members of a type class are also called *instances* of the class, and these two terms will be used interchangeably in the remainder of the text. The $Num\ a \Rightarrow\cdots$ part of the type signature is often called a *context*, or *constraint*.

Details: It is important not to confuse *Num* with a data type or a constructor within a data type, even though the same syntax ("*Num a*") is used. *Num* is a *type class*, and the context of its use (namely, to the left of a $\Rightarrow$) is always sufficient to determine this fact.

Recall now the type signature given for the function *simple* in Chapter 1:

$$\begin{aligned} &simple && :: Integer \to Integer \to Integer \to Integer \\ &simple\ x\ y\ z && = x * (y + z) \end{aligned}$$

Note that *simple* uses the operator $(+)$ discussed above. It also uses $(*)$, whose type is the same as that for $(+)$:

$$(*) :: Num\ a \Rightarrow a \to a \to a$$

This suggests that a more general type for *simple* is:

$$\begin{aligned} &simple && :: Num\ a \Rightarrow a \to a \to a \to a \\ &simple\ x\ y\ z && = x * (y + z) \end{aligned}$$

Indeed, this is the preferred, most general type that can be given for *simple*. It can now be used with any type that is a member of the *Num* class, which includes *Integer*, *Int*, *Rational*, *Float* and *Double*, among others.

The ability to qualify polymorphic types is a unique feature of Haskell, and, as you will soon see, provides great expressiveness. In the following sections the idea is explored much more thoroughly, and in particular it is

shown how a programmer can define his or her own type classes and their instances. To begin, a closer look is taken of one of the pre-defined type classes in Haskell, having to do with equality.

7.2 Equality

Equality between two expressions e_1 and e_2 in Haskell means that the value of e_1 is the same as the value of e_2. Another way to view equality is that you should be able to substitute e_1 for e_2, or vice versa, wherever they appear in a program, without affecting the result of that program.

In general, however, it is not possible for a program to determine the equality of two expressions—consider, for example, determining the equality of two infinite lists, two infinite *Music* values, or two functions of type $Integer \rightarrow Integer$.[1] The ability to compute the equality of two values is called *computational equality*. Even though by the above simple examples it is clear that computational equality is strictly weaker than full equality, it is still an operation that one would like to use in many ordinary programs.

Haskell's operator for computational equality is (==). Partly because of the problem mentioned above, there are many types for which one would like equality defined, but some for which it might not make sense. For example, it is common to compare two characters, two integers, two floating-point numbers, etc. On the other hand, comparing the equality of infinite data structures, or functions, is difficult, and in general not possible. Thus Haskell has a type class called *Eq*, so that the equality operator (==) can be given the qualified type:

$$(==) :: Eq\ a \Rightarrow a \rightarrow a \rightarrow Bool$$

In other words, (==) is a function that, for any type a in the class *Eq*, tests two values of type a for equality, returning a Boolean (*Bool*) value as a result. Amongst *Eq*'s instances are the types *Char* and *Integer*, so that the following calculations hold:

$$\begin{aligned}
42 &== 42 &&\Rightarrow True\\
42 &== 43 &&\Rightarrow False\\
\texttt{'a'} &== \texttt{'a'} &&\Rightarrow True\\
\texttt{'a'} &== \texttt{'b'} &&\Rightarrow False
\end{aligned}$$

Furthermore, the expression $42 == \texttt{'a'}$ is *ill-typed*; Haskell is clever enough

[1] This is the same as determining *program equivalence*, a well-known example of an *undecideable problem* in the theory of algoriithms.

to know when qualified types are ill-formed.

One of the nice things about qualified types is that they work in the presence of ordinary polymorphism. In particular, the type constraints can be made to propagate through polymorphic data types. For example, because *Integer* and *Float* are members of *Eq*, so are the types $(Integer, Char)$, $[Integer]$, $[Float]$, etc. Thus:

$$[42, 43] == [42, 43] \Rightarrow True$$
$$[4.2, 4.3] == [4.3, 4.2] \Rightarrow False$$
$$(42, \texttt{'a'}) == (42, \texttt{'a'}) \Rightarrow True$$

This will be elaborated upon in a later section.

Type constraints also propagate through function definitions. For example, consider this definition of the function $\in$ that tests for membership in a list:

$$x \in [\,] = False$$
$$x \in (y : ys) = x == y \vee x \in ys$$

Details: $(\in)$ is actually written *elem* in Haskell; i.e. it is a normal function, not an infix operator. Of course it can be used in an infix manner (and it often is) by enclosing it in backquotes.

Note the use of $(==)$ on the right-hand side of the second equation. The principal type for $(\in)$ is thus:

$$\in :: Eq\ a \Rightarrow a \to [a] \to Bool$$

This should be read, "For every type a that is an instance of the class *Eq*, $(\in)$ has type $a \to [a] \to Bool$." This is exactly what one would hope fort—it expresses the fact that $(\in)$ is not defined on all types, just those for which computational equality is defined.

The above type for $(\in)$ is also its principal type, and Haskell will infer this type if no signature is given. Indeed, if one were to write the type signature:

$$(\in) :: a \to [a] \to Bool$$

a type error would result, because this type is fundamentally *too general*, and the Haskell type system will complain.

Details: On the other hand, you could write:

$(\in) :: Integer \to [Integer] \to Bool$

if you expect to use $(\in)$ only on lists of integers. In other words, using a type signature to constrain a value to be less general than its principal type is Ok.

As another example of this idea, a function that squares its argument:

$square\ x = x * x$

has principal type $Num\ a \Rightarrow a \to a$, since $(*)$, like $(+)$, has type $Num\ a \Rightarrow a \to a \to a$. Thus:

$square\ 42 \Rightarrow 1764$
$square\ 4.2 \Rightarrow 17.64$

The *Num* class will be discusssed in greater detail shortly.

7.3 Defining Your Own Type Classes

Haskell provides a mechanism whereby you can create your own qualified types, by defining a new type class and specifying which types are members, or "instances" of it. Indeed, the type classes *Num* and *Eq* are not built-in as primitives in Haskell, but rather are simply predefined in the Standard Prelude.

To see how this is done, consider the *Eq* class. It is created by the following *type class declaration*:

class *Eq a* **where**
$(==) :: a \to a \to Bool$

The connection between $(==)$ and *Eq* is important: the above declaration should be read, "a type a is an instance of the class *Eq* only if there is an operation $(==) :: a \to a \to Bool$ defined on it." $(==)$ is called an *operation* in the class *Eq*, and in general more than one operation is allowed in a class. More examples of this will be introduced shortly.

So far so good. But how does one specify which types are instances of the class *Eq*, and the actual behavior of $(==)$ on each of those types? This is done with an *instance declaration*. For example:

instance *Eq Integer* **where**
$x == y = integereq\ x\ y$

The definition of (==) is called a *method*. The function *integerEq* happens to be the primitive function that compares integers for equality, but in general any valid expression is allowed on the right-hand side, just as for any other function definition. The overall instance declaration is essentially saying: "The type *Integer* is an instance of the class *Eq*, and here is the method corresponding to the operation (==)." Given this declaration, one can now compare fixed-precision integers for equality using (==). Similarly:

```
instance Eq Float where
  x == y = floatEq x y
```

allows one to compare floating-point numbers using (==).

More importantly, datatypes that you have defined on your own can also be made instances of the class *Eq*. Consider, for example, the *PitchClass* data type defined in Chapter 2:

```
data PitchClass = Cff | Cf | C | Dff | Cs | Df | Css | D | Eff | Ds
                | Ef | Fff | Dss | E | Es | Ff | F | Gff | Ess | Fs
                | Gf | Fss | G | Aff | Gs | Af | Gss | A | Bff | As
                | Bf | Ass | B | Bs | Bss
```

One can declare *PitchClass* to be an instance of *Eq* as follows:

```
instance Eq PitchClass where
  Cff == Cff = True
  Cf  == Cf  = True
  C   == C   = True
  ...
  Bs  == Bs  = True
  Bss == Bss = True
  _   == _   = False
```

where ... refers to the other thirty equations to make this definition of (==) complete. Indeed, this is rather tedious! It is not only tedious, it is also dead obvious how (==) should be defined. Therefore Haskell provides a convenient way to *automatically derive* such instance declarations from data type declarations, for certain predefined type classes, using a **deriving** clause. For example, in the case of *PitchClass* one simply writes:

```
data PitchClass = Cff | Cf | C | Dff | Cs | Df | Css | D | Eff | Ds
                | Ef | Fff | Dss | E | Es | Ff | F | Gff | Ess | Fs
                | Gf | Fss | G | Aff | Gs | Af | Gss | A | Bff | As
                | Bf | Ass | B | Bs | Bss
```

```
  deriving Eq
```

With this declaration, Haskell will automatically derive the instance declaration given above, so that (==) behaves in the way one would expect it to.

Consider now a polymorphic type, such as the *Primitive* type from Chapter 2:

```
data Primitive a = Note Dur a
                 | Rest Dur
```

What should an instance for this type in the class *Eq* look like? Here's a first attempt:

```
instance Eq (Primitive a) where
  Note d1 x1 == Note d2 x2 = (d1 == d2) ∧ (x1 == x2)
  Rest d1    == Rest d2    = d1 == d2
  _          == _          = False
```

Note the use of (==) on the right-hand side, in several places. Two of those places involve *Dur*, which a type synonym for *Rational*. The *Rational* type is in fact a predefined instance of *Eq*, so all is well there. (If it were not an instance of *Eq*, a type error would result.)

But what about the term $x_1 == x_2$? x_1 and x_2 are values of the polymorphic type a, but how does one know that equality is defined on a, i.e. that the type a is an instance of *Eq*? In fact this is not known in general. The simple fix is to add a constraint to the instance declaration, as follows:

```
instance Eq a ⇒ Eq (Primitive a) where
  Note d1 x1 == Note d2 x2 = (d1 == d2) ∧ (x1 == x2)
  Rest d1    == Rest d2    = d1 == d2
  _          == _          = False
```

This can be read, "For any type a in the class *Eq*, the type *Primitive a* is also in the class *Eq*, and here is the definition of (==) for that type." Indeed, if one had written the original type declaration like this:

```
data Primitive a = Note Dur a
                 | Rest Dur
  deriving Eq
```

then Haskell would have derived the above correct instance declaration automatically.

So, for example, (==) is defined on the type *Primitive Pitch*, because *Pitch* is a type synonym for (*PitchClass*, *Octave*), and (a) *PitchClass* is an

instance of *Eq* by the effort above, (b) *Octave* is a synonym for *Int*, which is a predefined instance of *Eq*, and (c) as mentioned earlier the pair type is a predefined instance of *Eq*. Indeed, now that an instance for a polymorphic type has been seen, one can understand what the predefined instance for polymorphic pairs must look like, namely:

$$\begin{aligned}&\textbf{instance}\ (Eq\ a, Eq\ b) \Rightarrow Eq\ (a, b)\ \textbf{where}\\ &\quad (x_1, y_1) == (x_2, y_2) = (x_1 == x_2) \wedge (y_1 == y_2)\end{aligned}$$

About the only thing not considered is a *recursive* data type. For example, recall the *Music* data type, also from Chapter 2:

$$\begin{aligned}&\textbf{data}\ Music\ a = Prim\ (Primitive\ a)\\ &\qquad\qquad\ \mid Music\ a :+: Music\ a\\ &\qquad\qquad\ \mid Music\ a :=: Music\ a\\ &\qquad\qquad\ \mid Modify\ Control\ (Music\ a)\end{aligned}$$

Its instance declaration for *Eq* seems obvious:

$$\begin{aligned}&\textbf{instance}\ Eq\ a \Rightarrow Eq\ (Music\ a)\ \textbf{where}\\ &\quad Prim\ p_1 \qquad == Prim\ p_2 \qquad = p_1 == p_2\\ &\quad (m_1 :+: m_2) \quad == (m_3 :+: m_4) \quad = (m_1 == m_3) \wedge (m_2 == m_4)\\ &\quad (m_1 :=: m_2) \quad == (m_3 :=: m_4) \quad = (m_1 == m_3) \wedge (m_2 == m_4)\\ &\quad Modify\ c_1\ m_1 == Modify\ c_2\ m_2 = (c_1 == c_2) \quad \wedge (m_1 == m_2)\end{aligned}$$

Indeed, assuming that *Control* is an instance of *Eq*, this is just what is expected, and can be automatically derived by adding a **deriving** clause to the data type declaration for *Music*.

In reality, the class *Eq* as defined in Haskell's Standard Prelude is slightly richer than what is defined above. Here is its exact form:

$$\begin{aligned}&\textbf{class}\ Eq\ a\ \textbf{where}\\ &\quad (==), (\neq) :: a \to a \to Bool\\ &\quad x \neq y \qquad = \neg\ (x == y)\\ &\quad x == y \qquad = \neg\ (x \neq y)\end{aligned}$$

This is an example of a class with two operations, one for equality, the other for inequality. It also demonstrates the use of a *default method*, one for each operator. If a method for a particular operation is omitted in an instance declaration, then the default one defined in the class declaration, if it exists, is used instead. For example, all of the instances of *Eq* defined earlier will work perfectly well with the above class declaration, yielding just the right definition of inequality that one would want: the logical negation of equality.

Details: Both the inequality and the logical negation operators are shown here using the mathematical notation, $\neq$ and $\neg$, respectively. When writing your Haskell programs, you instead will have to use the operator `/=` and the function name `not`, respectively.

A useful slogan that helps to distinguish type classes from ordinary polymorphism is this: "polymorphism captures similar structure over different values, while type classes capture similar operations over different structures." For example, a sequences of integers, sequence of characters, etc. can be captured as a polymorphic *List*, whereas equality of integers, equality of trees, etc. can be captured by a type class such as *Eq*.

7.4 Inheritance

Haskell also supports a notion called *inheritance*. For example, one may wish to define a class *Ord* that "inherits" all of the operations in *Eq*, but in addition has a set of comparison operations and minimum and maximum functions (a fuller definition of *Ord*, as taken from the Standard Prelude, is given in Appendix B):

$$\begin{array}{l}\textbf{class } \mathit{Eq}\ a \Rightarrow \mathit{Ord}\ a \textbf{ where}\\ \quad (<), (\leqslant), (\geqslant), (>) :: a \to a \to \mathit{Bool}\\ \quad \mathit{max}, \mathit{min} \qquad\quad :: a \to a \to a\end{array}$$

Note the constraint $\mathit{Eq}\ a \Rightarrow$ in the **class** declaration. *Eq* is a *superclass* of *Ord* (conversely, *Ord* is a *subclass* of *Eq*), and any type that is an instance of *Ord* must also be an instance of *Eq*. The reason that this extra constraint makes sense is that to perform comparisons such as $a \leqslant b$ and $a \geqslant b$ implies that one knows how to compute $a == b$.

For example, following the strategy used for *Eq*, one could declare *Music* an instance of *Ord* as follows (note the constraint $\mathit{Ord}\ a \Rightarrow ...$):

$$\begin{array}{l}\textbf{instance } \mathit{Ord}\ a \Rightarrow \mathit{Ord}\ (\mathit{Music}\ a) \textbf{ where}\\ \quad \begin{array}{lll}\mathit{Prim}\ p_1 & < \mathit{Prim}\ p_2 & = p_1 < p_2\\ (m_1 :+: m_2) & < (m_3 :+: m_4) & = (m_1 < m_3) \wedge (m_2 < m_4)\\ (m_1 :=: m_2) & < (m_3 :=: m_4) & = (m_1 < m_3) \wedge (m_2 < m_4)\\ \mathit{Modify}\ c_1\ m_1 & < \mathit{Modify}\ c_2\ m_2 & = (c_1 < c_2) \quad \wedge (m_1 < m_2)\end{array}\\ \quad ...\end{array}$$

Although this is a perfectly well-defined definition for $<$, it is not clear that it exhibits the desired behavior, an issue that will be returned to in Section 7.7.

Another benefit of inheritance is shorter constraints. For example, the type of a function that uses operations from both the *Eq* and *Ord* classes can use just the constraint $(Ord\ a)$ rather than $(Eq\ a, Ord\ a)$, since *Ord* "implies" *Eq*.

As an example of the use of *Ord*, a generic *sort* function should be able to sort lists of any type that is an instance of *Ord*, and thus its most general type should be:

$$sort :: Ord\ a \Rightarrow [a] \rightarrow [a]$$

This typing for *sort* would naturally arise through the use of comparison operators such as $<$ and $\geqslant$ in its definition.

Details: Haskell also permits *multiple inheritance*, since classes may have more than one superclass. Name conflicts are avoided by the constraint that a particular operation can be a member of at most one class in any given scope. For example, the declaration

$$\textbf{class}\ (Eq\ a, Show\ a) \Rightarrow C\ a\ \textbf{where}\ \ldots$$

creates a class C that inherits operations from both Eq and $Show$.

Finally, class methods may have additional class constraints on any type variable except the one defining the current class. For example, in this class:

$$\begin{array}{l}\textbf{class}\ C\ a\ \textbf{where}\\ \quad m :: Eq\ b \Rightarrow a \rightarrow b\end{array}$$

the method m requires that type b is in class Eq. However, additional class constraints on type a are not allowed in the method m; these would instead have to be part of the overall constraint in the class declaration.

7.5 Haskell's Standard Type Classes

The Standard Prelude defines many useful type classes, including *Eq* and *Ord*. They are described in detail in Appendix B. In addition, the Haskell Report and the Library Report contain useful examples and discussions of type classes; you should feel encouraged to read through them.

Most of the standard type classes in Haskell are shown in Figure 7.1,

Type Class	**Key functions**	**Key instances**
Num	$(+), (-), (*) :: Num\ a \Rightarrow a \to a \to a$ $negate :: Num\ a \Rightarrow a \to a$	*Integer*, *Int*, *Float*, *Double*, *Rational*
Eq	$(==), (\neq) :: Eq\ a \Rightarrow a \to a \to Bool$	*Integer*, *Int*, *Float*, *Double*, *Rational*, *Char*, *Bool*, ...
Ord	$(>), (<), (\geqslant), (\leqslant)::$ $Ord\ a \Rightarrow a \to a \to Bool$ $max, min :: Ord\ a \Rightarrow a \to a \to Bool$	*Integer*, *Int*, *Float*, *Double*, *Rational*, *Char*, *Bool*, ...
Enum	$succ, pred :: Enum\ a \Rightarrow a \to a$ also enables arithmetic sequences $fromEnum :: Enum\ a \Rightarrow a \to Int$ $toEnum :: Enum\ a \Rightarrow Int \to a$	*Integer*, *Int*, *Float*, *Double*, *Rational*, *Char*, *Bool*, ...
Show	$show :: Show\ a \Rightarrow a \to String$	Almost every type except for functions
Read	$read :: Read\ a \Rightarrow String \to a$	Almost every type except for functions

Figure 7.1: Common Type Classes and Their Instances

along with their key instances.

The *Num* class, which has been used implicitly throughout much of the text, is described in more detail below. With this explanation a few more of Haskell's secrets will be revealed.

7.5.1 The *Num* Class

As you know, Haskell provides several kinds of numbers, some of which have already been introduced: *Int*, *Integer*, *Rational*, and *Float*. These numbers are instances of various type classes arranged in a rather complicated hierarchy. The reason for this is that there are many operations, such as $(+)$, *abs*, and *sin*, that are common amongst some of these number types. For example, one would expect $(+)$ to be defined on every kind of number, whereas *sin* might only be applicable to either single precision (*Float*) or double-precision (*Double*) floating-point numbers.

Control over which numerical operations are allowed and which aren't is the purpose of the numeric type class hierarchy. At the top of the hierarchy, and therefore containing operations that are valid for all numbers, is the class *Num*. It is defined as:

$$
\begin{array}{l}
\textbf{class}\ (Eq\ a, Show\ a) \Rightarrow Num\ a\ \textbf{where} \\
\quad (+), (-), (*) :: a \to a \to a \\
\quad negate \qquad :: a \to a \\
\quad abs, signum\ :: a \to a \\
\quad fromInteger\ :: Integer \to a
\end{array}
$$

Note that $(/)$ is *not* an operation in this class. *negate* is the negation function; *abs* is the absolute value function; and *signum* is the sign function, which returns -1 if its argument is negative, 0 if it is 0, and 1 if it is positive. *fromInteger* converts an *Integer* into a value of type $Num\ a \Rightarrow a$, which is useful for certain coercion tasks.

Details: Haskell also has a negation operator, which is Haskell's only prefix operator. However, it is just shorthand for *negate*. That is, $-e$ in Haskell is shorthand for $negate\ e$.

The operation *fromInteger* also has a special purpose. one might wonder how it is that one can write the constant 42, say, both in a context requiring an *Int* and in one requiring a *Float* (say). Somehow Haskell "knows" which version of 42 is required in a given context. But, what is the type of 42 itself? The answer is that it has type $Num\ a \Rightarrow a$, for some a to be determined by its context. (If this seems strange, remember that [] by itself is also somewhat ambiguous; it is a list, but a list of what? The most one can say about its type is that it is $[a]$ for some a yet to be determined.)

The way this is achieved in Haskell is that literal numbers such as 42 are actually considered to be shorthand for $fromInteger\ 42$. Since *fromInteger* has type $Num\ a \Rightarrow Integer \to a$, then $fromInteger\ 42$ has type $Num\ a \Rightarrow a$.

The complete hierarchy of numeric classes is shown in Figure 7.2; note that some of the classes are subclasses of certain non-numeric classes, such as *Eq* and *Show*. The comments below each class name refer to the Standard Prelude types that are instances of that class. See Appendix B for more detail.

The Standard Prelude actually defines only the most basic numeric types: *Int*, *Integer*, *Float* and *Double*. Other numeric types such as rational numbers ($Ratio\ a$) and complex numbers ($Complex\ a$) are defined in libraries. The connection between these types and the numeric classes is given in Figure 7.3. The instance declarations implied by this table can be found in the Haskell Report.

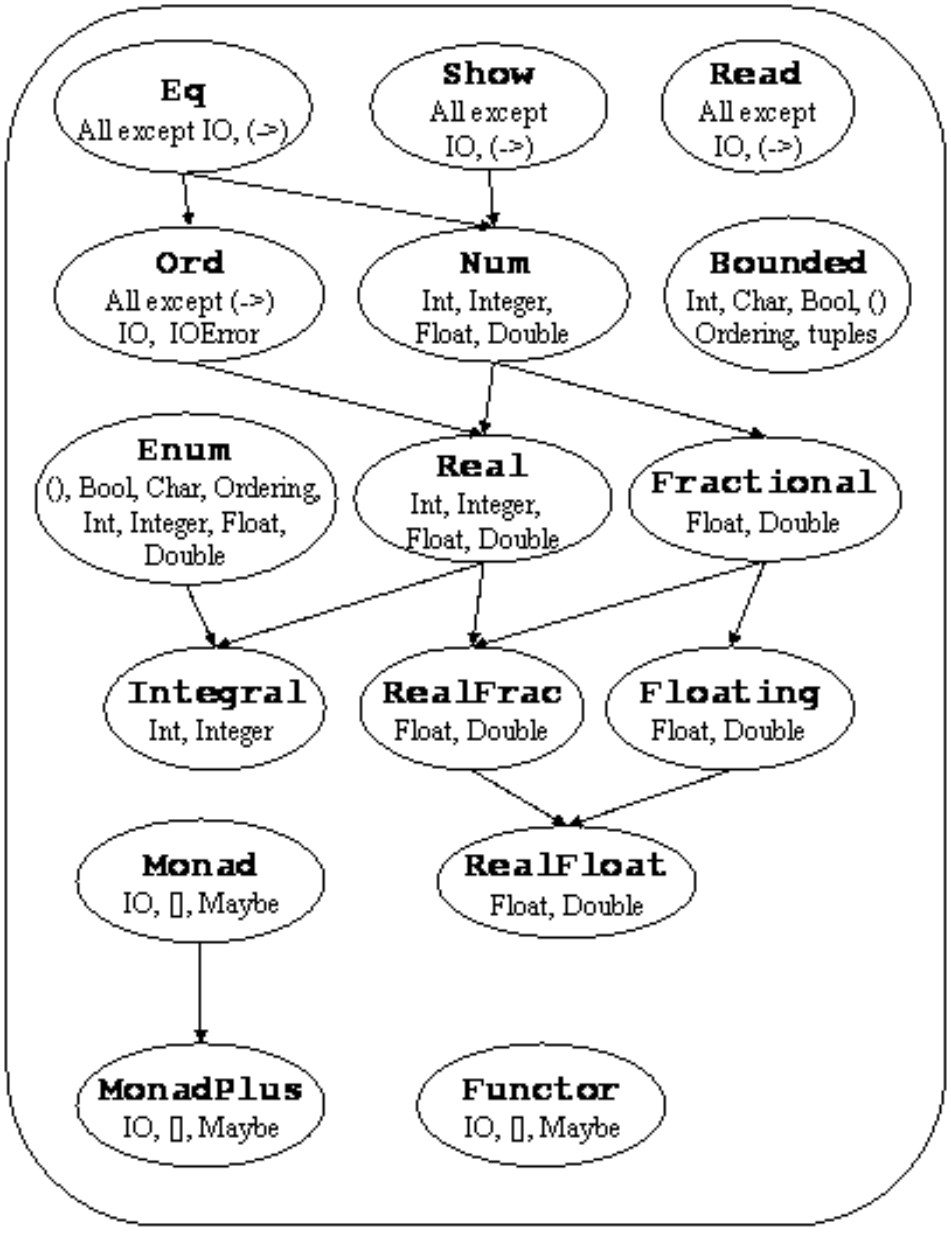


Figure 7.2: Numeric Class Hierarchy

Numeric Type	Type Class	Description
Int	*Integral*	Fixed-precision integers
Integer	*Integral*	Arbitrary-precision integers
Integral a ⇒ *Ratio a*	*RealFrac*	Rational numbers
Float	*RealFloat*	Real floating-point, single precision
Double	*RealFloat*	Real floating-point, double precision
RealFloat a ⇒ *Complex a*	*Floating*	Complex floating-point

Figure 7.3: Standard Numeric Types

7.5.2 The *Show* Class

It is very common to want to convert a data type value into a string. In fact, it happens all the time when one interacts with GHCi at the command prompt, and GHCi will complain if it does not "know" how to "show" a value. The type of anything that GHCi prints must be an instance of the *Show* class.

Not all of the operations in the *Show* class will be discussed here, in fact the only one of interest is *show*:

class *Show a* **where**
  show :: *a* → *String*
  ...

Instances of *Show* can be derived, so normally one doesn't have to worry about the details of the definition of *show*.

Lists also have a *Show* instance, but it is not derived, since, after all, lists have special syntax. Also, when *show* is applied to a string such as `"Hello"`, it should generate a string that, when printed, will look like `"Hello"`. This means that it must include characters for the quotation marks themselves, which in Haskell is achieved by prefixing the quotation mark with the "escape" character \. Given the following data declaration:

data *Hello* = *Hello*
  deriving *Show*

it is then instructive to ponder over the following calculations:

show Hello ⟹ `"Hello"`
show (*show Hello*) ⟹ *show* `"Hello"` ⟹ `"\"Hello\""`

$show\ (show\ (show\ Hello)) \Longrightarrow$ `"\"\\\"Hello\\\"\""`

Details: To refer to the escape character itself, it must also be escaped; thus `"\\"` prints as \.

For further pondering, consider the following program. See if you can figure out what it does, and why![2]

```
main   = putStr (quine q)
quine s = s ++ show s
q      = "main = putStr (quine q)\nquine s = s ++ show s\nq = "
```

Derived *Show* instances are possible for all types whose component types also have *Show* instances. *Show* instances for most of the standard types are provided in the Standard Prelude.

7.6 Derived Instances

In addition to *Eq* and *Ord*, instances of *Enum*, *Bounded*, *Ix*, *Read*, and *Show* (see Appendix B) can also be generated by the **deriving** clause. These type classes are widely used in Haskell programming, making the deriving mechanism very useful.

The textual representation defined by a derived *Show* instance is consistent with the appearance of constant Haskell expressions (i.e. values) of the type involved. For example, from:

```
data Color = Red | Orange | Yellow | Green | Blue | Indigo | Violet
  deriving (Eq, Enum, Show)
```

one can expect that:

```
show [Red ..]
 ⟹ "[Red,Orange,Yellow,Green,Blue,Indigo,Violet]"
```

Further details about derived instances can be found in the Haskell Report.

Many of the predefined data types in Haskell have **deriving** clauses, even ones with special syntax. For example, if one could write a data type

[2] The essence of this idea is due to Willard Van Orman Quine [Qui66], and its use in a computer program is discussed by Hofstadter [Hof79]. It was adapted to Haskell by Jón Fairbairn.

declaration for lists (the reason one cannot do this is that lists have special syntax, both at the value and type level) it would look something like this:

$$\begin{aligned}&\textbf{data}\ [a] = [\,]\\ &\qquad\quad \mid\ a : [a]\\ &\quad \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord})\end{aligned}$$

The derived *Eq* and *Ord* instances for lists are the usual ones; in particular, character strings, as lists of characters, are ordered as determined by the underlying *Char* type, with an initial sub-string being less than a longer string; for example, `"cat"` $<$ `"catalog"` is *True*.

In practice, *Eq* and *Ord* instances are almost always derived, rather than user-defined. In fact, you should provide your own definitions of equality and ordering predicates only with some trepidation, being careful to maintain the expected algebraic properties of equivalence relations and total orders, respectively (more on this later). An intransitive (==) predicate, for example, would be problematic, confusing readers of the program who expect (==) to be transitive. Nevertheless, it is sometimes necessary to provide *Eq* or *Ord* instances different from those that would be derived.

The data type declarations for *PitchClass*, *Primitive*, *Music* and *Control* given in Chapter 1 are not the ones actually used in Eutperpea. The actual definitions use a **deriving** clause, and are shown in Figure 7.4.

Note that with single and double sharps and flats, there are many enharmonic equivalences. Thus in the data declaration for *PitchClass*, the constructors are ordered such that, if $pc_1 < pc_2$, then $pcToInt\ pc_1 < pcToInt\ pc_2$.

Details: When instances of more than one type class are derived for the same data type, they appear grouped in parentheses as in Figure 7.4. Also, in this case *Eq must* appear if *Ord* does (unless an explicit instance for *Eq* is given), since *Eq* is a superclass of *Ord*.

For example, the *Show* class allows one to convert values to strings:

$$\begin{aligned}&\mathit{show}\ \mathit{Cs} &&\Longrightarrow \texttt{"Cs"}\\ &\mathit{show}\ \mathit{concertA} &&\Longrightarrow \texttt{"(A,4)"}\end{aligned}$$

The *Read* class allows one to go the other way around:

$$\begin{aligned}&\mathit{read}\ \texttt{"Cs"} &&\Longrightarrow \mathit{Cs}\\ &\mathit{read}\ \texttt{"(A,4)"} &&\Longrightarrow (A, 4)\end{aligned}$$

```
data PitchClass = Cff | Cf | C | Dff | Cs | Df | Css | D | Eff | Ds
                | Ef | Fff | Dss | E | Ff | Es | F | Gff | Ess | Fs
                | Gf | Fss | G | Aff | Gs | Af | Gss | A | Bff | As
                | Bf | Ass | B | Bs | Bss
  deriving (Eq, Ord, Show, Read, Enum)

data Primitive a = Note Dur a
                 | Rest Dur
  deriving (Show, Eq, Ord)

data Music a =
    Prim (Primitive a)            -- primitive value
  | Music a :+: Music a           -- sequential composition
  | Music a :=: Music a           -- parallel composition
  | Modify Control (Music a)      -- modifier
  deriving (Show, Eq, Ord)

data Control =
    Tempo      Rational           -- scale the tempo
  | Transpose  AbsPitch           -- transposition
  | Instrument InstrumentName     -- instrument label
  | Phrase     [PhraseAttribute]  -- phrase attributes
  | Player     PlayerName         -- player label
  deriving (Show, Eq, Ord)
```

Figure 7.4: Euterpea's Data Types with Deriving Clauses

The *Eq* class allows testing values for equality:

$$concertA == a440 \implies True$$
$$concertA == (A, 5) \implies False$$

And the *Ord* class has relational operators for types whose values can be ordered:

$$C < G \implies True$$
$$max\ C\ G \implies G$$

The *Enum* class allows one to use *arithmetic sequences*, which will be explained in a later chapter.

7.7 Reasoning With Type Classes

Type classes often imply a set of *laws* that govern the use of the operators in the class. For example, for the *Eq* class, one can expect the following laws to hold for every instance of the class:

$$\begin{array}{ll} x == x & \\ x == y & \supseteq y == x \\ (x == y) \wedge (y == z) & \supseteq x == z \\ (x \neq y) & \supseteq \neg\,(x == y) \end{array}$$

where $\supseteq$ should be read "implies that."[3]

However, there is no way to guarantee these laws. A user may create an instance of *Eq* that violates them, and in general Haskell has no way to enforce them. Nevertheless, it is useful to state the laws of interest for a particular class, and to state the expectation that all instances of the class be "law-abiding." Then as a diligent functional programmer, one should ensure that every instance that is defined, whether for one's own class or someone else's, is in fact law-abiding.

As another example, consider the *Ord* class, whose instances are intended to be *totally ordered*, which means that the following laws should hold, for all a, b, and c:

$$\begin{array}{ll} a \leqslant a & \\ (a \leqslant b) \wedge (b \leqslant c) & \supseteq (a \leqslant c) \\ (a \leqslant b) \wedge (b \leqslant a) & \supseteq (a == b) \\ (a \neq b) & \supseteq (a < b) \vee (b < a) \end{array}$$

[3] Mathematically, the first three of these laws are the same as those for an *equivalence relation*.

Similar laws should hold for $(>)$.

But alas, the instance of *Music* in the class *Ord* given in Section 7.4 does not satisfy all of these laws! To see why, consider two *Primitive* values p_1 and p_2 such that $p_1 < p_2$. Now consider these two *Music* values:

$$m_1 = Prim\ p_1 :+: Prim\ p_2$$
$$m_2 = Prim\ p_2 :+: Prim\ p_1$$

Clearly $m_1 == m_2$ is false, but the problem is, so are $m_1 < m_2$ and $m_2 < m_1$, thus violating the last law above.

To fix the problem, a *lexicographic ordering* should be used on the *Music* type, such as used in a dictionary. For example, "polygon" comes before "polymorphic," using a left-to-right comparison of the letters. The new instance declaration looks like this:

instance $Ord\ a \Rightarrow Ord\ (Music\ a)$ **where**

$$\begin{array}{lll}
Prim\ p_1 & < Prim\ p_2 & = p_1 < p_2 \\
Prim\ p_1 & < _ & = True \\
(m_1 :+: m_2) & < Prim\ _ & = False \\
(m_1 :+: m_2) & < (m_3 :+: m_4) & = (m_1 < m_3) \vee \\
 & & \quad (m_1 == m_3) \wedge (m_2 < m_4) \\
(m_1 :+: m_2) & < _ & = True \\
(m_1 :=: m_2) & < Prim\ _ & = False \\
(m_1 :=: m_2) & < (m_3 :+: m_4) & = False \\
(m_1 :=: m_2) & < (m_3 :=: m_4) & = (m_1 < m_3) \vee \\
 & & \quad (m_1 == m_3) \wedge (m_2 < m_4) \\
(m_1 :=: m_2) & < _ & = True \\
Modify\ c_1\ m_1 & < Modify\ c_2\ m_2 & = (c_1 < c_2) \vee \\
 & & \quad (c_1 == c_2) \wedge (m_1 < m_2) \\
Modify\ c_1\ m_1 & < _ & = False
\end{array}$$

This example shows the value of checking to be sure that each instance obeys the laws of its class. Of course, that check should come in the way of a proof. This example also highlights the utility of derived instances, since the derived instance of *Music* for the class *Ord* is equivalent to that above, yet is done automatically.

Exercise 7.1 Prove that the instance of *Music* in the class *Eq* satisfies the laws of its class. Also prove that the modified instance of *Music* in the class *Ord* satisfies the laws of its class.

Exercise 7.2 Write out appropriate instance declarations for the *Color*

type in the classes *Eq*, *Ord*, and *Enum*.

Chapter 8

Interpretation and Performance

```
{-# LANGUAGE FlexibleInstances, TypeSynonymInstances #-}
module Euterpea.Music.Note.Performance
  where
import Euterpea.Music.Note.Music
import Euterpea.Music.Note.MoreMusic
instance Show (a → b) where
  showsPrec p f = showString "<<function>>"
```

8.1 Abstract Performance

So far, our presentation of musical values in Haskell has been entirely structural, i.e. *syntactic*. But what do these musical values actually *mean*, i.e. what is their *semantics*, or *interpretation*? The formal process of giving a semantic interpretation to syntactic constructs is very common in computer science, especially in programming language theory. But it is obviously also common in music: the interpretation of music is the very essence of musical performance. However, in conventional music this process is usually informal, appealing to aesthetic judgments and values. What we would like to do is make the process formal in Euterpea—but still flexible, so that more than one interpretation is possible, just as in music.

To begin, we need to say exactly what an abstract *performance* is. Our approach is to consider a performance to be a time-ordered sequence of

musical *events*, where each event captures the playing of one individual note. In Haskell:

$$\begin{aligned}
&\textbf{type } Performance = [Event]\\
&\textbf{data } Event = Event\ \{\ eTime \quad :: PTime,\\
&\qquad\qquad\qquad\quad\ \ eInst \quad :: InstrumentName,\\
&\qquad\qquad\qquad\quad\ \ ePitch \quad :: AbsPitch,\\
&\qquad\qquad\qquad\quad\ \ eDur \quad :: DurT,\\
&\qquad\qquad\qquad\quad\ \ eVol \quad :: Volume,\\
&\qquad\qquad\qquad\quad\ \ eParams :: [Double]\ \}\\
&\quad \textbf{deriving } (Eq, Ord, Show)\\
&\textbf{type } PTime\ = Rational\\
&\textbf{type } DurT\ \ = Rational\\
&\textbf{type } Volume = Integer
\end{aligned}$$

Details: The data declaration for *Event* uses Haskell's *field label* syntax, also called *record* syntax, and is equivalent to:

$$\begin{aligned}
&\textbf{data } Event = Event\ PTime\ InstrumentName\\
&\qquad\qquad\qquad\quad\ AbsPitch\ DurT\ Volume\ [Float]\\
&\quad \textbf{deriving } (Eq, Ord, Show)
\end{aligned}$$

except that the former also defines "field labels" *eTime*, *eInst*, *ePitch*, *eDur*, *eVol*, and *eParams*, which can be used to create, update, and select from *Event* values. For example, this equation:

$$e = Event\ 0\ Cello\ 27\ (1/4)\ 50\ [\,]$$

is equivalent to:

$$\begin{aligned}
e = Event\ \{\ &eTime = 0, ePitch = 27, eDur = 1/4,\\
&eInst = Cello, eVol = 50, eParams = [\,]\ \}
\end{aligned}$$

Although more verbose, the latter is also more descriptive, and the order of the fields does not matter (indeed the order here is not the same as above).

Details: Field labels can be used to *select* fields from an *Event* value; for example, using the value of e above, $eInst\ e \Rightarrow Cello$, $eDur\ e \Rightarrow 1/4$, and so on. They can also be used to selectively *update* fields of an existing *Event* value. For example:

$$e\ \{eInst = Flute\} \Rightarrow Event\ 0\ Flute\ 27\ (1/4)\ 50\ []$$

Finally, they can be used selectively in pattern matching:

$$f\ (Event\ \{eDur = d, ePitch = p\}) = ...d\ ...\ p\ ...$$

Field labels do not change the basic nature of a data type; they are simply a convenient syntax for referring to the components of a data type by name rather than by position.

An event $Event\ \{eTime = s, eInst = i, ePitch = p, eDur = d, eVol = v\}$ captures the fact that at start time s, instrument i sounds pitch p with volume v for a duration d (where now duration is measured in seconds, rather than beats). (The *pField* of an event is for special instruments that require extra parameters, and will not be discussed much further in this chapter.)

An abstract performance is the lowest of our music representations not yet committed to MIDI, csound, or some other low-level computer music representation. In Chapter 14 we will discuss how to map a performance into MIDI.

To generate a complete performance of, i.e. give an interpretation to, a musical value, we must know the time to begin the performance, and the proper instrument, volume, key and tempo. In addition, to give flexibility to our interpretations, we must also know what *player* to use; that is, we need a mapping from the *PlayerName*s in a *Music* value to the actual players to be used.[1] We capture these ideas in Haskell as a "context" and "player map," respectively:

$$\begin{aligned}\textbf{data}\ Context\ a = Context\ \{&cTime\ \ :: PTime,\\ &cPlayer :: Player\ a,\\ &cInst\ \ \ :: InstrumentName,\\ &cDur\ \ \ \ :: DurT,\\ &cKey\ \ \ \ :: Key,\\ &cVol\ \ \ \ :: Volume\}\end{aligned}$$

deriving *Show*

[1] We don't need a mapping from *InstrumentNames* to instruments, since this is handled in the translation from a performance into MIDI, which is discussed in Chapter 14.

```
perform :: PMap a → Context a → Music a → Performance
perform pm
  c@Context {cTime = t, cPlayer = pl, cDur = dt, cKey = k} m =
  case m of
    Prim (Note d p)          → playNote pl c d p
    Prim (Rest d)            → []
    m1 :+: m2                →
      let c' = c {cTime = t + dur m1 * dt}
      in perform pm c m1 ++ perform pm c' m2
    m1 :=: m2                → merge (perform pm c m1)
                                     (perform pm c m2)
    Modify (Tempo r) m       → perform pm (c {cDur = dt/r}) m
    Modify (Transpose p) m   → perform pm (c {cKey = k + p}) m
    Modify (Instrument i) m  → perform pm (c {cInst = i}) m
    Modify (Player pn) m     → perform pm (c {cPlayer = pm pn}) m
    Modify (Phrase pa) m     → interpPhrase pl pm c pa m
```

Figure 8.1: An abstract *perform* function

```
type PMap a = PlayerName → Player a
type Key    = AbsPitch
```

Finally, we are ready to give an interpretation to a piece of music, which we do by defining a function *perform*, which is conceptually perhaps the most important function defined in this book, and is shown in Figure 8.1.

Some things to note about *perform*:

1. The *Context* is the running "state" of the performance, and gets updated in several different ways. For example, the interpretation of the *Tempo* constructor involves scaling *dt* appropriately and updating the *DurT* field of the context.

2. The interpretation of notes and phrases is player dependent. Ultimately a single note is played by the *playNote* function, which takes the player as an argument. Similarly, phrase interpretation is also player dependent, reflected in the use of *interpPhrase*. Precisely how these two functions work is described in Section 8.2.

3. The *DurT* component of the context is the duration, in seconds, of one whole note. To make it easier to compute, we can define a "metronome" function that, given a standard metronome marking (in

beats per minute) and the note type associated with one beat (quarter note, eighth note, etc.) generates the duration of one whole note:

$$\begin{aligned} &\mathit{metro} && :: \mathit{Int} \to \mathit{Dur} \to \mathit{DurT} \\ &\mathit{metro}\ \mathit{setting}\ \mathit{dur} && = 60/(\mathit{fromIntegral}\ \mathit{setting} * \mathit{dur}) \end{aligned}$$

Thus, for example, *metro* 96 *qn* creates a tempo of 96 quarter notes per minute.

Details: $\mathit{fromIntegral} :: (\mathit{Integral}\ a, \mathit{Num}\ b) \Rightarrow a \to b$ coerces a value whose type is a member of the *Integral* class to a value whose type is a member of the *Num* class. As used here, it is effectively converting the *Int* value *setting* to a *Rational* value, because *dur* is a *Rational* value, *Rational* is a member of the *Num* class, and multiplication has type $(*) :: \mathit{Num}\ a \Rightarrow a \to a \to a$.

4. In the treatment of (:+:), note that the sub-sequences are appended together, with the start time of the second argument delayed by the duration of the first. The function *dur* (defined in Section 6.5) is used to compute this duration. However, this results in a quadratic time complexity for *perform*. A more efficient solution is to have *perform* compute the duration directly, returning it as part of its result. This version of *perform* is shown in Figure 8.2.

5. The sub-sequences derived from the arguments to (:=:) are merged into a time-ordered stream. The definition of *merge* is given below.

$$\begin{aligned} &\mathit{merge} :: \mathit{Performance} \to \mathit{Performance} \to \mathit{Performance} \\ &\mathit{merge}\ a@(e_1 : es_1)\ b@(e_2 : es_2) = \textbf{if}\ e_1 < e_2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \textbf{then}\ e_1 : \mathit{merge}\ es_1\ b \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \textbf{else}\ e_2 : \mathit{merge}\ a\ es_2 \\ &\mathit{merge}\ [\,]\qquad\quad es_2\qquad\quad = es_2 \\ &\mathit{merge}\ es_1\qquad\quad [\,]\qquad\quad = es_1 \end{aligned}$$

Note that *merge* compares entire events rather than just start times. This is to ensure that it is commutative, a desirable condition for some of the proofs used later in the text.

```
perform :: PMap a → Context a → Music a → Performance
perform pm c m = fst (perf pm c m)

perf :: PMap a → Context a → Music a → (Performance, DurT)
perf pm
  c@Context {cTime = t, cPlayer = pl, cDur = dt, cKey = k} m =
  case m of
    Prim (Note d p)          → (playNote pl c d p, d * dt)
    Prim (Rest d)            → ([], d * dt)
    m1 :+: m2                →
        let (pf1, d1) = perf pm c m1
            (pf2, d2) = perf pm (c {cTime = t + d1}) m2
        in (pf1 ++ pf2, d1 + d2)
    m1 :=: m2                →
        let (pf1, d1) = perf pm c m1
            (pf2, d2) = perf pm c m2
        in (merge pf1 pf2, max d1 d2)
    Modify (Tempo r) m       → perf pm (c {cDur = dt/r}) m
    Modify (Transpose p) m   → perf pm (c {cKey = k + p}) m
    Modify (Instrument i) m  → perf pm (c {cInst = i}) m
    Modify (Player pn) m     → perf pm (c {cPlayer = pm pn}) m
    Modify (Phrase pas) m    → interpPhrase pl pm c pas m
```

Figure 8.2: A more efficient *perform* function

Here is a more efficient version of *merge* that will work just as well in practice:

```
merge a@(e1 : es1) b@(e2 : es2) = if eTime e1 < eTime e2
                                  then e1 : merge es1 b
                                  else e2 : merge a es2
merge []           es2           = es2
merge es1          []            = es1
```

8.2 Players

Recall from Section 2.2 the definition of the *Control* data type:

```
data Control =
    Tempo      Rational            -- scale the tempo
  | Transpose  AbsPitch            -- transposition
  | Instrument InstrumentName      -- intrument label
  | Phrase     [PhraseAttribute]   -- phrase attributes
  | Player     PlayerName          -- player label
  deriving (Show, Eq, Ord)
type PlayerName = String
```

We mentioned, but did not define, the *PhraseAttribute* data type, shown now fully in Figure 8.3. These attributes give us great flexibility in the interpretation process, because they can be interpreted by different players in different ways. For example, how should "legato" be interpreted in a performance? Or "diminuendo?" Different players interpret things in different ways, of course, but even more fundamental is the fact that a pianist, for example, realizes legato in a way fundamentally different from the way a violinist does, because of differences in their instruments. Similarly, diminuendo on a piano and diminuendo on a harpsichord are different concepts.

With a slight stretch of the imagination, we can even consider a "notator" of a score as a kind of player: exactly how the music is rendered on the written page may be a personal, stylized process. For example, how many, and which staves should be used to notate a particular instrument?

In any case, to handle these issues, Euterpea has a notion of a *player* that "knows" about differences with respect to performance and notation. A Euterpea player is a 4-tuple consisting of a name and three functions: one for interpreting notes, one for phrases, and one for producing a properly notated score.

```
data Player a = MkPlayer {pName        :: PlayerName,
                          playNote     :: NoteFun a,
                          interpPhrase :: PhraseFun a,
                          notatePlayer :: NotateFun a}
  deriving Show
type NoteFun a    = Context a -> Dur -> a -> Performance
type PhraseFun a  = PMap a -> Context a -> [PhraseAttribute]
                    -> Music a -> (Performance, DurT)
type NotateFun a = ()
```

```
data PhraseAttribute = Dyn Dynamic
                     | Tmp Tempo
                     | Art Articulation
                     | Orn Ornament
  deriving (Eq, Ord, Show)
data Dynamic = Accent Rational | Crescendo Rational
             | Diminuendo Rational | StdLoudness StdLoudness
             | Loudness Rational
  deriving (Eq, Ord, Show)
data StdLoudness = PPP | PP | P | MP | SF | MF | NF | FF | FFF
  deriving (Eq, Ord, Show, Enum)
data Tempo = Ritardando Rational | Accelerando Rational
  deriving (Eq, Ord, Show)
data Articulation = Staccato Rational | Legato Rational
                  | Slurred Rational | Tenuto | Marcato | Pedal
                  | Fermata | FermataDown | Breath | DownBow
                  | UpBow | Harmonic | Pizzicato | LeftPizz
                  | BartokPizz | Swell | Wedge | Thumb | Stopped
  deriving (Eq, Ord, Show)
data Ornament = Trill | Mordent | InvMordent | DoubleMordent
              | Turn | TrilledTurn | ShortTrill
              | Arpeggio | ArpeggioUp | ArpeggioDown
              | Instruction String | Head NoteHead
  deriving (Eq, Ord, Show)
data NoteHead = DiamondHead | SquareHead | XHead | TriangleHead
              | TremoloHead | SlashHead | ArtHarmonic | NoHead
  deriving (Eq, Ord, Show)
```

Figure 8.3: Phrase Attributes

Note that *NotateFun* is just the unit type; this is because notation is currently not implemented in Euterpea.

8.2.1 Example of Player Construction

In this section we define a "default player" called *defPlayer* (not to be confused with a "deaf player"!) for use when none other is specified in a score; it also functions as a basis from which other players can be derived.

In order to provide the most flexibility, we exploit poymorphism to define a version of *Music* that in addition to pitch, carries a list of "note attributes" for each individual note:

```
data NoteAttribute =
    Volume Int   -- MIDI convention: 0=min, 127=max
  | Fingering Integer
  | Dynamics String
  | Params [Double]
 deriving (Eq, Show)
```

Our goal then is to define a player for music values of type:

```
type Music1 = Music Note1
type Note1  = (Pitch, [NoteAttribute])
```

At the upper-most level, *defPlayer* is defined as a four-tuple:

```
defPlayer :: Player Note1
defPlayer = MkPlayer
            {pName        = "Default",
             playNote     = defPlayNote     defNasHandler,
             interpPhrase = defInterpPhrase defPasHandler,
             notatePlayer = ()}
```

The remaining functions are defined in Figure 8.4. Before reading this code, first review how players are invoked by the *perform* function defined in the last section; in particular, note the calls to *playNote* and *interpPhrase*. We will define *defPlayer* to respond only to the *Volume* note attribute and to the *Accent*, *Staccato*, and *Legato* phrase attributes.

Then note:

1. *defPlayNote* is the only function (even in the definition of *perform*) that actually generates an event. It also modifies that event based on an interpretation of each note attribute by the function *defNasHandler*.

```
defPlayNote :: (Context (Pitch, [a]) → a → Event → Event)
                → NoteFun (Pitch, [a])
defPlayNote nasHandler
  c@(Context cTime cPlayer cInst cDur cKey cVol) d (p, nas) =
    let initEv = Event { eTime   = cTime, eInst = cInst,
                         ePitch  = absPitch p + cKey,
                         eDur    = d * cDur, eVol = cVol,
                         eParams = [] }
    in [foldr (nasHandler c) initEv nas]

defNasHandler :: Context a → NoteAttribute → Event → Event
defNasHandler c (Volume v)  ev = ev { eVol = v }
defNasHandler c (Params pms) ev = ev { eParams = pms }
defNasHandler _ _           ev = ev

defInterpPhrase ::
  (PhraseAttribute → Performance → Performance) → PhraseFun a
defInterpPhrase pasHandler pm context pas m =
  let (pf, dur) = perf pm context m
  in (foldr pasHandler pf pas, dur)

defPasHandler :: PhraseAttribute → Performance → Performance
defPasHandler (Dyn (Accent x)) =
    map (λe → e { eVol = round (x * fromIntegral (eVol e)) })
defPasHandler (Art (Staccato x)) =
    map (λe → e { eDur = x * eDur e })
defPasHandler (Art (Legato x)) =
    map (λe → e { eDur = x * eDur e })
defPasHandler _ = id
```

Figure 8.4: Definition of default player *defPlayer*.

2. *defNasHandler* only recognizes the *Volume* attribute, which it uses to set the event volume accordingly.

3. *defInterpPhrase* calls (mutually recursively) *perform* to interpret a phrase, and then modifies the result based on an interpretation of each phrase attribute by the function *defPasHandler*.

4. *defPasHandler* only recognizes the *Accent*, *Staccato*, and *Legato* phrase attributes. For each of these it uses the numeric argument as a "scaling" factor of the volume (for *Accent*) and duration (for *Staccato* and *Legato*). Thus *Modify* (*Phrase* [*Legato* (5/4)]) m effectively increases the duration of each note in m by 25% (without changing the tempo).

8.2.2 Deriving New Players From Old Ones

It should be clear that much of the code in Figure 8.4 can be re-used in defining a new player. For example, to define a player *newPlayer* that interprets note attributes just like *defPlayer* but behaves differently with respect to certain phrase attributes, we could write:

```
newPlayer :: Player (Pitch, [NoteAttribute])
newPlayer = MkPlayer
  { pName       = "NewPlayer",
    playNote    = defPlayNote defNasHandler,
    interpPhrase = defInterpPhrase myPasHandler,
    notatePlayer = () }
```

and then supply a suitable definition of *myPasHandler*. Better yet, we could just do this:

```
newPlayer :: Player (Pitch, [NoteAttribute])
newPlayer = defPlayer
  { pName       = "NewPlayer",
    interpPhrase = defInterpPhrase myPasHandler }
```

This version uses the "record update" syntax to directly derive the new player from *defPlayer*.

The definition of *myPasHandler* can also re-use code, in the following sense: suppose we wish to add an interpretation for *Crescendo*, but otherwise have *myPasHandler* behave just like *defPasHandler*.

```
myPasHandler :: PhraseAttribute -> Performance -> Performance
myPasHandler (Dyn (Crescendo x)) pf = ...
myPasHandler pa                  pf = defPasHandler pa pf
```

[**To do:** Expain more... in particular, how "inheritance" works.]

8.2.3 A Fancy Player

Figure 8.5 defines a relatively sophisticated player called *fancyPlayer* that knows all that *defPlayer* knows, and more. Note that *Slurred* is different from *Legato* in that it doesn't extend the duration of the *last* note(s). The behavior of *Ritardando x* can be explained as follows. We'd like to "stretch" the time of each event by a factor from 0 to x, linearly interpolated based on how far along the musical phrase the event occurs. I.e., given a start time t_0 for the first event in the phrase, total phrase duration D, and event time t, the new event time t' is given by:

$$t' = (1 + \frac{t - t_0}{D}x)(t - t_0) + t_0$$

Further, if d is the duration of the event, then the end of the event $t + d$ gets stretched to a new time t'_d given by:

$$t'_d = (1 + \frac{t + d - t_0}{D}x)(t + d - t_0) + t_0$$

The difference $t'_d - t'$ gives us the new, stretched duration d', which after simplification is:

$$d' = (1 + \frac{2(t - t_0) + d}{D}x)\ d$$

Accelerando behaves in exactly the same way, except that it shortens event times rather than lengthening them. And a similar but simpler strategy explains the behaviors of *Crescendo* and *Diminuendo*.

8.3 Putting it all Together

The *play* function in Euterpea ueses a default player map and a default context that are defined as follows:

```
defPMap              :: PMap Note1   -- = PlayerName -¿ Player Note1
defPMap "Fancy"      = fancyPlayer
defPMap "Default"    = defPlayer
defPMap n            = defPlayer {pName = n}
defCon :: Context Note1
```

$$\begin{aligned} defCon = Context\ \{\ cTime &= 0,\\ cPlayer &= fancyPlayer,\\ cInst &= AcousticGrandPiano,\\ cDur &= metro\ 120\ qn,\\ cKey &= 0,\\ cVol &= 127\} \end{aligned}$$

Note that if anything other than a `"Fancy"` player is specified in the *Music* value, such as *player* `"Strange"` *m*, then the default player *defPlayer* is used.

If instead one wishes to use her own player, say *newPlayer* defined in Section 8.2.2, then a new player map can be defined, such as:

```
myPMap              :: PlayerName → Player Note1
myPMap "NewPlayer"  = newPlayer
myPMap p            = defPMap p
```

Similarly, different versions of the context can be defined based on a user's needs.

One could, then, use these versions of player maps and contexts to invoke the *perform* function to generate an abstract *Performance*. Of course, we ultimately want to hear our music, not just see an abstract *Performance*. Recall that *play*'s type signature is:

```
play :: Performable a ⇒ Music a → IO ()
```

To allow using different player maps and contexts, Eutperpea also has a version of *play* called *playA* whose type signature is:

```
playA :: Performable a ⇒
         PMap Note1 → Context Note1 → Music a → IO ()
```

For example, to play a *Music* value *m* using *myPMap* defined above and the default context *defCon*, one can do:

```
playA myPMap defCon m
```

In later chapters we will learn more about *play*, and how it converts a *Performance* into MIDI events that eventually are heard through your computer's sound card.

Exercise 8.1 Fill in the ... in the definition of *myPasHandler* according to the following strategy: Gradually scale the volume of each event in the performance by a factor of 1 through $1 + x$, using linear interpolation.

Exercise 8.2 Choose some of the other phrase attributes and provide interpretations for them.

Exercise 8.3 Define a player *myPlayer* that appropriately handles the *Pedal* articulation and both the *ArpeggioUp* and *ArpeggioDown* ornamentations. You should define *myPlayer* as a derivative of *defPlayer*.

Exercise 8.4 Define a player *jazzMan* (or *jazzWoman* if you prefer) that plays a melody using a jazz "swing" feel. Since there are different kinds and degrees of swing, we can be more specific as follows: whenever there is a sequence of two eighth notes, they should be interpreted instead as a quarter note followed by an eighth note, but with tempo 3/2. So in essence, the first note is lengthened, and the second note is shortened, so that the first note is twice as long as the second, but they still take up the same amount of overall time.

To do this at the *Player* level, some assumptions need to be made, such as what is an eighth note, where is the downbeat, etc.

[**To do:** This code has errors and needs to be fixed.]

```
fancyPlayer :: Player (Pitch, [NoteAttribute])
fancyPlayer = MkPlayer { pName       = "Fancy",
                         playNote    = defPlayNote defNasHandler,
                         interpPhrase = fancyInterpPhrase,
                         notatePlayer = () }

fancyInterpPhrase                :: PhraseFun a
fancyInterpPhrase pm c [] m = perf pm c m
fancyInterpPhrase pm
  c@Context { cTime = t, cPlayer = pl, cInst = i,
              cDur = dt, cKey = k, cVol = v }
  (pa : pas) m =
  let pfd@(pf, dur) = fancyInterpPhrase pm c pas m
      loud x        = fancyInterpPhrase pm c (Dyn (Loudness x) : pas) m
      stretch x     = let  t0 = eTime (head pf); r = x / dur
                           upd (e@Event { eTime = t, eDur = d }) =
                             let dt = t - t0
                                 t' = (1 + dt * r) * dt + t0
                                 d' = (1 + (2 * dt + d) * r) * d
                             in e { eTime = t', eDur = d' }
                      in (map upd pf, (1 + x) * dur)
      inflate x     = let  t0 = eTime (head pf);
                           r  = x / dur
                           upd (e@Event { eTime = t, eVol = v }) =
                             e { eVol = round ((1 + (t - t0) * r) *
                                               fromIntegral v) }
                      in (map upd pf, dur)
  in case pa of
       Dyn (Accent x) ->
         (map (λe -> e { eVol = round (x * fromIntegral (eVol e)) }) pf, dur)
       Dyn (StdLoudness l) ->
         case l of
           PPP -> loud 40;  PP -> loud 50;  P  -> loud 60
           MP  -> loud 70;  SF -> loud 80;  MF -> loud 90
           NF  -> loud 100; FF -> loud 110; FFF -> loud 120
       Dyn (Loudness x)    -> fancyInterpPhrase pm c
                              { cVol = (round ∘ fromRational) x } pas m
       Dyn (Crescendo x)   -> inflate x; Dyn (Diminuendo x) -> inflate (-x)
       Tmp (Ritardando x)  -> stretch x; Tmp (Accelerando x) -> stretch (-x)
       Art (Staccato x)    -> (map (λe -> e { eDur = x * eDur e }) pf, dur)
       Art (Legato x)      -> (map (λe -> e { eDur = x * eDur e }) pf, dur)
       Art (Slurred x)     ->
         let  lastStartTime = foldr (λe t -> max (eTime e) t) 0 pf
              setDur e      = if eTime e < lastStartTime
                              then e { eDur = x * eDur e }
                              else e
         in (map setDur pf, dur)
       Art _               -> pfd
       Orn _               -> pfd
```

[**To do:** Design Bug: To do things right we need to keep the key signature around to determine, for example, what the trill note is. Alternatively, provide an argument to Trill to carry this info.]

Figure 8.5: Definition of Player *fancyPlayer*.

Chapter 9

Self-Similar Music

```
module Euterpea.Examples.SelfSimilar where
import Euterpea
```

In this chapter we will explore the notion of *self-similar* music—i.e. musical structures that have patterns that repeat themselves recursively in interesting ways. There are many approaches to generating self-similar structures, the most general being *fractals*, which have been used to generate not just music, but also graphical images. We will delay a general treatment of fractals, however, and will instead focus on more specialized notions of self-similarity, notions that we conceive of musically, and then manifest as Haskell programs.

9.1 Self-Similar Melody

Here is the first notion of self-similar music that we will consider: Begin with a very simple melody of n notes. Now duplicate this melody n times, playing each in succession, but first perform the following transformations: transpose the ith melody by an amount proportional to the pitch of the ith note in the original melody, and scale its tempo by a factor proportional to the duration of the ith note. For example, Figure 9.1 shows the result of applying this process once to a four-note melody. Now imagine that this process is repeated infinitely often. For a melody whose notes are all shorter than a whole note, it yields an infinitely dense melody of infinitesimally shorter notes. To make the result playable, however, we will stop the process at some pre-determined level.

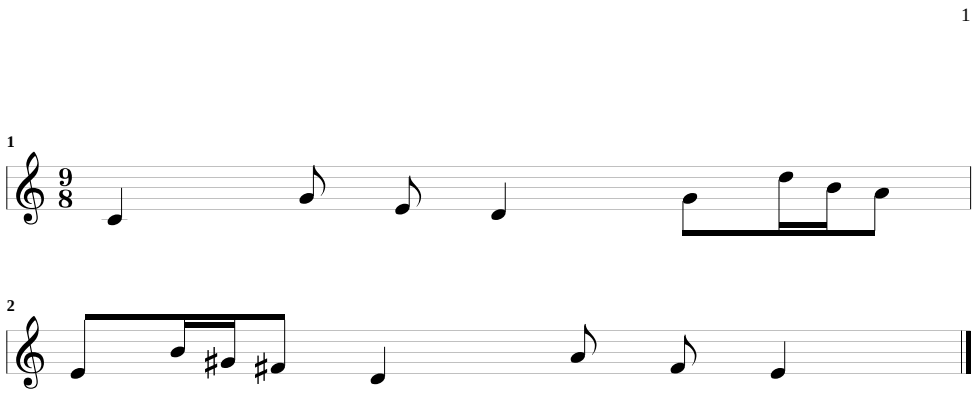


Figure 9.1: An Example of Self-Similar Music

How can this be represented in Haskell? A *tree* seems like it would be a logical choice; let's call it a *Cluster*:

$$\textbf{data}\ \mathit{Cluster} = \mathit{Cluster}\ \mathit{SNote}\ [\mathit{Cluster}]$$
$$\textbf{type}\ \mathit{SNote} = (\mathit{Dur}, \mathit{AbsPitch})$$

This particular kind of tree happens to be called a *rose tree* []. An *SNote* is just a "simple note," a duration paired with an absolute pitch. We prefer to stick with absolute pitches in creating the self-similar structure, and will convert the result into a normal *Music* value only after we are done.

The sequence of *SNote*s at each level of the cluster is the melodic fragment for that level. The very top cluster will contain a "dummy" note, whereas the next level will contain the original melody, the next level will contain one iteration of the process described above (e.g. the melody in Figure 9.1), and so forth.

To achieve this we will define a function *selfSim* that takes the initial melody as argument and generates an infinitely deep cluster:

$$\begin{aligned}
&\mathit{selfSim} :: [\mathit{SNote}] \to \mathit{Cluster}\\
&\mathit{selfSim}\ \mathit{pat} = \mathit{Cluster}\ (0,0)\ (\mathit{map}\ \mathit{mkCluster}\ \mathit{pat})\\
&\quad \textbf{where}\ \mathit{mkCluster}\ \mathit{note} =\\
&\qquad \mathit{Cluster}\ \mathit{note}\ (\mathit{map}\ (\mathit{mkCluster} \circ \mathit{addMult}\ \mathit{note})\ \mathit{pat})\\
&\mathit{addMult} :: \mathit{SNote} \to \mathit{SNote} \to \mathit{SNote}\\
&\mathit{addMult}\ (d_0, p_0)\ (d_1, p_1) = (d_0 * d_1, p_0 + p_1)
\end{aligned}$$

Note that *selfSim* itself is not recursive, but *mkCluster* is. This code should be studied carefully. In particualr, the recursion in *mkCluster* is different from what we have seen before, as it is not a direct invocation of

mkCluster, but rather it is a high-order argument to map (which in turn invokes *mkCluster* an aribtrary number of times).

Next, we define a function to skim off the notes at the nth level, or nth "fringe," of a cluster:

```
fringe                          :: Int → Cluster → [SNote]
fringe 0 (Cluster note cls)     = [note]
fringe n (Cluster note cls)     = concatMap (fringe (n − 1)) cls
```

Details: *concatMap* is defined in the Standard Prelude as:

```
concatMap   :: (a → [b]) → [a] → [b]
concatMap f = concat ∘ map f
```

Recall that *concat* appends together a list of lists, and is defined in the Prelude as:

```
concat :: [[a]] → [a]
concat = foldr (++) []
```

All that is left to do is convert this into a *Music* value that we can play:

```
simToMusic     :: [SNote] → Music Pitch
simToMusic ss = let mkNote (d, ap) = note d (pitch ap)
                in line (map mkNote ss)
```

We can define this with a bit more elegance as follows:

```
simToMusic     :: [SNote] → Music Pitch
simToMusic     = line ∘ map mkNote

mkNote         :: (Dur, AbsPitch) → Music Pitch
mkNote (d, ap) = note d (pitch ap)
```

The increased modularity will allow us to reuse *mkNote* later in the chapter.

Putting it all together, we can define a function that takes an initial pattern, a level, a number of pitches to transpose the result, and a tempo scaling factor, to yield a final result:

```
ss pat n tr te =
  transpose tr $ tempo te $ simToMusic $ fringe n $ selfSim pat
```

9.1.1 Sample Compositions

Let's start with a melody with no rhythmic variation.

$$
\begin{aligned}
m_0 &:: [SNote] \\
m_0 &= [(1,2),(1,0),(1,5),(1,7)] \\
tm_0 &= instrument\ Vibraphone\ (ss\ m_0\ 4\ 50\ 20)
\end{aligned}
$$

One fun thing to do with music like this is to combine it with variations of itself. For example:

$$ttm_0 = tm_0 :=: transpose\ (12)\ (revM\ tm_0)$$

We could also try the opposite: a simple percussion instrument with no melodic variation, i.e. all rhythm:

$$
\begin{aligned}
m_1 &:: [SNote] \\
m_1 &= [(1,0),(0.5,0),(0.5,0)] \\
tm_1 &= instrument\ Percussion\ (ss\ m_1\ 4\ 43\ 2)
\end{aligned}
$$

Note that the pitch is transposed by 43, which is the MIDI Key number for a "high floor tom" (i.e. percussion sound *HighFloorTom*—recall the discussion in Section 6.10).

Here is a very simple melody, two different pitches and two different durations:

$$
\begin{aligned}
m_2 &:: [SNote] \\
m_2 &= [(dqn,0),(qn,4)] \\
tm_2 &= ss\ m_2\ 6\ 50\ (1/50)
\end{aligned}
$$

Here are some more exotic compositions, combining both melody and rhythm:

$$
\begin{aligned}
m_3 &:: [SNote] \\
m_3 &= [(hn,3),(qn,4),(qn,0),(hn,6)] \\
tm_3 &= ss\ m_3\ 4\ 50\ (1/4) \\
ttm_3 &= \mathbf{let}\ l_1 = instrument\ Flute\ tm_3 \\
&\qquad\quad l_2 = instrument\ AcousticBass\ \$ \\
&\qquad\qquad\quad transpose\ (-9)\ (revM\ tm_3) \\
&\quad\ \mathbf{in}\ l_1 :=: l_2 \\
m_4 &:: [SNote] \\
m_4 &= [(hn,3),(hn,8),(hn,22),(qn,4),(qn,7),(qn,21), \\
&\quad\ (qn,0),(qn,5),(qn,15),(wn,6),(wn,9),(wn,19)] \\
tm_4 &= ss\ m_4\ 3\ 50\ 8
\end{aligned}
$$

Exercise 9.1 Experiment with this idea futher, using other melodic seeds, exploring different depths of the clusters, and so on.

Exercise 9.2 Note that *concat* is defined as *foldr* (++) [], which means that it takes a number of steps proportional to the sum of the lengths of the lists being concatenated; we cannot do any better than this. (If *foldl* were used instead, the number of steps would be proportional to the number of lists times their average length.)

However, *fringe* is not very efficient, for the following reason: *concat* is being used over and over again, like this:

$$concat\ [concat\ [...], concat\ [...], concat\ [...]]$$

This causes a number of steps proportional to the depth of the tree times the length of the sub-lists; clearly not optimal.

Define a version of *fringe* that is linear in the total length of the final list.

9.2 Self-Similar Harmony

In the last section we used a melody as a seed, and created longer melodies from it. Another idea is to stack the melodies vertically. Specifically, suppose we redefine *fringe* in such a way that it does not concatenate the sub-clusters together:

$$\begin{aligned} &fringe' && :: Int \to Cluster \to [[SNote]] \\ &fringe'\ 0\ (Cluster\ note\ cls) && = [[note]] \\ &fringe'\ n\ (Cluster\ note\ cls) && = map\ (fringe\ (n-1))\ cls \end{aligned}$$

Note that this strategy is only applied to the top level—below that we use fringe. Thus the type of the result is $[[SNote]]$, i.e. a list of lists of notes.

We can convert the individual lists into melodies, and play the melodies all together, like this:

$$\begin{aligned} &simToMusic' :: [[SNote]] \to Music\ Pitch \\ &simToMusic' = chord \circ map\ (line \circ map\ mkNote) \end{aligned}$$

Finally, we can define a function akin to *ss* defined earlier:

$$\begin{aligned} &ss'\ pat\ n\ tr\ te = \\ &\quad transpose\ tr\ \$\ tempo\ te\ \$\ simToMusic'\ \$\ fringe'\ n\ \$\ selfSim\ pat \end{aligned}$$

Using some of the same patterns used earlier, here are some sample compositions (with not necessarily a great outcome...):

$ss_1 = ss'\ m_2\ 4\ 50\ (1/8)$
$ss_2 = ss'\ m_3\ 4\ 50\ (1/2)$
$ss_3 = ss'\ m_4\ 3\ 50\ 2$

Here is a new one, based on a major triad:

$m_5 = [(en, 4), (sn, 7), (en, 0)]$
$ss_5 = ss\ m_5\ 4\ 45\ (1/500)$
$ss_6 = ss'\ m_5\ 4\ 45\ (1/1000)$

Note the need to scale the tempo back drastically, due to the short durations of the starting notes.

9.3 Other Self-Similar Structures

The reader will observe that our notion of "self-similar harmony" does not involve changing the structure of the *Cluster* data type, nor the algorithm for computing the sub-structures (as captured in *selfSim*). All that we do is interpret the result differently. This is a common characteristic of algorithmic music composition—the same mathematical or computational structure is interpreted in different ways to yield musically different results.

For example, instead of the above strategy for playing melodies in parallel, we could play entire levels of the *Cluster* in parallel, where the number of levels that we choose is given as a parameter. If alligned properly in time there will be a harmonic relationship between the levels, which could yield pleasing results.

The *Cluster* data type is conceptually useful in that is represents the infinite solution space of self-simlar melodies. And it is computationally useful in that it is computed to a desired depth only once, and thus can be inspected and reused without recomputing each level of the tree. This idea might be useful in the application mentioned above, namely combining two or more levels of the result in interesting ways.

However, the *Cluster* data type is strictly unnecessary, in that, for example, if we are interested in computing a specific level, we could define a function that recursed to that level and gave the result directly, without saving the intermediate levels.

A final point about the notion of self-similarity captured in this chapter is that the initial pattern is used as the basis with which to transform each successive level. Another strategy would be to use the entirety of each new level as the seed for transforming itself into the next level. This will result in an exponential blow-up in the size of each level, but may be worth pursuing—in some sense it is a simpler notion of self-similarity than what we have used in this chapter.

All of the ideas in this section, and others, we leave as exercises for the reader.

Exercise 9.3 Experiment with the self-similar programs in this chapter. Compose an interesting piece of music through a judicious choice of starting melody, depth of recursion, instrumentation, etc.

Exercise 9.4 Devise an interpretation of a *Cluster* that plays multiple levels of the *Cluster* in parallel. Try to get the levels to align properly in time so that each level has the same duration. You may choose to play all the levels up to a certain depth in parallel, or levels within a certain range, say levels 3 through 5.

Exercise 9.5 Define an alternative version of *simToMusic* that interprets the music differently. For example:

- Interpret the pitch as an index into a scale—e.g., as an index into the C major scale, so that 0 corresponds to C, 1 to D, 2 to E, 3 to F, ..., 6 to B, 7 to C in the next octave, and so on.
- Interpret the pitch as duration, and the duration as pitch.

Exercise 9.6 Modify the self-similar code in the following ways:

- Add a Volume component to *SNote* (in other words, define it as a triple instead of a pair), and redefine *addmult* so that it takes two of these triples and combines them in a suitable way. Then modify the rest of the code so that the result is a *Music1* value. With these modifications, compose something interesting that highlights the changes in volume.
- Change the *Pitch* field in *SNote* to be a list of *Pitch*, to be interpreted ultimately as a chord. Figure out some way to combine them in *addmult*, and compose something interesting.

Exercise 9.7 Devise some other variant of self-similar music, and encode it in Haskell. In particular, consider structures that are different from those generated by the *selfSim* function.

Exercise 9.8 Define a function that gives the same result as *ss*, but without using a data type such as *Cluster*.

Exercise 9.9 Define a version of self-similarity similar to that defined in this chapter, but that uses the entire melody generated at one level to transform itself into the next level (rather than using the original seed pattern).

Chapter 10

Proof by Induction

In this chapter we will study a powerful proof technique based on *mathematical induction*. With it we will be able to prove complex and important properties of programs that cannot be accomplished with proof-by-calculation alone. The inductive proof method is one of the most powerful and common methods for proving program properties.

10.1 Induction and Recursion

Induction is very closely related to *recursion*. In fact, in certain contexts the terms are used interchangeably; in others, one is preferred over the other primarily for historical reasons. Think of them as being duals of one another: induction is used to describe the process of starting with something small and simple, and building up from there, whereas recursion describes the process of starting with something large and complex, and working backward to the simplest case.

For example, although we have previously used the phrase *recursive data type*, in fact data types are often described *inductively*, such as a list:

> A *list* is either empty, or it is a pair consisting of a value and another list.

On the other hand, we usually describe functions that manipulate lists, such as *map* and *foldr*, as being recursive. This is because when you apply a function such as *map*, you apply it initially to the whole list, and work backwards toward [].

But these differences between induction and recursion run no deeper: they are really just two sides of the same coin.

This chapter is about *inductive properties* of programs (but based on the above argument could just as rightly be called *recursive properties*) that are not usually proven via calculation alone. Proving inductive properties usually involves the inductive nature of data types and the recursive nature of functions defined on the data types.

As an example, suppose that p is an inductive property of a list. In other words, $p\ (l)$ for some list l is either true or false (no middle ground!). To prove this property inductively, we do so based on the length of the list: starting with length 0, we first prove $p\ ([\,])$ (using our standard method of proof-by-calculation).

Now for the key step: assume for the moment that $p\ (xs)$ is true for any list xs whose length is less than or equal to n. Then if we can prove (via calculation) that $p\ (x : xs)$ is true for any x—i.e. that p is true for lists of length $n + 1$—then the claim is that p is true for lists of *any* (finite) length.

Why is this so? Well, from the first step above we know that p is true for length 0, so the second step tells us that it's also true for length 1. But if it's true for length 1 then it must also be true for length 2; similarly for lengths 3, 4, etc. So p is true for lists of any length!

(It it important to realize, however, that a property being true for every finite list does not necessarily imply that it is true for every infinite list. The property "the list is finite" is a perfect example of this! We will see how to prove properties of infinite lists in Chapter **??**.)

To summarize, to prove a property p by induction on the length of a list, we proceed in two steps:

1. Prove $p\ ([\,])$ (this is called the *base step*).
2. Assume that $p\ (xs)$ is true (this is called the *induction hypothesis*, and prove that $p\ (x : xs)$ is true (this is called the *induction step*).

10.2 Examples of List Induction

Ok, enough talk, let's see this idea in action. Recall in Section 3.1 the following property about *foldr*:

$$(\forall xs)\ \ foldr\ (:)\ [\,]\ xs \Longrightarrow xs$$

We will prove this by induction on the length of xs. Following the ideas above, we begin with the base step by proving the property for length 0; i.e. for $xs = [\,]$:

$$foldr\ (:)\ [\,]\ [\,] \Rightarrow [\,]$$

This step is immediate from the definition of *foldr*. Now for the induction step: we first *assume* that the property is true for all lists xs of length n, and then prove the property for list $x : xs$. Again proceeding by calculation:

$$\begin{aligned} & foldr\ (:)\ [\,]\ (x : xs) \\ & \Rightarrow x : foldr\ (:)\ [\,]\ xs \\ & \Rightarrow x : xs \end{aligned}$$

And we are done; the induction hypothesis is what justifies the second step.

Now let's do something a bit harder. Suppose we are interested in proving the following property:

$$(\forall xs, ys)\ \ length\ (xs +\!\!+\ ys) = length\ xs + length\ ys$$

Our first problem is to decide which list to perform the induction over. A little thought (in particular, a look at how the definitions of *length* and $(+\!\!+)$ are structured) should convince you that xs is the right choice. (If you do not see this, you are encouraged to try the proof by induction over the length of ys!) Again following the ideas above, we begin with the base step by proving the property for length 0; i.e. for $xs = [\,]$:

$$\begin{aligned} & length\ ([\,] +\!\!+\ ys) \\ & \Rightarrow length\ ys \\ & \Rightarrow 0 + length\ ys \\ & \Rightarrow length\ [\,] + length\ ys \end{aligned}$$

For the induction step, we first assume that the property is true for all lists xs of length n, and then prove the property for list $x : xs$. Again proceeding by calculation:

$$\begin{aligned} & length\ ((x : xs) +\!\!+\ ys) \\ & \Rightarrow length\ (x : (xs +\!\!+\ ys)) \\ & \Rightarrow 1 + length\ (xs +\!\!+\ ys) \\ & \Rightarrow 1 + (length\ xs + length\ ys) \end{aligned}$$

$\Rightarrow (1 + length\ xs) + length\ ys$
$\Rightarrow length\ (x : xs) + length\ ys$

And we are done. The transition from the 3rd line to the 4th is where we used the induction hypothesis.

10.3 Proving Function Equivalences

At this point it is a simple matter to return to Chapter 3 and supply the proofs that functions defined using *map* and *fold* are equivalent to the recursively defined versions. In particular, let's prove first that:

$$toAbsPitches = map\ absPitch$$

where *toAbsPitch* is the original recursively defined function:

$$\begin{aligned} toAbsPitches\ [\,] &= [\,] \\ toAbsPitches\ (p : ps) &= absPitch\ p : toAbsPitches\ ps \end{aligned}$$

To prove this, we first use the extensionality principle (briefly discussed in Section 3.6.1), which says that two functions are equal if, when applied to the same value, they always yield the same result. We can change the specification slightly to reflect this. For any finite list *ps*, we want to prove:

$$toAbsPitches\ ps = map\ absPitch\ ps$$

We proceed by induction, starting with the base case $ps = [\,]$:

$toAbsPitches\ [\,]$
$\Rightarrow [\,]$
$\Rightarrow map\ absPitch\ [\,]$

Next we assume that $toAbsPitches\ ps = map\ absPitch\ ps$ holds, and try to prove that $toAbsPitches\ (p : ps) = map\ absPitch\ (p : ps)$ (note the use of the induction hypothesis in the second step):

$toAbsPitches\ (p : ps)$
$\Rightarrow absPitch\ p : toAbsPitches\ ps$
$\Rightarrow absPitch\ p : map\ absPitch\ ps$
$\Rightarrow map\ absPitch\ (p : ps)$

The proof that $toPitches\ aps = map\ pitch\ aps$ is very similar, and is left as an exercise.

For a proof involving *foldr*, recall from Section 3.4 this recursive definition of *line*:

$$line\ [\,] = rest\ 0$$

$$line\ (m : ms) = m :+: line\ ms$$

and this non-recursive version:

$$line = foldr\ (:+:)\ (rest\ 0)$$

We can prove that these definitions are equivalent by induction. First the base case:

$$\begin{aligned} &line\ [\,] \\ &\Rightarrow rest\ 0 \\ &\Rightarrow foldr\ (:+:)\ (rest\ 0)\ [\,] \end{aligned}$$

Then the induction step:

$$\begin{aligned} &line\ (m : ms) \\ &\Rightarrow m :+: line\ ms \\ &\Rightarrow m :+: foldr\ (:+:)\ (rest\ 0)\ ms \\ &\Rightarrow foldr\ (:+:)\ (rest\ 0)\ (m : ms) \end{aligned}$$

The proofs of equivalence of the definitions of *toPitches*, *chord*, *maxPitch*, and *hList* from Section 3.4 are similar, and left as an exercise.

Exercise 10.1 From Chapter 3, prove that the original recursive versions of the following functions are equivalent to the versions using *map* or *fold*: *toPitches*, *chord*, *maxPitch*, and *hList*.

10.3.1 [Advanced] Reverse

The proofs of function equivalence in the last section were fairly straightforward. For something more challenging, consider the definition of *reverse* given in Section 3.5:

$$\begin{aligned} reverse1\ [\,] &= [\,] \\ reverse1\ (x : xs) &= reverse1\ xs +\!\!+ [x] \end{aligned}$$

and the version given in Section 3.6:

$$reverse2\ xs = foldl\ (flip\ (:))\ [\,]\ xs$$

We would like to show that these are the same; i.e. that $reverse1\ xs = reverse2\ xs$ for any finite list xs. In carrying out this proof two new ideas will be demonstrated, the first being that induction can be used to prove the equivalence of two programs. The second is the need for an *auxiliary property* which is proved independently of the main result.

The base case is easy, as it often is:

$$reverse1\ [\,]$$

$\Rightarrow [\,]$
$\Rightarrow \mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ [\,]$
$\Rightarrow \mathit{reverse2}\ [\,]$

Assume now that $\mathit{reverse1}\ xs = \mathit{reverse2}\ xs$. The induction step proceeds as follows:

$\mathit{reverse1}\ (x : xs)$
$\Rightarrow \mathit{reverse1}\ xs +\!\!+ [x]$
$\Rightarrow \mathit{reverse2}\ xs +\!\!+ [x]$
$\Rightarrow \mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ xs +\!\!+ [x]$
$\Rightarrow ???$

But now what do we do? Intuitively, it seems that the following property, which we will call property (1), should hold:

$\mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ xs +\!\!+ [x]$
$\Rightarrow \mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ (x : xs)$

in which case we could complete the proof as follows:

...
$\Rightarrow \mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ xs +\!\!+ [x]$
$\Rightarrow \mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ (x : xs)$
$\Rightarrow \mathit{reverse2}\ (x : xs)$

The ability to see that if we could just prove one thing, then perhaps we could prove another, is a useful skill in conducting proofs. In this case we have reduced the overall problem to one of proving property (1), which simplifies the structure of the proof, although not necessarily the difficulty. These auxiliary properties are often called *lemmas* in mathematics, and in many cases their proofs become the most important contributions, since they are often at the heart of a problem.

In fact if you try to prove property (1) directly, you will run into a problem, namely that it is not *general* enough. So first let's generalize property (1) (while renaming x to y), as follows:

$\mathit{foldl}\ (\mathit{flip}\ (:))\ ys\ xs +\!\!+ [y]$
$\Rightarrow \mathit{foldl}\ (\mathit{flip}\ (:))\ (ys +\!\!+ [y])\ xs$

Let's call this property (2). If (2) is true for any finite xs and ys, then property (1) is also true, because:

$\mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ xs +\!\!+ [x]$
$\Rightarrow \{\,\mathit{property}\ (2)\,\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ ([\,] +\!\!+ [x])\ xs$

$\Rightarrow \{\mathit{unfold}\ (+\!\!+)\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ [x]\ xs$
$\Rightarrow \{\mathit{fold}\ (\mathit{flip}\ (:))\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ (\mathit{flip}\ (:)\ [\,]\ x)\ xs$
$\Rightarrow \{\mathit{fold}\ \mathit{foldl}\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ [\,]\ (x : xs)$

You are encouraged to try proving property (1) directly, in which case you will likely come to the same conclusion, namely that the property needs to be generalized. This is not always easy to see, but is sometimes an important step is constructing a proof, because, despite being somewhat counterintuitive, it is often the case that making a property more general (and therefore more powerful) makes it easier to prove.

In any case, how do we prove property (2)? Using induction, of course! Setting xs to $[\,]$, the base case is easy:

$\mathit{foldl}\ (\mathit{flip}\ (:))\ ys\ [\,] +\!\!+ [y]$
$\Rightarrow \{\mathit{unfold}\ \mathit{foldl}\}$
$ys +\!\!+ [y]$
$\Rightarrow \{\mathit{fold}\ \mathit{foldl}\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ (ys +\!\!+ [y])\ [\,]$

and the induction step proceeds as follows:

$\mathit{foldl}\ (\mathit{flip}\ (:))\ ys\ (x : xs) +\!\!+ [y]$
$\Rightarrow \{\mathit{unfold}\ \mathit{foldl}\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ (\mathit{flip}\ (:)\ ys\ x)\ xs +\!\!+ [y]$
$\Rightarrow \{\mathit{unfold}\ \mathit{flip}\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ (x : ys)\ xs +\!\!+ [y]$
$\Rightarrow \{\mathit{induction\ hypothesis}\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ ((x : ys) +\!\!+ [y])\ xs$
$\Rightarrow \{\mathit{unfold}\ (+\!\!+)\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ (x : (ys +\!\!+ [y]))\ xs$
$\Rightarrow \{\mathit{fold}\ \mathit{foldl}\}$
$\mathit{foldl}\ (\mathit{flip}\ (:))\ (ys +\!\!+ [y])\ (x : xs)$

10.4 Useful Properties on Lists

There are many useful properties of functions on lists that require inductive proofs. Tables 10.1 and 10.2 list a number of them involving functions used in this text, but their proofs are left as exercises (except for one; see below).

Properties of *map*:

$$
\begin{aligned}
map\ (\lambda x \to x) &= \lambda x \to x \\
map\ (f \circ g) &= map\ f \circ map\ g \\
map\ f \circ tail &= tail \circ map\ f \\
map\ f \circ reverse &= reverse \circ map\ f \\
map\ f \circ concat &= concat \circ map\ (map\ f) \\
map\ f\ (xs +\!\!+ ys) &= map\ f\ xs +\!\!+ map\ f\ ys
\end{aligned}
$$

For all strict f:

$$f \circ head = head \circ map\ f$$

Properties of the *fold* **functions:**

1. If op is associative, and $e\ `op`\ x = x$ and $x\ `op`\ e = x$ for all x, then for all finite xs:

   $$foldr\ op\ e\ xs = foldl\ op\ e\ xs$$

2. If the following are true:

   $$
   \begin{aligned}
   x\ `op1`\ (y\ `op2`\ z) &= (x\ `op1`\ y)\ `op2`\ z \\
   x\ `op1`\ e &= e\ `op2`\ x
   \end{aligned}
   $$

   then for all finite xs:

   $$foldr\ op1\ e\ xs = foldl\ op2\ e\ xs$$

3. For all finite xs:

   $$foldr\ op\ e\ xs = foldl\ (flip\ op)\ e\ (reverse\ xs)$$

Table 10.1: Some Useful Properties of *map* and *fold*.

You may assume that these properties are true, and use them freely in proving other properties of your programs. In fact, some of these properties can be used to simplify the proof that *reverse1* and *reverse2* are the same; see if you can find them![1]

(Note, by the way, that in the first rule for *map* in Figure 10.1, the type of $\lambda x \to x$ on the left-hand side is $a \to b$, whereas on the right-hand side it is $[a] \to [b]$; i.e. these are really two different functions.)

[1] More thorough discussions of these properties and their proofs may be found in [BW88, Bir98].

Properties of $(+\!\!+)$**:**

For all xs, ys, and zs:

$$(xs +\!\!+ ys) +\!\!+ zs = xs +\!\!+ (ys +\!\!+ zs)$$
$$xs +\!\!+ [\,] \quad = [\,] +\!\!+ xs = xs$$

Properties of *take* **and** *drop***:**

For all finite non-negative m and n, and finite xs:

$$take\ n\ xs +\!\!+ drop\ n\ xs = xs$$
$$take\ m \circ take\ n \quad = take\ (min\ m\ n)$$
$$drop\ m \circ drop\ n \quad = drop\ (m + n)$$
$$take\ m \circ drop\ n \quad = drop\ n \circ take\ (m + n)$$

For all finite non-negative m and n such that $n \geqslant m$:

$$drop\ m \circ take\ n = take\ (n - m) \circ drop\ m$$

Properties of *reverse***:**

For all finite xs:

$$reverse\ (reverse\ xs) = xs$$
$$head\ (reverse\ xs) \quad = last\ xs$$
$$last\ (reverse\ xs) \quad = head\ xs$$

Table 10.2: Useful Properties of Other Functions Over Lists

10.4.1 [Advanced] Function Strictness

Note that the last rule for *map* in Figure 10.1 is only valid for *strict* functions. A function f is said to be strict if $f\ \bot = \bot$. Recall from Section 1.4 that $\bot$ is the value associated with a non-terminating computation. So another way to think about a strict function is that it is one that, when applied to a non-terminating computation, results in a non-terminating computation. For example, the successor function $(+1)$ is strict, because $(+1)\ \bot = \bot + 1 = \bot$. In other words, if you apply $(+1)$ to a non-terminating computation, you end up with a non-terminating computation.

Not all functions in Haskell are strict, and we have to be careful to say on which argument a function is strict. For example, $(+)$ is strict on both of its arguments, which is why the section $(+1)$ is also strict. On the other hand, the constant function:

$$const\ x\ y = x$$

is strict on its first argument (why?), but not its second, because $const\ x\ \bot = x$, for any x.

Details: Understanding strictness requires a careful understanding of Haskell's pattern-matching rules. For example, consider the definition of $(\wedge)$ from the Standard Prelude:

$$\begin{aligned}(\wedge) \quad &:: Bool \to Bool \to Bool\\ True \wedge x &= x\\ False \wedge \_ &= False\end{aligned}$$

When choosing a pattern to match, Haskell starts with the top, left-most pattern, and works to the right and downward. So in the above, $(\wedge)$ first evaluates its left argument. If that value is $True$, then the first equation succeeds, and the second argument gets evaluated because that is the value that is returned. But if the first argument is $False$, the second equation succeeds. In particular, *it does not bother to evaluate the second argument at all*, and simply returns $False$ as the answer. This means that $(\wedge)$ is strict in its first argument, but not its second.

A more detailed discussion of pattern matching is found in Appendix D.

Let's now look more closely at the last law for *map*, which says that for all strict f:

$$f \circ head = head \circ map\ f$$

Let's try to prove this property, starting with the base case, but ignoring

for now the strictness constraint on f:

$$\begin{aligned} & f\ (head\ [\,]) \\ \Rightarrow\ & f\ \bot \end{aligned}$$

head [ ] is an error, which you will recall has value $\bot$. So you can see immediately that the issue of strictness might play a role in the proof, because without knowing anything about f, there is no further calculation to be done here. Similarly, if we start with the right-hand side:

$$\begin{aligned} & head\ (map\ f\ [\,]) \\ \Rightarrow\ & head\ [\,] \\ \Rightarrow\ & \bot \end{aligned}$$

It should be clear that for the base case to be true, it must be that $f\ \bot = \bot$; i.e., f must be strict. Thus we have essentially "discovered" the constraint on the theorem through the process of trying to prove it! (This is not an uncommon phenomenon.)

The induction step is less problematic:

$$\begin{aligned} & f\ (head\ (x : xs)) \\ \Rightarrow\ & f\ x \\ \Rightarrow\ & head\ (f\ x : map\ f\ xs) \\ \Rightarrow\ & head\ (map\ f\ (x : xs)) \end{aligned}$$

and we are done.

Exercise 10.2 Prove as many of the properties in Tables 10.1 and 10.2 as you can.

Exercise 10.3 Which of the following functions are strict (if the function takes more than one argument, specify on which arguments it is strict): *reverse*, *simple*, *map*, *tail*, *dur*, *revM*, $(\wedge)$, $(True\ \wedge)$, $(False\ \wedge)$, and the following function:

$$\begin{aligned} & ifFun && :: Bool \to a \to a \to a \\ & ifFun\ pred\ cons\ alt = \textbf{if}\ pred\ \textbf{then}\ cons\ \textbf{else}\ alt \end{aligned}$$

10.5 Induction on the Music Data Type

Proof by induction is not limited to lists. In particular, we can use it to reason about *Music* values.

For example, we will show that:

$$mFold\ (:+:)\ (:=:)\ Prim\ Modify = id$$

To prove this, we again use the extensionality principle, and then proceed by induction. But what is the base case? Recall that the *Music* data type is defined as:

```
data Music a =
    Prim (Primitive a)
  | Music a :+: Music a
  | Music a :=: Music a
  | Modify Control (Music a)
```

The only constructor that does not take a *Music* value as an argument is *Prim*, so that in fact is the only base case.

So, starting with this base case:

```
mFold (:+:) (:=:) Prim Modify (Prim p)
⇒ Prim p
⇒ id (Prim p)
```

That was easy! Next, we develop an induction step for each of the three non-base cases:

```
mFold (:+:) (:=:) Prim Modify (m1 :+: m2)
⇒ mFold (:+:) (:=:) Prim Modify m1 :+:
  mFold (:+:) (:=:) Prim Modify m2
⇒ m1 :+: m2
⇒ id (m1 :+: m2)
```

```
mFold (:+:) (:=:) Prim Modify (m1 :=: m2)
⇒ mFold (:+:) (:=:) Prim Modify m1 :=:
  mFold (:+:) (:=:) Prim Modify m2
⇒ m1 :=: m2
⇒ id (m1 :=: m2)
```

```
mFold (:+:) (:=:) Prim Modify (Modify c m)
⇒ Modify c (mFold (:+:) (:=:) Prim Modify m)
⇒ Modify c m
⇒ id (Modify c m)
```

These three steps were quite easy as well, but is not something we could have done without induction.

For something more challenging, let's consider the following:

```
dur (revM m) = dur m
```

Again we proceed by induction, starting with the base case:

```
dur (revM (Prim p))
⇒ dur (Prim p)
```

Sequential composition is straightforward:

$$
\begin{array}{l}
\mathit{dur}\ (\mathit{revM}\ (m_1 :+: m_2)) \\
\Rightarrow \mathit{dur}\ (\mathit{revM}\ m_2 :+: \mathit{revM}\ m_1) \\
\Rightarrow \mathit{dur}\ (\mathit{revM}\ m_2) + \mathit{dur}\ (\mathit{revM}\ m_1) \\
\Rightarrow \mathit{dur}\ m_2 + \mathit{dur}\ m_1 \\
\Rightarrow \mathit{dur}\ m_1 + \mathit{dur}\ m_2 \\
\Rightarrow \mathit{dur}\ (m_1 :+: m_2)
\end{array}
$$

But things get more complex with parallel composition:

$$
\begin{array}{l}
\mathit{dur}\ (\mathit{revM}\ (m_1 :=: m_2)) \\
\Rightarrow \mathit{dur}\ (\textbf{let}\ d_1 = \mathit{dur}\ m_1 \\
\qquad\qquad\quad d_2 = \mathit{dur}\ m_2 \\
\qquad\quad \textbf{in if}\ d_1 > d_2\ \textbf{then}\ \mathit{revM}\ m_1 :=: (\mathit{rest}\ (d_1 - d_2) :+: \mathit{revM}\ m_2) \\
\qquad\qquad\qquad\quad \textbf{else}\ (\mathit{rest}\ (d_2 - d_1) :+: \mathit{revM}\ m_1) :=: \mathit{revM}\ m_2) \\
\Rightarrow \textbf{let}\ d_1 = \mathit{dur}\ m_1 \\
\qquad\quad d_2 = \mathit{dur}\ m_2 \\
\quad \textbf{in if}\ d_1 > d_2\ \textbf{then}\ \mathit{dur}\ (\mathit{revM}\ m_1 :=: (\mathit{rest}\ (d_1 - d_2) :+: \mathit{revM}\ m_2)) \\
\qquad\qquad\quad \textbf{else}\ \mathit{dur}\ ((\mathit{rest}\ (d_2 - d_1) :+: \mathit{revM}\ m_1) :=: \mathit{revM}\ m_2) \\
\ldots
\end{array}
$$

At this point, to make things easier to understand, we will consider each branch of the conditional in turn. First the consequent branch:

$$
\begin{array}{l}
\mathit{dur}\ (\mathit{revM}\ m_1 :=: (\mathit{rest}\ (d_1 - d_2) :+: \mathit{revM}\ m_2)) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ (\mathit{revM}\ m_1))\ (\mathit{dur}\ (\mathit{rest}\ (d_1 - d_2) :+: \mathit{revM}\ m_2)) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ m_1)\ (\mathit{dur}\ (\mathit{rest}\ (d_1 - d_2) :+: \mathit{revM}\ m_2)) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ m_1)\ (\mathit{dur}\ (\mathit{rest}\ (d_1 - d_2)) + \mathit{dur}\ (\mathit{revM}\ m_2)) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ m_1)\ ((d_1 - d_2) + \mathit{dur}\ m_2) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ m_1)\ (\mathit{dur}\ m_1) \\
\Rightarrow \mathit{dur}\ m_1
\end{array}
$$

And then the alternative:

$$
\begin{array}{l}
\mathit{dur}\ ((\mathit{rest}\ (d_2 - d_1) :+: \mathit{revM}\ m_1) :=: \mathit{revM}\ m_2) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ ((\mathit{rest}\ (d_2 - d_1) :+: \mathit{revM}\ m_1))\ (\mathit{dur}\ (\mathit{revM}\ m_2)) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ ((\mathit{rest}\ (d_2 - d_1) :+: \mathit{revM}\ m_1))\ (\mathit{dur}\ m_2) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ (\mathit{rest}\ (d_2 - d_1)) + \mathit{dur}\ (\mathit{revM}\ m_1))\ (\mathit{dur}\ m_2) \\
\Rightarrow \mathit{max}\ ((d_2 - d_1) + \mathit{dur}\ m_1)\ (\mathit{dur}\ m_2) \\
\Rightarrow \mathit{max}\ (\mathit{dur}\ m_2)\ (\mathit{dur}\ m_2) \\
\Rightarrow \mathit{dur}\ m_2
\end{array}
$$

Now we can continue the proof from above:

$$
\begin{array}{l}
\ldots \\
\Rightarrow \textbf{let}\ d_1 = \mathit{dur}\ m_1
\end{array}
$$

$$\begin{aligned}
&\quad d_2 = dur\ m_2 \\
&\textbf{in if } d_1 > d_2 \textbf{ then } dur\ m_1 \\
&\qquad\qquad \textbf{else } dur\ m_2 \\
&\Rightarrow max\ (dur\ m_1)\ (dur\ m_2) \\
&\Rightarrow dur\ (m_1 :=: m_2)
\end{aligned}$$

The final inductive step involves the *Modify* constructor, but recall that *dur* treats a *Tempo* modification specially, and thus we treat it specially as well:

$$\begin{aligned}
&dur\ (revM\ (Modify\ (Tempo\ r)\ m)) \\
&\Rightarrow dur\ (Modify\ (Tempo\ r)\ (revM\ m)) \\
&\Rightarrow dur\ (revM\ m)/r \\
&\Rightarrow dur\ m/r \\
&\Rightarrow dur\ (Modify\ (Tempo\ r)\ m)
\end{aligned}$$

Finally, we consider the case that $c \neq Tempo\ r$:

$$\begin{aligned}
&dur\ (revM\ (Modify\ c\ m)) \\
&\Rightarrow dur\ (Modify\ c\ (revM\ m)) \\
&\Rightarrow Modify\ c\ (dur\ (revM\ m)) \\
&\Rightarrow Modify\ c\ (dur\ m) \\
&\Rightarrow dur\ (Modify\ c\ m)
\end{aligned}$$

And we are done.

Exercise 10.4 Recall Exercises 3.9 and 3.10. Prove that, if $p_2 \geqslant p_1$:

$$\begin{aligned}
chrom\ p_1\ p_2 = mkScale\ p_1\ (&take\ (absPitch\ p_2 - absPitch\ p_1) \\
&(repeat\ 1))
\end{aligned}$$

using the lemma:

$$[m\,..\,n] = scanl\ (+)\ m\ (take\ (n-m)\ (repeat\ 1))$$

Exercise 10.5 Prove the following facts involving *dur*:

$$\begin{aligned}
dur\ (timesM\ n\ m) &= n * dur\ m \\
dur\ (cut\ d\ m) &= d, \textbf{if } d \leqslant dur\ m
\end{aligned}$$

Exercise 10.6 Prove the following facts involving *mMap*:

$$\begin{aligned}
mMap\ id\ m &= m \\
mMap\ f\ (mMap\ g\ m) &= mMap\ (f \circ g)\ m
\end{aligned}$$

Exercise 10.7 For that, for all *pmap*, c, and m_2:

$$perf\ pmap\ c\ m_2 = (perform\ pmap\ c\ m_2, dur\ m_2)$$

where *perform* is the function defined in Figure 8.1.

10.5.1 The Need for Musical Equivalence

In Chapter 1 we discussed the need for a notion of *musical equivalence*, noting that, for example, $m :+: rest\ 0$ "sounds the same" as m, even if the two *Music* values are not equal as Haskell values. That same issue can strike us here as we try to prove intuitively natural properties such as:

$$revM\ (revM\ m) = m$$

To see why this property cannot be proved without a notion of musical equivalence, note that:

$$revM\ (c\ 4\ en :=: d\ 4\ qn)$$
$$\Rightarrow (rest\ en :+: c\ 4\ en) :=: d\ 4\ qn$$

and therefore:

$$revM\ (revM\ (c\ 4\ en :=: d\ 4\ qn))$$
$$\Rightarrow (rest\ 0 :+: c\ 4\ en :+: rest\ en) :=: d\ 4\ qn$$

Clearly the last line above is not equal, as a Haskell value, to $c\ 4\ en :=: d\ 4\ qn$. But somehow we need to show that these two values "sound the same" as musical values. In the next chapter we will formally develop the notion of musical equivalence, and with it be able to prove the validity of our intuitions regarding *revM*, as well as many other important musical properties.

10.6 [Advanced] Induction on Other Data Types

Proof by induction can be used to reason about many data types. For example, we can use it to reason about natural numbers.[2] Suppose we define an exponentiation function as follows:

$$(\hat{}) :: Integer \to Integer \to Integer$$
$$x \hat{} 0 = 1$$
$$x \hat{} n = x * x \hat{} (n-1)$$

[2]Indeed, one could argue that a proof by induction over finite lists is really an induction over natural numbers, since it is an induction over the *length* of the list, which is a natural number.

Details: $(*)$ is defined in the Standard Prelude to have precedence level 7, and recall that if no **infix** declaration is given for an operator it defaults to precedence level 9, which means that $(\hat{})$ has precedence level 9, which is higher than that for $(*)$. Therefore no parentheses are needed to disambiguate the last line in the definition above, which corresponds nicely to mathematical convention.

Now suppose that we want to prove that:

$$(\forall x, n \geqslant 0, m \geqslant 0) \quad x\hat{}(n+m) = x\hat{}n * x\hat{}m$$

We proceed by induction on n, beginning with $n = 0$:

$$\begin{aligned} & x\hat{}(0+m) \\ & \Rightarrow x\hat{}m \\ & \Rightarrow 1 * (x\hat{}m) \\ & \Rightarrow x\hat{}0 * x\hat{}m \end{aligned}$$

Next we assume that the property is true for numbers less than or equal to n, and prove it for $n+1$:

$$\begin{aligned} & x\hat{}((n+1)+m) \\ & \Rightarrow x * x\hat{}(n+m) \\ & \Rightarrow x * (x\hat{}n * x\hat{}m) \\ & \Rightarrow (x * x\hat{}n) * x\hat{}m \\ & \Rightarrow x\hat{}(n+1) * x\hat{}m \end{aligned}$$

and we are done.

Or are we? What if, in the definition of $(\hat{})$, x or n is *negative*? Since a negative integer is not a natural number, we could dispense with the problem by saying that these situations fall beyond the bounds of the property we are trying to prove. But let's look a little closer. If x is negative, the property we are trying to prove still holds (why?). But if n is negative, $x\hat{}n$ will not terminate (why?). As diligent programmers we may wish to defend against the latter situation by writing:

$$\begin{aligned} (\hat{}) \quad & :: \mathit{Integer} \to \mathit{Integer} \to \mathit{Integer} \\ x\hat{}0 \quad & = 1 \\ x\hat{}n \mid n < 0 \quad & = \mathit{error}\ \texttt{"negative exponent"} \\ \mid \mathit{otherwise} \quad & = x * x\hat{}(n-1) \end{aligned}$$

If we consider non-terminating computations and ones that produce an error to both have the same value, namely $\perp$, then these two versions of $(\hat{})$ are

equivalent. Pragmatically, however, the latter is clearly superior.

Note that the above definition will test for $n < 0$ on every recursive call, when actually the only call in which it could happen is the first. Therefore a slightly more efficient version of this program would be:

$$
\begin{aligned}
&(\hat{}) :: \mathit{Integer} \to \mathit{Integer} \to \mathit{Integer} \\
&x \hat{} n \mid n < 0 \quad = \mathit{error}\ \texttt{"negative exponent"} \\
&\quad\ \mid \mathit{otherwise} = f\ x\ n \\
&\quad \textbf{where}\ f\ x\ 0 = 1 \\
&\qquad\quad\ \ f\ x\ n = x * f\ x\ (n-1)
\end{aligned}
$$

Proving the property stated earlier for this version of the program is straightforward, with one minor distinction: what we really need to prove is that the property is true for f; that is:

$$(\forall x, n \geqslant 0, m \geqslant 0)\ \ f\ x\ (n+m) = f\ x\ n * f\ x\ m$$

from which the proof for the whole function follows trivially.

10.6.1 A More Efficient Exponentiation Function

But in fact there is a more serious inefficiency in our exponentiation function: we are not taking advantage of the fact that, for any even number n, $x^n = (x * x)^{n/2}$. Using this fact, here is a more clever way to accomplish the exponentiation task, using the names $(\hat{}!)$ and *ff* for our functions to distinguish them from the previous versions:

$$
\begin{aligned}
&(\hat{}!) :: \mathit{Integer} \to \mathit{Integer} \to \mathit{Integer} \\
&x\ \hat{}!\ n \mid n < 0 \quad = \mathit{error}\ \texttt{"negative exponent"} \\
&\quad\ \mid \mathit{otherwise} = \mathit{ff}\ x\ n \\
&\quad \textbf{where}\ \mathit{ff}\ x\ n \mid n == 0 \quad = 1 \\
&\qquad\qquad\qquad\ \mid \mathit{even}\ n \quad = \mathit{ff}\ (x * x)\ (n\ `\mathit{quot}`\ 2) \\
&\qquad\qquad\qquad\ \mid \mathit{otherwise} = x * \mathit{ff}\ x\ (n-1)
\end{aligned}
$$

Details: *quot* is Haskell's *quotient* operator, which returns the integer quotient of the first argument divided by the second, rounded toward zero.

You should convince yourself that, intuitively at least, this version of exponentiation is not only correct, but also more efficient. More precisely,

($\hat{}$) executes a number of steps proportional to n, whereas ($\hat{}!$) executes a number of steps proportional to the $\log_2$ of n. The Standard Prelude defines ($\hat{}$) similarly to the way in which ($\hat{}!$) is defined here.

Since intuition is not always reliable, let's *prove* that this version is equivalent to the old. That is, we wish to prove that $x \hat{} n = x \hat{}! n$ for all x and n.

A quick look at the two definitions reveals that what we really need to prove is that $f\ x\ n = ff\ x\ n$, from which it follows immediately that $x \hat{} n = x \hat{}! n$. We do this by induction on n, beginning with the base case $n = 0$:

$$f\ x\ 0 \Rightarrow 1 \Rightarrow ff\ x\ 0$$

so the base step holds trivially. The induction step, however, is considerably more complicated. We must consider two cases: $n + 1$ is either even, or it is odd. If it is odd, we can show that:

$$\begin{aligned} & f\ x\ (n+1) \\ & \Rightarrow x * f\ x\ n \\ & \Rightarrow x * ff\ x\ n \\ & \Rightarrow ff\ x\ (n+1) \end{aligned}$$

and we are done (note the use of the induction hypothesis in the second step).

If $n + 1$ is even, we might try proceeding in a similar way:

$$\begin{aligned} & f\ x\ (n+1) \\ & \Rightarrow x * f\ x\ n \\ & \Rightarrow x * ff\ x\ n \end{aligned}$$

But now what shall we do? Since n is odd, we might try unfolding the call to ff:

$$\begin{aligned} & x * ff\ x\ n \\ & \Rightarrow x * (x * ff\ x\ (n-1)) \end{aligned}$$

but this doesn't seem to be getting us anywhere. Furthermore, *folding* the call to ff (as we did in the odd case) would involve *doubling* n and taking the square root of x, neither of which seems like a good idea!

We could also try going in the other direction:

$$\begin{aligned} & ff\ x\ (n+1) \\ & \Rightarrow ff\ (x * x)\ ((n+1)\ `quot`\ 2) \\ & \Rightarrow f\ (x * x)\ ((n+1)\ `quot`\ 2) \end{aligned}$$

The use of the induction hypothesis in the second step needs to be justified, because the first argument to f has changed from x to $x * x$. But recall that the induction hypothesis states that for *all* values x, and all natural numbers up to n, $f\ x\ n$ is the same as $ff\ x\ n$. So this is OK.

But even allowing this, we seem to be stuck again!

Instead of pushing this line of reasoning further, let's pursue a different tact based on the (valid) assumption that if m is even, then:

$$m = m\ `quot`\ 2 + m\ `quot`\ 2$$

Let's use this fact together with the property that we proved in the last section:

$$\begin{aligned} & f\ x\ (n+1) \\ & \Rightarrow f\ x\ ((n+1)\ `quot`\ 2 + (n+1)\ `quot`\ 2) \\ & \Rightarrow f\ x\ ((n+1)\ `quot`\ 2) * f\ x\ ((n+1)\ `quot`\ 2) \end{aligned}$$

Next, as with the proof in the last section involving *reverse*, let's make an assumption about a property that will help us along. Specifically, what if we could prove that $f\ x\ n * f\ x\ n$ is equal to $f\ (x * x)\ n$? If so, we could proceed as follows:

$$\begin{aligned} & f\ x\ ((n+1)\ `quot`\ 2) * f\ x\ ((n+1)\ `quot`\ 2) \\ & \Rightarrow f\ (x * x)\ ((n+1)\ `quot`\ 2) \\ & \Rightarrow ff\ (x * x)\ ((n+1)\ `quot`\ 2) \\ & \Rightarrow ff\ x\ (n+1) \end{aligned}$$

and we are finally done. Note the use of the induction hypothesis in the second step, as justified earlier. The proof of the auxiliary property is not difficult, but also requires induction; it is shown in Figure 10.1.

Aside from improving efficiency, one of the pleasant outcomes of proving that $(\hat{})$ and $(\hat{}!)$ are equivalent is that *anything that we prove about one function will be true for the other*. For example, the validity of the property that we proved earlier:

$$x \hat{} (n+m) = x \hat{} n * x \hat{} m$$

immediately implies the validity of:

$$x \hat{}! (n+m) = x \hat{}! n * x \hat{}! m$$

Base case $(n = 0)$:
$f\ x\ 0 * f\ x\ 0$
$\Rightarrow 1 * 1$
$\Rightarrow 1$
$\Rightarrow f\ (x * x)\ 0$
Induction step $(n + 1)$:
$f\ x\ (n + 1) * f\ x\ (n + 1)$
$\Rightarrow (x * f\ x\ n) * (x * f\ x\ n)$
$\Rightarrow (x * x) * (f\ x\ n * f\ x\ n)$
$\Rightarrow (x * x) * f\ (x * x)\ n$
$\Rightarrow f\ (x * x)\ (n + 1)$

Figure 10.1: Proof that $f\ x\ n * f\ x\ n = f\ (x * x)\ n$.

Although (^!) is more efficient than (^), it is also more complicated, so it makes sense to try proving new properties for (^), since the proofs will likely be easier.

The moral of this story is that you shouldn't throw away old code that is simpler but less efficient than a newer version. That old code can serve at least two good purposes: First, if it is simpler, it is likely to be easier to understand, and thus serves a useful role in documenting your effort. Second, as we have just discussed, if it is provably equivalent to the new code, then it can be used to simplify the task of proving properties about the new code.

Exercise 10.8 The function (^!) can be made more efficient by noting that in the last line of the definition of *ff*, n is odd, and therefore $n - 1$ must be even, so the test for n being even on the next recursive call could be avoided. Redefine (^!) so that it avoids this (minor) inefficiency.

Exercise 10.9 Consider this definition of the *factorial* function:[3]

$fac_1 :: Integer \to Integer$
$fac_1\ 0 = 1$
$fac_1\ n = n * fac_1\ (n - 1)$

[3]The factorial function is defined mathematically as:

$$factorial(n) = \begin{cases} 1 & \text{if } n = 0 \\ n * factorial(n-1) & \text{otherwise} \end{cases}$$

and this alternative definition:

$$
\begin{aligned}
&fac_2 \quad :: Integer \to Integer \\
&fac_2\ n = fac'\ n\ 1 \\
&\quad \textbf{where}\ fac'\ 0\ x\ = x \\
&\qquad\qquad fac'\ n\ x = fac'\ (n-1)\ (n * x)
\end{aligned}
$$

Prove that $fac_1\ n = fac_2\ n$ for all non-negative integers n.

Chapter 11

An Algebra of Music

In this chapter we will explore a number of properties of the *Music* data type and functions defined on it, properties that collectively form an *algebra of music*. With this algebra we can reason about, transform, and optimize computer music programs in a meaning preserving way.

11.1 Musical Equivalance

Suppose we have two values $m_1 :: Music\ Pitch$ and $m_2 :: Music\ Pitch$, and we want to know if they are equal. If we treat them simply as Haskell values, we could easily write a function that compares their structures recursively to see if they are the same at every level, all the way down to the *Primitive* rests and notes. This is in fact what the Haskell function (==) does. For example, if:

$$m_1 = c\ 4\ en :+: d\ 4\ qn$$
$$m_2 = revM\ (revM\ m_1)$$

Then $m_1 == m_2$ is *True*.

Unfortunately, as we saw in the last chapter, if we reverse a parallel composition, things don't work out as well. For example:

$$revM\ (revM\ (c\ 4\ en :=: d\ 4\ qn))$$
$$\Rightarrow (rest\ 0 :+: c\ 4\ en :+: rest\ en) :=: d\ 4\ qn$$

In addition, as we discussed briefly in Chapter 1, there are musical properties for which standard Haskell equivalence is insufficient to capture. For example, we would expect the following two musical values to *sound* the

same, regardless of the actual values of m_1, m_2, and m_3:

$$(m_1 :+: m_2) :+: m_3$$
$$m_1 :+: (m_2 :+: m_3)$$

In other words, we expect the operator (:+:) to be *associative*.

The problem is that, as data structures, these two values are *not* equal in general, in fact there are no finite values that can be assigned to m_1, m_2, and m_3 to make them equal.[1]

The obvious way out of this dilemma is to define a new notion of equality that captures the fact that the *performances* are the same—i.e. if two things *sound* the same, they must be musically equivalent. And thus we define a formal notion of musical equivalence:

Definition: Two musical values m_1 and m_2 are *equivalent*, written $m_1 \equiv m_2$, if and only if:

$$(\forall pm, c) \;\; perf\ pm\ c\ m_1 = perf\ pm\ c\ m_2$$

We will study a number of properties in this chapter that capture musical equivalences, similar in spirit to the associativity of (:+:) above. Each of them can be thought of as an *axiom*, and the set of valid axioms collectively forms an *algebra of music*. By proving the validity of each axiom we not only confirm our intuitions about how music is interpreted, but also gain confidence that our *perform* function actually does the right thing. Furthermore, with these axioms in hand, we can *transform* musical values in meaning-preserving ways.

Speaking of the *perform* function, recall from Chapter 8 that we defined *two* versions of *perform*, and the definition above uses the function *perf*, which includes the duration of a musical value in its result. The following Lemma captures the connection between these functions:

Lemma 11.1.1 For all pm, c, and m:

$$perf\ pm\ c\ m = (perform\ pm\ c\ m, dur\ m * cDur\ c)$$

where *perform* is the function defined in Figure 8.1.

To see the importance of including duration in the definition of equivalence, we first note that if two musical values are equivalent, we should

[1] If $m_1 = m_1 :+: m_2$ and $m_3 = m_2 :+: m_3$ then the two expressions are equal, but these are infinite values that cannot be reversed or even performed.

be able to substitute one for the other in any valid musical context. But if duration is not taken into account, then all rests are equivalent (because their performances are just the empty list). This means that, for example, $m_1 :+: \mathit{rest}\ 1 :+: m_2$ is equivalent to $m_1 :+: \mathit{rest}\ 2 :+: m_2$, which is surely not what we want.

Note that we could have defined *perf* as above, i.e. in terms of *perform* and *dur*, but as mentioned in Section 8.1 it would have been computationally inefficient to do so. On the other hand, if the Lemma above is true, then our proofs might be simpler if we first proved the property using *perform*, and then using *dur*. That is, to prove $m_1 \equiv m_2$ we need to prove:

$$\mathit{perf}\ \mathit{pm}\ c\ m_1 = \mathit{perf}\ \mathit{pm}\ c\ m_2$$

Instead of doing this directly using the definition of *perf*, we could instead prove both of the following:

$$\mathit{perform}\ \mathit{pm}\ c\ m_1 = \mathit{perform}\ \mathit{pm}\ c\ m_2$$
$$\mathit{dur}\ m_1 = \mathit{dur}\ m_2$$

11.2 Some Simple Axioms

Let's look at a few simple axioms, and see how we can prove each of them using the proof techniques that we have developed so far.

(Note: In the remainder of this chapter we will use the functions *tempo r* and *trans p* to represent their unfolded versions, *Modify* (*Tempo r*) and *Modify* (*Transpose t*), respectively. In the proofs we will not bother with the intermediate steps of unfolding these functions.)

Here is the first axiom that we will consider:

Axiom 11.2.1 For any r_1, r_2, and m:

$$\mathit{tempo}\ r_1\ (\mathit{tempo}\ r_2\ m) \equiv \mathit{tempo}\ (r_1 * r_2)\ m$$

In other words, *tempo scaling is multiplicative.*

We can prove this by calculation, starting with the definition of musical equivalence. For clarity we will first prove the property for *perform*, and then for *dur*, as suggested in the last section:

let $dt = \mathit{cDur}\ c$

$\mathit{perform}\ \mathit{pm}\ c\ (\mathit{tempo}\ r_1\ (\mathit{tempo}\ r_2\ m))$
$\Rightarrow \{\, \mathit{unfold}\ \mathit{perform} \,\}$
$\mathit{perform}\ \mathit{pm}\ (c\ \{\, \mathit{cDur} = dt / r_1 \,\})\ (\mathit{tempo}\ r_2\ m)$

$\Rightarrow \{\, unfold\ perform \,\}$
$perform\ pm\ (c\ \{\, cDur = (dt\,/\,r_1)\,/\,r_2 \,\})\ m$
$\Rightarrow \{\, arithmetic \,\}$
$perform\ pm\ (c\ \{\, cDur = dt\,/\,(r_1 * r_2) \,\})\ m$
$\Rightarrow \{\, fold\ perform \,\}$
$perform\ pm\ c\ (tempo\ (r_1 * r_2)\ m)$

$dur\ (tempo\ r_1\ (tempo\ r_2\ m))$
$\Rightarrow \{\, unfold\ dur \,\}$
$dur\ (tempo\ r_2\ m)\,/\,r_1$
$\Rightarrow \{\, unfold\ dur \,\}$
$(dur\ m\,/\,r_2)\,/\,r_1$
$\Rightarrow \{\, arithmetic \,\}$
$dur\ m\,/\,(r_1 * r_2)$
$\Rightarrow \{\, fold\ dur \,\}$
$dur\ (tempo\ (r_1 * r_2)\ m)$

Here is another useful axiom and its proof:

Axiom 11.2.2 For any r, m_1, and m_2:

$$tempo\ r\ (m_1 :+: m_2) \equiv tempo\ r\ m_1 :+: tempo\ r\ m_2$$

In other words, *tempo scaling distributes over sequential composition.*

Proof:

let $t = cTime\ c; dt = cDur\ c$
$t_1 = t + dur\ m_1 * (dt\,/\,r)$
$t_2 = t + (dur\ m_1\,/\,r) * dt$
$t3 = t + dur\ (tempo\ r\ m_1) * dt$
$perform\ pm\ c\ (tempo\ r\ (m_1 :+: m_2))$
$\Rightarrow \{\, unfold\ perform \,\}$
$perform\ pm\ (c\ \{\, cDur = dt\,/\,r \,\})\ (m_1 :+: m_2)$
$\Rightarrow \{\, unfold\ perform \,\}$
$perform\ pm\ (c\ \{\, cDur = dt\,/\,r \,\})\ m_1$
$+\!\!+\ perform\ pm\ (c\ \{\, cTime = t_1, cDur = dt\,/\,r \,\})\ m_2$
$\Rightarrow \{\, fold\ perform \,\}$
$perform\ pm\ c\ (tempo\ r\ m_1)$
$+\!\!+\ perform\ pm\ (c\ \{\, cTime = t_1 \,\})\ (tempo\ r\ m_2)$
$\Rightarrow \{\, arithmetic \,\}$
$perform\ pm\ c\ (tempo\ r\ m_1)$
$+\!\!+\ perform\ pm\ (c\ \{\, cTime = t_2 \,\})\ (tempo\ r\ m_2)$

$\Rightarrow \{ fold\ dur \}$
$perform\ pm\ c\ (tempo\ r\ m_1)$
$\quad +\!\!+ perform\ pm\ (c\ \{ cTime = t3 \})\ (tempo\ r\ m_2)$
$\Rightarrow \{ fold\ perform \}$
$perform\ pm\ c\ (tempo\ r\ m_1 :+: tempo\ r\ m_2)$

$dur\ (tempo\ r\ (m_1 :+: m_2))$
$\Rightarrow dur\ (m_1 :+: m_2)/r$
$\Rightarrow (dur\ m_1 + dur\ m_2)/r$
$\Rightarrow dur\ m_1/r + dur\ m_2/r$
$\Rightarrow dur\ (tempo\ r\ m_1) + dur\ (tempo\ r\ m_2)$
$\Rightarrow dur\ (tempo\ r\ m_1 :+: tempo\ r\ m_2)$

An even simpler axiom is given by:

Axiom 11.2.3 For any m, $tempo\ 1)\ m \equiv m$.

In other words, *unit tempo scaling is the identity function for type Music.*

Proof:

let $dt = cDur\ c$

$perform\ pm\ c\ (tempo\ 1)\ m)$
$\Rightarrow \{ unfold\ perform \}$
$perform\ pm\ (c\ \{ cDur = dt/1 \})\ m$
$\Rightarrow \{ arithmetic \}$
$perform\ pm\ c\ m$

$dur\ (tempo\ 1)\ m)$
$\Rightarrow dur\ m/1$
$\Rightarrow dur\ m$

Note that the above three proofs, being used to establish axioms, all involve the definitions of *perform* and *dur*. In contrast, we can also establish *theorems* whose proofs involve only the axioms. For example, Axioms 1, 2, and 3 are all needed to prove the following:

Theorem 11.2.1 For any r, m_1, and m_2:

$tempo\ r\ m_1 :+: m_2 \equiv tempo\ r\ (m_1 :+: tempo\ (1/r)\ m_2)$

Proof:

$tempo\ r\ m_1 :+: m_2$
$\Rightarrow \{ Axiom\ 3 \}$

$tempo\ r\ m_1 :+: tempo\ 1)\ m_2$
$\Rightarrow \{\, arithmetic \,\}$
$tempo\ r\ m_1 :+: tempo\ (r * (1/r))\ m_2$
$\Rightarrow \{\, Axiom\ 1 \,\}$
$tempo\ r\ m_1 :+: tempo\ r\ (tempo\ (1/r)\ m_2)$
$\Rightarrow \{\, Axiom\ 2 \,\}$
$tempo\ r\ (m_1 :+: tempo\ (1/r)\ m_2)$

11.3 The Axiom Set

There are many other useful axioms, but we do not have room to include all of their proofs here. They are listed below, which include the axioms from the previous section as special cases, and the proofs are left as exercises.

Axiom 11.3.1 *Tempo* is *multiplicative* and *Transpose* is *additive*. That is, for any r_1, r_2, p_1, p_2, and m:

$tempo\ r_1\ (tempo\ r_2\ m) \equiv tempo\ (r_1 * r_2)\ m$

$trans\ p_1\ (trans\ p_2\ m) \equiv trans\ (p_1 + p_2)\ m$

Axiom 11.3.2 Function composition is *commutative* with respect to both tempo scaling and transposition. That is, for any r_1, r_2, p_1 and p_2:

$tempo\ r_1 \circ tempo\ r_2 \equiv tempo\ r_2 \circ tempo\ r_1$

$trans\ p_1 \circ trans\ p_2 \equiv trans\ p_2 \circ trans\ p_1$

$tempo\ r_1 \circ trans\ p_1 \equiv trans\ p_1 \circ tempo\ r_1$

Axiom 11.3.3 Tempo scaling and transposition are *distributive* over both sequential and parallel composition. That is, for any r, p, m_1, and m_2:

$tempo\ r\ (m_1 :+: m_2) \equiv tempo\ r\ m_1 :+: tempo\ r\ m_2$

$tempo\ r\ (m_1 :=: m_2) \equiv tempo\ r\ m_1 :=: tempo\ r\ m_2$

$trans\ p\ (m_1 :+: m_2) \equiv trans\ p\ m_1 :+: trans\ p\ m_2$

$trans\ p\ (m_1 :=: m_2) \equiv trans\ p\ m_1 :=: trans\ p\ m_2$

Axiom 11.3.4 Sequential and parallel composition are *associative*. That is, for any m_0, m_1, and m_2:

$m_0 :+: (m_1 :+: m_2) \equiv (m_0 :+: m_1) :+: m_2$

$m_0 :=: (m_1 :=: m_2) \equiv (m_0 :=: m_1) :=: m_2$

Axiom 11.3.5 Parallel composition is *commutative*. That is, for any m_0 and m_1:

$$m_0 :=: m_1 \equiv m_1 :=: m_0$$

Axiom 11.3.6 *rest* 0 is a *unit* for *tempo* and *trans*, and a *zero* for sequential and parallel composition. That is, for any r, p, and m:

$$tempo\ r\ (rest\ 0) \equiv rest\ 0$$

$$trans\ p\ (rest\ 0) \equiv rest\ 0$$

$$m :+: rest\ 0 \equiv m \equiv rest\ 0 :+: m$$

$$m :=: rest\ 0 \equiv m \equiv rest\ 0 :=: m$$

Axiom 11.3.7 A rest can be used to "pad" a parallel composition. That is, for any m_1, m_2, such that $dur\ m_1 > dur\ m_2$, and any $d \leqslant dur\ m_1 - dur\ m_2$:

$$m_1 :=: m_2 \equiv m_1 :=: (m_2 :+: rest\ d)$$

Axiom 11.3.8 There is a duality between (:+:) and (:=:), namely that, for any m_0, m_1, m_2, and m_3 such that $dur\ m_0 = dur\ m_2$:

$$(m_0 :+: m_1) :=: (m_2 :+: m_3) \equiv (m_0 :=: m_2) :+: (m_1 :=: m_3)$$

Exercise 11.1 Prove Lemma 11.1.1.

Exercise 11.2 Establish the validity of each of the above axioms.

Exercise 11.3 Recall the function *revM* defined in Chapter 2, and note that, in general, $revM\ (revM\ m)$ is not equal to m. However, the following is true:

$$revM\ (revM\ m) \equiv m$$

Prove this fact by calculation.

Exercise 11.4 Prove that $timesM\ a\ m :+: timesM\ b\ m \equiv timesM\ (a + b)\ m$.

11.4 Soundness and Completeness

TBD

Chapter 12

Musical L-Systems

```
module Euterpea.Examples.LSystems where
import Data.List
import System.Random
import Euterpea
```

12.1 Generative Grammars

A *grammar* describes a *formal language*. One can either design a *recognizer* (or *parser*) for that language, or design a *generator* that generates sentences in that language. We are interested in using grammars to generate music, and thus we are only interested in generative grammars.

A generative grammar is a four-tuple (N, T, n, P), where:

- N is the set of *non-terminal symbols*.
- T is the set of *terminal symbols*.
- n is the *initial symbol*.
- P is a set of *production rules*, where each production rule is a pair (X, Y), often written $X \rightarrow Y$, where X and Y are words over the alphabet $N \cup T$, and X contains at least one non-terminal.

A *Lindenmayer system*, or *L-system*, is an example of a generative grammer, but is different in two ways:

1. The *sequence* of sentences is as important as the individual sentences, and

2. A new sentence is generated from the previous one by applying as many productions as possible on each step—a kind of "parallel production."

Lindenmayer was a biologist and mathematician, and he used L-systems to describe the growth of certain biological organisms (such as plants, and in particular algae).

We will limit our discussion to L-systems that have the following additional characteristics:

1. They are *context-free*: the left-hand side of each production (i.e. X above) is a single non-terminal.

2. No distinction is made between terminals and non-terminals (with no loss of expressive power—why?).

We will consider both *deterministic* and *non-deterministic* grammars. A deterministic grammar has exactly one production corresponding to each non-terminal symbol in the alphabet, whereas a non-deterministic grammar may have more than one, and thus we will need some way to choose between them.

12.2 A Simple Implementation

A very simple context-free, deterministic grammar can be designed as follows. We represent the set of productions as a list of symbol/list-of-symbol pairs:

```
data DetGrammar a = DetGrammar a              -- start symbol
                               [(a, [a])]     -- productions
  deriving Show
```

To generate a succession of "sentential forms," we need to define a function that, given a grammar, returns a list of lists of symbols:

```
detGenerate :: Eq a ⇒ DetGrammar a → [[a]]
detGenerate (DetGrammar st ps) = iterate (concatMap f) [st]
  where f a = maybe [a] id (lookup a ps)
```

Details: *maybe* is a convenient function for conditionally giving a result based on the structure of a value of type $Maybe\ a$. It is defined in the Standard Prelude as:

$$\begin{aligned} &maybe && :: b \to (a \to b) \to Maybe\ a \to b \\ &maybe\ \_\ f\ (Just\ x) &&= f\ x \\ &maybe\ z\ \_\ Nothing &&= z \end{aligned}$$

$lookup :: Eq\ a \Rightarrow a \to [(a, b)] \to Maybe\ b$ is a convenient function for finding the value associated with a given key in an association list. For example:

$$\begin{aligned} &lookup\ \texttt{'b'}\ [(\texttt{'a'}, 0), (\texttt{'b'}, 1), (\texttt{'c'}, 2)] \Rightarrow Just\ 1 \\ &lookup\ \texttt{'d'}\ [(\texttt{'a'}, 0), (\texttt{'b'}, 1), (\texttt{'c'}, 2)] \Rightarrow Nothing \end{aligned}$$

Note that we expand each symbol "in parallel" at each step, using *concatMap*. The repetition of this process at each step is achieved using *iterate*. Note also that a list of productions is essentially an *association list*, and thus the library function *lookup* works quite well in finding the production rule that we seek. Finally, note once again how the use of higher-order functions makes this definition concise yet efficient.

As an example of the use of this simple program, a Lindenmayer grammer for red algae (taken from []) is given by:

```
redAlgae = DetGrammar 'a'
  [('a', "b|c"),
   ('b', "b"),
   ('c', "b|d"),
   ('d', "e\\d"),
   ('e', "f"),
   ('f', "g"),
   ('g', "h(a)"),
   ('h', "h"),
   ('|', "|"),
   ('(', "("),
   (')', ")"),
   ('/', "\\"),
   ('\\', "/")
  ]
```

Then *detGenerate redAlgae* gives us the result that we want—or, to make it look nicer, we could do:

```
t n g = sequence_ (map putStrLn (take n (detGenerate g)))
```

For example, the 10th element of *t* 10 *redAlgae* is:

```
"b|b|h(b|b|e\d)\h(b|b|d)/h(b|c)\h(a)/g\f/e\d"
```

Exercise 12.1 Design a function *testDet* :: *Grammar a* → *Bool* such that *testDet g* is *True* if *g* has exactly one rule for each of its symbols; i.e. it is deterministic. Then modify the *generate* function above so that it returns an error if a grammer not satisfying this constraint is given as argument.

12.3 Grammars in Haskell

The design given in the last section only captures deterministic context-free grammars. We would also like to consider non-deterministic grammars, where a user can specify the probability that a particular rule is selected, as well as possibly non-context free (i.e. context sensitive) grammars. Thus we will represent a generative grammar a bit more abstractly, as a data structure that has a starting sentence in an (implicit, polymorphic) alphabet, and a list of production rules:

```
data Grammar a = Grammar a          -- start sentence
                         (Rules a)  -- production rules
  deriving Show
```

The production rules are instructions for converting sentences in the alphabet to other sentences in the alphabet. A rule set is either a set of uniformly distributed rules (meaning that those with the same left-hand side have an equal probability of being chosen), or a set of stochastic rules (each of which is paired with a probabilty). A specific rule consists of a left-hand side and a right-hand side.

```
data Rules a = Uni [Rule a]
             | Sto [(Rule a, Prob)]
  deriving (Eq, Ord, Show)
data Rule a = Rule {lhs :: a, rhs :: a}
  deriving (Eq, Ord, Show)
type Prob = Float
```

One of the key sub-problems that we will have to solve is how to probabilistically select a rule from a set of rules, and use that rule to expand a

non-terminal. We define the following type to capture this process:

```
type ReplFun a = [[(Rule a, Prob)]] → (a, [Rand]) → (a, [Rand])
type Rand      = Float
```

The idea here is that a function $f :: \mathit{ReplFun}\ a$ is such that $f\ \mathit{rules}\ (s, \mathit{rands})$ will return a new sentence s' in which each symbol in s has been replaced according to some rule in *rules* (which are grouped by common left-hand side). Each rule is chosen probabilitically based on the random numbers in *rands*, and thus the result also includes a new list of random numbers to account for those "consumed" by the replacement process.

With such a function in hand, we can now define a function that, given a grammar, generates an infinite list of the sentences produced by this replacement process. Because the process is non-deterministic, we also pass a seed (an integer) to generate the initial pseudo-random number sequence to give us repeatable results.

```
gen :: Ord a ⇒ ReplFun a → Grammar a → Int → [a]
gen f (Grammar s rules) seed =
  let Sto newRules = toStoRules rules
      rands        = randomRs (0.0, 1.0) (mkStdGen seed)
  in if checkProbs newRules
     then generate f newRules (s, rands)
     else (error "Stochastic rule-set is malformed.")
```

toStoRules converts a list of uniformly distributed rules to an equivalent list of stochastic rules. Each set of uniform rules with the same LHS is converted to a set of stochastic rules in which the probability of each rule is one over the number of uniform rules.

```
toStoRules :: (Ord a, Eq a) ⇒ Rules a → Rules a
toStoRules (Sto rs) = Sto rs
toStoRules (Uni rs) =
  let rs' = groupBy (λr1 r2 → lhs r1 == lhs r2) (sort rs)
  in Sto (concatMap insertProb rs')

insertProb :: [a] → [(a, Prob)]
insertProb rules = let prb = 1.0/fromIntegral (length rules)
                   in zip rules (repeat prb)
```

checkProbs takes a list of production rules and checks whether, for every rule with the same LHS, the probabilities sum to one (plus or minus some epsilon, currenty set to 0.001).

```
checkProbs :: (Ord a, Eq a) ⇒ [(Rule a, Prob)] → Bool
```

```
checkProbs rs = and (map checkSum (groupBy sameLHS (sort rs)))
eps = 0.001
checkSum :: [(Rule a, Prob)] → Bool
checkSum rules = let mySum = sum (map snd rules)
                 in abs (1.0 − mySum) ⩽ eps
sameLHS :: Eq a ⇒ (Rule a, Prob) → (Rule a, Prob) → Bool
sameLHS (r1, f1) (r2, f2) = lhs r1 == lhs r2
```

generate takes a list of rules, a replacement function, a starting sentence, and a source of random numbers. It returns an infinite list of sentences.

```
generate :: Eq a ⇒
            ReplFun a → [(Rule a, Prob)] → (a, [Rand]) → [a]
generate f rules xs =
  let newRules     = map probDist (groupBy sameLHS rules)
      probDist rrs = let (rs, ps) = unzip rrs
                     in zip rs (tail (scanl (+) 0 ps))
  in map fst (iterate (f newRules) xs)
```

12.4 An L-System Grammar for Music

The above is all for a generic grammar. For a musical L-system we will define a specific grammar, whose sentences are defined as follows. A musical L-system sentence is either:

- A non-terminal symbol $(N\ a)$.
- A sequential composition $s_1 :+ s_2$.
- A functional composition $s_1 :. s_2$.
- The symbol Id, which will eventually interpeted as the identity function.

We capture this in the *LSys* data type:

```
data LSys a = N a
            | LSys a :+ LSys a
            | LSys a :. LSys a
            | Id
  deriving (Eq, Ord, Show)
```

We also need to define a replacement function for this grammar. We treat (:+) and (:.) as binary branches, and recursively traverse each of their arguments. We treat *Id* as a constant that never gets replaced. Most importantly, each non-terminal of the form $N\ x$ could each be the left-hand side of a rule, so we call the function *getNewRHS* to generate the replacement term for it.

```
replFun :: Eq a ⇒ ReplFun (LSys a)
replFun rules (s, rands) =
  case s of
    a :+ b → let (a', rands') = replFun rules (a, rands)
                 (b', rands'') = replFun rules (b, rands')
             in (a' :+ b', rands'')
    a :. b → let (a', rands') = replFun rules (a, rands)
                 (b', rands'') = replFun rules (b, rands')
             in (a' :. b', rands'')
    Id     → (Id, rands)
    N x    → (getNewRHS rules (N x) (head rands), tail rands)
```

Note the use of *filter* to select only the rules whose left-hand side matches the non-terminal. A key aspect of the algorithm is to generate the *probability density* of the successive rules, which is basically the sum of its probability plus the probabilities of all rules that precede it. This modified rule-set is then given to *getNewRHS* as an argument. *getNewRHS* is defined as:

```
getNewRHS :: Eq a ⇒ [[(Rule a, Prob)]] → a → Rand → a
getNewRHS rrs ls rand =
  let loop ((r, p) : rs) = if rand ⩽ p then rhs r else loop rs
      loop []            = error "getNewRHS anomaly"
  in case (find (λ((r, p): _) → lhs r == ls) rrs) of
    Just rs  → loop rs
    Nothing → error "No rule match"
```

12.5 Examples

The final step is to interpret the resulting sentence (i.e. a value of type *LSys a*) as music. The intent of the *LSys* design is that a value is interpreted as a *function* that is applied to a single note (or, more generally, a single *Music* value). The specific constructors are interpreted as follows:

```
type IR a b = [(a, Music b → Music b)]   -- "interpetation rules"
```

```
interpret :: (Eq a) ⇒ LSys a → IR a b → Music b → Music b
interpret (a :. b)  r m = interpret a r (interpret b r m)
interpret (a :+ b) r m = interpret a r m :+: interpret b r m
interpret Id        r m = m
interpret (N x)     r m = case (lookup x r) of
                            Just f    → f m
                            Nothing → error "No interpetation rule"
```

For example, we could define the following interpretation rules:

```
data LFun = Inc | Dec | Same
  deriving (Eq, Ord, Show)

ir :: IR LFun Pitch
ir = [(Inc, Euterpea.transpose 1),
  (Dec, Euterpea.transpose (−1)),
  (Same, id)]

inc, dec, same :: LSys LFun
inc  = N Inc
dec  = N Dec
same = N Same
```

In other words, *inc* transposes the music up by one semitone, *dec* transposes it down by a semitone, and *same* does nothing.

Now let's build an actual grammar. *sc* increments a note followed by its decrement—the two notes are one whole tone apart:

```
sc = inc :+ dec
```

Now let's define a bunch of rules as follows:

```
r1a = Rule inc (sc :. sc)
r1b = Rule inc sc
r2a = Rule dec (sc :. sc)
r2b = Rule dec sc
r3a = Rule same inc
r3b = Rule same dec
r3c = Rule same same
```

and the corresponding grammar:

```
g1 = Grammar same (Uni [r1b, r1a, r2b, r2a, r3a, r3b])
```

Finally, we generate a sentence at some particular level, and interpret it as music:

$t_1\ n = instrument\ Vibraphone\ \$$
$\quad interpret\ (gen\ replFun\ g1\ 42\ !!\ n)\ ir\ (c\ 5\ tn)$

Try "*play* (t_1 3)" or "*play* (t_1 4)" to hear the result.

Exercise 12.2 Play with the L-System grammar defined above. Change the production rules. Add probabilities to the rules, i.e. change it into a *Sto* grammar. Change the random number seed. Change the depth of recursion. And also try changing the "musical seed" (i.e. the note $c\ 5\ tn$).

Exercise 12.3 Define a new L-System structure. In particular, (a) define a new version of *LSys* (for example, add a parallel constructor) and its associated interpretation, and/or (b) define a new version of *LFun* (perhaps add something to control the volume) and its associated interpretation. Then define some grammars with the new design to generate interesting music.

Chapter 13

Random Numbers, Probability Distributions, and Markov Chains

```
module Euterpea.Examples.RandomMusic where
import Euterpea
import System.Random
import System.Random.Distributions
import qualified Data.MarkovChain as M
```

The use of randomness in composition can be justified by the somewhat random, exploratory nature of the creative mind, and indeed it has been used in computer music composition for many years. In this chapter we will explore several sources of random numbers and how to use them in generating simple melodies. With this foundation you will hopefully be able to use randomness in more sophisticated ways in your compositions. Music relying at least to some degree on randomness is said to be *stochastic*, or *aleatoric*.

13.1 Random Numbers

This section describes the basic functionality of Haskell's *System.Random* module, which is a library for random numbers. The library presents a fairly abstract interface that is structured in two layers of type classes: one

that captures the notion of a *random generator*, and one for using a random generator to create *random sequences*.

We can create a random number generator using the built-in *mkStdGen* function:

```
mkStdGen :: Int → StdGen
```

which takes an *Int* seed as argument, and returns a "standard generator" of type *StdGen*. For example, we can define:

```
sGen :: StdGen
sGen = mkStdGen 42
```

We will use this single random generator quite extensively in the remainder of this chapter.

StdGen is an instance of *Show*, and thus its values can be printed—but they appear in a rather strange way, basically as two integers. Try typing *sGen* to the GHCi prompt.

More importantly, *StdGen* is an instance of the *RandomGen* class:

```
class RandomGen g where
  genRange :: g → (Int, Int)
  next     :: g → (Int, g)
  split    :: g → (g, g)
```

The reason that *Int*s are used here is that essentially all pseudo-random number generator algorithms are based on a fixed-precision binary number, such as *Int*. We will see later how this can be coerced into other number types.

For now, try applying the operators in the above class to the *sGen* value above. The *next* function is particularly important, as it generates the next random number in a sequence as well as a new random number generator, which in turn can be used to generate the next number, and so on. It should be clear that we can then create an infinite list of random *Int*s like this:

```
randInts :: StdGen → [Int]
randInts g = let (x, g') = next g
             in x : randInts g'
```

Look at the value *take* 10 (*randInts sGen*) to see a sample output.

To support other number types, the *Random* library defines this type class:

```
class Random a where
  randomR  :: RandomGen g ⇒ (a, a) → g → (a, g)
```

$$
\begin{aligned}
&\mathit{random} &&:: \mathit{RandomGen}\ g \Rightarrow g \rightarrow (a, g)\\
&\mathit{randomRs} &&:: \mathit{RandomGen}\ g \Rightarrow (a, a) \rightarrow g \rightarrow [a]\\
&\mathit{randoms} &&:: \mathit{RandomGen}\ g \Rightarrow g \rightarrow [a]\\
&\mathit{randomRIO} &&:: (a, a) \rightarrow \mathit{IO}\ a\\
&\mathit{randomIO} &&:: \mathit{IO}\ a
\end{aligned}
$$

Built-in instances of *Random* are provided for *Int*, *Integer*, *Float*, *Double*, *Bool*, and *Char*.

The set of operators in the *Random* class is rather daunting, so let's focus on just one of them for now, namely the third one, *RandomRs*, which is also perhaps the most useful one. This function takes a random number generator (such as *sGen*), along with a range of values, and generates an infinite list of random numbers within the given range (the pair representing the range is treated as a closed interval). Here are several examples of this idea:

$$
\begin{aligned}
&\mathit{randFloats} :: [\mathit{Float}]\\
&\mathit{randFloats} = \mathit{randomRs}\ (-1, 1)\ \mathit{sGen}\\
&\mathit{randIntegers} :: [\mathit{Integer}]\\
&\mathit{randIntegers} = \mathit{randomRs}\ (0, 100)\ \mathit{sGen}\\
&\mathit{randString} :: \mathit{String}\\
&\mathit{randString} = \mathit{randomRs}\ (\texttt{'a'}, \texttt{'z'})\ \mathit{sGen}
\end{aligned}
$$

Recall that a string is a list of characters, so we choose here to use the name *randString* for our infinite list of characters. If you believe the story about a monkey typing a novel, then you might believe that *randString* contains something interesting to read.

So far we have used a seed to initialize our random number generators, and this is good in the sense that it allows us to generate repeatable, and therefore more easily testable, results. If instead you prefer a non-repeatable result, in which you can think of the seed as being the time of day when the program is executed, then you need to use a function that is in the IO monad. The last two operators in the *Random* class serve this purpose. For example, consider:

$$
\begin{aligned}
&\mathit{randIO} :: \mathit{IO}\ \mathit{Float}\\
&\mathit{randIO} = \mathit{randomRIO}\ (0, 1)
\end{aligned}
$$

If you repeatedly type *randIO* at the GHCi prompt, it will return a different random number every time. This is clearly not purely "functional," and is why it is in the IO monad. As another example:

$$
\begin{aligned}
&\mathit{randIO}' :: \mathit{IO}\ () \\
&\mathit{randIO}' = \mathbf{do}\ r_1 \leftarrow \mathit{randomRIO}\ (0,1) :: \mathit{IO}\ \mathit{Float} \\
&\qquad\qquad\ \ r_2 \leftarrow \mathit{randomRIO}\ (0,1) :: \mathit{IO}\ \mathit{Float} \\
&\qquad\qquad\ \ \mathit{print}\ (r_1 == r_2)
\end{aligned}
$$

will almost always return *False*, because the chance of two randomly generated floating point numbers being the same is exceedingly small. (The type signature is needed to ensure that the value generated has an unambigous type.)

Details: $\mathit{print} :: \mathit{Show}\ a \Rightarrow a \rightarrow \mathit{IO}\ ()$ converts any showable value into a string, and displays the result in the standard output area.

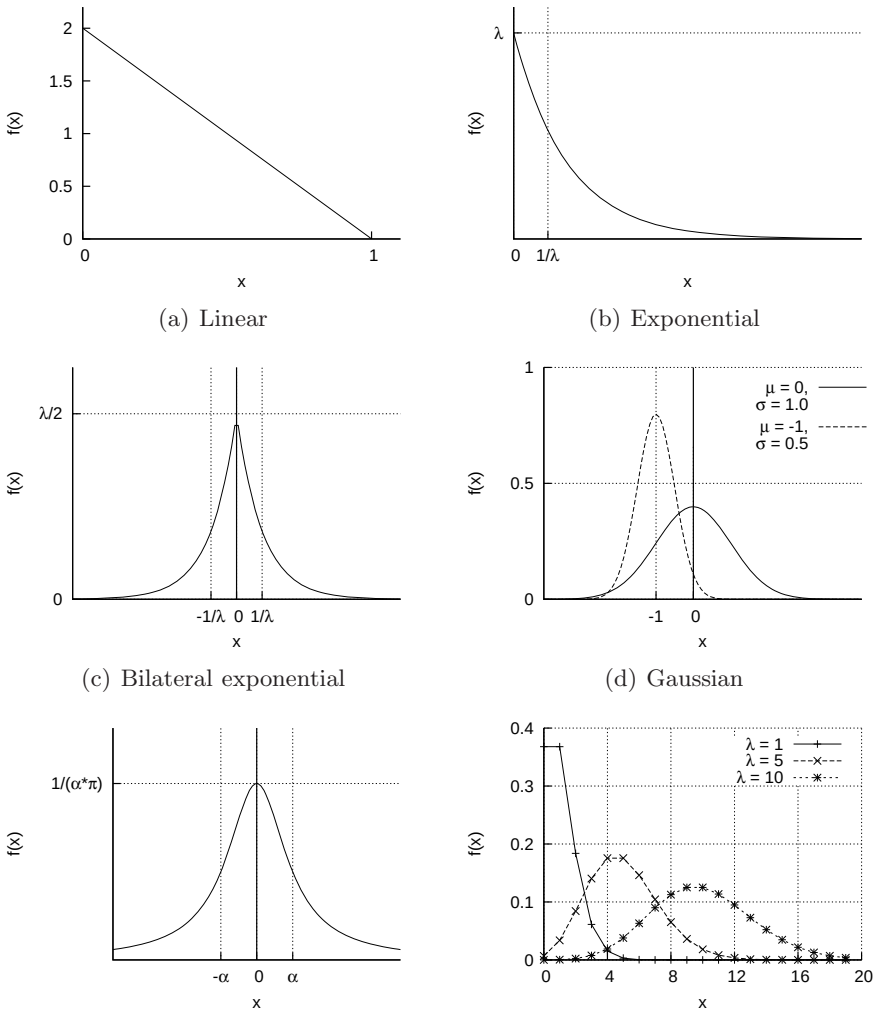
13.2 Probability Distributions

The random number generators described in the previous section are assumed to be *uniform*, meaning that the probability of generating a number within a given interval is the same everywhere in the range of the generator. For example, in the case of *Float* (that purportedly represents *continuous* real numbers), suppose we are generating numbers in the range 0 to 10. Then we would expect the probability of a number appearing in the range 2.3-2.4 to be the same as the probability of a number appearing in the range 7.6-7.7, namely 0.01, or 1% (i.e. 0.1/10). In the case of *Int* (a *discrete* or *integral* number type), we would expect the probability of generating a 5 to be the same as generating an 8. In both cases, we say that we have a *uniform distribution*.

But we don't always want a uniform distribution. In generating music, in fact, it's often the case that we want some kind of a non-uniform distribution. Mathematically, the best way to describe a distribution is by plotting how the probability changes over the range of values that it produces. In the case of continuous numbers, this is called the *probability density function*, which has the property that its integral over the full range of values is equal to 1.

The *System.Random.Distributions* library provides a number of different probability distributions, which are described below. Figure 13.1 shows the probability density functions for each of othem.

Here is a list and brief description of each random number generator:



(a) Linear (b) Exponential

(c) Bilateral exponential (d) Gaussian

(e) Cauchy (f) Poisson

Figure 13.1: Various Probability Density Functions

linear Generates a *linearly* distributed random variable between 0 and 1. The probability density function is given by:

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

The type signature is:

$$\begin{aligned} &linear :: (RandomGen\ g, Floating\ a, Random\ a, Ord\ a) \Rightarrow \\ &\qquad g \to (a, g) \end{aligned}$$

The mean value of the linear distribution is 1/3.

exponential Generates an *exponentially* distributed random variable given a spread parameter λ. A larger spread increases the probability of generating a small number. The mean of the distribution is $1/\lambda$. The range of the generated number is conceptually 0 to ∞, although the chance of getting a very large number is very small. The probability density function is given by:

$$f(x) = \lambda e^{-\lambda x}$$

The type signature is:

$$\begin{aligned} &exponential :: (RandomGen\ g, Floating\ a, Random\ a) \Rightarrow \\ &\qquad a \to g \to (a, g) \end{aligned}$$

The first argument is the parameter λ.

bilateral exponential Generates a random number with a *bilateral exponential* distribution. It is similar to exponential, but the mean of the distribution is 0 and 50% of the results fall between $-1/\lambda$ and $1/\lambda$. The probability density function is given by:

$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$$

The type signature is:

$$\begin{aligned} &bilExp :: (Floating\ a, Ord\ a, Random\ a, RandomGen\ g) \Rightarrow \\ &\qquad a \to g \to (a, g) \end{aligned}$$

Gaussian Generates a random number with a *Gaussian*, also called *normal*, distribution, given mathematically by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where σ is the *standard deviation*, and μ is the *mean*. The type signature is:

$$\begin{aligned} &gaussian :: (Floating\ a, Random\ a, RandomGen\ g) \Rightarrow \\ &\qquad a \to a \to g \to (a, g) \end{aligned}$$

The first argument is the standard deviation σ and the second is the mean μ. Probabilistically, about 68.27% of the numbers in a Gaussian distribution fall within $\pm\sigma$ of the mean; about 95.45% are within $\pm 2\sigma$, and 99.73% are within $\pm 3\sigma$.

Cauchy Generates a *Cauchy*-distributed random variable. The distribution is symmetric with a mean of 0. The density function is given by:

$$f(x) = \frac{\alpha}{\pi(\alpha^2 + x^2)}$$

As with the Gaussian distribution, it is unbounded both above and below the mean, but at its extremes it approaches 0 more slowly than the Gaussian. The type signature is:

$$\begin{aligned} &cauchy :: (Floating\ a, Random\ a, RandomGen\ g) \Rightarrow \\ &\qquad a \to g \to (a, g) \end{aligned}$$

The first argument corresponds to α above, and is called the *density*.

Poisson Generates a *Poisson*-distributed random variable. The Poisson distribution is discrete, and generates only non-negative numbers. λ is the mean of the distribution. If λ is an integer, the probability that the result is $j = \lambda - 1$ is the same as that of $j = \lambda$. The probability of generating the number j is given by:

$$P\{X = j\} = \frac{\lambda^j}{j!} e^{-\lambda}$$

The type signature is:

$$\begin{aligned} &poisson :: (Num\ t, Ord\ a, Floating\ a, Random\ a \\ &\qquad RandomGen\ g) \Rightarrow \\ &\qquad a \to g \to (t, g) \end{aligned}$$

Custom Sometimes it is useful to define one's own discrete probability distribution function, and to generate random numbers based on it. The function *frequency* does this—given a list of weight-value pairs, it generates a value randomly picked from the list, weighting the probability of choosing each value by the given weight.

$$\begin{aligned} &\mathit{frequency} :: (\mathit{Floating}\ w, \mathit{Ord}\ w, \mathit{Random}\ w, \mathit{RandomGen}\ g) \Rightarrow \\ &\qquad [(w, a)] \to g \to (a, g) \end{aligned}$$

13.2.1 Random Melodies and Random Walks

Note that each of the non-uniform distribution random number generators described in the last section takes zero or more parameters as arguments, along with a uniform random number generator, and returns a pair consisting of the next random number and a new generator. In other words, the tail end of each type signature has the form:

$$\ldots \to g \to (a, g)$$

where g is the type of the random number generator, and a is the type of the next value generated.

Given such a function, we can generate an infinite sequence of random numbers with the given distribution in a way similar to what we did earlier for *randInts*. In fact the following function is defined in the *Distributions* library to make this easy:

$$\begin{aligned} &\mathit{rands} :: (\mathit{RandomGen}\ g, \mathit{Random}\ a) \Rightarrow \\ &\qquad (g \to (a, g)) \to g \to [a] \\ &\mathit{rands}\ f\ g = x : \mathit{rands}\ f\ g' \ \textbf{where}\ (x, g') = f\ g \end{aligned}$$

Let's work through a few musical examples. One thing we will need to do is convert a floating point number to an absolute pitch:

$$\begin{aligned} &\mathit{toAbsP}_1 :: \mathit{Float} \to \mathit{AbsPitch} \\ &\mathit{toAbsP}_1\ x = \mathit{round}\ (40 * x + 30) \end{aligned}$$

This function converts a number in the range 0 to 1 into an absolute pitch in the range 30 to 70.

And as we have often done, we will also need to convert an absolute pitch into a note, and a sequence of absolute pitches into a melody:

$$\begin{aligned} &\mathit{mkNote}_1 :: \mathit{AbsPitch} \to \mathit{Music\ Pitch} \\ &\mathit{mkNote}_1 = \mathit{note}\ \mathit{tn} \circ \mathit{pitch} \\ &\mathit{mkLine}_1 :: [\mathit{AbsPitch}] \to \mathit{Music\ Pitch} \\ &\mathit{mkLine}_1\ \mathit{rands} = \mathit{line}\ (\mathit{take}\ 32\ (\mathit{map}\ \mathit{mkNote}_1\ \mathit{rands})) \end{aligned}$$

With these functions in hand, we can now generate sequences of random numbers with a variety of distributions, and convert each of them into a melody. For example:

```
-- uniform distribution
```

```
m1 :: Music Pitch
m1 = mkLine1 (randomRs (30, 70) sGen)
  -- linear distribution
m2 :: Music Pitch
m2 = let rs1 = rands linear sGen
     in mkLine1 (map toAbsP1 rs1)
  -- exponential distribution
m3      :: Float → Music Pitch
m3 lam = let rs1 = rands (exponential lam) sGen
         in mkLine1 (map toAbsP1 rs1)
  -- Gaussian distribution
m4          :: Float → Float → Music Pitch
m4 sig mu = let rs1 = rands (gaussian sig mu) sGen
            in mkLine1 (map toAbsP1 rs1)
```

Exercise 13.1 Try playing each of the above melodies, and listen to the musical differences. For *lam*, try values of 0.1, 1, 5, and 10. For *mu*, a value of 0.5 will put the melody in the central part of the scale range—then try values of 0.01, 0.05, and 0.1 for *sig*.

Exercise 13.2 Do the following:

- Try using some of the other probability distributions to generate a melody.
- Instead of using a chromatic scale, try using a diatonic or pentatonic scale.
- Try using randomness to control parameters other than pitch—in particular, duration and/or volume.

Another approach to generating a melody is sometimes called a *random walk*. The idea is to start on a particular note, and treat the sequence of random numbers as *intervals*, rather than as pitches. To prevent the melody from wandering too far from the starting pitch, one should use a probability distribution whose mean is zero. This comes for free with something like the bilateral exponential, and is easily obtained with a distribution that takes the mean as a parameter (such as the Gaussian), but is also easily achieved

for other distributions by simply subtracting the mean. To see these two situations, here are random melodic walks using first a Gaussian and then an exponential distribution:

```
  -- Gaussian distribution with mean set to 0
m5       :: Float -> Music Pitch
m5 sig   = let rs1 = rands (gaussian sig 0) sGen
           in mkLine2 50 (map toAbsP2 rs1)

  -- exponential distribution with mean adjusted to 0
m6       :: Float -> Music Pitch
m6 lam = let rs1 = rands (exponential lam) sGen
         in mkLine2 50 (map (toAbsP2 . subtract (1/lam)) rs1)

toAbsP2    :: Float -> AbsPitch
toAbsP2 x = round (5 * x)

mkLine2 :: AbsPitch -> [AbsPitch] -> Music Pitch
mkLine2 start rands =
  line (take 64 (map mkNote1 (scanl (+) start rands)))
```

Note that $toAbsP_2$ does something reasonable to interpret a floating-point number as an interval, and $mkLine_2$ uses *scanl* to generate a "running sum" that represents the melody line.

13.3 Markov Chains

Each number in the random number sequences that we have described thus far is *independent* of any previous values in the sequence. This is like flipping a coin—each flip has a 50% chance of being heads or tails, i.e. it is independent of any previous flips, even if the last ten flips were all heads.

Sometimes, however, we would like the probability of a new choice to depend upon some number of previous choices. This is called a *conditional probability*. In a discrete system, if we look only at the previous value to help determine the next value, then these conditional probabilities can be conveniently represented in a matrix. For example, if we are choosing between the pitches C, D, E, and F, then Table 13.1 might represent the conditional probabilities of each possible outcome. The previous pitch is found in the left column—thus note that the sum of each row is 1.0. So, for example, the probability of choosing a D given that the previous pitch was an E is 0.6, and the probability of an F occurring twice in succession is 0.2. The resulting stochastic system is called a *Markov Chain*.

	C	D	E	F
C	0.4	0.2	0.2	0.2
D	0.3	0.2	0.0	0.5
E	0.1	0.6	0.1	0.2
F	0.2	0.3	0.3	0.2

Table 13.1: Second-Order Markov Chain

This idea can of course be generalized to arbitrary numbers of previous events, and in general an $(n+1)$-dimensional array can be used to store the various conditional probabilities. The number of previous values observed is called the *order* of the Markov Chain.

[TO DO: write the Haskell code to implement this]

13.3.1 Training Data

Instead of generating the conditional probability table ourselves, another approach is to use *training data* from which the conditional probabilities can be *inferred*. This is handy for music, because it means that we can feed in a bunch of melodies that we like, including melodies written by the masters, and use that as a stochastic basis for generating new melodies.

[TO DO: Give some pointers to the literatue, in particular David Cope's work.]

The *Data.MarkovChain* library provides this functionality through a function called *run*, whose type signature is:

```
run :: (Ord a, RandomGen g) ⇒
       Int       -- order of Markov Chain
       → [a]     -- training sequence (treated as circular list)
       → Int     -- index to start within the training sequence
       → g       -- random number generator
       → [a]
```

The *runMulti* function is similar, except that it takes a list of training sequences as input, and returns a list of lists as its result, each being an independent random walk whose probabilities are based on the training data. The following examples demonstrate how to use these functions.

```
   -- some sample training sequences
ps0, ps1, ps2 :: [Pitch]
```

$ps_0 = [(C,4),(D,4),(E,4)]$
$ps_1 = [(C,4),(D,4),(E,4),(F,4),(G,4),(A,4),(B,4)]$
$ps_2 = [(C,4),(E,4),(G,4),(E,4),(F,4),(A,4),(G,4),(E,4),$
$\quad (C,4),(E,4),(G,4),(E,4),(F,4),(D,4),(C,4)]$

-- functions to package up *run* and *runMulti*
$mc \quad ps \quad n = mkLine_3\ (M.run\ n\ ps\ 0\ (mkStdGen\ 42))$
$mcm\ pss\ n = mkLine_3\ (concat\ (M.runMulti\ n\ pss\ 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (mkStdGen\ 42)))$

-- music-making functions
$mkNote_3 \quad :: Pitch \rightarrow Music\ Pitch$
$mkNote_3 \quad = note\ tn$
$mkLine_3 \quad :: [Pitch] \rightarrow Music\ Pitch$
$mkLine_3\ ps = line\ (take\ 64\ (map\ mkNote_3\ ps))$

Here are some things to try with the above definitions:

- *mc* ps_0 0 will generate a completely random sequence, since it is a "zeroth-order" Markov Chain that does not look at any previous output.
- *mc* ps_0 1 looks back one value, which is enough in the case of this simple training sequence to generate an endless sequence of notes that sounds just like the training data. Using any order higher than 1 generates the same result.
- *mc* ps_1 1 also generates a result that sounds just like its training data.
- *mc* ps_2 1, on the other hand, has some (random) variety to it, because the training data has more than one occurrence of most of the notes. If we increase the order, however, the output will sound more and more like the training data.
- *mcm* $[ps_0, ps_2]$ 1 and *mcm* $[ps_1, ps_2]$ 1 generate perhaps the most interesting results yet, in which you can hear aspects of both the ascending melodic nature of ps_0 and ps_1, and the harmonic structure of ps_2.
- *mcm* $[ps_1, reverse\ ps_1]$ 1 has, not suprisingly, both ascending and descending lines in it, as reflected in the training data.

Exercise 13.3 Play with Markov Chains. Use them to generate more melodies, or to control other aspects of the music, such as rhythm. Also consider other kinds of training data rather than simply sequences of pitches.

Chapter 14

From Performance to Midi

```
module Euterpea.IO.MIDI.ToMidi (toMidi, UserPatchMap, defST,
  defUpm, testMidi, testMidiA,
  test, testA, play, playM, playA,
  makeMidi, mToMF, gmUpm, gmTest) where
import Euterpea.Music.Note.Music
import Euterpea.Music.Note.MoreMusic
import Euterpea.Music.Note.Performance
import Euterpea.IO.MIDI.GeneralMidi
import Euterpea.IO.MIDI.MidiIO
import Sound.PortMidi
import Data.List (partition)
import Data.Char (toLower, toUpper)
import Codec.Midi
```

Midi is shorthand for "Musical Instrument Digital Interface," and is a standard protocol for controlling electronic musical instruments. This chapter describes how to convert an abstract *performance* as defined in Chapter 8 into a *standard Midi file* that can be played on any modern PC with a standard sound card.

14.1 An Introduction to Midi

Midi is a standard adopted by most, if not all, manufacturers of electronic instruments and personal computers. At its core is a protocol for communicating *musical events* (note on, note off, etc.) and so-called *meta events*

(select synthesizer patch, change tempo, etc.). Beyond the logical protocol, the Midi standard also specifies electrical signal characteristics and cabling details, as well as a *standard Midi file* which any Midi-compatible software package should be able to recognize.

Most "sound-blaster"-like sound cards on conventional PC's know about Midi. However, the sound generated by such modules, and the sound produced from the typically-scrawny speakers on most PC's, is often quite poor. It is best to use an outboard keyboard or tone generator, which are attached to a computer via a Midi interface and cables. It is possible to connect several Midi instruments to the same computer, with each assigned to a different *channel*. Modern keyboards and tone generators are quite good. Not only is the sound excellent (when played on a good stereo system), but they are also *multi-timbral*, which means they are able to generate many different sounds simultaneously, as well as *polyphonic*, meaning that simultaneous instantiations of the same sound are possible.

14.1.1 General Midi

Over the years musicians and manufacturers decided that they also wanted a standard way to refer to commonly used instrument sounds, such as "acoustic grand piano," "electric piano," "violin," and "acoustic bass," as well as more exotic sounds such as "chorus aahs," "voice oohs," "bird tweet," and "helicopter." A simple standard known as *General Midi* was developed to fill this role. The General Midi standard establishes standard names for 128 common instrument sounds (also called "patches") and assigns an integer called the *program number* (also called "program change number"), to each of them. The instrument names and their program numbers are grouped into "familes" of instrument sounds, as shown in Table 14.1.

Now recall that in Chapter 2 we defined a set of instruments via the *InstrumentName* data type (see Figure 2.1). All of the names chosen for that data type come directly from the General Midi standard, except for two, *Percussion* and *Custom*, which were added for convenience and extensibility. By listing the constructors in the order that reflects this assignment, we can derive an *Enum* instance for *InstrumentName* that defines the method *toEnum* that essentially does the conversion from instrument name to program number for us. We can then define a function:

$$toGM :: InstrumentName \to ProgNum$$
$$toGM\ Percussion = 0$$
$$toGM\ (Custom\ name) = 0$$

Family	Program #		Family	Program #
Piano	1-8		Reed	65-72
Chromatic Percussion	9-16		Pipe	73-80
Organ	17-24		Synth Lead	81-88
Guitar	25-32		Synth Pad	89-96
Bass	33-40		Synth Effects	97-104
Strings	41-48		Ethnic	105-112
Ensemble	49-56		Percussive	113-120
Brass	57-64		Sound Effects	121-128

Table 14.1: General Midi Instrument Families

$toGM$ **in** $= fromEnum$ **in**

type $ProgNum = Int$

that takes care of the two extra cases, which are simply assigned to program number 0.

The derived *Enum* instance also defines a function *fromEnum* that converts program numbers to instrument names. We can then define:

$fromGM :: ProgNum \to InstrumentName$
$fromGM\ pn \mid pn \geqslant 0 \wedge pn \leqslant 127 = fromEnum\ pn$
$fromGM\ pn = error\ (\texttt{"fromGM: "} +\!\!+ show\ pn +\!\!+$
  `" is not a valid General Midi program number"`)

Details: Design bug: Because the *IntrumentName* data type contains a nonnullary constructor, namely *Custom*, the *Enum* instance cannot be derived. For now it is defined in the module *GeneralMidi*, but a better solution is to redefine *InstrumentName* in such a way as to avoid this.

14.1.2 Channels and Patch Maps

A Midi *channel* is in essence a programmable instrument. You can have up to 16 channels, numbered 0 through 15, each assigned a different program number (corresponding to an instrument sound, see above). All of the dynamic "Note On" and "Note Off" messages (to be defined shortly) are tagged with a channel number, so up to 16 different instruments can be controlled independently and simultaneously.

The assignment of Midi channels to instrument names is called a *patch map*, and we define a simple association list to capture its structure:

```
type UserPatchMap = [(InstrumentName, Channel)]

type Channel = Int
```

The only thing odd about Midi Channels is that General Midi specifies that Channel 10 (9 in Euterpea's 0-based numbering) is dedicated to *percussion* (which is different from the "percussive instruments" described in Table 14.1). When Channel 10 is used, any program number to which it is assigned is ignored, and instead each note corresponds to a different percussion sound. In particular, General Midi specifies that the notes corresponding to Midi Keys 35 through 82 correspond to specific percussive sounds. Indeed, recall that in Chapter 6 we in fact captured these percussion sounds through the *PercussionSound* data type, and we defined a way to convert such a sound into an absolute pitch (i.e. *AbsPitch*). Euterpea's absolute pitches, by the way, are in one-to-one correspondence with Midi Key nunmbers.

Except for percussion, the Midi Channel used to represent a particular instrument is completely arbitrary. Indeed, it is tedious to explicitly define a new patch map every time the instrumentation of a piece of music is changed. Therefore it is convenient to define a function that automatically creates a *UserPatchMap* from a list of instrument names:

```
makeGMMap :: [InstrumentName] → UserPatchMap
makeGMMap ins = mkGMMap 0 ins
  where mkGMMap _ [] = []
    mkGMMap n _ | n ⩾ 15 =
      error "MakeGMMap: Too many instruments."
    mkGMMap n (Percussion : ins) =
      (Percussion, 9) : mkGMMap n ins
    mkGMMap n (i : ins) =
      (i, chanList !! n) : mkGMMap (n + 1) ins
    chanList = [0 .. 8] ++ [10 .. 15]   -- channel 9 is for percussion
```

Note that, since there are only 15 Midi channels plus percussion, we can handle only 15 different instruments, and an error is signaled if this limit is exceeded.[1]

[1] It is conceivable to define a function to test whether or not two tracks can be combined with a Program Change (tracks can be combined if they don't overlap), but this remains for future work.

Finally, we define a function to look up an *InstrumentName* in a *UserPatchMap*, and return the associated channel as well as its program number:

```
upmLookup :: UserPatchMap → InstrumentName
                          → (Channel, ProgNum)
upmLookup upm iName = (chan, toGM iName)
  where chan = maybe (error ("instrument " ++ show iName ++
                             " not in patch map"))
                     id (lookup iName upm)
```

14.1.3 Standard Midi Files

The Midi standard defines the precise format of a *standard Midi file*. At the time when the Midi standard was first created, disk space was at a premium, and thus a compact file structure was important. Standard Midi files are thus defined at the bit and byte level, and are quite compact. We are not interested in this low-level representation (any more than we are interested in the signals that run on Midi cables), and thus in Euterpea we take a more abstract approach: We define an algebraic data type called *Midi* to capture the abstract structure of a standard Midi file, and then define functions to convert values of this data type to and from actual Midi files. This separation of concerns makes the structure of the Midi file clearer, makes debugging easier, and provides a natural path for extending Euterpea's functionality with direct Midi capability.

We will not discuss the details of the functions that read and write the actual Midi files; the interested reader may find them in the modules *ReadMidi* and *OutputMidi*, respectively. Instead, we will focus on the *Midi* data type, which is defined in the module *Codec.Midi*. We do not need all of its functionality, and thus we show in Figure 14.1 only those parts of the module that we need for this chapter. Here are the salient points about this data type and the structure of Midi files:

1. There are three types of Midi files:
   - A Format 0, or *SingleTrack*, Midi file stores its information in a single track of events, and is best used only for monophonic music.
   - A Format 1, or *MultiTrack*, Midi file stores its information in multiple tracks that are played simultaneously, where each track normally corresponds to a single Midi Channel.

```
  -- From the Codec.Midi module
data Midi = Midi {fileType :: FileType,
  timeDiv :: TimeDiv
  tracks :: [Track Ticks]}
  deriving (Eq, Show)
data FileType = SingleTrack | MultiTrack | MultiPattern
  deriving (Eq, Show)
type Track a = [(a, Message)]
data TimeDiv = TicksPerBeat Int   -- 1 through (2^15 - 1)
  | ...
  deriving (Show, Eq)
type Ticks    = Int   -- 0 through (2^28 - 1)
type Time     = Double
type Channel = Int   -- 0 through 15
type Key      = Int   -- 0 through 127
type Velocity = Int   -- 0 through 127
type Pressure = Int   -- 0 through 127
type Preset   = Int   -- 0 through 127
type Tempo    = Int   -- microseconds per beat, 1 through (2^24 - 1)
data Message =
    -- Channel Messages
    NoteOff {channel :: !Channel, key :: !Key, velocity :: !Velocity}
  | NoteOn  {channel :: !Channel, key :: !Key, velocity :: !Velocity}
  | ProgramChange {channel :: !Channel, preset :: !Preset}
  | ...
    -- Meta Messages
  | TempoChange ! Tempo |
  | ...
  deriving (Show, Eq)
fromAbsTime :: (Num a) => Track a -> Track a
fromAbsTime trk = zip ts' ms
  where (ts, ms) = unzip trk
    (_, ts') = mapAccumL (λacc t -> (t, t - acc)) 0 ts
```

Figure 14.1: Partial Definition of the *Midi* Data Type

- A Format 2, or *MultiPattern*, Midi file also has multiple tracks, but they are temporally independent.

In this chapter we only use *SingleTrack* and *MultiTrack* Midi files, depending on how many Channels we need.

2. The *TimeDiv* field refers to the *time-code division* used by the Midi file. We will always use 96 time divisions, or "ticks," per quarternote, and thus this field will always be *TicksPerBeat* 96.

3. The main body of a Midi file is a list of *Track*s, each of which in turn is a list of time-stamped (in number of ticks) *Message*s (or "events").

4. There are two kinds of *Message*s: *channel messages* and *meta messages*. Figure 14.1 shows just those messages that we are interested in:

   (a) *NoteOn ch k v* turns on key (pitch) k with velocity (volume) v on Midi channel *ch*. The velocity is an integer in the range 0 to 127.

   (b) *NoteOff ch k v* performs a similar function in turning the note off.

   (c) *ProgChange ch pr* sets the program number for channel *ch* to *pr*. This is how an instrument is selected.

   (d) *TempoChange t* sets the tempo to t, which is the time, in microseconds, of one whole note. Using 120 beats per minute as the norm, or 2 beats per second, that works out to 500,000 microseconds per beat, which is the default value that we will use.

14.2 Converting a Performance into Midi

Our goal is to convert a value of type *Performance* into a value of type *Midi*. We can summarize the situation pictorially as follows ...

Given a *UserPatchMap*, a *Performance* is converted into a *Midi* value by the *toMidi* function. If the given *UserPatchMap* is invalid, it creates a new one using *makeGMMap* described earlier.

```
toMidi :: Performance → UserPatchMap → Midi
toMidi pf upm =
  let split      = splitByInst pf
```

```
    insts      = map fst split
    rightMap = if (allValid upm insts) then upm
                 else (makeGMMap insts)
  in Midi (if length split == 1 then SingleTrack
                                else MultiTrack)
          (TicksPerBeat division)
          (map (fromAbsTime ∘ performToMEvs rightMap) split)
division = 96 :: Int
```

The following function is used to test whether or not every instrument in a list is found in a *UserPatchMap*:

```
allValid :: UserPatchMap → [InstrumentName] → Bool
allValid upm = and ∘ map (lookupB upm)

lookupB :: UserPatchMap → InstrumentName → Bool
lookupB upm x = or (map ((== x) ∘ fst) upm)
```

The strategy is to associate each channel with a separate track. Thus we first partition the event list into separate lists for each instrument, and signal an error if there are more than 16:

```
splitByInst :: Performance → [(InstrumentName, Performance)]
splitByInst [] = []
splitByInst pf = (i, pf_1) : splitByInst pf_2
  where i = eInst (head pf)
    (pf_1, pf_2) = partition (λe → eInst e == i) pf
```

Note how *partition* is used to group into pf_1 those events that use the same instrument as the first event in the performance. The rest of the events are collected into pf_2, which is passed recursively to *splitByInst*.

Details: *partition* takes a predicate and a list and returns a pair of lists: those elements that satisfy the predicate, and those that do not, respectively. *partition* is defined in the *List* Library as:

```
partition :: (a → Bool) → [a] → ([a], [a])
partition p xs =
  foldr select ([], []) xs
    where select x (ts, fs) | p x = (x : ts, fs)
      | otherwise = (ts, x : fs)
```

The crux of the conversion process is in *performToMEvs*, which converts a *Performance* into a stream of time-stamped messages, i.e. a stream of (*Tick*, *Message*) pairs:

```
type MEvent = (Ticks, Message)
defST = 500000
performToMEvs :: UserPatchMap
                 → (InstrumentName, Performance)
                 → [MEvent]
performToMEvs upm (inm, pf) =
  let (chan, progNum) = upmLookup upm inm
      setupInst       = (0, ProgramChange chan progNum)
      setTempo        = (0, TempoChange defST)
      loop []         = []
      loop (e : es)   = let (mev1, mev2) = mkMEvents chan e
                        in mev1 : insertMEvent mev2 (loop es)
  in setupInst : setTempo : loop pf
```

A source of incompatibilty between Euterpea and Midi is that Euterpea represents notes with an onset and a duration, while Midi represents them as two separate events, a note-on event and a note-off event. Thus *MkMEvents* turns a Euterpea *Event* into two *MEvents*, a *NoteOn* and a *NoteOff*.

```
mkMEvents :: Channel → Event → (MEvent, MEvent)
mkMEvents mChan (Event { eTime = t, ePitch = p,
                         eDur = d, eVol = v })
        = ((toDelta t, NoteOn mChan p v'),
           (toDelta (t + d), NoteOff mChan p v'))
  where v' = max 0 (min 127 (fromIntegral v))
toDelta t = round (t * 2.0 * fromIntegral division)
```

The time-stamp associated with an event in Midi is called a *delta-time*, and is the time at which the event should occur expressed in time-code divisions since the beginning of the performance. Since there are 96 time-code divisions per quarter note, there are 4 times that many in a whole note; multiplying that by the time-stamp on one of our *Event*s gives us the proper delta-time.

In the code for *performToMEvs*, note that the location of the first event returned from *mkMEvents* is obvious; it belongs just where it was created. However, the second event must be inserted into the proper place in the rest of the stream of events; there is no way to know of its proper position ahead

of time. The function *insertMEvent* is thus used to insert an *MEvent* into an already time-ordered sequence of *MEvent*s.

$$
\begin{aligned}
&\mathit{insertMEvent} :: \mathit{MEvent} \to [\mathit{MEvent}] \to [\mathit{MEvent}] \\
&\mathit{insertMEvent}\ \mathit{mev1}\ [\,] = [\mathit{mev1}\,] \\
&\mathit{insertMEvent}\ \mathit{mev1}@(t_1, \_)\ \mathit{mevs}@(\mathit{mev2}@(t_2, \_) : \mathit{mevs}') = \\
&\quad \textbf{if}\ t_1 \leqslant t_2\ \textbf{then}\ \mathit{mev1} : \mathit{mevs} \\
&\qquad \textbf{else}\ \mathit{mev2} : \mathit{insertMEvent}\ \mathit{mev1}\ \mathit{mevs}'
\end{aligned}
$$

14.3 Putting It All Together

[**To do:** Move the code for the *PerformanceDefault* type class, the family of *play* functions, and so on, to this section.]

Chapter 15

Basic Input/Output

So far the only input/output (IO) that we have seen in Euterpea is the use of the *play* function to generate the Midi output corresponding to a *Music* value. But we've said very little about the *play* function itself. What is its type? How does it work? How does one do IO in a purely functional language such as Haskell? Our goal in this chapter is to answer these questions. Then in the next chapter we will describe an elegant way to do IO involving a "musical user interface," or *MUI*.

15.1 IO in Haskell

The Haskell Report defines the result of a program to be the value of the variable *main* in the module *Main*. This is a mere technicality, however, only having relevance when you compile a program as a stone-alone executable (see the GHC documentation for a discussion of how to do that).

The way most people run Haskell programs, especially during program development, is through the GHCi command prompt. As you know, the GHCi implementation of Haskell allows you to type whatever expression you wish to the command prompt, and it will evaluate it for you.

In both cases, the Haskell system "executes a program" by evaluating an expression, which (for a well-behaved program) eventually yields a value. The system must then display that value on your computer screen in some way that makes sense to you. GHC does this by insisting that the type of the value be an instance of the *Show* class—in which case it "shows" the result

by converting it to a string using the *show* function (recall the discussion in Section 7.1). So an integer is printed as an integer, a string as a string, a list as a list, and so on. We will refer to the area of the computer screen where this result is printed as the *standard output area*, which may vary from one implementation to another.

But what if a program is intended to write to a file? Or print a file on a printer? Or, the main topic of this book, to play some music through the computer's sound card, or an external Midi device? These are examples of *output*, and there are related questions about *input*: for example, how does a program receive input from the computer keyboard or mouse, or receive input from a Midi keyboard?

In general, how does Haskell's "expression-oriented" notion of "computation by calculation" accommodate these various kinds of input and output?

The answer is fairly simple: in Haskell there is a special kind of value called an *action*. When a Haskell system evaluates an expression that yields an action, it knows not to try to display the result in the standard output area, but rather to "take the appropriate action." There are primitive actions—such as writing a single character to a file or receiving a single character from a Midi keyboard—as well as compound actions—such as printing an entire string to a file or playing an entire piece of music. Haskell expressions that evaluate to actions are commonly called *commands*.

Some commands return a value for subsequent use by the program: a character from the keyboard, for instance. A command that returns a value of type T has type $IO\ T$. If no useful value is returned, the command has type $IO\ ()$. The simplest example of a command is $return\ x$, which for a value $x :: T$ immediately returns x and has type $IO\ T$.

Details: The type () is called the *unit type*, and has exactly one value, which is also written (). Thus *return* () has type $IO\ ()$, and is often called a "noop" because it is an operation that does nothing and returns no useful result. Despite the negative connotation, it is used quite often!

Remember that all expressions in Haskell must be well-typed before a program is run, so a Haskell implementation knows ahead of time, by looking at the type, that it is evaluating a command, and is thus ready to "take action."

15.2 do Syntax

To make these ideas clearer, let's consider a few examples. One useful IO command is *putStr*, which prints a string argument to the standard output area, and has type *String* → *IO* (). The () simply indicates that there is no useful result returned from this action; its sole purpose is to print its argument to the standard output area. So the program:

```
module Main where
main = putStr "Hello World\n"
```

is the canonical "Hello World" program that is often the first program that people write in a new language.

Suppose now that we want to perform *two* actions, such as first writing to a file named `"testFile.txt"`, then printing to the standard output area. Haskell has a special keyword, **do**, to denote the beginning of a sequence of commands such as this, and so we can write:

```
do writeFile "testFile.txt" "Hello File System"
   putStr "Hello World\n"
```

where the file-writing function *writeFile* has type:

```
writeFile       :: FilePath → String → IO ()
type FilePath = String
```

Details: A **do** expression allows one to sequence an arbitrary number of commands, each of type *IO* (), using layout to distinguish them (just as in a **let** or **where** expression). When used in this way, the result of a **do** expression also has type *IO* ().

So far we have only used actions having type *IO* (); i.e. output actions. But what about input? As above, we will consider input from both the user and the file system.

To receive a line of input from the user (which will be typed in the *standard input area* of the computer screen, usually the same as the standard output area) we can use the function:

```
getLine :: IO String
```

Suppose, for example, that we wish to read a line of input using this function, and then write that line (a string) to a file. To do this we write the

compound command:

do $s \leftarrow$ *getLine*
  writeFile `"testFile.txt"` s

Details: Note the syntax for binding s to the result of executing the *getLine* command—when doing this in your program, you will have to type `<-`. Since the type of *getLine* is *IO String*, the type of s is *String*. Its value is then used in the next line as an argument to the *writeFile* command.

Similarly, we can read the entire contents of a file using the command *readFile* :: *FilePath* $\rightarrow$ *IO String*, and then print the result to standard output:

do $s \leftarrow$ *readFile* `"testFile.txt"`
  putStr s

Details: Any type that is an instance of the *Monad* type class can be used with the **do** syntax to sequence actions. The *Monad* class is discussed in detail in Chapter **??**. It suffices to say for now that the *IO* type is an instance of the *Monad* class, as is the *UI* type to be described in the next chapter, which is part of Eutperpea's MUI design.

15.3 Actions are Just Values

There are many other commands available for file, system, and user IO, some in the Standard Prelude, and some in various libararies (such as *IO*, *Directory*, *System*, and *Time*). We will not discuss any of these here; rather, in the next chapter will concentrate on Midi input and output as well as a collection of graphical input widgets (such as sliders and pushbuttons) that we collectively refer to as Euterpea's musical user interface (MUI).

Before that, however, we wish to emphasize that, despite the special **do** syntax, Haskell's IO commands are no different in status from any other Haskell function or value. For example, it is possible to create a *list* of actions, such as:

```
actionList = [putStr "Hello World\n",
              writeFile "testFile.txt" "Hello File System",
              putStr "File successfully written."]
```

However, a list of actions is just a list of values: they actually don't *do* anything until they are sequenced appropriately using a **do** expression, and then returned as the value of the overall program (either as the variable *main* in the module *Main*, or typed at the GHCi prompt). Still, it is often convenient to place actions into a list as above, and the Haskell Report and Libraries have some useful functions for turning them into single commands. In particular, the function *sequence_* in the Standard Prelude, when used with IO, has type:

```
sequence_ :: [IO a] → IO ()
```

and can thus be applied to the *actionList* above to yield the single command:

```
main :: IO ()
main = sequence_ actionList
```

For a more interesting example of this idea, we first note that Haskell's strings are really just *lists of characters*. Indeed, *String* is a type synonym for a list of characters:

```
type String = [Char]
```

Because strings are used so often, Haskell allows you to write `"Hello"` instead of `['H','e','l','l','o']`. But keep in mind that this is just syntax—strings really are just lists of characters, and these two ways of writing them are identical from Haskell's perspective.

(Earlier the type synonym *FilePath* was defined for *String*. This shows that type synonyms can be created using other type synonyms.)

Now back to the example. From the function *putChar* :: *Char* → *IO* (), which prints a single character to the standard output area, we can define the function *putStr* used earlier, which prints an entire string. To do this, let's first define a function that converts a list of characters (i.e. a string) into a list of IO actions:

```
putCharList :: String → [IO ()]
putCharList = map putChar
```

With this, *putStr* is easily defined:

```
putStr :: String → IO ()
putStr = sequence_ ∘ putCharList
```

Or, more succinctly:

```
putStr :: String → IO ()
putStr = sequence_ ∘ map putStr
```

Of course, *putStr* can also be defined directly as a recursive function, which we do here just to emphasize that actions are just values, so we can use all of the functional programming skills that we normally use:

```
putStr          :: String → IO ()
putStr []       = return ()
putStr (c : cs) = do putChar c
                     putStr cs
```

IO processing in Haskell is consistent with everything we have learned about programming with expressions and reasoning through calculation, although that may not be completely obvious yet. Indeed, it turns out that a **do** expression is just syntax for a more primitive way of combining actions using functions, namely a *monad*, to be revealed in full in Chapter **??**.

15.4 Reading and Writing Midi Files

[TODO: Explain Midi-file IO functions defined in *Codec.Midi*, as well as the Euterpea functions for writing Midi files.]

Chapter 16

Musical User Interface

```
module Euterpea.Examples.MUI where
import Euterpea
import Euterpea.IO.MUI
import Data.Maybe
import Euterpea.IO.MIDI.MidiIO
import qualified Codec.Midi as Midi
```

(This module is not part of the Euterpea module hierarchy, but can be found in the *Examples* folder in the Euterpea distribution.)

Most music software packages have a graphical user interface (aka "GUI") that provides varying degrees of functionality to the user. In Euterpea a basic set of widgets is provided that are collectively referred to as the *musical user interface*, or MUI. This interface has two levels of abstraction: At the *user interface (UI) level*, basic IO-like commands are provided for creating graphical sliders, pushbuttons, and so on for input, and textual displays and graphic images for output (in the future, other kinds of graphic input and output, including virtual keyboards, plots, and so on, will be provided). In addition to these graphical widgets, the UI level also provides an interface to standard MIDI input and output devices.

The second level of abstraction of the MUI is the *signal level*. A *signal* is a time-varying quantity that nicely captures the behavior of many MUI widgets. A special case of a signal is an *event*, and a special case of an event is a *MIDI event*, such as a $Note - On$ or $Note - Off$ message.

We begin our discussion with a description of signals and events.

16.1 Signals

A value of type *Signal T* is a time-varying value of type T. For example, *Signal Float* is a time-varying floating-point number, *Signal AbsPitch* is a time-varing absolute pitch, and so on. Abstractly, one can think of a signal as a function:

$$Signal\ a = Time \to a$$

where *Time* is some suitable representation of time (currently *Double* in Euterpea).

However, this is not how signals are actually implemented in Euterpea, indeed the above is not even valid Haskell syntax. Nevertheless it is helpful to think of signals in this way. Indeed, for pedagogical purposes, we can go one step further and write the above as a Haskell data declaration:

$$\textbf{data}\ Signal\ a = Sig\ (Time \to a)$$

and then describe in more detail how signals are manipulated once this concrete representation is in hand.

First of all, the following functions can be used to "lift" static values and functions to the time-varying domain of signals:

$$lift0 :: a \to Signal\ a$$
$$lift0\ x = Sig\ (\lambda t \to x)$$

$$lift1 :: (a \to b) \to (Signal\ a \to Signal\ b)$$
$$lift1\ f = \lambda(Sig\ g) \to Sig\ (\lambda t \to f\ (g\ t))$$

$$lift2 :: (a \to b \to c) \to (Signal\ a \to Signal\ b \to Signal\ c)$$
$$lift2\ f = \lambda(Sig\ g)\lambda(Sig\ h) \to Sig\ (\lambda t \to f\ (g\ t)\ (h\ t))$$

...

For example, *lift0* 42 is a "constant signal" that, at every point in time, returns the value 42:

$$lift0\ 42$$
$$\Rightarrow Sig\ (\lambda t \to 42)$$

And *lift1 sin* is a function that converts one signal into another signal that, at every point in time, is the sine of the value of the first signal:

$$lift1\ sin$$
$$\Rightarrow \lambda(Sig\ g) \to Sig\ (\lambda t \to sin\ (g\ t))$$

So, for example, *lift1 sin* (*lift0* 42) is the constant signal that returns the sine of 42, and in that sense is the same as *lift0* (*sin* 42):

$$
\begin{aligned}
&\mathit{lift1}\ \mathit{sin}\ (\mathit{lift0}\ 42)\\
&\Rightarrow (\lambda(\mathit{Sig}\ g) \rightarrow \mathit{Sig}\ (\lambda t \rightarrow \mathit{sin}\ (g\ t)))\ (\mathit{Sig}\ (\lambda t \rightarrow 42))\\
&\Rightarrow \mathit{Sig}\ (\lambda t \rightarrow \mathit{sin}\ ((\lambda t \rightarrow 42)\ t))\\
&\Rightarrow \mathit{Sig}\ (\lambda t \rightarrow \mathit{sin}\ 42)\\
&\Rightarrow \mathit{lift0}\ (\mathit{sin}\ 42)
\end{aligned}
$$

16.1.1 Numeric Signals

It is inconvenient to write "lifts" everywhere when we want to work at the signal level. One solution to this would be to define new function names, such as *sinS* for a signal-level version of the sine function, and so on. Better yet, we can take advantage of Haskell's overloaded numeric system, through the use of type classes. For example, we'd like to add, subtract, and multiply signals, as well as apply transcendental functions such as sine, cosine, and exponentiation. Haskell's numeric classes provide a delightfully convenient way to do this.

(Keep in mind while reading this section that signals aren't actually implemented in this way, but conceptually this should give you a good idea of the desired behavior.)

For starters, we can declare *Signal* to be an instance of the class *Num*:

$$
\begin{aligned}
&\textbf{instance}\ \mathit{Num}\ a \Rightarrow \mathit{Num}\ (\mathit{Signal}\ a)\ \textbf{where}\\
&\quad \mathit{Sig}\ f_1 + \mathit{Sig}\ f_2 = \mathit{Sig}\ (\lambda t \rightarrow f_1\ t + f_2\ t)\\
&\quad \mathit{Sig}\ f_1 * \mathit{Sig}\ f_2\ = \mathit{Sig}\ (\lambda t \rightarrow f_1\ t * f_2\ t)\\
&\quad \mathit{Sig}\ f_1 - \mathit{Sig}\ f_2 = \mathit{Sig}\ (\lambda t \rightarrow f_1\ t - f_2\ t)\\
&\quad \mathit{negate}\ (\mathit{Sig}\ f) = \mathit{Sig}\ (\lambda t \rightarrow \mathit{negate}\ f)\\
&\quad \mathit{fromInteger}\ i\ = \mathit{Sig}\ (\lambda t \rightarrow \mathit{fromInteger}\ i)\\
&\quad \ldots
\end{aligned}
$$

Better yet, we can write this using the lifting functions defined above:

$$
\begin{aligned}
&\textbf{instance}\ \mathit{Num}\ a \Rightarrow \mathit{Num}\ (\mathit{Signal}\ a)\ \textbf{where}\\
&\quad (+) = \mathit{lift2}\ (+)\\
&\quad (*) = \mathit{lift2}\ (*)\\
&\quad (-) = \mathit{lift2}\ (-)\\
&\quad \mathit{negate} = \mathit{lift1}\ \mathit{negate}\\
&\quad \mathit{fromInteger}\ i = \mathit{lift0}\ (\mathit{fromInteger}\ i)\\
&\quad \ldots
\end{aligned}
$$

(See Section 7.5.1 for a full list of the operators in this class.)

Euterpea also defines instances of *Signal* for the classes *Fractional* and *Floating*. Using our pedagogical representation:

```
instance Num a ⇒ Fractional (Signal a) where
  (/)          = lift2 (/)
  fromRational r = lift0 (fromRational r)
  ...
instance Fractional a ⇒ Floating (Signal a) where
  pi  = lift0 pi
  sin = lift1 sin
  exp = lift1 exp
  ...
```

(See the Haskell Report for the full list of operators in these classes.)

With these definitions in place, we can now write, for example, *sin* 42 instead of *lift1 sin* (*lift0* 42). Of course, the type of *sin* 42 by itself is ambiguous, but in a context where a *Signal* is expected, it will be interpreted properly.

16.1.2 Time

It is convenient to have a signal that represents the current time. Using our pedagogical representation of signals, we can think of the time t being defined as follows:

```
t :: Signal Time
t = Sig id
```

For example, in physics a *sine wave* is described by the following mathematical formula:

$$x(t) = \sin(\omega t + \phi)$$

where ω is the angular frequency (in radians) and ϕ is the phase angle. We can write this using the overloaded operators from the last section, and the time t as defined above, quite succinctly:

```
x :: Signal Double
x = sin (omega * t + phi)
```

To see how this works, let's proceed by calculation, with the assumption that *omega* and *phi* are constants. We will work from the inner expressions outward—first *omega* ∗ *t*:

```
omega * t
 ⇒ lift2 (*) (lift0 omega) (Sig id)
 ⇒ (λ(Sig g)λ(Sig h) → Sig (λt → g t * h t))
```

$$
\begin{aligned}
&(\lambda t \to omega)\ (Sig\ id)\\
&\Rightarrow Sig\ (\lambda t \to (\lambda t \to omega)\ t * id\ t))\\
&\Rightarrow Sig\ (\lambda t \to omega * t)
\end{aligned}
$$

Now $omega * t + phi$:

$$
\begin{aligned}
&|\ omega * t + phi\ |\\
&\Rightarrow lift2\ (+)\ (Sig\ (\lambda t \to omega * t))\ (lift0\ phi)\\
&\Rightarrow (\lambda(Sig\ g)\lambda(Sig\ h) \to Sig\ (\lambda t \to g\ t + h\ t))\\
&\quad (Sig\ (\lambda t \to omega * t))\ (Sig\ (\lambda t \to phi))\\
&\Rightarrow Sig\ (\lambda t \to (\lambda t \to omega * t)\ t + (\lambda t \to phi)\ t)\\
&\Rightarrow Sig\ (\lambda t \to omega * t + phi)
\end{aligned}
$$

And finally the whole thing:

$$
\begin{aligned}
&sin\ (omega * t + phi)\\
&\Rightarrow lift1\ sin\ (Sig\ (\lambda t \to omega * t + phi))\\
&\Rightarrow (\lambda(Sig\ g) \to Sig\ (\lambda t \to sin\ (g\ t)))\ (Sig\ (\lambda t \to omega * t + phi))\\
&\Rightarrow Sig\ (\lambda t \to sin\ ((\lambda t \to omega * t + phi)\ t))\\
&\Rightarrow Sig\ (\lambda t \to sin\ (omega * t + phi))
\end{aligned}
$$

16.1.3 Musical Signals

Of course, any function can be lifted. For example, consider the *pitch* function from Chapter 2. Its type is $AbsPitch \to Pitch$, therefore the function *lift1 pitch* must have type

$Signal\ AbsPitch \to Signal\ Pitch$. Since functions such as this are not members of a type class, we cannot overload them, and thus may decide to use new function names, such as:

$$
\begin{aligned}
&pitchS &&:: Signal\ AbsPitch \to Signal\ Pitch\\
&pitchS &&= lift1\ pitch\\
&absPitchS &&:: Signal\ Pitch \to Signal\ AbsPitch\\
&absPitchS &&= lift1\ absPitch
\end{aligned}
$$

We will see a larger example using this idea shortly.

16.1.4 Useful Signal Operators

Sometimes we need to zip and unzip values at the signal level. The following functions facilitate this:

$$\begin{aligned}
&\mathit{join} &&:: \mathit{Signal}\ a \to \mathit{Signal}\ b \to \mathit{Signal}\ (a, b)\\
&\mathit{split} &&:: \mathit{Signal}\ (a, b) \to (\mathit{Signal}\ a, \mathit{Signal}\ b)\\
&\mathit{fstS} &&:: \mathit{Signal}\ (a, b) \to \mathit{Signal}\ a\\
&\mathit{sndS} &&:: \mathit{Signal}\ (a, b) \to \mathit{Signal}\ b
\end{aligned}$$

The behavior of these functions should be clear from their type signatures. For example, *join* takes two signals and zips their values together pointwise; *split* takes a signal of pairs and returns a pair of signals; and so on.

We would also like to compare signals, but it is not as easy to overload the relational operators as we did the arithmetic operators, since they do not have a uniform type structure, and are not members of a convenient type class. Therefore the following special operators are defined:

$$\begin{aligned}
&(<*), (>*), (\leqslant*), (\geqslant*) &&:: \mathit{Ord}\ a \Rightarrow \mathit{Signal}\ a \to \mathit{Signal}\ a \to \mathit{Signal}\ \mathit{Bool}\\
&(==*), (\neq*) &&:: \mathit{Eq}\ a \Rightarrow \mathit{Signal}\ a \to \mathit{Signal}\ a \to \mathit{Signal}\ \mathit{Bool}\\
&(\&\&*), (||*) &&:: \mathit{Signal}\ \mathit{Bool} \to \mathit{Signal}\ \mathit{Bool} \to \mathit{Signal}\ \mathit{Bool}\\
&\mathit{notS} &&:: \mathit{Signal}\ \mathit{Bool} \to \mathit{Signal}\ \mathit{Bool}
\end{aligned}$$

For example, if $s_1, s_2 :: \mathit{Signal}\ \mathit{AbsPitch}$ are two signals of absolute pitches, then $s_1 ==* s_2$ is a signal of Boolean values that represents the pointwise equality comparison of the two signals.

16.1.5 Stateful Signals

Some signals are *stateful*, meaning that they depend on past values in some way. A particularly important example of a stateful signal is the *integral* of a signal, which can be computed with the following function:

$$\mathit{integral} :: \mathit{Signal}\ \mathit{Time} \to \mathit{Signal}\ \mathit{Double} \to \mathit{Signal}\ \mathit{Double}$$

The first argument to *integral* is a signal that represents the current time. We will say more about this in the next section, but conceptually you can think of $\mathit{integral}\ t\ s$ as the integral of s with respect to t.

Another kind of stateful signal can be generated with the functions:

$$\mathit{initS}, \mathit{initS}' :: a \to \mathit{Signal}\ a \to \mathit{Signal}\ a$$

initS conceptually introduces an infinitesimally small delay in a signal by "initializing" it with a given value. In other words, the signal $\mathit{initS}\ v\ s$ behaves just like s, but it has the value v at time 0, and takes on the values

of s henceforth. $initS'\ v\ s$ behaves similarly, except that it *replaces* the initial value in s with v. In the limit, these are mathematically the same, but in practice, there is sometimes reason to choose one over the other.

As an example of the use of *initS*, suppose we have a signal $s :: Signal\ AbsPitch$, and we wish to know when its value changes—i.e. we would like a value of type *Signal Bool* that is *True* just at those moments when s has changed.[1] Using *initS* we can compute the desired result by comparing the value of the signal to its values an infinitesimally short time in the past; that is:

$$s \neq* initS\ 0\ s$$

From the type of $\neq*$ it is easy to see that the type of this result is *Signal Bool*.

16.2 Events and Reactivity

Although signals are a nice abstraction of time-varying entities, and the world is arguably full of such entities, there are some things that happen at discrete points in time, like a mouse click, or a MIDI keyboard press, and so on. We call these *events*. To represent events, and have them coexist with signals, recall the *Maybe* type defined in the Standard Prelude:

data $Maybe\ a = Nothing \mid Just\ a$

We define an event simply as a value of type $Signal\ (Maybe\ a)$, and in this sense events in Euterpea are really event *streams*, since more than one may occur over time. We say that the value associated with an event is "attached to" or "carried by" that event. For clarity we define *EventS* as a type synonym:

type $EventS\ a = Signal\ (Maybe\ a)$

"*EventS*" can be read either as "event stream" or "event signal," although we will often just write "event."

16.2.1 Manipulating Event Streams

There are many things that we would like to do with events. For example, it is convenient to be able to apply a function to each value attached to an

[1] Mathematically, if s were truly a continuous numeric function of time, this would be the same as asking when the derivative is non-zero. But we would also like to compute this result for integral types such as *AsbPitch*, as well as for non-numeric types.

event, just like *map* does for a list. In the context of events, however, we prefer to use an infix operator:

$$(\Longrightarrow) :: EventS\ a \to (a \to b) \to EventS\ b$$

Mnemonically, an expression of the form $e \Longrightarrow f$ can be read as "send the event stream e through the function f." For examnple, if $s :: Event\ AbsPitch$ is a stream of absolute pitch events, then $s \Longrightarrow pitch$ is a stream of pitch events, with type $Event\ Pitch$.

For convenience we also define a version of $(\Longrightarrow)$ that ignores its input value:

$$\begin{aligned}&(-\!\gg) \quad :: EventS\ a \to b \to EventS\ b\\ &s_1 -\!\gg v = s \Longrightarrow (\backslash\_ \to v)\end{aligned}$$

We can merge two event streams using:

$$(.|.) :: EventS\ a \to EventS\ a \to EventS\ a$$

If two events happen at the same time, preference is given to the one in the first argument.

16.2.2 Turning Signals into Events

It is sometimes useful to turn a Boolean signal into an event stream, which can be done in two different ways:

$$edge, when :: Signal\ Bool \to EventS\ ()$$

edge s generates an event whenever the Boolean signal s changes from *False* to *True*—in signal processing this is called an "edge detector," and thus the name chosen here. *when s* is also an edge detector, but it generates an event whenever s changes either from *False* to *True* or from *True* to *False*.

A related operation is:

$$unique :: Eq\ a \Rightarrow Signal\ a \to EventS\ a$$

which generates an event whenever the signal argument changes, and attaches the value of the signal at that time to the event. For example, if $ap :: Signal\ AbsPitch$ changes its pitch once every second, starting with absolute pitch 0 and incrementally moving upward, then *unique ap* will generate an event stream whose attached values are successively 0, 1, 2, and so on. Furthermore, the signal:

$$unique\ ap \Longrightarrow pitch$$

generates events once per second, with the attached values being the pitches

$$
\begin{array}{ll}
\mathit{snapshot} & :: \mathit{EventS}\ a \to \mathit{Signal}\ b \to \mathit{EventS}\ (a, b) \\
\mathit{snapshot\_} & :: \mathit{EventS}\ a \to \mathit{Signal}\ b \to \mathit{EventS}\ b \\
\mathit{hold} & :: a \to \mathit{EventS}\ a \to \mathit{Signal}\ a \\
\mathit{accum} & :: a \to \mathit{EventS}\ (a \to a) \to \mathit{Signal}\ a
\end{array}
$$

Table 16.1: Signal Samplers

$(C, 0)$, $(Cs, 0)$, $(D, 0)$, and so on.

16.2.3 Signal Samplers

The useful collection of functions shown in Table 16.1 can be thought of as "signal samplers." For example, if *ticks* :: *EventS* () generates unit events at some sampling rate, then *snapshot_ ticks ap* is a stream of events at the same rate, but the value attached to each event is the value of the absolute pitch *ap* at that time. *snapshot* behaves similarly, but pairs the sampled value with the original event value.

hold v e is a signal whose initial value is v, which it "holds" until the first event in e happens, at which point it changes to the value attached to that event, which it then "holds" until the next event, and so on. *accum* is a bit like *scan*. The signal *accum v e* starts with the value v, but then applies the function attached to the first event to that value to get the next value, and so on.

16.2.4 Switches and Reactivity

More generally, perhaps the most fundamental set of operations on events are the ones that introduce *reactivity*: the ability to change a signal's behavior in response to an event. There are two operations for this purpose:

$$\mathit{switch}, \mathit{untilS} :: \mathit{Signal}\ a \to \mathit{EventS}\ (\mathit{Signal}\ a) \to \mathit{Signal}\ a$$

The signal s `` `untilS` `` e initially behaves just like s, until the first event in e occurs. It then behaves forever after like the behavior attached to that event. s `` `switch` `` e behaves similarly, except that each subsequent event after the first will change the behavior to the event's new attached signal value.

It is important to note that the second argument to these functions, whose type is *EventS* (*Signal a*), reflects a kind of "higher-order" signal, that is, a signal of signals. Nothing like this can appear in an electrical circuit, for example, since it it not possible carry entire circuits as values on

label	:: *String* → *UI* ()
display	:: *Signal String* → *UI* ()
button	:: *String* → *UI* (*Signal Bool*)
checkbox	:: *String* → *Bool* → *UI* (*Signal Bool*)
radio	:: [*String*] → *Int* → *UI* (*Signal Int*)
hSlider, *vSlider*	:: (*RealFrac a*) ⇒ (*a*, *a*) → *a* → *UI* (*Signal a*)
hiSlider, *viSlider*	:: (*Integral a*) ⇒ *a* → (*a*, *a*) → *a* → *UI* (*Signal a*)
canvas	:: *Dimension* → *EventS Graphic* → *UI* ()

Table 16.2: MUI Input Widgets

a wire.

[**To do:** Insert example or two here]

16.3 The UI Level

It is at the UI level that "graphical widgets" are actually created, using a style very similar to the way we did IO in Chapter 15. But instead of values of type *IO T*, which we referred to as *IO actions*, we will use values of type *UI T*, which we refer to as *UI actions*. Just like *IO*, the *UI* type is fully abstract (meaning its implementation is hidden), and is an instance of the *Monad* class, which means that it can be used with the **do** syntax to sequence UI actions.

16.3.1 Input Widgets

Euterpea's basic input widgets are shown in Table 16.2. Note that each of them returns, ultimately, a value of type *UI T*, for some *T*, and therefore must be used with the **do** syntax to properly sequence their execution. The names and type signatures of these functions suggest their functionality, which we elaborate on more below:

- A simple (static) text string can be displayed using:

  label :: *String* → *UI* ()

- Alternatively, a time-varying string can be be displayed using:

  display :: *Signal String* → *UI* ()

For convenience, Euterpea defines the following useful variations of *display*:

```
displaySig :: Show a ⇒ Signal a → UI ()
displaySig = display ∘ lift1 show

withDisplay          :: Show a ⇒ UI (Signal a) → UI (Signal a)
withDisplay widget = do w ← widget
                        displaySig w
                        return w
```

- *button*, *checkbox*, and *radio* are three kinds of "pushbuttons." A *button* (or *checkbox*) is pressed and unpressed (or checked and unchecked) independently of others. In contrast, a *radio* button is dependent upon other radio buttons—specifically, only one can be "on" at a time, so pressing one will turn off the others. The string argument to these functions is the label attached to the button. *radio* takes a list of strings, each being the label of one of the buttons in the mutually-exclusive group; indeed the length of the list determines how many buttons are in the group.

- *hSlider*, *vSlider*, *hiSlider* and *viSlider* are four kinds of sliders—the first two yield floating-point numbers in a given range, oriented horizontally and vertically, respectively, whereas the latter two return integral numbers. For the integral sliders, the first argument is the size of the step taken when the slider is clicked at any point on either side of the slider "handle." In each of the four cases, the other two arguments are the range and initial setting of the slider, respectively.

- *canvas* is a graphical canvas on which images can be drawn. More will be said boit ut later.

As a very simple example, let's combine our running example of absolute pitches with a slider. We will define a MUI program that has a single slider representing the absolute pitch, and a display widget that displays the pitch corresponding to the current setting of the slider:

```
ui0 :: UI ()
ui0 = do ap ← hiSlider 1 (0,100) 0
         displaySig (pitchS ap)
```

Note how the use of signals makes this dynamic MUI trivial to write.

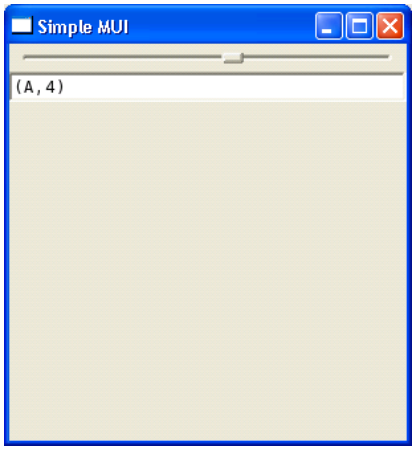
We can execute this example using the function:

$runUI :: String \to UI\ a \to IO\ ()$

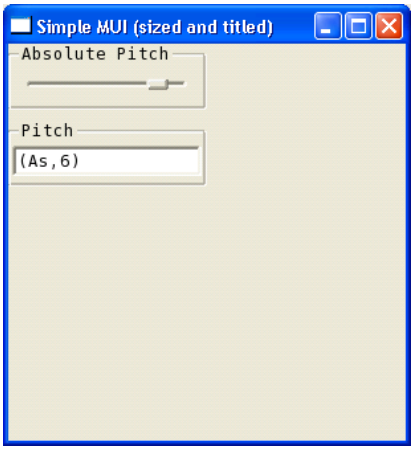
where the string argument is a title displayed in the window holding the widget. So our first running example of a MUI is:

$mui0 = runUI$ `"Simple MUI"` $ui0$

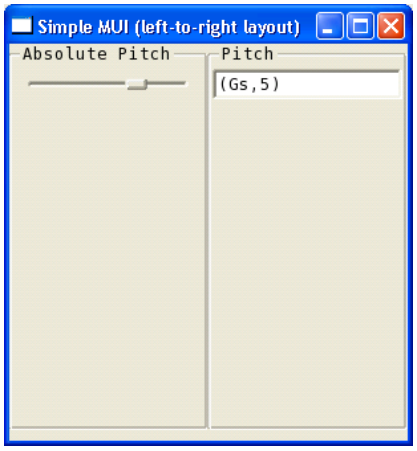
The resulting MUI, once the slider has been moved a bit, is shown in Figure 16.1(a).



(a) Very Simple



(b) With Titles and Sizing



(c) With Alternate (left-to-right) Layout

Figure 16.1: Several Simple MUIs

$$
\begin{aligned}
&\mathit{title} &&:: \mathit{String} \to \mathit{UI}\ a \to \mathit{UI}\ a\\
&\mathit{setSize} &&:: \mathit{Dimension} \to \mathit{UI}\ a \to \mathit{UI}\ a\\
&\mathit{pad} &&:: (\mathit{Int}, \mathit{Int}, \mathit{Int}, \mathit{Int}) \to \mathit{UI}\ a \to \mathit{UI}\ a\\
&\mathit{topDown}, \mathit{bottomUp}, \mathit{leftRight}, \mathit{rightLeft} &&:: \mathit{UI}\ a \to \mathit{UI}\ a\\
&\textbf{type}\ \mathit{Dimension} = (\mathit{Int}, \mathit{Int})
\end{aligned}
$$

Table 16.3: MUI Layout Widget Transformers

16.3.2 UI Transformers

Table 16.3 shows a set of "UI transformers"–functions that take UI values as input, and return modified UI values as output.

title simply attaches a title (a string) to a UI, and *setSize* establishes a fixed size (in pixels that represent two sides of a rectangle) for a UI. For example we can modify the previous example:

```
ui1  :: UI ()
ui1  = setSize (150, 150) $
         do ap ← title "Absolute Pitch" (hiSlider 1 (0, 100) 0)
            title "Pitch" (displaySig (pitchS ap))
mui1 = runUI "Simple MUI (sized and titled)" ui1
```

This MUI is shown in Figure 16.1(b).

pad (w, n, e, s) *ui* adds w pixels of space to the "west" of the UI *ui*, and n, e, and s pixels of space to the north, east, and south, respectively. The remaining four functions are used to control the relative layout of the widgets within a UI. By default widgets are arranged top-to-bottom, but, for example, we could modify the previous UI program to arrange the two widgets left-to-right:

```
ui2  :: UI ()
ui2  = leftRight $
         do ap ← title "Absolute Pitch" (hiSlider 1 (0, 100) 0)
            title "Pitch" (display (lift1 (show ∘ pitch) ap))
mui2 = runUI "Simple MUI (left-to-right layout)" ui2
```

This MUI is shown in Figure 16.1(c). Layout transformers can be nested (as demonstrated in some later examples), so a fair amount of flexibility is available.

16.3.3 MIDI Input and Output

There are two widgets for MIDI, one for input and the other for output, but neither of them displays anything graphically:

```
midiIn  :: Signal DeviceID → UI (EventS [MidiMessage])
midiOut :: Signal DeviceID → EventS [MidiMessage] → UI ()
```

Except for the *DeviceID* (about which more will be said shortly), these functions are fairly straigtforward: *midiOut* takes a stream of *MidiMessage* events and sends them to the MIDI output device, whereas *midiIn* generates a stream of *MidiMessage* events corresponding to the messages sent by the MIDI input device. The *MidiMessage* data type is defined as:

```
data MidiMessage = ANote { channel :: Channel, key :: Key,
                           velocity :: Velocity, duration :: Time }
                 | Std Message
  deriving Show
```

A *MidiMessage* is either an *ANote*, which allows one to specify a note with duration, or is a standard MIDI *Message*. Recall that MIDI does not have a notion of duration, but rather has separate *NoteOn* and *NoteOff* events. With *ANote*, the design above is a bit more convenient, although what happens "behind-the-scenes" is that each *ANote* is transformed into a *NoteOn* and *NoteOff* event.

The *Message* data type was described in Chapter 14, and is defined in the *Codec.Midi* module. Its most important functionality is summarized here:

```
data Message =
    -- Channel Messages
    NoteOff { channel :: !Channel, key :: !Key, velocity :: !Velocity }
  | NoteOn  { channel :: !Channel, key :: !Key, velocity :: !Velocity }
  | ProgramChange { channel :: !Channel, preset :: !Preset }
  | ...
    -- Meta Messages
  | TempoChange ! Tempo |
  | ...
  deriving (Show, Eq)
```

As an example of the use of *midiOut*, we will modify our previous MUI program to output an *ANote* message every time the absolute pitch changes:

```
ui3 = do ap ← title "Absolute Pitch" (hiSlider 1 (0,100) 0)
```

```
        title "Pitch" (displaySig (pitchS ap))
        let ns = unique ap =>> (λk → [ANote 0 k 100 0.1])
        midiOut 0 ns
mui3 = runUI "Pitch Player" ui3
```

Note the use of *unique* to generate an event when the pitch changes, and the use of (=≫) to convert those events into *ANote*s.

16.3.4 Midi Device IDs

Note in the previous example that the *DeviceID* argument to *midiOut* is set to 0. The MIDI device ID is a system-dependent concept that provides an operating system with a simple way to uniquely identify various MIDI devices that may be attached to a computer. Indeed, as devices are dynamically connected and disconnected from a computer, the mapping of these IDs to a particular device may change. If you try to run the above code, it may or may not work, depending on whether the MIDI device with ID 0 corresponds to the preferred MIDI output device on your machine.

To overcome this problem, most MIDI software programs allow the user to select the preferred MIDI input and output devices. The user usually has the best knowledge of which devices are connected, and which devices to use. In Euterpea, the easiest way to do this is using the UI widgets:

```
selectInput, selectOutput :: UI (Signal DeviceID)
```

Each of these widgets automatically queries the operating system to obtain a list of connected MIDI devices, and then displays the list as a set of radio buttons, thus allowing the user to select one of them. Note that the result is a signal, and the *DeviceID* arguments to *midiIn* and *midiOut* are also signals. This makes wiring up the user choice very easy. For example, we can modify the previous program to look like this:

```
ui4  :: UI ()
ui4  = do devid ← selectOutput
          ap ← title "Absolute Pitch" (hiSlider 1 (0,100) 0)
          title "Pitch" (displaySig (pitchS ap))
          let ns = unique ap =>> (λk → [ANote 0 k 100 0.1])
          midiOut devid ns
mui4 = runUI "Pitch Player with MIDI Device Select" ui4
```

We suggest that this approach is always taken when dealing with MIDI, even if you think you know the exact device ID.

For an example using MIDI input as well, here is a simple program that copies each MIDI message verbatim from the selected input device to the selected output device:

```
ui5  :: UI ()
ui5  = do mi ← selectInput
          mo ← selectOutput
          m  ← midiIn mi
          midiOut mo m
mui5 = runUI "MIDI Input / Output UI" ui5
```

16.3.5 Timer Widgets

Remember that there is no "hidden" time in the MUI—anything that depends on the notion of time (such as *integral* discussed earlier) takes a time signal explicitly as an argument. For this purpose, the following function generates a signal corresponding to the current time:

```
time :: UI (Signal Time)
```

Besides *integral*, another function that depends explicitly on time is the following, which creates a *timer*:

```
timer :: Signal Time → Signal Double → EventS ()
```

timer t i takes a time source *t* and a signal *i* that represents the timer interval (in seconds), and generates a stream of events, with each pair of consecutive events separated by the timer interval. Note that the timer interval is itself a signal, so the timer output can have varying frequency.

As an example of this, let's modify our previous UI so that, instead of playing a note everytime the absolute pitch changes, we will output a note continuously, at a rate controlled by a second slider:

```
ui6  :: UI ()
ui6  = do devid ← selectOutput
          ap ← title "Absolute Pitch" (hiSlider 1 (0,100) 0)
          title "Pitch" (displaySig (pitchS ap))
          t  ← time
          f  ← title "Tempo" (hSlider (1,10) 1)
          let ticks = timer t (1/f)
          let ns    = snapshot_ ticks ap =≫
                        (λk → [ANote 0 k 100 0.1])
          midiOut devid ns
mui6 = runUI "Pitch Player with Timer" ui6
```

Note that:

- The time t is needed solely to drive the timer.
- The rate of *ticks* is controlled by the slider. A higher slider value causes a lower time between ticks, and thus a higher frequency, or tempo.
- *snapshot_* uses the timer output to control the sample rate of the absolute pitch.

Finally, an event stream can be delayed by a given (variable) amount of time using the following function:

$$delayt :: Signal\ Time \to Signal\ Double \to EventS\ a \to EventS\ a$$

The second argument specifies the amount of delay to be applied to the third argument.

16.4 Putting It All Together

Recall that a Haskell program must eventually be a value of type $IO\ ()$, and thus we need a function to turn a UI value into a IO value—i.e. the UI needs to be "run." We can do this using one of the following two functions, the first of which we have already been using:

$$\begin{aligned} runUI &:: String \to UI\ a \to IO\ () \\ runUIEx &:: Dimension \to String \to UI\ a \to IO\ () \end{aligned}$$

Both of these functions take a string argument that is displayed in the title bar of the graphical window that is generated. $runUIEx$ additionally takes the dimensions of the window as an argument. Executing $runUI\ s\ ui$ or $runUIEx\ d\ s\ ui$ will create a single MUI window whose behavior is governed by the argument $ui :: UI\ a$.

16.5 Musical Examples

In this section we work through three larger musical examples that use Euterpea's MUI in interesting ways.

16.5.1 Chord Builder

This MUI will display a collection of chord types (Maj, Maj7, Maj9, min, min7, min9, and so on), one of which is selectable via a radio button. Then

when a key is pressed on a MIDI keyboard, the selected chord is built and played using that key as the root.

To begin, we define a mapping between chord types and their intervals starting with the root note:

$$
\begin{aligned}
&\mathit{chordIntervals} :: [(\mathit{String}, [\mathit{Int}])] \\
&\mathit{chordIntervals} = [(\texttt{"Maj"}, [4,3,5]), (\texttt{"Maj7"}, [4,3,4,1]), \\
&\qquad (\texttt{"Maj9"}, [4,3,4,3]), (\texttt{"Maj6"}, [4,3,2,3]), \\
&\qquad (\texttt{"min"}, [3,4,5]), (\texttt{"min7"}, [3,4,3,2]), \\
&\qquad (\texttt{"min9"}, [3,4,3,4]), (\texttt{"min7b5"}, [3,3,4,2]), \\
&\qquad (\texttt{"mMaj7"}, [3,4,4,1]), (\texttt{"dim"}, [3,3,3]), \\
&\qquad (\texttt{"dim7"}, [3,3,3,3]), (\texttt{"Dom7"}, [4,3,3,2]), \\
&\qquad (\texttt{"Dom9"}, [4,3,3,4]), (\texttt{"Dom7b9"}, [4,3,3,3])]
\end{aligned}
$$

We will display the list of extensions on the screen as radio buttons for the user to click on.

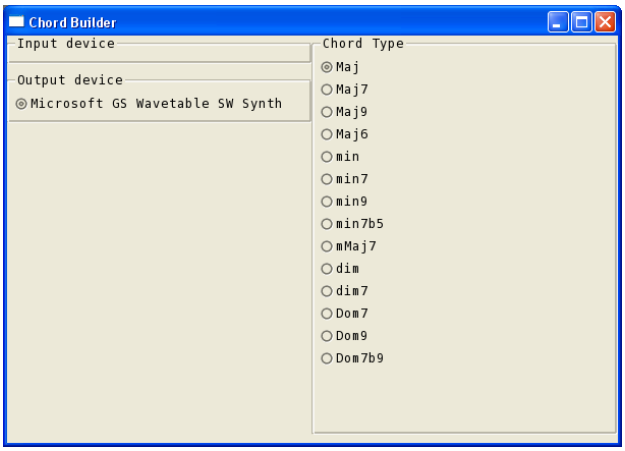


Figure 16.2: A Chord Builder MUI

The *toChord* function takes an input MIDI message as the root note and the index of the selected chord extension, and outputs the notes of the selected chord based on the root note. For simplicity, we only process the head of the message list and ignore everything else.

$$
\begin{aligned}
&\mathit{toChord} :: ([\mathit{MidiMessage}], \mathit{Int}) \to [\mathit{MidiMessage}] \\
&\mathit{toChord}\ (\mathit{ms}@(m\colon \_), i) = \\
&\quad \textbf{case}\ m\ \textbf{of} \\
&\qquad \mathit{Std}\ (\mathit{Midi.NoteOn}\ c\ k\ v) \to f\ \mathit{Midi.NoteOn}\ c\ k\ v \\
&\qquad \mathit{Std}\ (\mathit{Midi.NoteOff}\ c\ k\ v) \to f\ \mathit{Midi.NoteOff}\ c\ k\ v
\end{aligned}
$$

```
    _ → ms
  where f g c k v = map (λk' → Std (g c k' v))
                        (scanl (+) k (snd (chordIntervals !! i)))
```

The MUI is arranged in the following way. On the left side, the list of input and output devices are displayed top-down. On the right is the list of chord types. We take the name of each chord type from the *chordIntervals* list to create the radio buttons.

When a MIDI input event occurs, the input message and the currently selected index to the list of chords is sent to the *toChord* function, and the resulting chord is then sent to the Midi output device.

```
buildChord :: UI ()
buildChord = leftRight $
  do (mi, mo) ← topDown $
                do mi ← selectInput
                   mo ← selectOutput
                   return (mi, mo)
     m        ← midiIn mi
     i        ← topDown $ title "Chord Type" $
                radio (fst (unzip chordIntervals)) 0
     midiOut mo (snapshot m i ⋙= toChord)
chordBuilder = runUIEx (600, 400) "Chord Builder" buildChord
```

Figure 16.2 shows this MUI in action.

16.5.2 Bifurcate Me, Baby!

Here is an example with some ideas borrowed from Gary Lee Nelson's composition "Bifurcate Me, Baby!"

The basic idea is to evaluate the logistic growth function at different points and convert the value to a musical note. The growth function is given by the equation:

$$x_{n+1} = rx_n(1 - x_n)$$

Mathematically, we start with an initial population x_0 and iteratively apply the growth function to it, where r is the growth rate. For certain values of r, the population stablizes to a certain value, but as r increases,

the period doubles, quadruples, and eventually leads to chaos. It is one of the classic examples in chaos theory.

First we define the growth function in Haskell, which, given a rate r and current population x, generates the next population.

```
grow      :: Double → Double → Double
grow r x = r * x * (1 - x)
```

Then we define a signal *ticks* that pulsates at a given frequency specified by slider f. This is the signal that will drive the simulation.

The next thing we need is a time-varying population. This is where *accum* comes in handy. *accum* takes an initial value and an event signal carrying a modifying function, and updates the current value by applying the function to it. Since we want the growth rate to be time-varying, we lift the growth function to the signal level and pass in the growth rate signal r. This gives us a value of type *Signal* (*Double* → *Double*), that is, a signal of functions that will update a population at the current growth rate. Then, at every tick, we take a snapshot of this signal, producing values of type *EventS* (*Double* → *Double*). This is given to *accum* with an initial value of 0.1, and we get back our population signal *pop* driven by the clock ticks.

We can now write a simple function that maps a population value to a musical note:

```
popToNote :: Double → [MidiMessage]
popToNote x = [ANote 0 n 64 0.05]
              where n = truncate (x * 127)
```

Finally, to play the note at every tick, we again take a snapshot of the current population at every tick and send the result to *popToNote*. The resulting event signal is played through the selected MIDI output device.

```
bifurcateUI :: UI ()
bifurcateUI = do
  t  ← time
  mo ← selectOutput
  f  ← title "Frequency" $ withDisplay (hSlider (1,10) 1)
  r  ← title "Growth rate" $ withDisplay (hSlider (2.4,4.0) 2.4)
  let ticks :: EventS ()
      ticks = timer t (1.0/f)
      pop   :: Signal Double
      pop   = accum 0.1 (snapshot_ ticks (lift1 grow r))
```

```
    title "Population" $ displaySig pop
    midiOut mo (snapshot_ ticks pop =>> popToNote)
bifurcate = runUIEx (300,500) "Bifurcate!" $ bifurcateUI
```

16.5.3 MIDI Echo Effect

As a final example we present a program that receives a MIDI event stream and, in addition to playing each note received from the input device, it also echoes the note at a given rate, while playing each successive note more softly until the velocity reduces to 0.

The key component we need for this problem is a delay function that can delay a given event signal for a certain amount of time. Recall that the function *delayt* takes a time signal, the amount of time to delay, and an input signal, and returns a delayed version of the input signal.

There are two signals we want to attenuate, or "decay." One is the signal coming from the input device, and the other is the delayed and decayed signal containing the echoes. In the code shown below, they are denoted as m and s, respectively. First we merge the two event streams into one, and then remove events with empty MIDI messages by replacing them with Nothing. The resulting signal m' is then processed further as follows.

Whenever there is an event in m', we take a snapshot of the current decay rate specified by a slider r. The MIDI messages and the current decay rate are passed to a function k, which softens each note in the list of messages. We define a function called *decay* that reduces the velocity of each note by the given rate. If the velocity drops to 0, the note is removed. The resulting signal is then delayed by the amount of time determined by another slider f, producing signal s. s is then fed back to the *mergeM* function, closing the loop of the recursive signal. At the same time, m' is sent to the output device.

```
echoUI :: UI ()
echoUI = do
  mi ← selectInput
  mo ← selectOutput
  m  ← midiIn mi
  t  ← time
  r  ← title "Decay rate" $ withDisplay $ hSlider (0,0.9) 0.5
  f  ← title "Echoing frequency" $ withDisplay $
                                       hSlider (1,10) 10
```

```
  let m'       = lift1 removeNull $ lift2 mergeM m s
      s        = delayt t (1.0/f) (snapshot m' r >>= k)
      k (ns,r) = mapMaybe (decay 0.1 r) ns
  midiOut mo m'
echo = runUIEx (500,500) "Echo" echoUI

mergeM :: Maybe [MidiMessage] -> Maybe [MidiMessage] ->
          Maybe [MidiMessage]
mergeM (Just ns1) (Just ns2) = Just (ns1 ++ ns2)
mergeM n1         Nothing    = n1
mergeM Nothing    n2         = n2

removeNull :: Maybe [MidiMessage] -> Maybe [MidiMessage]
removeNull (Just []) = Nothing
removeNull mm        = mm

decay :: Time -> Double -> MidiMessage -> Maybe MidiMessage
decay dur r m =
  let f c k v d = if v > 0
                  then let v' = truncate (fromIntegral v * r)
                       in Just (ANote c k v' d)
                  else Nothing
  in case m of
    ANote c k v d             -> f c k v d
    Std (Midi.NoteOn c k v) -> f c k v dur
    _                         -> Nothing
```

Chapter 17

Sound and Signals

In this chapter we study the fundamental nature of sound and its basic mathematical representation as a signal. We also discuss discrete digital representations of a signal, which form the basis of modern sound synthesis and audio processing.

17.1 The Nature of Sound

Before studying digital audio, it's important that we first know what *sound* is. In essence, sound is the rapid compression and relaxation of air, which travels as a *wave* through the air from the physical source of the sound to, ultimately, our ears. The physical source of the sound could be the vibration of our vocal chords (resulting in speech or singing), the vibration of a speaker cone, the vibration of a car engine, the vibration of a string in a piano or violin, the vibration of the reed in a saxophone or of the lips when playing a trumpet, or even the (brief and chaotic) vibrations that result when our hands come together as we clap. The "compression and relaxation" of the air (or of a coiled spring) is called a *longitudinal* wave, in which the vibrations occur parallel to the direction of travel of the wave. In contrast, a rope that is fixed at one end and being shaken at the other, and a wave in the ocean, are examples of a *transverse* wave, in which the rope's and water's movement is perpendicular to the direction the wave is traveling.

[Note: There are some great animations of these two kinds of waves at: `http://www.computermusicresource.com/what.is.sound.html`.]

If the rate and amplitude of the sound are within a suitable range, we

can *hear* the sound—i.e. it is *audible sound*. "Hearing" results when the vibrating air waves cause our ear drum to vibrate, in turn stimulating nerves that enter our brain. Sound above our hearing range (i.e. vibration that is too quick to induce any nerve impulses) is called *ultrasonic sound*, and sound below our hearing range is said to be *infrasonic*.

Staying within the analog world, sound can also be turned into an *electrical* signal using a *microphone* (or "mic" for short). Several common kinds of microphones are:

1. Carbon microphone. Based on the resistance of a pocket of carbon particles that are compressed and relaxed by the sound waves hitting a diaphram.

2. Condenser microphone. Based on the capacitance between two diaphrams, one being vibrated by the sound.

3. Dynamic microphone. Based on the inductance of a coil of wire suspended in a magnetic field (the inverse of a speaker).

4. Piezoelectric microphone. Based on the property of certain crystals to induce current when they are bent.

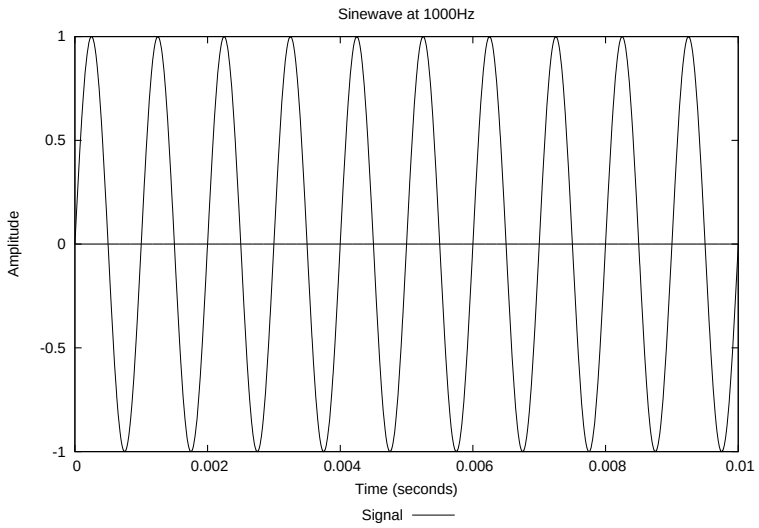


Figure 17.1: A Sine Wave

Perhaps the most common and natural way to represent a wave diagrammatically, whether it be a sound wave or electrical wave, longitudinal

or transverse, is as a *graph* of its amplitude vs. time. For example, Figure 17.1 shows a *sinusiodal wave* of 1000 cycles per second, with an amplitude that varies beween +1 and -1. A sinusoidal wave follows precisely the definition of the mathematical sine function, but also relates strongly, as we shall soon see, to the vibration of sound produced by most musical instruments. In the remainder of this text, we will refer to a sinusoidal wave simply as a sine wave.

Acoustics is the study of the properties, in particular the propagation and reflection, of sound. *Psychoacoustics* is the study of the mind's interpretation of sound, which is not always as tidy as the physical properties that are manifest in acoustics. Obviously both of these are important areas of study for music in general, and therefore play an important role in generating or simulating music with a computer.

The speed of sound can vary considerably, depending on the material, the temperature, the humidity, and so on. For example, in dry air at room temperature (68 degrees Farenheit), sound travels at a rate of 1,125 feet (343 meters) per second, or 768 miles (1,236 kilometers) per hour. Perhaps surprisingly, the speed of sound varies little with respect to air pressure, although it does vary with temperature.

The reflection and absorbtion of sound is a much more difficult topic, since it depends so much on the material, the shape and thickness of the material, and the frequency of the sound. Modeling well the acoustics of a concert hall, for example, is quite challenging. To understand how much such reflections can affect the overall sound that we hear, consider a concert hall that is 200 feet long and 100 feet wide. Based on the speed of sound given above, it will take a sound wave $2 \times 200 / 1125 = 0.355$ seconds to travel from the front of the room to the back of the room and back to the front again. That $1/3$ of a second, if loud enough, would result in a significant distortion of the music, and corresponds to about one beat with a metronome set at 168.

With respect to our interpretation of music, sound has (at least) three key properties:

1. *Frequency* (perceived as *pitch*).
2. *Amplitude* (perceived as *loudness*).
3. *Spectrum* (perceived as *timbre*).

We discuss each of these in the sections that follow.

17.1.1 Frequency and Period

The *frequency* f is simply the rate of the vibrations (or repetitions, or cycles) of the sound, and is the inverse of the *period* (or duration, or wavelength) p of each of the vibrations:

$$f = \frac{1}{p}$$

Frequency is measured in *Hertz* (abbreviated Hz), where 1 Hz is defined as one cycle per second. For example, the sound wave in Figure 17.1 has a frequency of 1000 Hz (i.e. 1 kHz) and a period of $1/1000$ second (i.e. 1 ms).

In trigonometry, functions like sine and cosine are typically applied to angles that range from 0 to 360 degrees. In audio processing (and signal processing in general) angles are instead usually measured in *radians*, where 2π radians is equal to 360°. Since the sine function has a period of 2π and a frequency of $1/2\pi$, it repeats itself every 2π radians:

$$\sin(2\pi k + \theta) = \sin\theta$$

for any integer k.

But for our purposes it is better to parameterize these functions over frequency as follows. Since $\sin(2\pi t)$ covers one full cycle in one second, i.e. has a frequency of 1 Hz, it makes sense that $\sin(2\pi f t)$ covers f cycles in one second, i.e. has a frequency of f. Indeed, in signal processing the quantity ω is defined as:

$$\omega = 2\pi f$$

That is, a pure sine wave as a function of time behaves as $\sin(\omega t)$.

Finally, it is convenient to add a *phase* (or *phase angle*) to our formula, which effectively shifts the sine wave in time. The phase is usually represented by ϕ. Adding a multiplicative factor A for amplitude (see next section), we arrive at our final formula for a sine wave as a function of time:

$$s(t) = A\sin(\omega t + \phi)$$

A negative value for ϕ has the effect of "delaying" the sine wave, whereas a positive value has the effect of "starting early." Note also that this equation holds for negative values of t.

All of the above can be related to cosine by recalling the following identity:

$$\sin(\omega t + \frac{\pi}{2}) = \cos(\omega t)$$

More generally:

$$A\sin(\omega t + \phi) = a\cos(\omega t) + b\sin(\omega t)$$

Given a and b we can solve for A and ϕ:

$$\begin{aligned} A &= \sqrt{a^2 + b^2} \\ \phi &= \tan^{-1}\frac{b}{a} \end{aligned}$$

Given A and ϕ we can also solve for a and b:

$$\begin{aligned} a &= A\cos(\phi) \\ b &= A\sin(\phi) \end{aligned}$$

17.1.2 Amplitude and Loudness

Amplitude can be measured in several ways. The *peak amplitude* of a signal is its maximum deviation from zero; for example our sine wave in Figure 17.1 has a peak amplitude of 1. But different signals having the same peak amplitude have more or less "energy," depending on their "shape." For example, Figure 17.2 shows four kinds of signals: a sine wave, a square wave, a sawtooth wave, and a triangular wave (whose names are suitably descriptive). Each of them has a peak amplitude of 1. But, intuitively, one would expect the square wave, for example, to have more "energy," or "power," than a sine wave, because it is "fatter." In fact, it's value is everywhere either +1 or -1.

To measure this characteristic of a signal, scientists and engineers often refer to the *root-mean-square* amplitude, or RMS. Mathematically, the root-mean-square is the square root of the mean of the squared values of a given quantity. If x is a discrete quantity given by the values $x_1, x_2, ..., x_n$, the formula for RMS is:

$$x_{\text{RMS}} = \sqrt{\frac{x_1^2 + x_2^2 + ... + x_n^2}{n}}$$

And if f is continuous function, its RMS value over the interval $T_1 \leqslant t \leqslant T_2$ is given by:

$$\sqrt{\frac{1}{T_2 - T_1}\int_{-T_1}^{T_2} f(t)^2 dt}$$

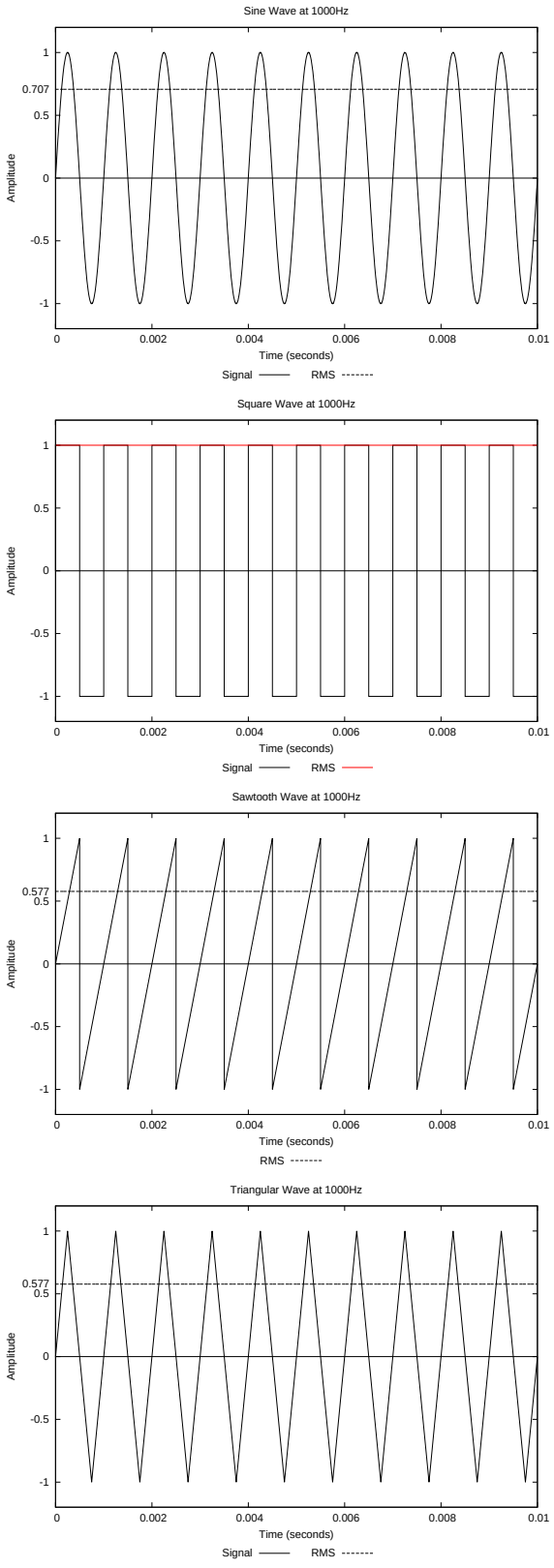


Figure 17.2: RMS Amplitude for Different Signals

For a sine wave, it can be shown that the RMS value is approximately 0.707 of the peak value. For a square wave, it is 1.0. And for both a sawtooth wave and a triangular wave, it is approximately 0.577. Figure 17.2 shows these RMS values superimposed on each of the four signals.

Another way to measure amplitude is to use a relative logarithmic scale that more aptly reflects how we hear sound. This is usually done by measuring the sound level (usually in RMS) with respect to some reference level. The number of *decibels* (dB) of sound is given by:

$$S_{dB} = 10 \log_{10} \frac{S}{R}$$

where S is the RMS sound level, and R is the RMS reference level. The accepted reference level for the human ear is 10^{-12} watts per square meter, which is roughly the threshold of hearing.

A related concept is the measure of how much useful information is in a signal relative to the "noise." The *signal-to-noise ratio*, or *SNR*, is defined as the ratio of the *power* of each of these signals, which is the square of the RMS value:

$$SNR = \left(\frac{S}{N}\right)^2$$

where S and N are the RMS values of the signal and noise, respectively. As is often the case, it is better to express this on a logarithmic scale, as follows:

$$\begin{aligned} SNR_{dB} &= 10 \log_{10} \left(\frac{S}{N}\right)^2 \\ &= 20 \log_{10} \frac{S}{N} \end{aligned}$$

The *dynamic range* of a system is the difference between the smallest and largest values that it can process. Because this range is often very large, it is usually measured in decibels, which is a logarithmic quantity. The ear, for example, has a truly remarkable dynamic range—about 130 dB. To get some feel for this, silence should be considered 0 dB, a whisper 30 dB, normal conversation about 60 dB, loud music 80 dB, a subway train 90 dB, and a jet plane taking off or a very loud rock concert 120 dB or higher.

Note that if you double the sound level, the decibels increase by about 3 dB, whereas a million-fold increase corresponds to 60 dB:

$$\begin{aligned} 10 \log_{10} 2 &= 10 \times 0.301029996 \cong 3 \\ 10 \log_{10} 10^6 &= 10 \times 6 = 60 \end{aligned}$$

So the ear is truly adaptive! (The eye also has a large dynamic range with respect to light intensity, but not quite as much as the ear, and its response time is much slower.)

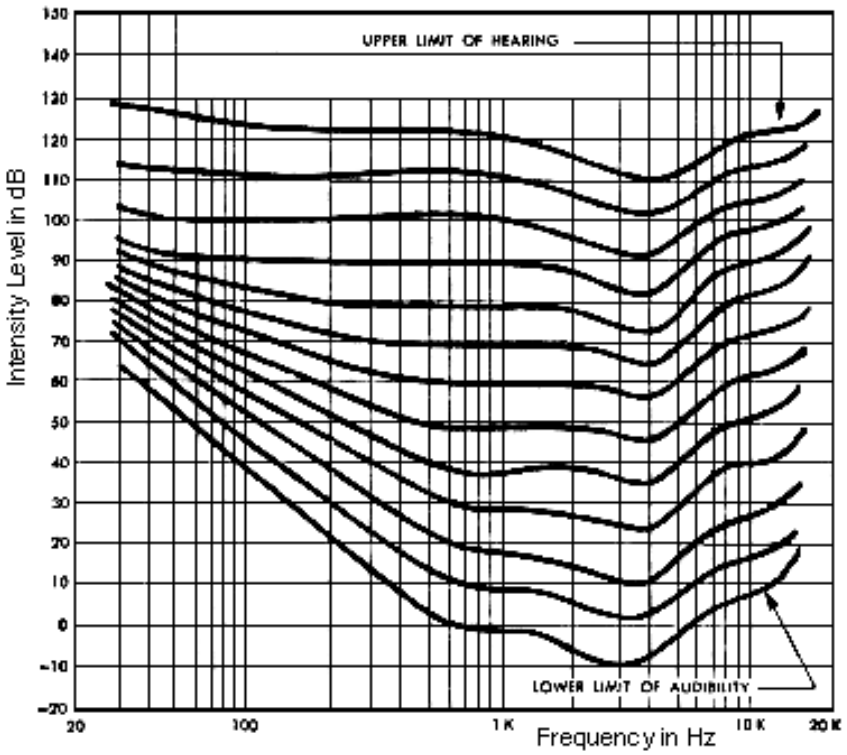


Figure 17.3: Fletcher-Munson Equal Loudness Contour

Loudness is the perceived measure of amplitude, or volume, of sound, and is thus subjective. It is most closely aligned with RMS amplitude, with one important exception: loudness depends somewhat on frequency! Of course that's obvious for really high and really low frequencies (since at some point we can't hear them at all), but in between things aren't constant either. Furthermore, no two humans are the same. Figure 17.3 shows the *Fletcher-Munson Equal-Loudness Contour*, which reflects the perceived equality of sound intensity by the average human ear with respect to frequency. Note from this figure that:

- The human ear is less sensitive to low frequencies.
- The maximum sensitivity is around 3-4 kHz, which roughly corre-

sponds to the resonance of the auditory canal.

Another important psychoacoustical property is captured in the *Weber-Fechner Law*, which states that the *just noticeable difference* (jnd) in a quantity—i.e. the minimal change necessary for humans to notice something in a cognitive sense—is a relative constant, independent of the absolute level. That is, the ratio of the change to the absolute measure of that quantity is constant:

$$\frac{\Delta q}{q} = k$$

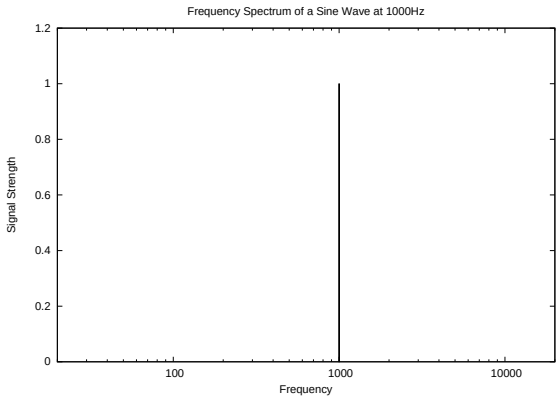
The jnd for loudness happens to be about 1 db, which is another reason why the decibel scale is so convenient. 1 db corresponds to a sound level ratio of 1.25892541. So, in order for a person to "just notice" an increase in loudness, one has to increase the sound level by about 25%. If that seems high to you, it's because your ear is so adaptive that you are not even aware of it.

17.1.3 Frequency Spectrum

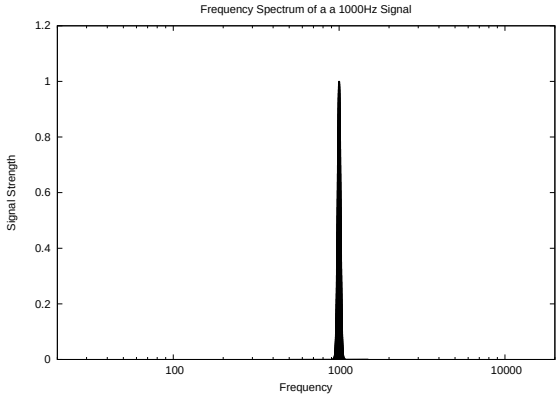
Humans can hear sound approximately in the range 20 Hz to 20,000 Hz = 20 kHz. This is a dynamic range in frequency of a factor of 1000, or 30 dB. Different people can hear different degrees of this range (I can hear very low tones well, but not very high ones). On a piano, the fundamental frequency of the lowest note is 27.5 Hz, middle (concert) A is 440 hz, and the top-most note is about 4 kHz. Later we will learn that these notes also contain *overtones*—multiples of the fundamental frequency—that contribute to the *timbre*, or sound quality, that distinguishes one instrument from another. (Overtones are also called *harmonics* or *partials*.)

The *phase*, or time delay, of a signal is important too, and comes into play when we start mixing signals together, which can happen naturally, deliberately, from reverberations (room acoustics), and so on. Recall that a pure sine wave can be expressed as $\sin(\omega t + \phi)$, where ϕ is the *phase angle*. Manipulating the phase angle is common in additive synthesis and amplitude modulation, topics to be covered in later chapters.

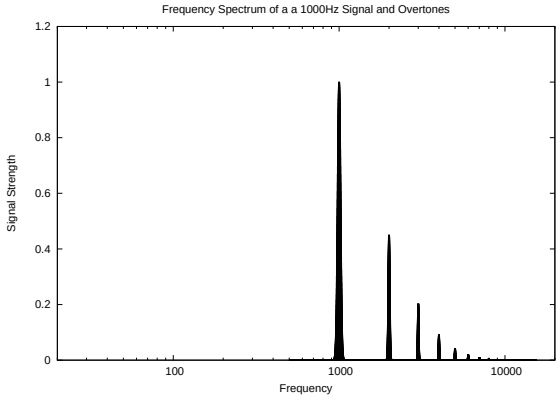
A key point is that most sounds do not consist of a single, pure sine wave—rather, they are a combination of many frequencies, and at varying



(a) Spectral plot of pure sine wave



(b) Spectral plot of a noisy sine wave



(c) Spectral plot of a musical tone

Figure 17.4: Spectral Plots of Different Signals

phases relative to one another. Thus it is helpful to talk of a signal's *frequency spectrum*, or spectral content. If we have a regular repetitive sound (called a *periodic signal*) we can plot its spectral content instead of its time-varying graph. For a pure sine wave, this looks like an impulse function, as shown in Figure 17.4a.

But for a richer sound, it gets more complicated. First, the distribution of the energy is not typically a pure impulse, meaning that the signal might vary slightly above and below a particular frequency, and thus its frequency spectrum typically looks more like Figure 17.4b.

In addition, a typical sound has many different frequencies associated with it, not just one. Even for an instrument playing a single note, this will include not just the perceived pitch, which is called the *fundamental frequency*, but also many *overtones* (or harmonics) which are multiples of the fundamental, as shown in Figure 17.4c. The *natural harmonic series* is one that is approximated often in nature, and has a harmonically decaying series of overtones.

What's more, the articulation of a note by a performer on an instrument causes these overtones to vary in relative size over time. There are several ways to visualize this graphically, and Figure 17.5 shows two of them. In 17.5a, shading is used to show the varying amplitude over time. And in 17.5b, a 3D projection is used.

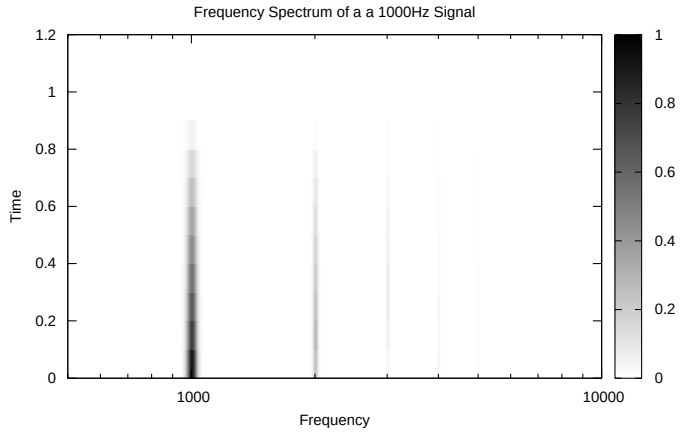
The precise blend of the overtones, their phases, and how they vary over time, is primarily what distinguishes a particular note, say concert A, on a piano from the same note on a guitar, a violin, a saxophone, and so on. We will have much more to say about these issues in later chapters.

[See pictures at:
`http://www.computermusicresource.com/spectrum.html`.]

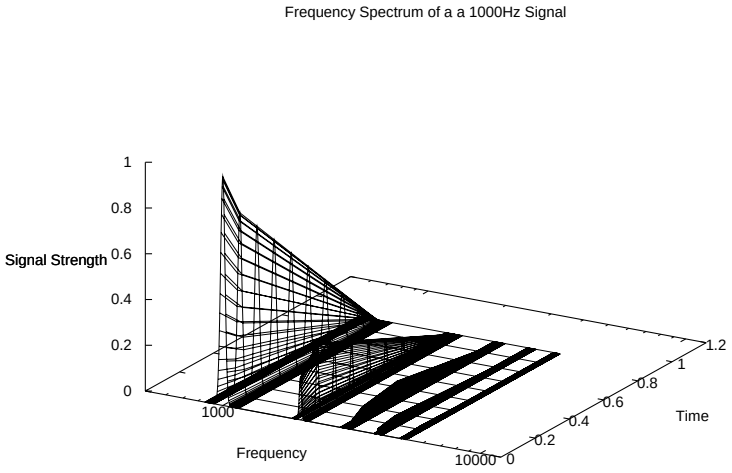
17.2 Digital Audio

The preceding discussion has assumed that sound is a continuous quantity, which of course it is, and thus we represent it using continuous mathematical functions. If we were using an analog computer, we could continue with this representation, and create electronic music accordingly. Indeed, the earliest electronic synthesizers, such as the *Moog synthesizer* of the 1960's, were completely analog.

However, most computers today are *digital*, which require representing



(a) Using shading



(b) Using 3D projection

Figure 17.5: Time-Varying Spectral Plots

sound (or signals in general) using digital values. The simplest way to do this is to represent a continuous signal as a *sequence of discrete samples* of the signal of interest. An *analog-to-digital converter*, or ADC, is a device that converts an instantaneous sample of a continuous signal into a binary value. The microphone input on a computer, for example, connects to an ADC.

Normally the discrete samples are taken at a fixed *sampling rate*. Choosing a proper sampling rate is quite important. If it is too low, we will not acquire sufficient samples to adequately represent the signal of interest. And if the rate is too high, it may be an overkill, thus wasting precious computing resources (in both time and memory consumption). Intuitively, it seems that the highest frequency signal that we could represent using a sampling rate r would have a frequency of $r/2$, in which case the result would have the appearance of a square wave, as shown in Figure 17.6a. Indeed, it is easy to see that problems could arise if we sampled at a rate significantly lower than the frequency of the signal, as shown in Figures 17.6b and 17.6c for sampling rates equal to, and one-half, of the frequency of the signal of interest—in both cases the result is a sampled signal of 0 Hz!

Indeed, this observation is captured in what is known as the *Nyquist-Shannon Sampling Theorm* that, stated informally, says that the accurate reproduction of an analog signal (no matter how complicated) requires a sampling rate that is at least twice the highest frequency of the signal of interest.

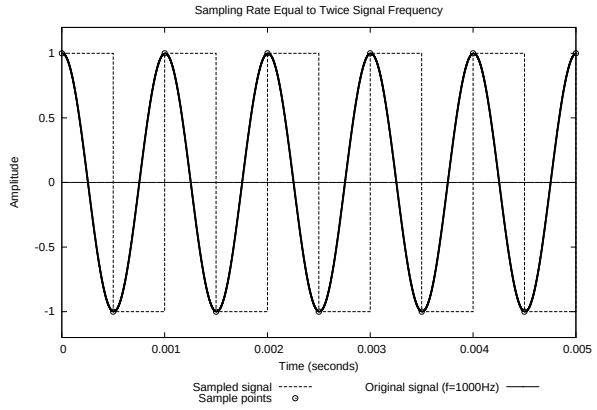
For example, for audio signals, if the highest frequency humans can hear is 20 kHz, then we need to sample at a rate of at least 40 kHz for a faithful reproduction of sound. In fact, CD's are recorded at 44.1 kHz. But many people feel that this rate is too low, as some people can hear beyond 20 kHz. Another recording studio standard is 48 kHz. Interestingly, a good analog tape recorder from generations ago was able to record signals with frequency content even higher than this—perhaps digital is not always better!

17.2.1 From Continuous to Discrete

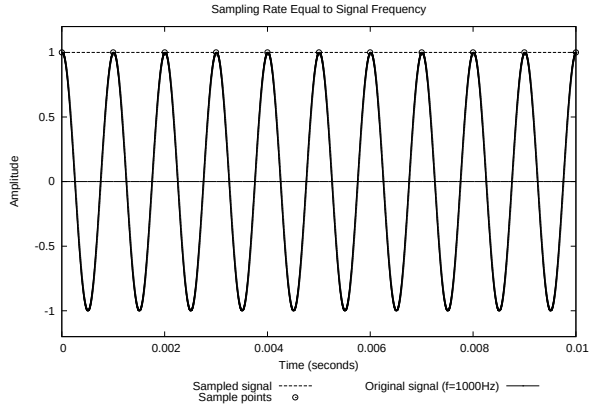
Recall the definition of a sine wave from Section 17.1.1:

$$s(t) = A\sin(\omega t + \phi)$$

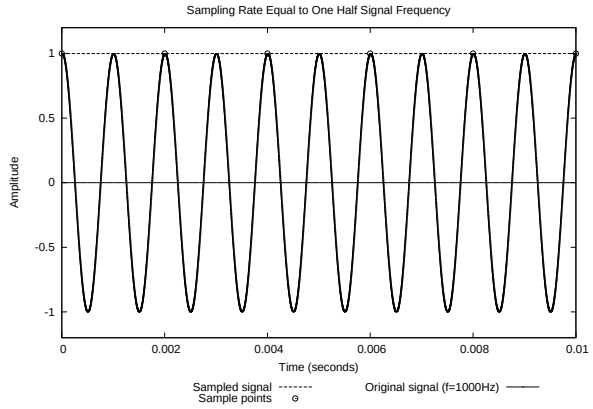
We can easily and intuitively convert this to the discrete domain by replacing the time t with the quantity n/r, where n is the integer index into the



(a)



(b)



(c)

Figure 17.6: Choice of Sampling Rate

sequence of discrete samples, and r is the sampling rate discussed above. If we use $s[n]$ to denote the $(n+1)^{\text{th}}$ sample of the signal, we have:

$$s[n] = A\sin\left(\frac{\omega n}{r} + \phi\right), \qquad n = 0, 1, ..., \infty$$

Thus $s[n]$ corresponds to the signal's value at time n/r.

17.2.2 Fixed-Waveform Table-Lookup Synthesis

One of the most fundamental questions in digital audio is how to generate a sine wave as efficiently as possible, or, in general, how to generate a fixed periodic signal of any form (sine wave, square wave, sawtooth wave, even a sampled sound bite). A common and efficient way to generate a periodic signal is through *fixed-waveform table-lookup synthesis*. The idea is very simple: store in a table the samples of a desired periodic signal, and then index through the table at a suitable rate to reproduce that signal at some desired frequency. The table is often called a *wavetable*.

In general, if we let:

$$\begin{aligned} L &= \text{table length} \\ f &= \text{resulting frequency} \\ i &= \text{indexing increment} \\ r &= \text{sample rate} \end{aligned}$$

then we have:

$$f = \frac{ir}{L}$$

For example, suppose the table contains 8196 samples. If the sample rate is 44.1 kHz, how do we generate a tone of, say, 440 Hz? Plugging in the numbers and solving the above equation for i, we get:

$$\begin{aligned} 440 &= \frac{i \times 44.1\text{kHz}}{8196} \\ i &= \frac{440 \times 8196}{44.1\text{kHz}} \\ &= 81.77 \end{aligned}$$

So, if we were to sample approximately every 81.77^{th} value in the table, we would generate a signal of 440 Hz.

Now suppose the table T is a vector, and $T[n]$ is the nth element. Let's call the exact index increment i into a continuous signal the *phase*, and the actual index into the corresponding table the *phase index* p. The computation of successive values of the phase index and output signal s is then captured by these equations:

$$\begin{aligned} p_o &= \lfloor \phi_0 + 0.5 \rfloor \\ p_{n+1} &= (p_n + i) \bmod L \\ s_n &= T[\ \lfloor p_n + 0.5 \rfloor\ ] \end{aligned}$$

$\lfloor a+0.5 \rfloor$ denotes the floor of $a+0.5$, which effectively rounds a to the nearest integer. ϕ_0 is the initial phase angle (recall earlier discussion), so p_0 is the initial index into the table that specifies where the fixed waveform should begin.

Instead of rounding the index, one could do better by *interpolating* between values in the table, at the expense of efficiency. In practice, rounding the index is often good enough. Another way to increase accuracy is to simply increase the size of the table.

17.2.3 Aliasing

Earlier we saw examples of problems that can arise if the sampling rate is not high enough. We saw that if we sample a sine wave at twice its frequency, we can suitably capture that frequency. If we sample at exactly its frequency, we get 0 Hz. But what happens in between? Consider a sampling rate ever-so-slightly higher or lower than the sine wave's fundamental frequency--in both cases, this will result in a frequency much lower than the original signal, as shown in Figures 17.7 and 17.8. This is analogous to the effect of seeing spinning objects under fluorescent or LED light, or old motion pictures of the spokes in the wheels of horse-drawn carriages.

These figures suggest the following. Suppose that m is one-half the sampling rate. Then:

Original signal	Reproduced signal
$0 - m$	$0 - m$
$m - 2m$	$m - 0$
$2m - 3m$	$0 - m$
$3m - 4m$	$m - 0$
$\ldots$	$\ldots$

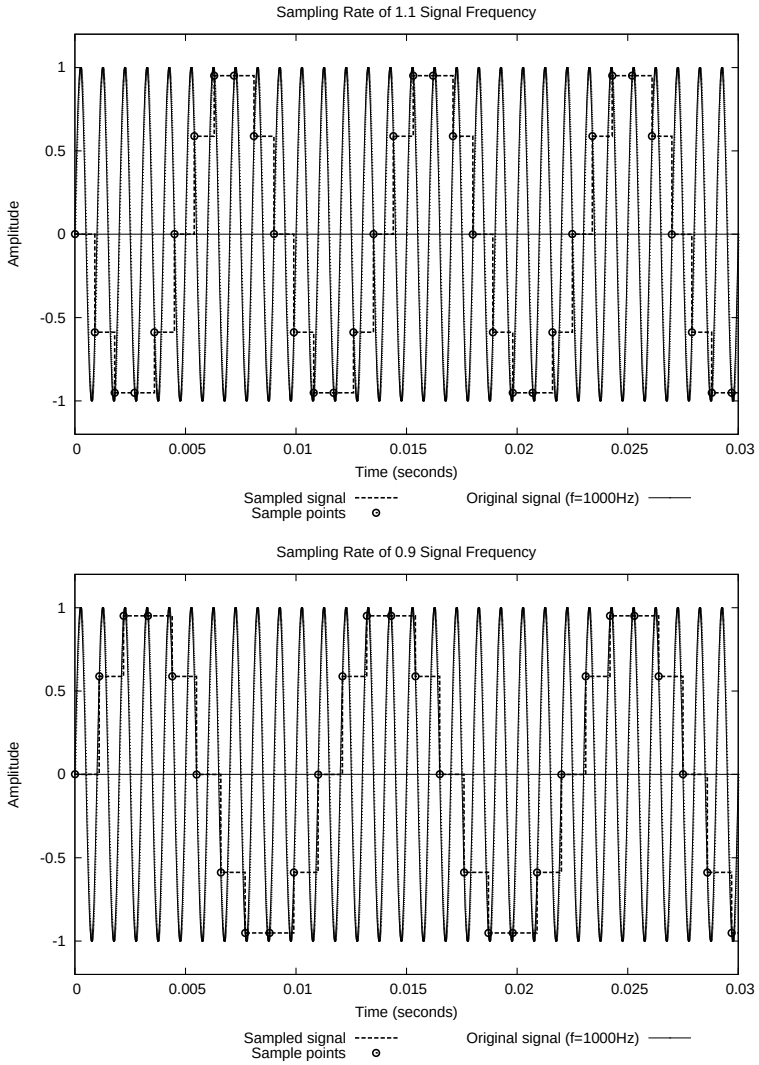


Figure 17.7: Aliasing 1

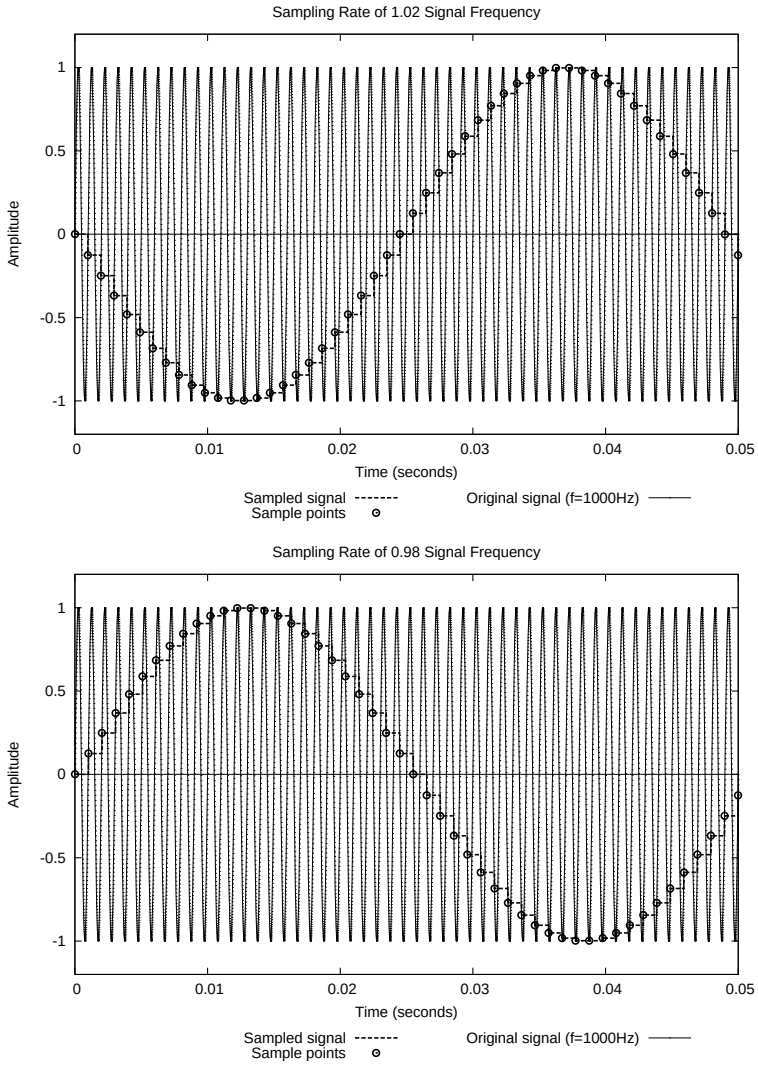


Figure 17.8: Aliasing 2

This phenomenon is called *aliasing*, or *foldover* of the signal onto itself.

This is not good! In particular, it means that audio signals in the ultrasonic range will get "folded" into the audible range. To solve this problem, we can add an analog *low-pass filter* in front of the ADC–usually called an *anti-aliasing* filter—to eliminate all but the audible sound before it is digitized. In practice, however, this can be tricky. For example, a steep analog filter introduces *phase distortion* (i.e. frequency-dependent time delays), and early digital recordings were notorious in the "harsh sound" that resulted. This can be fixed by using a filter with less steepness (but resulting in more aliasing), or using a time correlation filter to compensate, or using a technique called *oversampling*, which is beyond the scope of this text.

A similar problem occurs at the other end of the digital audio process—i.e. when we reconstruct an analog signal from a digital signal using a *digital-to-analog converter*, or DAC. The digital representation of a signal can be viewed mathematically as a stepwise approximation to the real signal, as shown in Figure 17.9, where the sampling rate is ten times the frequency of interest. As discussed earlier, at the highest frequency (i.e. at one-half the sampling rate), we get a square wave. As we will see in Chapter 19, a square wave can be represented mathematically as the sum of an infinite sequence of sine waves, consisting of the fundamental frequency and all of its odd harmonics. These harmonics can enter the ultrasonic region, causing potential havoc in the analog circuitry, or in a dog's ear (dogs can hear frequencies much higher than humans). The solution is to add yet another low-pass filter, called an *anti-imaging* or *smoothing* filter to the output of the DAC. In effect, this filter "connects the dots," or interpolates, between successive values of the stepwise approximation.

In any case, a basic block diagram of a typical digital audio system—from sound input to sound output—is shown in Figure 17.10.

17.2.4 Quantization Error

In terms of amplitude, remember that we are using digital numbers to represent an analog signal. For conventional CD's, 16 bits of precision are used. If we were to compute and then "listen to" the round-off errors that are induced, we would hear subtle imperfections, called *quantization error*, or more commonly, "noise."

One might compare this to "hiss" on a tape recorder (which is due to the molecular disarray of the magnetic recording medium), but there are

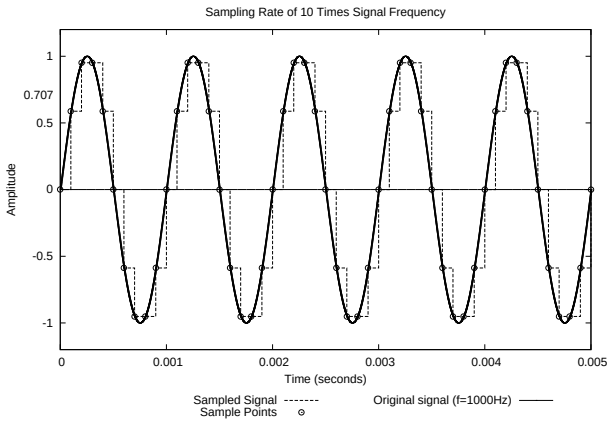


Figure 17.9: A Properly Sampled Signal

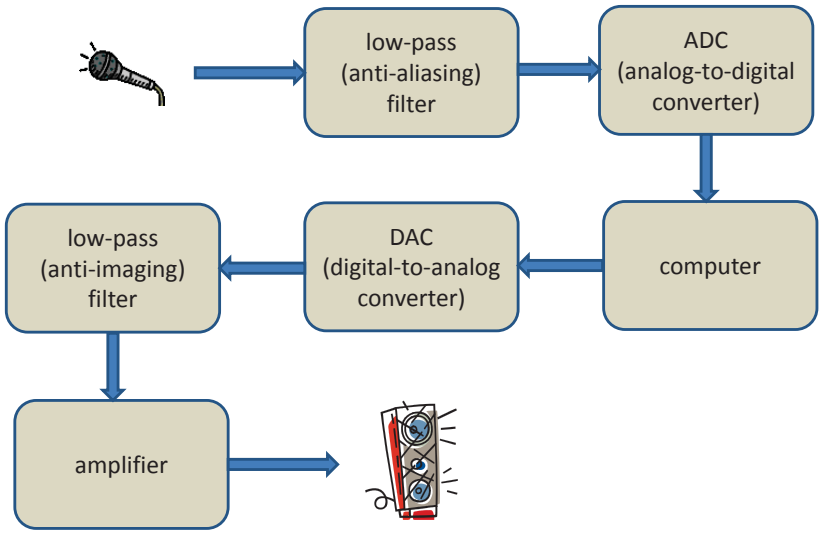


Figure 17.10: Block Diagram of Typical Digital Audio System

important differences. First of all, when there is no sound, there is no quantization error in a digital signal, but there is still hiss on a tape. Also, when the signal is very low and regular, the quantization error becomes somewhat regular as well, and is thus audible as something different from hiss. Indeed, it's only when the signal is loud and complex that quantization error compares favorably to tape hiss.

One solution to the problem of low signal levels mentioned above is to purposely introduce noise into the system to make the signal less predictable. This fortuitous use of noise deserves a better name, and indeed it is called *dither*.

17.2.5 Dynamic Range

What is the dynamic range of an n-bit digital audio system? If we think of quantization error as noise, it makes sense to use the equation for SNR_{dB} given in Section 17.1.2:

$$SNR_{dB} = 20 \log_{10} \frac{S}{N}$$

But what should N be, i.e. the quantization error? Given a signal amplitude range of $\pm a$, with n bits of resolution it is divided into $2a/2^n$ points. Therefore the dynamic range is:

$$\begin{aligned} 20 \log_{10} \left(\frac{2a}{2a/2^n} \right) &= 20 \times \log_{10}(2^n) \\ &= 20 \times n \times \log_{10}(2) \\ &\approx 20 \times n \times (0.3) \\ &= 6n \end{aligned}$$

For example, a 16-bit digital audio system results in a dynamic range of 96 dB, which is pretty good, although a 20-bit system yields 120 dB, corresponding to the dynamic range of the human ear.

Exercise 17.1 For each of the following, say whether it is a longitudinal wave or a transverse wave:

- A vibrating violin string.
- Stop-and-go traffic on a highway.

- "The wave" in a crowd at a stadium.
- "Water hammer" in the plumbing of your house.
- The wave caused by a stone falling in a pond.
- A radio wave.

Exercise 17.2 You see a lightning strike, and 5 seconds later you hear the thunder. How far away is the lightning?

Exercise 17.3 You clap your hands in a canyon, and 2 seconds later you hear an echo. How far away is the canyon wall?

Exercise 17.4 By what factor must one increase the RMS level of a signal to yield a 10 dB increase in sound level?

Exercise 17.5 A dog can hear in the range 60-45,000 Hz, and a bat 2,000-110,000 Hz. In terms of the frequency response, what are the corresponding dynamic ranges for these two animals, and how do they compare to that of humans?

Exercise 17.6 What is the maximum number of audible overtones in a note whose fundamental frequency is 100 Hz? 500 Hz? 1500 Hz? 5 kHz?

Exercise 17.7 Consider a continuous input signal whose frequency is f. Devise a formula for the frequency r of the reproduced signal given a sample rate s.

Exercise 17.8 How much memory is needed to record 3 minutes of stereo sound using 16-bit samples taken at a rate of 44.1 kHz?

Exercise 17.9 If we want the best possible sound, how large should the table be using fixed-waveform table-lookup synthesis, in order to cover the audible frequency range?

Exercise 17.10 The Doppler effect occurs when a sound source is in motion. For example, as a police car moves toward you its siren sounds higher than it really is, and as it goes past you, it gets lower. How fast would a police car have to go to change a siren whose frequency is the same as concert A, to a pitch an octave higher? (i.e. twice the frequency) At that speed, what frequency would we hear after the police car passes us?

Chapter 18

Euterpea's Signal Functions

```
{-# LANGUAGE Arrows #-}
module Euterpea.Music.Signal.SigFuns where
import Euterpea
import Control.Arrow ((>>>), (<<<), arr)
```

Details: The first line in the module header above is a *compiler pragma*, amd in this case is telling GHC to accept *arrow syntax*, which will be explained in Section 18.1.

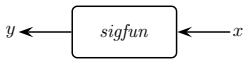
In this chapter we show how the theoretical concepts involving sound and signals studied in the last chapter are manifested in Euterpea. The techniques learned will lay the groundwork for doing two broad kinds of activities: *sound synthesis* and *audio processing*. Sound synthesis might include creating the sound of a footstep on dry leaves, simulating a conventional musical instrument, creating an entirely new instrument sound, or composing a single "soundscape" that stands alone as a musical composition. Audio processing includes such things as equalization, filtering, reverb, special effects, and so on. In future chapters we will study various techniques for achieving these goals.

18.1 Signals and Signal Functions

As we saw in Chapter 16, it would seem natural to represent a signal as an abstract type, say *Signal T* in Haskell, and then define functions to add, multiply, take the sine of, and so on, signals represented in this way. For example, *Signal Float* would be the type of a time-varying floating-point number, *Signal AbsPitch* would be the type of a time-varing absolute pitch, and so on. Then given $s_1, s_2 ::$ *Signal Float* we might simply write $s_1 + s_2$, $s_1 * s_2$, and *sin* s_1 as examples of applying the above operations. Haskell's numeric type class hierarchy would make this particularly easy to do. Indeed, several domain-specific languages based on this approach have been defined before, beginning with the language *Fran* [EH97] that was designed for writing computer animation programs.

But years of experience and theoretical study have revealed that such an approach leads to a language with subtle time- and space-leaks,[1] for reasons that are beyond the scope of this textbook [LH07]. Therefore Euterpea takes a somewhat different approach, as described below.

Perhaps the simplest way to understand Euterpea's approach to programming with signals is to think of it as a language for expressing *signal processing diagrams* (or equivalently, electrical circuits). We refer to the lines in a typical signal processing diagram as *signals*, and the boxes that convert one signal into another as *signal functions*. For example, this very simple diagram has two signals, x and y, and one signal function, *sigfun*:



Using Haskell's *arrow syntax* [Hug00, Pat01], this diagram can be expressed as a code fragment in Euterpea simply as:

$$y \leftarrow sigfun \prec\!\!- x$$

Details: The syntax $\leftarrow$ and $\prec\!\!-$ is typeset here in an attractive way, but the user will have to type `<-` and `-<`, respectively, in her source file.

[1] A time-leak in a real-time system occurs whenever a time-dependent computation falls behind the current time because its value or effect is not needed yet, but then requires "catching up" at a later point in time. This catching up process can take an arbitrarily long time, and may consume additional space as well. It can destroy any hope for real-time behavior if not managed properly.

Arrows and arrow syntax will be described in much more detail in Chapter **??**. For now, keep in mind that $\leftarrow$ and $\multimap$ are part of the *syntax*, and are not simply binary operators. Indeed, we can't just write the above code fragment anywhere. It has to be within an enclosing *proc* construct whose result type is that of a signal function. The *proc* construct begins with the keyword *proc* along with an argument, analogous to an anonymous function. For example, a signal function that takes a signal of type *Double* and adds 1 to every signal sample, and then applies *sigfun* to the resulting signal, can be written:

$$
\begin{array}{l}
\mathit{proc}\ y \rightarrow \mathbf{do} \\
\quad x \leftarrow \mathit{sigfun} \multimap y + 1 \\
\quad \mathit{outA} \multimap x
\end{array}
$$

Details: The **do** keyword in arrow syntax introduces layout, just as it does in monad syntax.

Note the analogy of this code to the following snippet involving an ordinary anonymous function:

$$
\begin{array}{l}
\lambda y \rightarrow \\
\quad \mathbf{let}\ x = \mathit{sigfun}\ (y + 1) \\
\quad \mathbf{in}\ x
\end{array}
$$

The important difference, however, is that *sigfun* works on a signal, which we can think of as a stream of values, whose representative values at the "point" level are the variables x and y above. So in reality we would have to write something like this:

$$
\begin{array}{l}
\lambda \mathit{ys} \rightarrow \\
\quad \mathbf{let}\ \mathit{xs} = \mathit{sigfun}\ (\mathit{map}\ (+1)\ \mathit{ys}) \\
\quad \mathbf{in}\ \mathit{xs}
\end{array}
$$

to achieve the effect of the arrow code above. The arrow syntax allows us to avoid worrying about the streams themselves. It also has other important advantages that are beyond the scope of the current discussion.

Arrow syntax is just that--syntactic sugar that is expanded into a set of conventional functions that work just as well, but are more cumbersome to program with (just as with monad syntax). This syntactic expansion will be described in more detail in Chapter **??**. To use the arrow syntax within a ∘*lhs* file, one must declare a compiler flag in GHC at the very beginning

of the file, as follows:

```
{-# LANGUAGE Arrows #-}
```

18.1.1 The Type of a Signal Function

Polymorphically speaking, a signal function has type:

```
Clock c ⇒ SigFun c a b
```

which should be read, "for some clock type (i.e. sampling rate) c, this is the type of signal functions that convert signals of type a into signals of type b."

The type variable c indicates what clock rate is being used, and for our purposes will always be one of two types: *AudRate* or *CtrRate* (for *audio rate* and *control rate*, respectively). Being able to express the sampling rate of a signal function is what we call *clock polymorphism*. Although we like to think of signals as continuous, time-varying quantities, in practice we know that they are sampled representations of continous quantities, as discussed in the last chapter. However, some signals need to be sampled at a very high rate—say, an audio signal—whereas other signals need not be sampled at such a high rate—say, a signal representing the setting of a slider. The problem is, we often want to mix signals sampled at different rates; for example, the slider might control the volume of the audio signal.

One solution to this problem would be to simply sample everything at the very highest rate, but this is computationally inefficient. A better approach is to sample signals at their most appropriate rate, and to perform coercions to "up sample" or "down sample" a signal when it needs to be combined with a signal sampled at a different rate. This is the approach used in Euterpea.

More specifically, the base type of each signal into and out of a signal function must satisfy the type class constraint *Clock c*, where c is a *clock type*. The *Clocked* class is defined as:

```
class Clock c where
  rate :: c → Double
```

The single method *rate* allows the user to extract the sampling rate from the type. In Euterpea, the *AudRate* is pre-defined to be 44.1 kHz, and the *CtrRate* is set at 4.41 kHz. Here are the definitions of *AudRate* and *CtrRate*, along with their instance declarations in the *Clock* class, to achieve this:

```
data AudRate
data CtrRate
```

```
instance Clock AudRate where
  rate _ = 44100
instance Clock CtrRate where
  rate _ = 4410
```

Because these two clock types are so often used, it is helpful to define a couple of type synonyms:

```
type AudSF a b = SigFun AudRate a b
type CtrSF a b = SigFun CtrRate a b
```

From these definitions it should be clear how to define your own clock type.

Details: Note that *AudRate* and *CtrRate* have no constructors—they are called *empty* data types. More precisely, they are each inhabited by exactly one value, namely $\perp$.

The sampling rate can be determined from a given clock type. In this way, a coercion function can be written to change a signal sampled at one rate to a signal sampled at some other rate. In Euterpea, there are two such functions that are pre-defined:

```
coerce, upsample :: (Clock c1, Clock c2) ⇒
                    SigFun c1 a b → SigFun c2 a b
```

The function *coerce* looks up the sampling rates of the input and output signals from the type variables c_1 and c_2. It then either stretches the input stream by duplicating the same element or contracts it by skipping elements. (It is also possible to define a more accurate coercion function that performs interpolation, at the expense of performance.)

For simpler programs, the overhead of calling *coerce* might not be worth the time saved by generating signals with lower resolution. (Haskells fractional number implementation is relatively slow.) The specialized coercion function *upsample* avoids this overhead, but only works properly when the output rate is an integral multiple of the input rate (which is true in the case of *AudRate* and *CtrRate*).

Keep in mind that one does not have to commit a signal function to a particular clock rate—it can be left *polymorphic*. Then that signal function will adapt its sampling rate to whatever is needed in the context in which

it is used.

Also keep in mind that a signal function is an abstract function. You cannot just apply it to an argument like an ordinary function—that is the purpose of the arrow syntax. There are no values that directly represent *signals* in Euterpea—there are only signal *functions*.

The arrow syntax provides a convenient way to compose signal functions together—i.e. to wire together the boxes that make up a signal processing diagram. By not giving the user direct access to signals, and providing a disciplined way to compose signal functions (namely arrow syntax), time- and space-leaks are avoided. In fact, the resulting framwework is highly amenable to optimization, although this requires using special features in Haskell, as described in Chapter **??**.

A signal function whose type is of the form $Clock\ c \Rightarrow SigFun\ c\ ()\ b$ essentially takes no input, but produces some output of type b. Because of this we often refer to such a signal function as a *signal source*.

18.1.2 Four Useful Functions

There are four useful auxiliary functions that will make writing signal functions a bit easier. The first two essentially "lift" constants and functions from the Haskell level to the arrow (signal function) level:

$$\begin{aligned} arr &:: Clock\ c \Rightarrow (a \to b) \to SigFun\ c\ a\ b \\ constA &:: Clock\ c \Rightarrow b \to SigFun\ c\ ()\ b \end{aligned}$$

For example, a signal function that adds one to every sample of its input can be written simply as $arr\ (+1)$, and a signal function that returns the constant 440 as its result can be written $constA\ 440$ (and is a signal source, as defined earlier).

The other two functions allow us to *compose* signal functions:

$$\begin{aligned} (\ggg) &:: Clock\ clk \Rightarrow \\ &\quad SigFun\ clk\ a\ b \to SigFun\ clk\ b\ c \to SigFun\ clk\ a\ c \\ (\lll) &:: Clock\ clk \Rightarrow \\ &\quad SigFun\ clk\ b\ c \to SigFun\ clk\ a\ b \to SigFun\ clk\ a\ c \end{aligned}$$

$(\lll)$ is analogous to Haskell's standard composition operator $(\circ)$, whereas $(\ggg)$ is like "reverse composition."

As an example that combines both of the ideas above, recall the very first example given in this chapter:

$proc\ y \to$ **do**

$$x \leftarrow sigfun \prec y + 1$$
$$outA \prec x$$

which essentially applies *sigfun* to one plus the input. This signal function can be written more succinctly as either $arr\ (+1) \ggg sigfun$ or $sigfun \lll arr\ (+1)$.

The functions $(\ggg)$, $(\lll)$, and *arr* are actually generic operators on arrows, and thus to use them one must import them from the *Arrow* library, as follows:

import *Control.Arrow* $((\ggg), (\lll), arr)$

18.1.3 Some Simple Examples

Let's now work through a few examples that focus on the behavior of signal functions, so that we can get a feel for how they are used in practice. Euterpea has many pre-defined signal functions, including ones for sine waves, numeric computations, transcendental functions, delay lines, filtering, noise generation, integration, and so on. Many of these signal functions are inspired by csound [Ver86], where they are called *unit generators*. Some of them are not signal functions *per se*, but take a few fixed arguments to yield a signal function, and it is important to understand this distinction.

For example, there are several pre-defined functions for generating sine waves and periodic waveforms in Euterpea. Collectively these are called *oscillators*, a name taken from electronic circuit design. They are summarized in Figure 18.1.

The two most common oscillators in Euterpea are:

$$\begin{aligned} osc &:: Clock\ c \Rightarrow \\ &\quad Table \rightarrow Double \rightarrow SigFun\ c\ Double\ Double \\ oscFixed &:: Clock\ c \Rightarrow \\ &\quad Double \rightarrow SigFun\ c\ ()\ Double \end{aligned}$$

osc uses fixed-waveform table-lookup synthesis as described in Section 17.2.2. The first argument is the fixed wavetable; we will see shortly how such a table can be generated. The second argument is the initial phase angle, represented as a fraction between 0 and 1. The resulting signal function then converts a signal representing the desired output frequency to a signal that has that output frequency.

oscFixed uses an efficient recurrence relation to compute a pure sinusoidal wave; the mathematics of this are described in Section **??**. In contrast

osc, *oscI* :: *Clock c* ⇒
Table → *Double* → *SigFun p Double Double*

osc tab ph is a signal function whose input is a frequency, and output is a signal having that frequency. The output is generated using fixed-waveform table-lookup, using the table *tab*, starting with initial offset (phase angle) *ph* expressed as a fraction of a cycle (0 to 1). *oscI* is the same, but uses linear interpolation between points.

oscFixed :: *Clock c* ⇒
Double → *SigFun c* () *Double*

oscFixed freq is a signal source whose sinusoidal output frequency is *freq*. Uses a recurrence relation that requires only one multiply and two add operations for each sample of output.

oscDur, *oscDurI* :: *Clock c* ⇒
Table → *Double* → *Double* → *SigFun* () *Double*

oscDur tab del dur samples just once through the table *tab* at a rate determined by *dur*. For the first *del* seconds, the point of scan will reside at the first location of the table; it will then move through the table at a constant rate, reaching the end in another *dur* seconds; from that time on (i.e. after *del* + *dur* seconds) it will remain pointing at the last location. *oscDurI* is similar but uses linear interpolation between points.

oscPartials :: *Clock c* ⇒
Table → *Double* → *SigFun c* (*Double*, *Int*) *Double*

oscPartials tab ph is a signal function whose pair of inputs determines the frequency (as with *osc*), as well as the number of harmonics of that frequency, of the output. *tab* is the table that is cycled through, and *ph* is the phase angle (as with *osc*).

Figure 18.1: Eutperea's Oscillators

with *osc*, its single argument is the desired output frequency. The resulting signal function is therefore a signal source (i.e. its input type is ()).

[**To do:** Discuss recurrence relations here or perhaps in the last chapter where the fixed-waveform table-lookup method is described.]

The key point here is that the frequency that is output by *osc* is an *input to the signal function*, and therefore can vary with time, whereas the frequency output by *oscFixed* is a *fixed argument*, and cannot vary with time. To see this concretely, let's define a signal source that generates a pure sine wave using *oscFixed* at a fixed frequency, say 440 Hz:

```
s1 :: Clock c => SigFun c () Double
s1 = proc () -> do
  s <- oscFixed 440 -< ()
  outA -< s
```

Since the resulting signal s is directly returned through *outA*, this example can also be written:

```
s1 = proc () -> do
  oscFixed 440 -< ()
```

Alternatively, we could simply write *oscFixed* 440.

To use *osc* instead, we first need to generate a wavetable that represents one full cycle of a sine wave. We can do this using one of Eutperpea's table generating functions, which are summarized in Figure 18.2. For example, using Euterpea's *tableSinesN* function, we can define:

```
tab1 :: Table
tab1 = tableSinesN 4096 [1]
```

This will generate a table of 4096 elements, consisting of one sine wave whose peak amplitude is 1.0. Then we can define the following signal source:

```
s2 :: Clock c => SigFun c () Double
s2 = proc () -> do
  osc tab1 0 -< 440
```

Alternatively, we could use the *const* and compostion operators to write either *constA* 440 >>> *osc* *tab*$_1$ 0 or *osc* *tab*$_2$ 0 <<< *constA* 440. s_1 and s_2 should be compared closely.

Keep in mind that *oscFixed* only generates a sine wave, whereas *osc* generates whatever is stored in the wavetable. Indeed, *tableSinesN* actually creates a table that is the sum of a series of overtones, i.e. multiples of the fundmental frequency (recall the discussion in Section 17.1.3). For example:

type *TableSize* = *Int*
type *PartialNum* = *Double*
type *PartialStrength* = *Double*
type *PhaseOffset* = *Double*
type *StartPt* = *Double*
type *SegLength* = *Double*
type *EndPt* = *Double*

tableLinear, *tableLinearN* ::
TableSize → *StartPt* → [(*SegLength*, *EndPt*)] → *Table*
tableLinear size sp pts is a table of size *size* whose starting point is $(0, sp)$ and that uses straight lines to move from that point to, successively, each of the points in *pts*, which are segment-length/endpoint pairs (segment lengths are projections along the x-axis). *tableLinearN* is a normalized version of the result.

tableExpon, *tableExponN* ::
TableSize → *StartPt* → [(*SegLength*, *EndPt*)] → *Table*
Just like *tableLinear* and *tableLinearN*, respectively, except that exponential curves are used to connect the points.

tableSines3, *tableSines3N* ::
TableSize → [(*PartialNum*, *PartialStrength*, *PhaseOffset*)] → *Table*
tableSines3 size triples is a table of size *size* that represents a sinusoidal wave and an arbitrary number of partials, whose relationship to the fundamental frequency, amplitude, and phase are determined by each of the triples in *triples*. *tableSines3N* is a normalized version of the result.

tableSines, *tableSinesN* ::
TableSize → [*PartialStrength*] → *Table*
Like *tableSines3* and *tableSines3N*, respectively, except that the second argument is an ordered list of the strengths of each partial, starting with the fundamental.

tableBesselN ::
TableSize → *Double* → *Table*
tableBesselN size x is a table representing the log of a modified Bessel function of the second kind, order 0, suitable for use in amplitude-modulated FM. x is the x-interval (0 to x) over which the function is defined.

Figure 18.2: Table Generating Functions

$tab_2 = tableSinesN\ 4096\ [1.0, 0.5, 0.33]$

generates a waveform consisting of the fundamental frequency with amplitude 1.0, the first overtone at amplitude 0.5, and the second overtone at amplitude 0.33. So a more complex sound can be synthesized just by changing the wavetable:

$s_3 :: Clock\ c \Rightarrow SigFun\ c\ ()\ Double$
$s_3 = proc\ () \rightarrow \mathbf{do}$
  $osc\ tab_2\ 0 \prec\!\!- 440$

To get the same effect using *oscFixed* we would have to write:

$s_4 :: Clock\ c \Rightarrow SigFun\ c\ ()\ Double$
$s_4 = proc\ () \rightarrow \mathbf{do}$
  $f_0 \leftarrow oscFixed\ 440 \prec\!\!- ()$
  $f_1 \leftarrow oscFixed\ 880 \prec\!\!- ()$
  $f_2 \leftarrow oscFixed\ 1320 \prec\!\!- ()$
  $outA \prec\!\!- (f_0 + 0.5 * f_1 + 0.33 * f_2)/1.83$

Not only is this more complex, it is less efficient. (The division by 1.83 is to normalize the result—if the peaks of the three signals f_0, f_1, and f_2 align properly, the peak amplitude will be 1.83 (or -1.83), which is outside the range ± 1.0 and may cause clipping (see discussion in Section 18.2).

So far in these examples we have generated a signal whose fundamental frequency is 440 Hz. But as mentioned, in the case of *osc*, the input to the oscillator is a signal, and can therefore itself be time-varying. As an example of this idea, let's implement *vibrato*—the performance effect whereby a musician slightly varies the frequency of a note in a pulsating rhythm. On a string instrument this is typically achieved by wiggling the finger on the fingerboard, on a reed instrument by an adjustment of the breath and embouchure to compress and relax the reed in a suitable way, and so on.

Specifically, let's define a function:

$vibrato :: Clock\ c \Rightarrow$
  $Double \rightarrow Double \rightarrow SigFun\ c\ Double\ Double$

such that *vibrato f d* is a signal function that takes a frequency argument (this is not a signal of a given frequency, it is the frequency itself), and generates a signal at that frequency, but with vibrato added, where f is the vibrato frequency, and d is the vibrato depth. We will consider "depth" to be a measure of how many Hz the input frequency is modulated.

Intuitively, it seems as if we need *two* oscillators, one to generate the fundamental frequency of interest, and the other to generate the vibrato

(much lower in frequency). Here is a solution:

```
vibrato :: Clock c ⇒
           Double → Double → SigFun c Double Double
vibrato vfrq dep = proc afrq → do
  vib  ← osc tab1 0 −≺ vfrq
  aud ← osc tab2 0 −≺ afrq + vib ∗ dep
  outA −≺ aud
```

Note that a pure sine wave is used for the vibrato signal, whereas tab_2, a sum of three sine waves, is chosen for the signal itself.

For example, to play a 1000 Hz tone with a vibrato frequency of 5 Hz and a depth of 20 Hz, we could write:

```
s5 :: AudSF () Double
s5 = constA 1000 >>> vibrato 5 20
```

Vibrato is actually an example of a more general sound synthesis technique called *frequency modulation* (since one signal is being used to vary, or modulate, the frequency of another signal), and will be explained in more detail in Chapter **??**. Other chapters include synthesis techniques such as additive and subtractive synthesis, plucked instruments using waveguides, physical modeling, granular synthesis, as well as audio processing techniques such as filter design, reverb, and other effects. Now that we have a basic understanding of signal functions, these techniques will be straighforward to express in Euterpea.

18.2 Generating Sound

Euterpea can execute some programs in real-time, but sufficiently complex programs require writing the result to a file. The function for achieving this is:

```
outFile :: (AudioSample a, Clock c) ⇒
           String → Double → SigFun c () a → IO ()
```

The first argument is the name of the WAV file to which the result is written. The second argument is the duration of the result, in seconds (remember that signals are conceptually infinite). The third argument is a signal function that takes no input and generates a signal of type a as output (i.e. a signal source), where a is required to be an instance of the *AudioSample* type class, which allows one to choose between mono, stereo, etc.

For example, the IO command *outfile* `"test.wav"` 5 *sf* generates 5 seconds of output from the signal function *sf*, and writes the result to the file `"test.wav"`. If *sf* has type *SigFun AudRate* () *Double* then the result will be monophonic; if the type is *SigFun AudRate* () (*Double*, *Double*) the result will be stereophonic; *SigFun AudRate* () (*Double*, *Double*, *Double*, *Double*) yields quadraphonic sound, and so on.

One might think that *outFile* should be restricted to *AudRate*. However, by allowing a signal of any clock rate to be written to a file, one can use external tools to analyze the result of control signals or other signals of interest as well.

An important detail in writing WAV files with *outFile* is that care must be taken to ensure that each sample falls in the range ± 1.0. If this range is exceeded, the output sound will be harshly distorted, a phenomenon known as *clipping*. The reason that clipping sounds especially bad is that once the maximum limit is exceeded, the subsequent samples are interpreted as the *negation* of their intended value—and thus the signal swings abruptly from its largest possible value to its smallest possible value. Of course, signals within your program may be well outside this range—it is only when you are ready to write the result to a file that clipping needs to be avoided.

One can easily write signal functions that deal with clipping in one way or another. For example here's one that simply returns the maximum (positive) or mininum (negative) value if they are exceeded. thus avoiding the abrupt change in magnitude described above, and degenerating in the worst case to a square wave:

```
simpleClip :: Clock c ⇒ SigFun c Double Double
simpleClip = arr f where
  f x = if abs x ⩽ 1.0 then x else signum x
```

Details: *abs* is the absolute value function in Haskell, and *signum* returns -1 for negative numbers, 0 for zero, and 1 for positive numbers.

[**To do:** Define some signal functions to deal with time—for example one that "takes" the first t seconds of a signal function, returning zero for all times beyond that. We could write a special function to do this, but using Occam's Razor suppose we have a signal function *time* :: *Clock c* ⇒ *SigFun c* () *Double* that returns the current time. Then we could write:

```
takeSF :: Clock c ⇒ Double → SigFun c Double Double
takeSF t = proc x do
  now ← time ⤙ ()
  outA ⤙ if now < t then x else 0
```

Indeed, time can be defined by:

```
time :: Clock c ⇒ SigFun c () Double
time = integral ⋘ constA 1
```

Or, we could take a Yampa-like approach and use a "switcher," but then we'd need some switcher signal functions. There is a collection-based switcher defined in *Euterpea.Audio.Render* called *pSwitch*, but we might want something simpler.

Even with all this, it seems desirable to have a "debug" function that takes a time and a signal function, and returns a Boolean indicating whether or not the signal function clipped or not during that period of time. Again using Occam's razor, it seems best to define a function *sfToList* that returns the infinite list underlying a signal source. If we know the clock rate, then "take"ing a suitable prefix of this list will return the desired result. Then, for example, *max* (*take* 44100 (*sfToList ss*)) yields the maximum value of the first 44100 samples of the signal source *ss*. One could then use this to normalize the *ss*.

Note that *sfToList* is not something that can be defined using Euterpea as a library—it would have to be defined within Euterpea's implementation of signal functions.]

18.3 Instruments

So far we have only considered signal functions as stand-alone values whose output we can write to a WAV file. But how do we connect the ideas in previous chapters about *Music* values, *Performance*s, and so on, to the ideas presented in this chapter? This section presents a bridge between the two worlds.

18.3.1 Turning a Signal Function into an Instruement

Suppose that we have a *Music* value that, previously, we would have played using a MIDI instrument, and now we want to play using an instrument that we have designed using signal functions. To do this, first recall from

Chapter 2 that the *InstrumentName* data type has a special constructor called *Custom*:

```
data InstrumentName =
    AcousticGrandPiano
  | BrightAcousticPiano
  | ...
  | Custom String
  deriving (Show, Eq, Ord)
```

With this constructor, names (represented as strings) can be given to instruments that we have designed using signal functions. For example:

```
simpleInstr :: InstrumentName
simpleInstr = Custom "Simple Instrument"
```

Now we need to define the instrument itself. Euterpea defines the following type synonym:

```
type Instr a = Dur → AbsPitch → Volume → [Double] → a
```

Although *Instr* is polymorphic, by far its most common instantiation is the type *Instr* (*AufSF* () *Double*). An instrument of this type is a function that takes a duration, absolute pitch, volume, and a list of parameters, and returns a signal source that generates the resulting sound.

The list of parameters (similar to the "pfields" in csound) are not used by MIDI instruments, and thus have not been discussed until now. They afford us unlimited expressiveness in controlling the sound of our signal-function based instruments. Recall from Chapter 8 the types:

```
type Music1 = Music Note1
type Note1  = (Pitch, [NoteAttribute])
data NoteAttribute =
    Volume Int
  | Fingering Integer
  | Dynamics String
  | Params [Double]
  deriving (Eq, Show)
```

Using the *Params* constructor, each individual note in a *Music1* value can be given a different list of parameters. It is up to the instrument designer to decide how these parameters are used.

There are three steps to playing a *Music* value using a user-defined instrument. First, we must coerce our signal function into an instrument hav-

ing the proper type *Instr* as described above. For example, let's turn the *vibrato* function from the last section into a (rather primitive) instrument:

```
myInstr :: Instr (AudSF () Double)
    -- Dur → AbsPitch → Volume → [Double] → (AudSF () Double)
myInstr dur ap vol [vfrq, dep] =
  proc () → do
    vib ← osc tab1 0 ⤙ vfrq
    aud ← osc tab2 0 ⤙ apToHz ap + vib * dep
    outA ⤙ aud
```

Aside from the re-shuffling of arguments, note the use of the function *apToHz*, which converts an absolute pitch into its corresponding frequency:

```
apToHz :: Floating a ⇒ AbsPitch → a
```

Next, we must connect our instrument name (used in the *Music* value) to the instrument itself (such as defined above). This is achieved using a simple association list, or *instrument map*:

```
type InstrMap a = [(InstrumentName, Instr a)]
```

Continuing the example started above:

```
myInstrMap :: InstrMap (AudSF () Double)
myInstrMap = [(simpleInstr, myInstr)]
```

Finally, we need a function that is analogous to *perform* from Chapter 8, except that instead of generating a *Performance*, it creates a single signal function that will "play" our *Music* value for us. In Euterpea that function is called *renderSF*:

```
renderSF :: (Performable a, AudioSample b, Clock c) ⇒
            Music a →
            InstrMap (SigFun p () b) →
            (Double, SigFun p () b)
```

The first element of the pair that is returned is the duration of the *Music* value, just as is returned by *perform*. That way we know how much of the signal function to render in order to hear the entire composition.

Using the simple melody *mel* in Figure 18.3, and the simple vibrato instrument defined above, we can generate our result and write it to a file, as follows:

```
(dr, sf) = renderSF mel myInstrMap
main = outFile "simple.wav" dr sf
```

For clarity we show in Figure 18.4 all of the pieces of this running example

```
mel :: Music1
mel =
  let m = Euterpea.line [na1 (c 4 en), na1 (ef 4 en), na1 (f 4 en),
                         na2 (af 4 qn), na1 (f 4 en),  na1 (af 4 en),
                         na2 (bf 4 qn), na1 (af 4 en), na1 (bf 4 en),
                         na1 (c 5 en),  na1 (ef 5 en), na1 (f 5 en),
                         na3 (af 5 wn)]
      na1 (Prim (Note d p)) = Prim (Note d (p, [Params [0,0]]))
      na2 (Prim (Note d p)) = Prim (Note d (p, [Params [5,10]]))
      na3 (Prim (Note d p)) = Prim (Note d (p, [Params [5,20]]))
  in instrument simpleInstr m
```

Figure 18.3: A Simple Melody

as one program.

18.3.2 Envelopes

Most instruments played by humans have a distinctive sound that is partially dependent on how the performer plays a particular note. For example, when a wind instrument is played (whether it be a flute, saxophone, or trumpet), the note does not begin instantaneously—it depends on how quickly and forcibly the performer blows into the instrument. This is called the "attack." Indeed, it is not uncommon for the initial pulse of energey to generate a sound that is louder than the "sustained" portion of the sound. And when the note ends, the airflow does not stop instantaneously, so there is variability in the "release" of the note.

The overall variability in the loudness of a note can be simulated by multiplying the output of a signal function by an *envelope*, which is a time-varying signal that captures the desired behavior. For example, ...

Euterpea provides several evelope-generating functions: see Figure 18.5.

Fifth arg to *envCSEnvplx*: A value greater than 1 causes exponential growth; a value less than 1 causes exponential decay; a value = 1 will maintain a true steady state at the last rise value. The attenuation is not by a fixed rate (as in a piano), but is sensitive to a note's duration. However, if this argument is less than 0 (or if steady state is less than 4 k-periods) a fixed attenuation rate of abs atss per second will be used. 0 is illegal.

Sixth arg to *envCSEnvplx*: Must be positive and is normally of the order

```
simpleInstr :: InstrumentName
simpleInstr = Custom "Simple Instrument"

myInstr :: Instr (AudSF () Double)
myInstr dur ap vol [vfrq, dep] =
  proc () → do
    vib ← osc tab1 0 ⤙ vfrq
    aud ← osc tab2 0 ⤙ apToHz ap + vib * dep
    outA ⤙ aud

myInstrMap :: InstrMap (AudSF () Double)
myInstrMap = [(simpleInstr, myInstr)]

(d, sf) = renderSF mel myInstrMap
main    = outFile "simple.wav" d sf
```

Figure 18.4: A Complete Example of a Signal-Function Based Instrument

of 0.01. A large or excessively small value is apt to produce a cutoff that is not audible. Values less than or equal to 0 are disallowed.

Exercise 18.1 Using the Euterpea function *osc*, create a simple sinusoidal wave, but using different table sizes, and different frequencies, and see if you can hear the differences (report on what you hear). Use *outFile* to write your results to a file, and be sure to use a decent set of speakers or headphones.

Exercise 18.2 The *vibrato* function varies a signals frequency at a given rate and depth. Define an analogous function *tremolo* that varies the volume at a given rate and depth. However, in a sense, *tremolo* is a kind of envelope (infinite in duration), so define it as a signal source, with which you can then shape whatever signal you wish. Consider the "depth" to be the fractional change to the volume; that is, a value of 0 would result in no tremolo, a value of 0.1 would vary the amplitude from 0.9 to 1.1, and so on. Test your result.

Exercise 18.3 Define an ADSR ("attack/decay/sustain/release") envelope generator (i.e. a signal source) called *envADSR*, with type:

```
type DPair = (Double, Double)   -- pair of duration and amplitude
envADSR    :: DPair → DPair → DPair → Double → AudSF () Double
```

The three *DPair* arguments are the duration and amplitude of the attack, decay, and release "phases," respectively, of the envelope. The sustain phase

```
envLine      :: Clock p ⇒
  Double  →    -- starting value
  Double  →    -- duration in seconds
  Double  →    -- value after dur seconds
  SigFun p () Double
envExpon     :: Clock p ⇒
  Double  →    -- starting value; zero is illegal for exponentials
  Double  →    -- duration in seconds
  Double  →    -- value after dur seconds (must be non-zero
               -- and agree in sign with first argument)
  SigFun p () Double
envLineSeg   :: Clock p ⇒
  [Double] →   -- list of points to trace through
  [Double] →   -- list of durations for each line segment
               -- (one element fewer than previous argument)
  SigFun p () Double
envExponSeg  :: Clock p ⇒
  [Double] →   -- list of points to trace through
  [Double] →   -- list of durations for each line segment
               -- (one element fewer than previous argument)
  SigFun p () Double
envASR       :: Clock p ⇒
  Double  →    -- rise time in seconds
  Double  →    -- overall duration in seconds
  Double  →    -- decay time in seconds
  SigFun p () Double
envCSEnvlpx  :: Clock p ⇒
  Double  →    -- rise time in seconds
  Double  →    -- overall duration in seconds
  Double  →    -- decay time in seconds
  Table   →    -- table of stored rise shape
  Double  →    -- attenuation factor, by which the last value
               -- of the envlpx rise is modified during the
               -- note's pseudo steady state
  Double  →    -- attenuation factor by which the closing
               -- steady state value is reduced exponentially
               -- over the decay period
  SigFun p () Double
```

Figure 18.5: Envelopes

should hold the last value of the decay phase. The fourth argument is the duration of the entire envelope, and thus the duration of the sustain phase should be that value minus the sum of the durations of the other three phases. (Hint: use Euterpeas *envLineSeg* function.) Test your result.

Exercise 18.4 Generate a signal that causes clipping, and listen to the result. Then use *simpleClip* to "clean it up" somewhat—can you hear the difference? Now write a more ambitious clipping function. In particular, one that uses some kind of non-linear reduction in the signal amplitude as it approaches plus or minus one (rather than abruptly "sticking" at plus or minus one, as in *simpleClip*).

Exercise 18.5 Define two instruments, each of type *Instr* (*AudSF* () *Double*). These can be as simple as you like, but each must take at least two *Params*. Define an *InstrMap* that uses these, and then use *renderSF* to "drive" your instruments from a *Music1* value. Test your result.

Chapter 19

Spectrum Analysis

```
{-# LANGUAGE Arrows #-}
module Euterpea.Music.Signal.SpectrumAnalysis where
import Euterpea hiding (Event)
import Euterpea.IO.MUI
import Euterpea.IO.MUI.SOE (Color (..))
import Data.Complex
import Data.Maybe (listToMaybe, catMaybes)
import Control.SF.AuxFunctions (fftA, FFTData, Event)
```

There are many situations where it is desirable to take an existing sound signal—in particular one that is recorded by a microphone—and analyze it for its spectral content. If one can do this effectively, it is then possible (at least in theory) to recreate the original sound, or to create novel variations of it. The thepry behind this approach is based on *Fourier's Theorem*, which states that any periodic signal can be decomposed into a weighted sum of (a potentially infinite number of) sine waves. In this chapter we discuss the theory as well as the pragmatics for doing spectrum analysis in Euterpea.

19.1 Fourier's Theorem

A *periodic signal* is a signal that repeats itself infinitely often. Mathematically, a signal x is periodic if there exists a real number T such that for all integers n:

$$x(t) = x(t + nT)$$

T is called the *period*, which may be just a few microseconds, a few seconds, or perhaps days—the only thing that matters is that the signal repeats itself. Usually we want to find the smallest value of T that satisfies the above property. For example, a sine wave is surely periodic; indeed, recall from Section 17.1.1 that:

$$\sin(2\pi k + \theta) = \sin\theta$$

for any integer k. In this case, $T = 2\pi$, and it is the smallest value that satisfies this property.

But in what sense is, for example, a single musical note periodic? Indeed it is not, unless it is repeated infinitely often, which would not be very interesting musically. Yet something we would like to know is the spectral content of that single note, or even of a small portion of that note, within an entire composition. This is one of the practical problems that we will address later in the chapter.

Recall from Section 17.1.1 that a sine wave can be represented by: $x(t) = A\sin(\omega t + \phi)$, where A is the amplitude, ω is the radian frequency, and ϕ is the phase angle. Joseph Fourier, a french mathematician and physicist, showed the following result. Any periodic signal $x(t)$ with period T can be represented as:

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_0 n t + \phi_n) \tag{19.1}$$

This is called *Fourier's Theorem*. $\omega_0 = {}^{2\pi}/_{T}$ is called the *fundamental frequency*. Note that the frequency of each cosine wave in the series is an integer multiple of the fundamental frequency. The above equation is also called the *Fourier series* or *harmonic series* (related, but not to be confused with, the mathematical definition of harmonic series, which has the precise form $1 + {}^1/_2 + {}^1/_3 + {}^1/_4 + \cdots$).

The trick, of course, is determining what the coefficients C_0, ..., C_n and phase angles ϕ_1, ..., ϕ_n are. Determining the above equation for a particular periodic signal is called *Fourier analysis*, and synthesizing a sound based on the above equation is called *Fourier synthesis*. Theoretically, at least, we should be able to use Fourier analysis to decompose a sound of interest into its composite sine waves, and then regenerate it by artificially generating those composite sine waves and adding them together (i.e. additive synthesis, to be described in Chapter 20). Of course, we also have to deal with the fact that the representation may involve an *infinite* number of composite signals.

As discussed somewhat in Chapter 17, many naturally occurring vibrations in nature—including the resonances of most musical instruments—are characterized as having a fundamental frequency (the perceived pitch) and some combination of multiples of that frequency, which are often called *harmonics*, *overtones* or *partials*. So Fourier's Theorem seems to be a good match for this musical application.

19.1.1 The Fourier Transform

When studying Fourier analysis, it is more convenient, mathematically, to use *complex exponentials*. We can relate working with complex exponentials back to sines and cosines using *Euler's Formula*:

$$\begin{aligned} e^{j\theta} &= cos(\theta) + jsin(\theta) \\ cos(\theta) &= \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \\ sin(\theta) &= \frac{1}{2}(e^{j\theta} - e^{-j\theta}) \end{aligned}$$

For a periodic signal $x(t)$, which we consider to be a function of time, we denote its *Fourier transform* by $\hat{x}(f)$, which is a function of frequency. Each point in $\hat{x}$ is a complex number that represents the magnitude and phase of the frequency f's presence in $x(t)$. Using complex exponentials, the formula for $\hat{x}(f)$ in terms of $x(t)$ is:

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

where $\omega = 2\pi f$, and j is the same as the imaginary unit i used in mathematics.[1] Intuitively, the Fourier transform at a particular frequency f is the integral of the product of the original signal and a pure sinusiodal wave $e^{-j\omega t}$. This latter process is related to the *convolution* of the two signals, and intuitively will be non-zero only when the signal has some content of that pure signal in it.

The above equation describes $\hat{x}$ in terms of x. We can also go the other way around—defining x in terms of $\hat{x}$:

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f)e^{j\hat{\omega}f}df$$

[1] Historically, engineers prefer to use the symbol j rather than i, because i is generally used to represent current in an electrical circuit.

where $\hat{\omega} = 2\pi t$. This is called the *inverse* Fourier transform.

If we expand the definitions of ω and $\hat{\omega}$ we can see how similar these two equations are:

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \tag{19.2}$$

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f) e^{j2\pi ft} df \tag{19.3}$$

These two equations, for the Fourier transform and its inverse, are remarkable in their simplicity and power. They are also remarkable in the following sense: *no information is lost when converting from one to the other.* In other words, a signal can be represented in terms of its time-varying behavior or its spectral content—they are equivalent!

A function that has the property that $f(x) = f(-x)$ is called an *even* function; if $f(x) = -f(-x)$ it is said to be *odd.* It turns out that, perhaps surprisingly, *any* function can be expressed as the sum of a single even function and a single odd function. This may help provide some intuition about the equations for the Fourier transform, because the complex exponential $e^{j2\pi ft}$ separates the waveform by which it is being multiplied into its even and odd parts (recall Euler's formula). The real (cosine) part affects only the even part of the input, and the imaginary (sine) part affects only the odd part of the input.

19.1.2 Examples

Let's consider some examples, which are illustrated in Figure 19.1:

- Intuitively, the Fourier transform of a pure cosine wave should be an impulse function—that is, the spectral content of a cosine wave should be concentrated completely at the frequency of the cosine wave. The only catch is that, when working in the complex domain, the Fourier transform also yields the mirror image of the spectral content, at a frequency that is the negation of the cosine wave's frequency, as shown in Figure 19.1a. In other words, in this case, $\hat{x}(f) = \hat{x}(-f)$, i.e. $\hat{x}$ is even. So the spectral content is the *real* part of the complex number returned from the Fourier transform (recall Euler's formula).
- In the case of a pure sine wave, we should expect a similar result. The only catch now is that the spectral content is contained in the *imaginary* part of the complex number returned from the Fourier transform

(recall Euler's formula), and the mirror image is negated. That is, $\hat{x}(f) = -\hat{x}(-f)$, i.e. $\hat{x}$ is odd. This is illustrated in Figure 19.1b.

- Conversely, consider what the spectral content of an impulse function should be. Because an impulse function is infinitely "sharp," it would seem that its spectrum should contain energy at every point in the frequency domain. Indeed, the Fourier transform of an impulse function centered at zero is a constant, as shown in Figure 19.1c.

- Consider now the spectral content of a square wave. It can be shown that the Fourier series representation of a square wave is the sum of the square wave's fundamental frequency plus its harmonically decreasing (in magnitude) odd harmonics. Specifically:

$$sq(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sin k\omega t, \quad \text{for odd } k \tag{19.4}$$

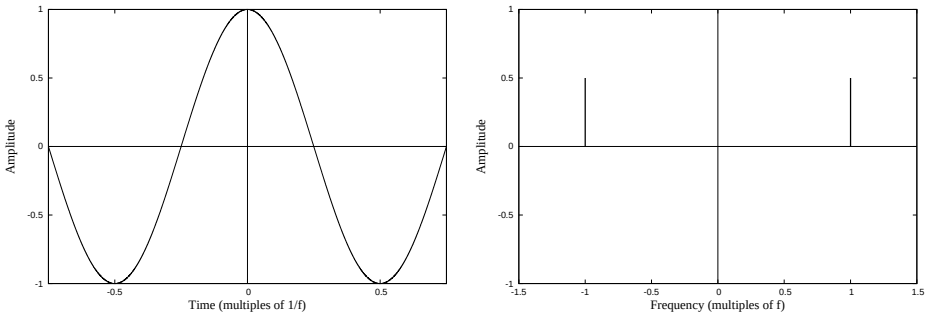
  The spectral content of this signal in shown in Figure 19.1d. Figure 19.2 also shows partial reconstruction of the square wave from a finite number of its composite signals.

It is worth noting that the diagrams in Figure 19.1 make no assumptions about time or frequency. Therefore, because the Fourier transform and its inverse are true mathematical inverses, we can read the diagrams as time domain / frequency domain pairs, or the other way around; i.e. as frequency domain / time domain pairs. For example, interpreting the diagram on the left of Figure 19.1a in the frequency domain, is to say that it is the Fourier transform of the signal on the right (interpreted in the time domain).

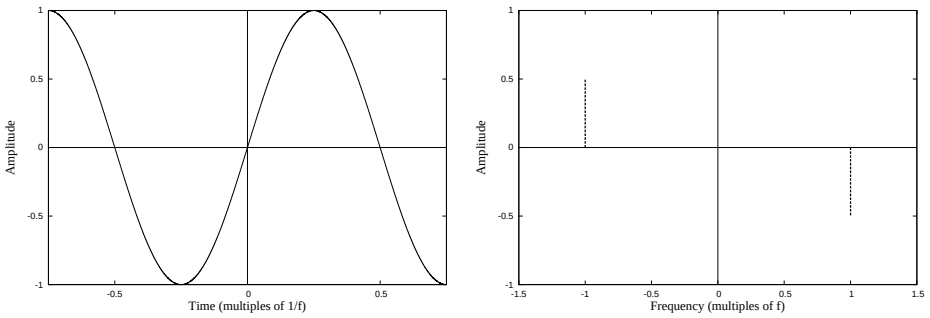
19.2 The Discrete Fourier Transform

Recall from Section 17.2.1 that we can move from the continuous signal domain to the discrete domain by replacing the time t with the quantity n/r, where n is the integer index into the sequence of discrete samples, and r is the sampling rate. Let us assume that we have done this for x, and we will use square brackets to denote the difference. That is, $x[n]$ denotes the n^{th} sample of the continuous signal $x(t)$, corresponding to the value $x(n/r)$.

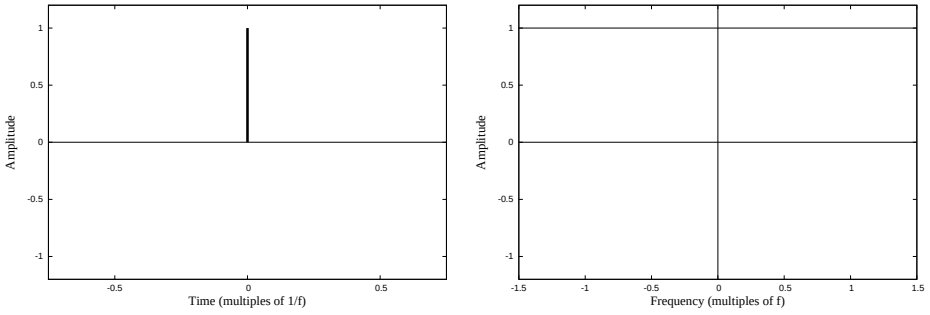
We would now like to compute the *Discrete Fourier Transform* (DFT) of our discrete signal. But instead of being concerned about the sampling rate (which can introduce aliasing, for example), our concern turns to the



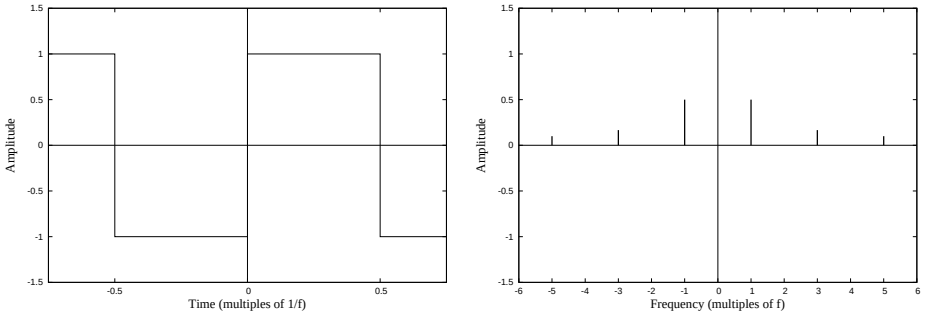
(a) Cosine wave



(b) Sine wave

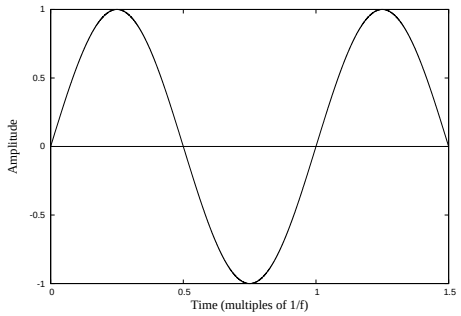


(c) Impulse function

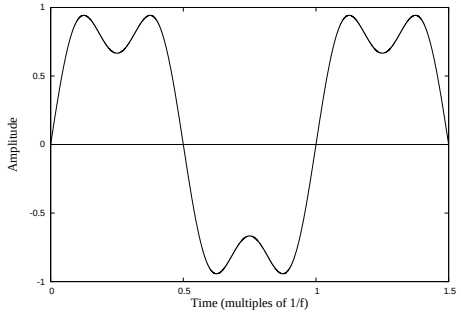


(d) Square wave

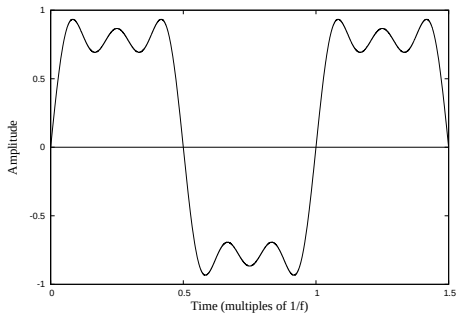
Figure 19.1: Examples of Fourier Transforms



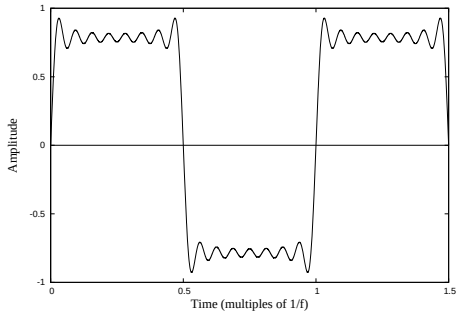
(a) Sine wave



(b) Sine wave + third harmonic



(c) Since wave + third and fifth harmonics



(d) Sum of first eight terms of the Fourier series of a square wave

Figure 19.2: Generating a Square Wave from Odd Harmonics

number of samples that we use in computing the DFT—let's call this N. Intuitively, the integrals used in our equations for the Fourier transform and its inverse should become sums over the range $0...N-1$. This leads to a reformulation of our two equations (19.2 and 19.3) as follows:[2]

$$\hat{x}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, ..., N-1 \tag{19.5}$$

$$x[n] = \sum_{k=0}^{N-1} \hat{x}[k]e^{j\frac{2\pi kn}{N}}, \quad n = 0, 1, ..., N-1 \tag{19.6}$$

Despite all of the mathematics up to this point, the reader may now realize that the discrete Fourier transform as expressed above is amenable to implementation—for example it should not be difficult to write Haskell functions that realize each of the above equations. But before addressing implementation issues, let's discuss a bit more what the results actually *mean*.

19.2.1 Interpreting the Frequency Spectrum

Just as $x[n]$ represents a sampled version of the continuous input signal, $\hat{x}[k]$ represents a sampled version of the continous frequency spectrum. Care must be taken when interpreting either of these results, keeping in mind the Nyquist-Shannon Sampling Theorem (recall Section 17.2) and aliasing (Section 17.2.3).

Also recall that the result of a Fourier transform of a periodic signal is a Fourier series (see Section 19.1), in which the signal being analyzed is expressed as multiples of a fundamental frequency. In equation 19.5 above, that fundamental frequency is the inverse of the duration of the N samples, i.e. the inverse of N/r, or r/N. For example, if the sampling rate is 44.1 kHz (the CD standard), then:

- If we take $N = 441$ samples, then the fundamental frequency will be $r/N = 100$ Hz.

[2] The purpose of the factor $1/N$ in Equation 19.5 is to ensure that the DFT and the inverse DFT are in fact inverses of each other. But it is just by convention that one equation has this factor and the other does not—it would be sufficient if it were done the other way around. In fact, all that matters is that the product of the two coefficients be $1/N$, and thus it would also be sufficient for each equation to have the same coefficient, namely $1/\sqrt{N}$. Similarly, the negative exponent in one equation and positive in the other is also by convention—it would be sufficient to do it the other way around.

- If we take $N = 4410$ samples, then the fundamental frequency will be $r/N = 10$ Hz.
- If we take $N = 44100$ samples, then the fundamental frequency will be $r/N = 1$ Hz.

Thus, as would be expected, taking more samples yields a *finer* resolution of the frequency spectrum. On the other hand, note that if we increase the sampling rate and keep the number of samples fixed, we get a *courser* resolution of the spectrum—this also should be expected, because if we increase the sampling rate we would expect to have to look at more samples to get the same accuracy.

Analogous to the Nyquist-Shannon Sampling Theorem, the representable points in the resulting frequency spectrum lie in the range $\pm r/2$, i.e. between plus and minus one-half of the sampling rate. For the above three cases, respectively, that means the points are:

- -22.0 kHz, -21.9 kHz, ..., -0.1 kHz, 0, 0.1 kHz, ..., 21.9 kHz, 22.0 kHz
- -22.05 kHz, -22.04 kHz, ..., -10 Hz, 0, 10 Hz, ..., 22.04 kHz, 22.05 kHz
- -22.05 kHz, -22.049 kHz, ..., -1 Hz, 0, 1 Hz, ..., 22.049 kHz, 22.05 kHz

For practical purposes, the first of these is usually too course, the third is too fine, and the middle one is useful for many applications.

Note that the first range of frequencies above does not quite cover the range $\pm r/2$. But remember that this is a discrete representation of the actual frequency spectrum, and the proper interpretation would include the frequences $+r/2$ and $-r/2$.

Also note that there are $N+1$ points in each of the above ranges, not N. Indeed, the more general question is, how do these points in the frequency spectrum correspond to the indices $i = 0, 1, ..., N-1$ in $\hat{x}[i]$? If we denote each of these frequencies as f, the answer is that:

$$f = \frac{ir}{N}, \qquad i = 0, 1, ..., N-1 \tag{19.7}$$

But note that this range of frequencies extends from 0 to $(N-1)(r/N)$, which exceeds the Nyquist-Shannon sampling limit of $r/2$. The way out of this dilemma is to realize that the DFT assumes that the input signal is periodic in time, and therefore the DFT is periodic in frequency. In

other words, values of f for indices i greater than $N/2$ can be interpreted as frequencies that are the *negation* of the frequency given by the formula above. Assuming even N, we can revise formula 19.7 as follows:

$$f = \begin{cases} i\frac{r}{N}, & i = 0, 1, ..., \frac{N}{2} \\ (i - N)\frac{r}{N} & i = \frac{N}{2}, \frac{N}{2} + 1, ..., N - 1 \end{cases} \tag{19.8}$$

Note that when $i = N/2$, both equations apply, yielding $f = r/2$ in the first case, and $f = -r/2$ in the second. Indeed, the magnitude of the DFT for each of these frequencies is the same (see discussion in the next section), reflecting the periodicity of the DFT, and thus is simply a form of redundancy.

The above discussion has assumed a periodic signal whose fundamental frequency is known, thus allowing us to parameterize the DFT with the same fundamental frequency. In practice this rarely happens. That is, the fundamental frequency of the DFT typically has no integral relationship to the period of the periodic signal. This raises the question, what happens to the frequencies that "fall in the gaps" between the frequencies discussed above? The answer is that the energy of that frequency component will be distributed amongst neighboring points in a way that makes sense mathematically, although the result may look a little funny compared to the ideal result (where every frequency component is an integer multiple of the fundamental). The important thing to remember is that these are digital representations of the exact spectra, just as a digitized signal is representative of an exact signal. Two digitized signals can look very different (depending on sample rate, phase angle, and so on), yet represent the same underlying signal—the same is true of a digitized spectrum.

In practice, for reasons of computational efficiency, N is usually chosen to be a power of two. We will return to this issue when we discuss implementing the DFT.

19.2.2 Amplitude and Power of Spectrum

We discussed above how each sample in the result of a DFT relates to a point in the frequency spectrum of the input signal. But how do we determine the amplitude and phase angle of each of those frequency components? In general each sample in the result of a DFT is a complex number, thus having both a real and imaginary part, of the form $a + jb$. We can visualize this number as a point in the complex Cartesian plane, where the abscissa

(x-axis) represents the real part, and the ordinate (y-axis) represents the imaginary part, as shown in Figure 19.3. It is easy to see that the line from the origin to the point of interest is a vector A, whose length is the *amplitude* of the frequency component in the spectrum:

$$A = \sqrt{a^2 + b^2} \tag{19.9}$$

The angle θ is the *phase*, and it is easily defined from the figure as:

$$\theta = tan^{-1}\frac{b}{a} \tag{19.10}$$

(This amplitude / phase pair is often called the *polar* representation of a complex number.)

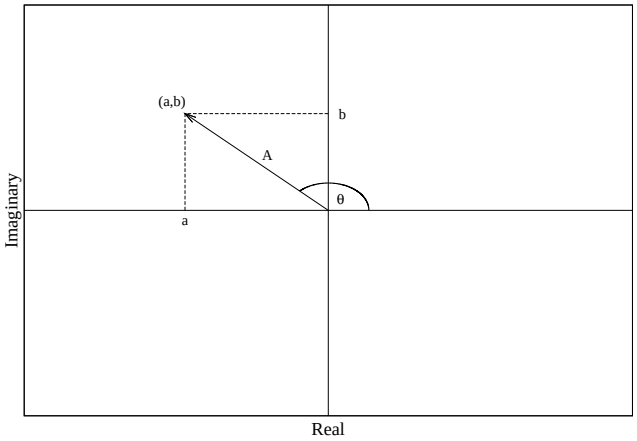


Figure 19.3: Complex and Polar Coordinates

Recall from Section 17.1.2 that power is proportional to the square of the amplitude. Since taking a square root adds computational expense, the square root is often omitted from Equation 19.9, thus yielding a *power spectrum* instead of an *amplitude spectrum*.

One subtle aspect of the resulting DFT is how to interpret *negative* frequencies. In the case of having an input whose samples are all real numbers (i.e. there are no imaginary components), which is true for audio applications, the negative spectrum is a mirror image of the positive spectrum, and the amplitude/power is distributed evenly between the two.

19.2.3 A Haskell Implementation of the DFT

From equation 19.5, which defines the DFT mathematically, we can write a Haskell program that implements the DFT.

The first thing we need to do is understand how complex numbers are handled in Haskell. They are captured in the *Complex* library, which must be imported into any program that uses them. The type *Complex T* is the type of complex numbers whose underlying numeric type is T. We will use, for example, *Complex Double* for testing our DFT. A complex number $a + jb$ is represented in Haskell as $a :+ b$, and since $(:+)$ is a constructor, such values can be pattern matched.

Details: Complex numbers in Haskell are captured in the *Complex* library, in which complex numbers are defined as a polymorphic data type:

$$\textbf{infix } 6 :+$$
$$\textbf{data } (\mathit{RealFloat}\ a) \Rightarrow \mathit{Complex}\ a = !a :+ !a$$

The "!" in front of the type variables declares that the constructor $(:+)$ is strict in its arguments. For example, the complex number $a + jb$ is represented by $a :+ b$ in Haskell. One can pattern match on complex number values to extract the real and imaginary parts, or use one of the predefined selectors defined in the *Complex* library:

$$\mathit{realPart}, \mathit{imagPart} :: \mathit{RealFloat}\ a \Rightarrow \mathit{Complex}\ a \rightarrow a$$

The *Complex* library also defines the following functions:

$$\begin{array}{ll} \mathit{conjugate} & :: \mathit{RealFloat}\ a \Rightarrow \mathit{Complex}\ a \rightarrow \mathit{Complex}\ a \\ \mathit{mkPolar} & :: \mathit{RealFloat}\ a \Rightarrow a \rightarrow a \rightarrow \mathit{Complex}\ a \\ \mathit{cis} & :: \mathit{RealFloat}\ a \Rightarrow a \rightarrow \mathit{Complex}\ a \\ \mathit{polar} & :: \mathit{RealFloat}\ a \Rightarrow \mathit{Complex}\ a \rightarrow (a, a) \\ \mathit{magnitude}, \mathit{phase} & :: \mathit{RealFloat}\ a \Rightarrow \mathit{Complex}\ a \rightarrow a \end{array}$$

The library also declares instances of *Complex* for the type classes *Num*, *Fractional*, and *Floating*.

Although not as efficient as arrays, for simplicity we choose to use lists to represent the vectors that are the input and output of the DFT. Thus if xs is the list that represents the signal x, then $xs\,!!\,n$ is the $n+1^{th}$ sample of that signal, and is equivalent to $x[n]$. Furthermore, using list comprehensions, we can make the Haskell code look very much like the mathematical definition captured in Equation 19.5. Finally, we adopt the convention that the length

of the input signal is the number of samples that we will use for the DFT.

Probably the trickiest part of writing a Haskell program for the DFT is dealing with the types! In particular, if you look closely at Equation 19.5 you will see that N is used in three different ways—as an integer (for indexing), as a real number (in the exponent of e), and as a complex number (in the expression $1/N$).

Here is a Haskell program that implements the DFT:

```
dft :: RealFloat a => [Complex a] -> [Complex a]
dft xs =
  let lenI = length xs
      lenR = fromIntegral lenI
      lenC = lenR :+ 0
  in [let i = -2 * pi * fromIntegral k / lenR
      in (1 / lenC) * sum [(xs !! n) * exp (0 :+ i * fromIntegral n)
                                     | n <- [0, 1 .. lenI - 1]]
     | k <- [0, 1 .. lenI - 1]]
```

Note that *lenI*, *lenR*, and *lenC* are the integer, real, and complex versions, respectively, of N. Otherwise the code is fairly straightforward—note in particular how list comprehensions are used to implement the ranges of n and k in Equation 19.5.

To test our program, let's first create a couple of waveforms. For example, recall that Equation 19.4 defines the Fourier series for a square wave. We can implement the first, first two, and first three terms of this series, corresponding respectively to Figures 19.2a, 19.2b, and 19.2c, by the following Haskell code:

```
mkTerm :: Int -> Double -> [Complex Double]
mkTerm num n = let f = 2 * pi / fromIntegral num
  in [sin (n * f * fromIntegral i) / n :+ 0
     | i <- [0, 1 .. num - 1]]

mkxa, mkxb, mkxc :: Int -> [Complex Double]
mkxa num = mkTerm num 1
mkxb num = zipWith (+) (mkxa num) (mkTerm num 3)
mkxc num = zipWith (+) (mkxb num) (mkTerm num 5)
```

Thus *mkTerm num n* is the n^{th} term in the series, using *num* samples.

Using the helper function *printComplexL* defined in Figure 19.4, which "pretty prints" a list of complex numbers, we can look at the result of our

```
printComplexL :: [Complex Double] → IO ()
printComplexL xs =
  let f (i, rl :+ im) =
        do putStr (spaces (3 − length (show i)))
           putStr (show i ++ ":  (")
           putStr (niceNum rl  ++ ", ")
           putStr (niceNum im ++ ")\n")
  in mapM_ f (zip [0 .. length xs − 1] xs)
niceNum :: Double → String
niceNum d =
  let d' = fromIntegral (round (1 e10 ∗ d))/1 e10
      (dec, fra)  = break (== '.') (show d')
      (fra', exp) = break (== 'e') fra
  in spaces (3 − length dec) ++ dec ++ take 11 fra'
     ++ exp ++ spaces (12 − length fra' − length exp)
spaces :: Int → String
spaces n = take n (repeat ' ')
```

Figure 19.4: Helper Code for Pretty-Printing DFT Results

DFT in a more readable form.[3]

For example, suppose we want to take the DFT of a 16-sample representation of the first three terms of the square wave series. Typing the following at the GHCi prompt:

```
printComplexL (dft (mkxc 16))
```

will yield the result of the DFT, pretty-printing each number as a pair, along with its index:

```
0:  (  0.0          ,   0.0          )
1:  (  0.0          ,  -0.5          )
2:  (  0.0          ,   0.0          )
3:  (  0.0          ,  -0.1666666667 )
4:  (  0.0          ,   0.0          )
5:  (  0.0          ,  -0.1          )
6:  (  0.0          ,   0.0          )
```

[3] "Pretty-printing" real numbers is a subtle task. The code in Figure 19.4 rounds the numnber to 10 decimal places of accuracy, and inserts spaces before and after to line up the decimal points and give a consistent string length. The fractional part is not padded with zeros, since that would give a false impression of its accuracy. (It is not necessary to understand this code in order to understand the concepts in this chapter.)

```
 7:  (  0.0          ,   0.0          )
 8:  (  0.0          ,   0.0          )
 9:  (  0.0          ,   0.0          )
10:  (  0.0          ,   0.0          )
11:  (  0.0          ,   0.1          )
12:  (  0.0          ,   0.0          )
13:  (  0.0          ,   0.1666666667 )
14:  (  0.0          ,   0.0          )
15:  (  0.0          ,   0.5          )
```

Let's study this result more closely. For sake of argument, assume a sample rate of 1.6 KHz. Then by construction using *mkxc*, our square-wave input's fundamental frequency is 100 Hz. Similarly, recall that the resolution of the DFT is r/N, which is also 100 Hz.

Now compare the overall result to Figure 19.1b. Recalling also Equation 19.8, we note that the above DFT results are non-zero precisely at 100, 300, 500, -500, -300, and -100 Hz. This is just what we would expect. Furthermore, the amplitudes are one-half of the corresponding harmonically decreasing weights dictated by Equation 19.4, namely the values 1, 1/6, and 1/10 (recall the discussion in Section 19.2.2).

Let's do another example. We can create an impulse function as follows:

$$mkPulse :: Int \rightarrow [Complex\ Double]$$
$$mkPulse\ n = 100 : take\ (n-1)\ (repeat\ 0)$$

and print its DFT with the command:

$$printComplexL\ (dft\ (mkPulse\ 16))$$

whose effect is:

```
 0:  (  6.25         ,   0.0          )
 1:  (  6.25         ,   0.0          )
 2:  (  6.25         ,   0.0          )
 3:  (  6.25         ,   0.0          )
 4:  (  6.25         ,   0.0          )
 5:  (  6.25         ,   0.0          )
 6:  (  6.25         ,   0.0          )
 7:  (  6.25         ,   0.0          )
 8:  (  6.25         ,   0.0          )
 9:  (  6.25         ,   0.0          )
10:  (  6.25         ,   0.0          )
11:  (  6.25         ,   0.0          )
12:  (  6.25         ,   0.0          )
13:  (  6.25         ,   0.0          )
```

```
14:  (  6.25          ,   0.0           )
15:  (  6.25          ,   0.0           )
```

Compare this to Figure 19.1c, and note how the original magnitude of the impulse (100) is distributed evenly among the 16 points in the DFT ($100/16 = 6.25$).

So far we have considered only input signals whose frequency components are integral multiples of the DFT's resolution. This rarely happens in practice, however, because music is simply too complex, and noisy. As mentioned in 19.2.1, the energy of the signals that "fall in the gaps" is distributed among neighboring points, although not in as simple a way as you might think. To get some perspective on this, let's do one other example. We define a function to generate a signal whose frequeny is π times the fundamental frequency:

$$x_1\ num = \textbf{let}\ f = pi * 2 * pi\,/\,fromIntegral\ num$$
$$\textbf{in}\ map\ (:+0)\ [sin\ (f * fromIntegral\ i)$$
$$\mid i \leftarrow [0, 1\,..\,num - 1]]$$

π is an irrational number, but any number that "falls in the gaps" between indices would do. We can see the result by typing the command:

$$printComplexL\ (dft\ x_1)$$

which yields:

```
 0:  ( -7.9582433e-3 ,   0.0           )
 1:  ( -5.8639942e-3 ,  -1.56630897e-2)
 2:  (  4.7412105e-3 ,  -4.56112124e-2)
 3:  (  0.1860052232 ,  -0.4318552865 )
 4:  ( -5.72962095e-2,   7.33993364e-2)
 5:  ( -3.95845728e-2,   3.14378088e-2)
 6:  ( -3.47994673e-2,   1.65400768e-2)
 7:  ( -3.29813518e-2,   7.4048103e-3 )
 8:  ( -3.24834325e-2,   0.0           )
 9:  ( -3.29813518e-2,  -7.4048103e-3 )
10:  ( -3.47994673e-2,  -1.65400768e-2)
11:  ( -3.95845728e-2,  -3.14378088e-2)
12:  ( -5.72962095e-2,  -7.33993364e-2)
13:  (  0.1860052232 ,   0.4318552865 )
14:  (  4.7412105e-3 ,   4.56112124e-2)
15:  ( -5.8639942e-3 ,   1.56630897e-2)
```

This is much more complicated than the previous examples! Not only do the points in the spectrum seem to have varying amounts of energy, they also

have both non-zero real and non-zero imaginary components, meaning that the magnitude and phase vary at each point. We can define a function that converts a list of complex numbers into a list of their polar representations as follows:

$$mkPolars :: [Complex\ Double] \to [Complex\ Double]$$
$$mkPolars = map\ ((\lambda(m, p) \to m :+ p) \circ polar)$$

which we can then use to reprint our result:

$$printComplexL\ (mkPolars\ (dft\ x_1))$$

```
 0:  (  7.9582433e-3 ,   3.1415926536 )
 1:  (  1.67247961e-2,  -1.9290259418 )
 2:  (  4.58569709e-2,  -1.4672199604 )
 3:  (  0.470209455  ,  -1.1640975898 )
 4:  (  9.31145435e-2,   2.2336013741 )
 5:  (  5.05497204e-2,   2.4704023271 )
 6:  (  3.85302097e-2,   2.6979021519 )
 7:  (  3.38023784e-2,   2.9207398294 )
 8:  (  3.24834325e-2,  -3.1415926536 )
 9:  (  3.38023784e-2,  -2.9207398294 )
10:  (  3.85302097e-2,  -2.6979021519 )
11:  (  5.05497204e-2,  -2.4704023271 )
12:  (  9.31145435e-2,  -2.2336013741 )
13:  (  0.470209455  ,   1.1640975898 )
14:  (  4.58569709e-2,   1.4672199604 )
15:  (  1.67247961e-2,   1.9290259418 )
```

If we focus on the magnitude (the first column), we can see that there is a peak near index 3 (corresponding roughly to the frequency π), with small amounts of energy elsewhere.

Exercise 19.1 Write a Haskell function *idft* that implements the *inverse* DFT as captured in Equation 19.3. Test your code by applying *idft* to one of the signals used earlier in this section. In other words, show empirically that, up to round-off errors, $idft\ (dft\ xs) == xs$.

Exercise 19.2 Use *dft* to analyze some of the signals generated using signal functions defined in Chapter 18.

[**To do:** To do the above exercise we need to provide a function that extracts N samples from a sigfun, and somehow keeps it in the sigfun world. Perhaps something like:

$$sample :: Rate \to Int \to Signal\ c\ a\ (Event\ (Table\ a))$$

such that *sample* r n is a sigfun that generates an event every $1/r$ seconds, each event being a table containing n samples of the input. These tables may or may not overlap, depending on the relationship between r, n, and the sampling rate.]

Exercise 19.3 Define a function $mkSqWave :: Int \to Int \to [Complex\ Double]$ such that $mkSqWave\ num\ n$ is the sum of the first n terms of the Fourier series of a square wave, having num samples in the result.

Exercise 19.4 Prove mathematically that x and $\hat{x}$ are inverses. Also prove, using equational reasoning, that *dft* and *idft* are inverses. (For the latter you may assume that Haskell numeric types obey the standard axioms of real arithmetic.)

19.3 The Fast Fourier Transform

In the last section a DFT program was developed in Haskell that was easy to understand, being a faithful translation of Equation 19.5. For pedogogical purposes, this effort served us well. However, for practical purposes, the program is inherently inefficient.

To see why, think of $x[n]$ and $\hat{x}[k]$ as vectors. Thus, for example, each element of $\hat{x}$ is the sum of N multiplications of a vector by a complex exponential (which can be represented as a pair, the real and imaginary parts). And this overall process must be repeated for each value of k, also N times. Therefore the overall time complexity of the implied algorithm is $O(N^2)$. For even moderate values of N, this can be computationally intractable. (Our choice of lists for the implementation of vectors makes the complexity even worse, because of the linear-time complexity of indexing, but the discussion below makes this a moot point.)

Fortunately, there exists a much faster algorithm called the *Fast Fourier Transform*, or FFT, that reduces the complexity to $O(N \log N)$. This difference is quite significant for large values of N, and is the standard algorithm used in most signal processing applications. We will not go into the details of the FFT algorithm, other than to note that it is a divide-and-conquer algorithm that depends on the vector size being a power of two.[4]

[4]The basic FFT algorithm was invented by James Cooley and John Tukey in 1965.

Rather than developing our own program for the FFT, we will instead use the Haskell library *Numeric.FFT* to import a function that will do the job for us. Specifically:

fft :: ...

With this function we could explore the use of the FFT on specific iinput vectors, as we did earlier with *dft*.

However, our ultimate goal is to have a version of FFT that works on *signals*. We would like to be able to specify the number of samples as a power of two (which we can think of as the "window size"), the clock rate, and how often we would like to take a snapshot of the current window (and thus successive windows may or may not overlap). The resulting signal function takes a signal as input, and outputs *events* at the specified rate. Events are discussed in more detail in Chapter 16.

Indeed, Euterpea provide this functionality for us in a function called *fftA*:

fftA :: *Int* → *Double* → *Int* → *SF Double* (*Event FFTData*)
type *FFTData* = *Map Double Double*

SF is a signal function type similar to *SigFun*, except that it is targeted for use in the Musical User Interface (MUI) discussed in detail in Chapter **??**, and thus, for example, does not have a clock rate. *Map* T_1 T_2 is an abstract type that maps values of type T_1 to values of type T_2, and is imported from *Data.Map*.

fftA winInt rate size is a signal function that, every *winInt* samples of the input, creates a window of size 2^*size*, and computes the FFT of that window. For every such result, it issues an *Event* that maps from frequency to magnitude (using the clock rate *rate* to determine the proper mapping).

Combining *fftA* with the MUI widgets discussed in Chapter **??**, we can write a simple program that generates a sine wave whose frequency is controlledd by a slider, and whose real-time graph as well as its FFT are displayed. The program to do this is shown in Figure 19.5.

19.4 Further Pragmatics

[**To do:** Discuss windowing.]

Exercise 19.5 Modify the program in Figure 19.5 in the following ways:

```
fftEx :: UISF () ()
fftEx = proc _ → do
    f       ← hSlider (1, 2000) 440 ⤙ ()
    (d, _) ← convertToUISF 100 simpleSig ⤙ f
    let (s, fft) = unzip d
    _ ← histogram (500, 150) 20 ⤙ listToMaybe (catMaybes fft)
    _ ← realtimeGraph' (500, 150) 200 20 Black ⤙ s
    outA ⤙ ()
  where
    simpleSig :: SigFun CtrRate Double (Double, Event FFTData)
    simpleSig = proc f → do
      s     ← osc (tableSinesN 4096 [1]) 0 ⤙ f
      fft   ← fftA 100 (rate (⊥ :: CtrRate)) 8 ⤙ s
      outA ⤙ (s, fft)
t0 = runUIEx (500, 600) "fft Test" fftEx
```

Figure 19.5: A Real-Time Display of FFT Results

1. Add a second slider, and use it to control the frequency of a second oscillator.

2. Let s_1 and s_2 be the names of the signals whose frequencies are controlled by the first and second sliders, respectively. Instead of displaying the FFT of just s_1, try a variety of combinations of s_1 and s_2, such as $s_1 + s_2$, $s_1 - s_2$, $s_1 * s_2$, $1/s_1 + 1/s_2$, and s_1/s_2. Comment on the results.

3. Use s_2 to control the frequency of s_1 (as was done with *vibrato* in Chapter 18). Plot the fft of s_1 and comment on the result.

4. Instead of using *osc* to generate a pure sine wave, try using other oscillators and/or table generators to create more complex tones, and plot their FFT's. Comment on the results.

19.5 References

Most of the ideas in this chapter can be found in any good textbook on signal processing, such as []. The particular arrangement of the material here, in particular Figure 19.1 and the development and demonstration of a

program for the DFT, is borrowed from the excellent text *Computer Music* by Moore [**?**].

Chapter 20

Additive Synthesis and Amplitude Modulation

```
{-# LANGUAGE Arrows #-}
module Euterpea.Music.Signal.Additive where
import Euterpea
import Control.Arrow ((>>>), (<<<), arr)
```

Additive synthesis is, conceptually at least, the simplest of many sound synthesis techniques. Simply put, the idea is to add signals (usually sine waves of differing amplitudes, frequencies and phases) together to form a sound of interest. It is based on Fourier's theorem as discussed in the previous chapter, and indeed is sometimes called *Fourier synthesis*.

20.1 Preliminaries

When doing pure additive synthesis it is often convenient to work with a *list of signal sources* whose elements are eventually summed together to form a result. To facilitate this, we define a few auxiliary functions, as shown in Figure 20.1.

constSF s sf simply lifts the value *s* to the signal function level, and composes that with *sf*, thus yielding a signal source.

foldSF f b sfs is analogous to *foldr* for lists: it returns the signal source *constA b* if the list is empty, and otherwise uses *f* to combine the results, pointwise, from the right. In other words, if *sfs* has the form:

```
constSF :: Clock c ⇒ a → SigFun c a b → SigFun c () b
constSF s sf = constA s ⋙ sf

foldSF :: Clock c ⇒
          (a → b → b) → b → [SigFun c () a] → SigFun c () b
foldSF f b sfs =
  foldr g (constA b) sfs where
    g sfa sfb =
      proc () → do
        s1 ← sfa ⤙ ()
        s2 ← sfb ⤙ ()
        outA ⤙ f s1 s2
```

Figure 20.1: Working With Lists of Signal Sources

sf1 : *sf2* : ... : *sfn* : [ ]

then the result will be:

```
proc () → do
  s1 ← sf1 ⤙ ()
  s2 ← sf2 ⤙ ()
  ...
  sn ← sfn ⤙ ()
  outA ⤙ f s1 (f s2 (...(f sn b)))
```

20.2 A Bell Sound

A bell, or gong, sound is a good example of the use of "brute force" additive synthesis. Physically, a bell or gong can be thought of as a bunch of concentric rings, each having a different resonant frequency because they differ in diameter depending on the shape of the bell. Some of the rings will be more dominant than others, but the important thing to note is that these resonant frequencies often do not have an integral relationship with each other, and sometimes the higher frequencies can be quite strong, rather than rolling off significantly as with many other instruments. Indeed, it is sometime difficult to say exactly what the pitch of a particular bell is (especially large bells), so complex is its sound. Of course, the pitch of a bell can be controlled by mimimizing the taper of its shape (especially for small bells), thus giving it more of a pitched sound.

```
bell1 :: Instr (Mono AudRate)
        -- Dur -¿ AbsPitch -¿ Volume -¿ AudSF () Double
bell1 dur ap vol [] =
  let f   = apToHz ap
      v   = fromIntegral vol/100
      d   = fromRational dur
      sfs = map (λp → constA (f * p) >>> osc tab1 0)
                [4.07, 3.76, 3, 2.74, 2, 1.71, 1.19, 0.92, 0.56]
  in proc () → do
      aenv ← envExponSeg [0, 1, 0.001] [0.003, d − 0.003] -≺ ()
      a1   ← foldSF (+) 0 sfs -≺ ()
      outA -≺ a1 * aenv * v/9
tab1 = tableSinesN 4096 [1]
test1 = outFile "bell1.wav" 6 (bell1 6 (absPitch (C, 5)) 100 [])
```

Figure 20.2: A Bell Instrument

In any case, a pitched instrument representing a bell sound can be designed using additive synthesis by using the instrument's absolute pitch to create a series of partials that are conspicuously non-integral multiples of the fundamental. If this sound is then shaped by an envelope having a sharp rise time and a relatively slow, exponentially decreasing decay, we get a decent result. A Euterpea program to achieve this is shown in Figure 20.2. Note the use of *map* to create the list of partials, and *foldSF* to add them together. Also note that some of the partials are expressed as *fractions* of the fundamental—i.e. their frequencies are less than that of the fundamental.

The reader might wonder why we don't just use one of Euterpea's table generating functions, such as:

```
tableSines3, tableSines3N ::
  TableSize → [(PartialNum, PartialStrength, PhaseOffset)] → Table
```

to generate a table with all the desired partials. The problem is, even though *PartialNum* is a *Double*, the intent is that the partial numbers all be integral. To see why, suppose 1.5 were one of the partial numbers—then 1.5 cycles of a sine wave would be written into the table. But the whole point of wavetable lookup synthesis is that the wavetable be a periodic representation of the desired sound—but that is certainly not true of 1.5 cycles of a sine wave. The situation gets worse with partials such as 4.07, 3.75, 2.74, 0.56, and so on.

```
bell2 :: Instr (Mono AudRate)
        -- Dur -¿ AbsPitch -¿ Volume -¿ AudSF () Double
bell2 dur ap vol [] =
  let f  = apToHz ap
      v  = fromIntegral vol/100
      d  = fromRational dur
      sfs = map (mySF f d)
                [4.07, 3.76, 3, 2.74, 2, 1.71, 1.19, 0.92, 0.56]
  in proc () → do
      a1  ← foldSF (+) 0 sfs ⤙ ()
      outA ⤙ a1 * v/9
mySF f d p = proc () → do
  s    ← osc tab1 0 ⋘ constA (f * p) ⤙ ()
  aenv ← envExponSeg [0, 1, 0.001] [0.003, d/p − 0.003] ⤙ ()
  outA ⤙ s * aenv
test2 = outFile "bell2.wav" 6 (bell2 6 (absPitch (C, 5)) 100 [])
```

Figure 20.3: A More Sophisticated Bell Instrument

In any case, we can do even better than *bell1*. An important aspect of a bell sound that is not captured by the program in Figure 20.2, is that the higher frequency partials tend to decay more quickly than the lower ones. We can remedy this by giving each partial its own envelope, and making the duration of the envelope inversely proportional to the partial number. Such a more sophisticated instrument is shown in Figure 20.3. This results in a much more pleasing and realistic sound.

Exercise 20.1 A problem with the more sophisticated bell sound in Figure 20.3 is that the duration of the resulting sound exceeds the specified duration of the note, because some of the partial numbers are less than one. Fix this.

Exercise 20.2 Neither of the bell sounds shown in Figures **??** and 20.3 actually contain the fundamental frequency—i.e. a partial number of 1.0. Yet they contain the partials at the integer multiples 2 and 3. How does this affect the result? What happens if you add in the fundamental?

20.3 Amplitude Modulation

Technically speaking, whenever the amplitude of a signal is dynamically changed, it is a form of *amplitude modulation*, or *AM* for short; that is, we are modulating the amplitude of a signal. So, for example, shaping a signal with an envelope, as well as adding tremolo, are both forms of AM. In this section more interesting forms of AM are explored, including their mathematical basis. To help distinguish these forms of AM from others, we define a few terms:

- The dynamically changing signal that is doing the modulation is called the *modulating signal*
- The signal being modulated is sometimes called the *carrier*.
- A *unipolar signal* is one that is always either positive or negative (usually positive).
- A *bipolar signal* is one that takes on both positive and negative values (that are often symmetric and thus average out to zero).

So, shaping a signal using an envelope is an example of amplitude modulation using a unipolar modulating signal whose frequency is very low (to be precise, $1/dur$, where *dur* is the length of the note), and in fact only one cyctle of that signal is used. Likewise, tremolo is an example of amplitude modulation with a unipolar modulating signal whose frequency is a bit higher than with envelope shaping, but still quite low (typically 2-10 Hz). In both cases, the modulating signal is infrasonic.

Note that a bipolar signal can be made unipolar (or the other way around) by adding or subtracting an offset (sometimes called a "DC offset," where DC is shorthand for "direct current"). This is readily seen if we try to mathematically formalize the notion of tremolo. Specifically, tremolo can be defined as adding an offset of 1 to an infrasonic sine wave whose frequency is f_t (typically 2-10Hz), multiplying that by a "depth" argument d (in the range 0 to 1), and using the result as the modulating signal; the carrier frequency is f:

$$(1 + d \times \sin(2\pi f_t t)) \times \sin(2\pi f t)$$

Based on this equation, here is a simple tremolo envelope generator written in Euterpea, and defined as a signal source (see Exercise 18.2):

```
tremolo :: Clock c ⇒
           Double → Double → SigFun c () Double
tremolo tfrq dep = proc () → do
  trem ← osc tab1 0 −≺ tfrq
  outA −≺ 1 + dep ∗ trem
```

tremolo can then be used to modulate an audible signal as follows:

...

20.3.1 AM Sound Synthesis

What happens when the modulating signal is audible, just like the carrier signal? This is where things get interesting from a sound synthesis point of view, and can result in a rich blend of sounds. To understand this mathematically, recall this trigonometric identity:

$$\sin(C) \times \sin(M) = \frac{1}{2}(\cos(C - M) - \cos(C + M))$$

or, sticking entirely with cosines:

$$\cos(C) \times \cos(M) = \frac{1}{2}(\cos(C - M) + \cos(C + M))$$

These equations demonstrate that AM is really just additive synthesis, which is why the two topics are included in the same chapter. Indeed, the equations imply two ways to implement AM in Euterpea: We can directly multiply the two outputs, as specified by the left-hand sides of the equations above, or we can add two signals as specified by the right-hand sides of the equations.

Note the following:

1. When the modulating frequency is the same as the carrier frequency, the right-hand sides above reduce to $1/2\cos(2C)$. That is, we essentially double the frequency.

2. Since multiplication is commutative, the following is also true:

$$\cos(C) \times \cos(M) = \frac{1}{2}(\cos(M - C) + \cos(M + C))$$

which is validated because $\cos(t) = \cos(-t)$.

3. Scaling the modulating signal or carrier just scales the entire signal, since multiplication is associative.

Also note that adding a third modulating frequency yields the following:

$$\begin{aligned}
&\cos(C) \times \cos(M1) \times cos(M2) \\
&\quad = (0.5 \times (\cos(C - M1) \times \cos(C + M1))) \times \cos(M2) \\
&\quad = 0.5 \times (\cos(C - M1) \times \cos(M2) + \cos(C + M1) \times \cos(M2)) \\
&\quad = 0.25 \times (\cos(C - M1 - M2) + \cos(C - M1 + M2) + \\
&\qquad \cos(C + M1 - M2) + \cos(C + M1 + M2))
\end{aligned}$$

In general, combining n signals using amplitude modulation results in 2^{n-1} signals. AM used in this way for sound synthesis is sometimes called *ring modulation*, because the analog circuit (of diodes) originally used to implement this technique took the shape of a ring. Some nice bell-like tones can be generated with this technique.

20.4 What do Tremolo and AM Radio Have in Common?

Combining the previous two ideas, we can use a bipolar carrier in the *electromagnetic spectrum* (i.e. the radio spectrum) and a unipolar modulating frequency in the *audible* range, which we can represent mathematically as:

$$\cos(C) \times (1 + \cos(M)) = \cos(C) + 0.5 \times (\cos(C - M) + \cos(C + M))$$

Indeed, this is how AM radio works. The above equation says that AM Radio results in a carrier signal plus two sidebands. To completely cover the audible frequency range, the modulating frequency would need to be as much as 20kHz, thus yielding sidebands of $\pm$20kHz, thus requiring station separation of at least 40 kHz. Yet, note that AM radio stations are separated by only 10kHz! (540 kHz, 550 kHz, ..., 1600 kHz). This is because, at the time Commercial AM Radio was developed, a fidelity of 5KHz was considered "good enough."

Also note now that the amplitude of the modulating frequency does matter:

$$\cos(C) \times (1 + A \times cos(M)) = cos(C) + 0.5 \times A \times (\cos(C - M) + \cos(C + M))$$

A, called the *modulation index*, controls the size of the sidebands. Note the similarity of this equation to that for tremolo.

Appendix A

The PreludeList Module

The use of lists is particularly common when programming in Haskell, and thus, not surprisingly, there are many pre-defined polymorphic functions for lists. The list data type itself, plus some of the most useful functions on it, are contained in the Standard Prelude's *PreludeList* module, which we will look at in detail in this chapter. There is also a Standard Library module called *List* that has additional useful functions. It is a good idea to become familiar with both modules.

Although this chapter may feel like a long list of "Haskell features," the functions described here capture many common patterns of list usage that have been discovered by functional programmers over many years of trials and tribulations. In many ways higher-order declarative programming with lists takes the place of lower-level imperative control structures in more conventional languages. By becoming familiar with these list functions you will be able to more quickly and confidently develop your own applications using lists. Furthermore, if all of us do this, we will have a common vocabulary with which to understand each others' programs. Finally, by reading through the code in this module you will develop a good feel for how to write proper function definitions in Haskell.

It is not necessary for you to understand the details of every function, but you should try to get a sense for what is available so that you can return later when your programming needs demand it. In the long run you are well-advised to read the rest of the Standard Prelude as well as the various Standard Libraries, to discover a host of other functions and data types that you might someday find useful in your own work.

A.1 The PreludeList Module

To get a feel for the *PreludeList* module, let's first look at its module declaration:

```
module PreludeList (
  map, (++), filter, concat,
  head, last, tail, init, null, length, (!!),
  foldl, foldl1, scanl, scanl1, foldr, foldr1, scanr, scanr1,
  iterate, repeat, replicate, cycle,
  take, drop, splitAt, takeWhile, dropWhile, span, break,
  lines, words, unlines, unwords, reverse, and, or,
  any, all, elem, notElem, lookup,
  sum, product, maximum, minimum, concatMap,
  zip, zip3, zipWith, zipWith3, unzip, unzip3)
  where
import qualified Char (isSpace)
infixl 9 !!
infixr 5 ++
infix 4  ∈, ∉
```

We will not discuss all of the functions listed above, but will cover most of them (and some were discussed in previous chapters).

A.2 Simple List Selector Functions

head and *tail* extract the first element and remaining elements, respectively, from a list, which must be non-empty. *last* and *init* are the dual functions that work from the end of a list, rather than from the beginning.

```
head          :: [a] → a
head (x: _)   = x
head []       = error "PreludeList.head: empty list"

last          :: [a] → a
last [x]      = x
last (_ : xs) = last xs
last []       = error "PreludeList.last: empty list"

tail          :: [a] → [a]
tail (_ : xs) = xs
tail []       = error "PreludeList.tail: empty list"
```

```
init          :: [a] → [a]
init [x]      = []
init (x : xs) = x : init xs
init []       = error "PreludeList.init: empty list"
```

Although *head* and *tail* were previously discussed in Section 3.1, the definitions here include an equation describing their behaviors under erroneous situations—such as selecting the head of an empty list—in which case the *error* function is called. It is a good idea to include such an equation for any definition in which you have not covered every possible case in pattern-matching; i.e. if it is possible that the pattern-matching could "run off the end" of the set of equations. The string argument that you supply to the *error* function should be detailed enough that you can easily track down the precise location of the error in your program.

Details: If such an error equation is omitted, and then during pattern-matching all equations fail, most Haskell systems will invoke the *error* function anyway, but most likely with a string that will be less informative than one you can supply on your own.

The *null* function tests to see if a list is empty.

```
null          :: [a] → Bool
null []       = True
null (_ : _)  = False
```

A.3 Index-Based Selector Functions

To select the nth element from a list, with the first element being the 0th element, we can use the indexing function (!!):

```
(!!)          :: [a] → Int → a
(x : _) !! 0 =              x
(_ : xs) !! n | n > 0 = xs !! (n − 1)
(_ : _) !! _ =              error "PreludeList.!!: negative index"
[]      !! _ =              error "PreludeList.!!: index too large"
```

Details: Note the definition of two error conditions; be sure that you understand under what conditions these two equations would succeed. In particular, recall that equations are matched in top-down order: the first to match is the one that is chosen.

take n *xs* returns the prefix of *xs* of length n, or *xs* itself if $n > \mathit{length}\ xs$. Similarly, *drop* n *xs* returns the suffix of *xs* after the first n elements, or $[\,]$ if $n > \mathit{length}\ xs$. Finally, *splitAt* n *xs* is equivalent to $(\mathit{take}\ n\ xs, \mathit{drop}\ n\ xs)$.

$$
\begin{aligned}
&\mathit{take} && :: \mathit{Int} \to [a] \to [a]\\
&\mathit{take}\ 0\ \_ && = [\,]\\
&\mathit{take}\ \_\ [\,] && = [\,]\\
&\mathit{take}\ n\ (x : xs) \mid n > 0 && = x : \mathit{take}\ (n-1)\ xs\\
&\mathit{take}\ \_\ \_ && =\\
&\quad \mathit{error}\ \texttt{"PreludeList.take: negative argument"}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{drop} && :: \mathit{Int} \to [a] \to [a]\\
&\mathit{drop}\ 0\ xs && = xs\\
&\mathit{drop}\ \_\ [\,] && = [\,]\\
&\mathit{drop}\ n\ (\_ : xs) \mid n > 0 && = \mathit{drop}\ (n-1)\ xs\\
&\mathit{drop}\ \_\ \_ && =\\
&\quad \mathit{error}\ \texttt{"PreludeList.drop: negative argument"}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{splitAt} && :: \mathit{Int} \to [a] \to ([a], [a])\\
&\mathit{splitAt}\ 0\ xs && = ([\,], xs)\\
&\mathit{splitAt}\ \_\ [\,] && = ([\,], [\,])\\
&\mathit{splitAt}\ n\ (x : xs) \mid n > 0 && = (x : xs', xs'')\\
&\qquad && \quad \textbf{where}\ (xs', xs'') = \mathit{splitAt}\ (n-1)\ xs\\
&\mathit{splitAt}\ \_\ \_ = &&\\
&\quad \mathit{error}\ \texttt{"PreludeList.splitAt: negative argument"}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{length} && :: [a] \to \mathit{Int}\\
&\mathit{length}\ [\,] && = 0\\
&\mathit{length}\ (\_ : l) && = 1 + \mathit{length}\ l
\end{aligned}
$$

For example:

$$
\begin{aligned}
&\mathit{take}\ 3\ [0, 1\,..\,5] \Rightarrow [0, 1, 2]\\
&\mathit{drop}\ 3\ [0, 1\,..\,5] \Rightarrow [3, 4, 5]\\
&\mathit{splitAt}\ 3\ [0, 1\,..\,5] \Rightarrow ([0, 1, 2], [3, 4, 5])
\end{aligned}
$$

A.4 Predicate-Based Selector Functions

takeWhile p xs returns the longest (possibly empty) prefix of *xs*, all of whose elements satisfy the predicate *p*. *dropWhile p xs* returns the remaining suffix. Finally, *span p xs* is equivalent to (*takeWhile p xs*, *dropWhile p xs*), while *break p* uses the negation of *p*.

$$
\begin{aligned}
&\mathit{takeWhile} && :: (a \to \mathit{Bool}) \to [a] \to [a] \\
&\mathit{takeWhile}\ p\ [\,] && = [\,] \\
&\mathit{takeWhile}\ p\ (x : xs) \\
&\quad \mid p\ x && = x : \mathit{takeWhile}\ p\ xs \\
&\quad \mid \mathit{otherwise} = && [\,] \\
&\mathit{dropWhile} && :: (a \to \mathit{Bool}) \to [a] \to [a] \\
&\mathit{dropWhile}\ p\ [\,] && = [\,] \\
&\mathit{dropWhile}\ p\ xs@(x : xs') \\
&\quad \mid p\ x && = \mathit{dropWhile}\ p\ xs' \\
&\quad \mid \mathit{otherwise} = && xs \\
&\mathit{span}, \mathit{break} && :: (a \to \mathit{Bool}) \to [a] \to ([a], [a]) \\
&\mathit{span}\ p\ [\,] && = ([\,], [\,]) \\
&\mathit{span}\ p\ xs@(x : xs') \\
&\quad \mid p\ x && = (x : xs', xs'')\ \textbf{where}\ (xs', xs'') = \mathit{span}\ p\ xs \\
&\quad \mid \mathit{otherwise} = && (xs, [\,]) \\
&\mathit{break}\ p && = \mathit{span}\ (\neg \circ p)
\end{aligned}
$$

filter removes all elements not satisfying a predicate:

$$
\begin{aligned}
&\mathit{filter} :: (a \to \mathit{Bool}) \to [a] \to [a] \\
&\mathit{filter}\ p\ [\,] = [\,] \\
&\mathit{filter}\ p\ (x : xs) \mid p\ x = x : \mathit{filter}\ p\ xs \\
&\quad \mid \mathit{otherwise} = \mathit{filter}\ p\ xs
\end{aligned}
$$

A.5 Fold-like Functions

foldl1 and *foldr1* are variants of *foldl* and *foldr* that have no starting value argument, and thus must be applied to non-empty lists.

$$
\begin{aligned}
&\mathit{foldl} && :: (a \to b \to a) \to a \to [b] \to a \\
&\mathit{foldl}\ f\ z\ [\,] && = z \\
&\mathit{foldl}\ f\ z\ (x : xs) && = \mathit{foldl}\ f\ (f\ z\ x)\ xs \\
&\mathit{foldl1} && :: (a \to a \to a) \to [a] \to a \\
&\mathit{foldl1}\ f\ (x : xs) && = \mathit{foldl}\ f\ x\ xs
\end{aligned}
$$

```
foldl1 _ []     = error "PreludeList.foldl1: empty list"
foldr           :: (a -> b -> b) -> b -> [a] -> b
foldr f z []    = z
foldr f z (x:xs) = f x (foldr f z xs)
foldr1          :: (a -> a -> a) -> [a] -> a
foldr1 f [x]    = x
foldr1 f (x:xs) = f x (foldr1 f xs)
foldr1 _ []     = error "PreludeList.foldr1: empty list"
```

foldl1 and *foldr1* are best used in cases where an empty list makes no sense for the application. For example, computing the maximum or mimimum element of a list does not make sense if the list is empty. Thus *foldl1 max* is a proper function to compute the maximum element of a list.

scanl is similar to *foldl*, but returns a list of successive reduced values from the left:

$$scanl\ f\ z\ [x_1, x_2, ...] == [z, z\ `f`\ x_1, (z\ `f`\ x_1)\ `f`\ x_2, ...]$$

For example:

$$scanl\ (+)\ 0\ [1,2,3] \Rightarrow [0,1,3,6]$$

Note that $last\ (scanl\ f\ z\ xs) = foldl\ f\ z\ xs$. *scanl1* is similar, but without the starting element:

$$scanl1\ f\ [x_1, x_2, ...] == [x_1, x_1\ `f`\ x_2, ...]$$

Here are the full definitions:

```
scanl           :: (a -> b -> a) -> a -> [b] -> [a]
scanl f q xs    = q : (case xs of
                         []   -> []
                         x:xs -> scanl f (f q x) xs)
scanl1          :: (a -> a -> a) -> [a] -> [a]
scanl1 f (x:xs) = scanl f x xs
scanl1 _ []     = error "PreludeList.scanl1: empty list"
scanr           :: (a -> b -> b) -> b -> [a] -> [b]
scanr f q0 []   =  [q0]
scanr f q0 (x:xs) = f x q : qs
                    where qs@(q: _) = scanr f q0 xs
scanr1          :: (a -> a -> a) -> [a] -> [a]
scanr1 f [x]    = [x]
scanr1 f (x:xs) = f x q : qs
                  where qs@(q: _) = scanr1 f xs
```

$scanr1\ \_\ [\,] \quad = error$ `"PreludeList.scanr1: empty list"`

A.6 List Generators

There are some functions which are very useful for generating lists from scratch in interesting ways. To start, *iterate* $f\ x$ returns an *infinite list* of repeated applications of f to x. That is:

$iterate\ f\ x \Rightarrow [x, f\ x, f\ (f\ x), ...]$

The "infinite" nature of this list may at first seem alarming, but in fact is one of the more powerful and useful features of Haskell.

[say more]

$$\begin{aligned} &iterate && :: (a \to a) \to a \to [a] \\ &iterate\ f\ x && = x : iterate\ f\ (f\ x) \end{aligned}$$

repeat x is an infinite list, with x the value of every element. *replicate* $n\ x$ is a list of length n with x the value of every element. And *cycle* ties a finite list into a circular one, or equivalently, the infinite repetition of the original list.

$$\begin{aligned} &repeat && :: a \to [a] \\ &repeat\ x && = xs\ \textbf{where}\ xs = x : xs \\ &replicate && :: Int \to a \to [a] \\ &replicate\ n\ x && = take\ n\ (repeat\ x) \\ &cycle && :: [a] \to [a] \\ &cycle\ [\,] && = error\ \texttt{"Prelude.cycle: empty list"} \\ &cycle\ xs && = xs'\ \textbf{where}\ xs' = xs +\!\!+ xs' \end{aligned}$$

A.7 String-Based Functions

Recall that strings in Haskell are just lists of characters. Manipulating strings (i.e. text) is a very common practice, so it makes sense that Haskell would have a few pre-defined functions to make this easier for you.

lines breaks a string at every newline character (written as `'\n'` in Haskell), thus yielding a *list* of strings, each of which contains no newline characters. Similary, *words* breaks a string up into a list of words, which were delimited by white space. Finally, *unlines* and *unwords* are the inverse operations: *unlines* joins lines with terminating newline characters, and *unwords* joins words with separating spaces. (Because of the potential

presence of multiple spaces and newline characters, however, these pairs of functions are not true inverses of each other.)

```
lines        :: String → [String]
lines ""     = []
lines s      = let (l, s') = break (== '\n') s
               in l : case s' of
                        [] → []
                        (_ : s'') → lines s''

words        :: String → [String]
words s      = case dropWhile Char.isSpace s of
                 "" → []
                 s' → w : words s''
                   where (w, s'') = break Char.isSpace s'

unlines      :: [String] → String
unlines      = concatMap (++"\n")

unwords      :: [String] → String
unwords []   = ""
unwords ws   = foldr1 (λw s → w ++ ' ' : s) ws
```

reverse reverses the elements in a finite list.

```
reverse :: [a] → [a]
reverse = foldl (flip (:)) []
```

A.8 Boolean List Functions

and and *or* compute the logical "and" and "or," respectively, of all of the elements in a list of Boolean values.

```
and, or :: [Bool] → Bool
and     = foldr (∧) True
or      = foldr (∨) False
```

Applied to a predicate and a list, *any* determines if any element of the list satisfies the predicate. An analogous behavior holds for *all*.

```
any, all :: (a → Bool) → [a] → Bool
any p    = or ∘ map p
all p    = and ∘ map p
```

A.9 List Membership Functions

elem is the list membership predicate, usually written in infix form, e.g., $x \in xs$ (which is why it was given a fixity declaration at the beginning of the module). *notElem* is the negation of this function.

```
elem, notElem :: (Eq a) => a -> [a] -> Bool
elem x        = any (== x)
notElem x     = all (/= x)
```

It is common to store "key/value" pairs in a list, and to access the list by finding the value associated with a given key (for this reason the list is often called an *association list*). The function *lookup* looks up a key in an association list, returning *Nothing* if it is not found, or *Just y* if *y* is the value associated with the key.

```
lookup          :: (Eq a) => a -> [(a, b)] -> Maybe b
lookup key []   = Nothing
lookup key ((x, y) : xys)
    | key == x = Just y
    | otherwise = lookup key xys
```

A.10 Arithmetic on Lists

sum and *product* compute the sum and product, respectively, of a finite list of numbers.

```
sum, product :: (Num a) => [a] -> a
sum          = foldl (+) 0
product      = foldl (*) 1
```

maximum and *minimum* return the maximum and minimum value, respectively from a non-empty, finite list whose element type is ordered.

```
maximum, minimum :: (Ord a) => [a] -> a
maximum []  = error "Prelude.maximum: empty list"
maximum xs = foldl1 max xs

minimum []  = error "Prelude.minimum: empty list"
minimum xs = foldl1 min xs
```

Note that even though *foldl1* is used in the definition, a test is made for the empty list to give an error message that more accurately reflects the source of the problem.

A.11 List Combining Functions

map and (++) were defined in previous chapters, but are repeated here for completeness:

```
map :: (a → b) → [a] → [a]
map f [] = []
map f (x : xs) = f x : map f xs

(++) :: [a] → [a] → [a]
[] ++ ys = ys
(x : xs) ++ ys = x : (xs ++ ys)
```

concat appends together a list of lists:

```
concat :: [[a]] → [a]
concat xss = foldr (++) [] xss
```

concatMap does what it says: it concatenates the result of mapping a function down a list.

```
concatMap  :: (a → [b]) → [a] → [b]
concatMap f = concat ∘ map f
```

zip takes two lists and returns a list of corresponding pairs. If one input list is short, excess elements of the longer list are discarded. *zip3* takes three lists and returns a list of triples. ("Zips" for larger tuples are contained in the List Library.)

```
zip  :: [a] → [b] → [(a, b)]
zip  = zipWith (,)

zip3 :: [a] → [b] → [c] → [(a, b, c)]
zip3 = zipWith3 (,,)
```

Details: The functions *(,)* and *(,,)* are the pairing and tripling functions, respectively:

```
(,)  ⇒ λx y → (x, y)
(,,) ⇒ λx y z → (x, y, z)
```

The *zipWith* family generalises the *zip* and *map* families (or, in a sense, combines them) by applying a function (given as the first argument) to each pair (or triple, etc.) of values. For example, *zipWith* (+) is applied to two lists to produce the list of corresponding sums.

```
zipWith        :: (a → b → c) → [a] → [b] → [c]
zipWith z (a : as) (b : bs)
                =     z a b : zipWith z as bs
zipWith _ _ _ =     []
zipWith3       :: (a → b → c → d) → [a] → [b] → [c] → [d]
zipWith3 z (a : as) (b : bs) (c : cs)
                =     z a b c : zipWith3 z as bs cs
zipWith3 _ _ _ _ = []
```

The following two functions perform the inverse operations of *zip* and *zip3*, respectively.

```
unzip   :: [(a, b)] → ([a], [b])
unzip   = foldr (λ(a, b)∼(as, bs) → (a : as, b : bs)) ([], [])
unzip3 :: [(a, b, c)] → ([a], [b], [c])
unzip3 = foldr (λ(a, b, c)∼(as, bs, cs) → (a : as, b : bs, c : cs))
               ([], [], [])
```

Appendix B

Haskell's Standard Type Classes

This provides a "tour" through the predefined standard type classes in Haskell, as was done for lists in Chapter A. We have simplified these classes somewhat by omitting some of the less interesting methods; the Haskell Report and Standard Library Report contain more complete descriptions.

B.1 The Ordered Class

The equality class *Eq* was defined precisely in Chapter 7, along with a simplified version of the class *Ord*. Here is its full specification of class *Ord*; note the many default methods.

$$
\begin{array}{l}
\textbf{class}\ (Eq\ a) \Rightarrow Ord\ a\ \textbf{where} \\
\quad compare\ \ :: a \to a \to Ordering \\
\quad (<), (\leqslant), (\geqslant), (>) :: a \to a \to Bool \\
\quad max, min :: a \to a \to a \\
\quad compare\ x\ y \\
\qquad\quad \mid x == y \quad = \quad EQ \\
\qquad\quad \mid x \leqslant y \quad = \quad LT \\
\qquad\quad \mid otherwise = \quad GT \\
\quad x \leqslant y \qquad\quad = \quad compare\ x\ y \neq GT \\
\quad x < y \qquad\quad = \quad compare\ x\ y == LT \\
\quad x \geqslant y \qquad\quad = \quad compare\ x\ y \neq LT \\
\quad x > y \qquad\quad = \quad compare\ x\ y == GT
\end{array}
$$

```
    max x y
        | x ⩾ y      =  x
        | otherwise =  y
    min x y
        | x <     y  =  x
        | otherwise =  y
data Ordering = LT | EQ | GT
    deriving (Eq, Ord, Enum, Read, Show, Bounded)
```

Note that the default method for *compare* is defined in terms of ($\leqslant$), and that the default method for ($\leqslant$) is defined in terms of *compare*. This means that an instance of *Ord* should contain a method for at least one of these for everything to be well defined. (Using *compare* can be more efficient for complex types.) This is a common idea in designing a type class.

B.2 The Enumeration Class

Class *Enum* has a set of operations that underlie the syntactic sugar of *arithmetic sequences*; for example, the arithmetic sequence $[1, 3\,..]$ is actually shorthand for *enumFromThen* 1 3. If this is true, then we should be able to generate arithmetic sequences for any type that is an instance of *Enum*. This includes not only most numeric types, but also *Char*, so that, for instance, `['a'..'z']` denotes the list of lower-case letters in alphabetical order. Furthermore, a user-defined enumerated type such as *Color*:

```
data Color = Red | Orange | Yellow | Green | Blue | Indigo | Violet
```

can easily be given an *Enum* instance declaration, after which we can calculate the following results:

```
[Red .. Violet]    ⟹  [ Red, Orange, Yellow, Green,
                        Blue, Indigo, Violet]
[Red, Yellow ..]   ⟹  [ Red, Yellow, Blue, Violet]
fromEnum Green     ⟹  3
toEnum 5 :: Color  ⟹  Indigo
```

Indeed, the derived instance will give this result. Note that the sequences are still *arithmetic* in the sense that the increment between values is constant, even though the values are not numbers.

The complete definition of the *Enum* class is given below:

```
class Enum a where
  succ, pred          :: a → a
  toEnum              :: Int → a
  fromEnum            :: a → Int
  enumFrom            :: a → [a]             -- [n..]
  enumFromThen        :: a → a → [a]         -- [n,n'..]
  enumFromTo          :: a → a → [a]         -- [n..m]
  enumFromThenTo :: a → a → a → [a]  -- [n,n'..m]
    -- Minimal complete definition: toEnum, fromEnum
  succ                = toEnum ∘ (+1) ∘ fromEnum
  pred                = toEnum ∘ (subtract 1) ∘ fromEnum
  enumFrom x          = map toEnum [fromEnum x ..]
  enumFromThen x y = map toEnum [fromEnum x, fromEnum y ..]
  enumFromTo x y      = map toEnum [fromEnum x .. fromEnum y]
  enumFromThenTo x y z =
    map toEnum [fromEnum x, fromEnum y .. fromEnum z]
```

The six default methods are sufficient for most applications, so when writing your own instance declaration it is usually sufficient to only provide methods for the remaining two operations: *toEnum* and *fromEnum*.

In terms of arithmetic sequences, the expressions on the left below are equivalent to those on the right:

enumFrom n	$[n\,..]$
enumFromThen $n\ n'$	$[n, n'\,..]$
enumFromTo $n\ m$	$[n\,..\,m]$
enumFromThenTo $n\ n'\ m$	$[n, n'\,..\,m]$

B.3 The Bounded Class

The class *Bounded* captures data types that are linearly bounded in some way; i.e. they have both a minimum value and a maximum value.

```
class Bounded a where
  minBound :: a
  maxBound :: a
```

B.4 The Show Class

Instances of the class *Show* are those types that can be converted to character strings. This is useful, for example, when writing a representation of a value to the standard output area or to a file. The class *Read* works in the other direction: it provides operations for parsing character strings to obtain the values that they represent. In this section we will look at the *Show* class; in the next we will look at *Read*.

For efficiency reasons the primitive operations in these classes are somewhat esoteric, but they provide good lessons in both algorithm and software design, so we will look at them in some detail.

First, let's look at one of the higher-level functions that is defined in terms of the lower-level primitives:

$$show :: (Show\ a) \Rightarrow a \rightarrow String$$

Naturally enough, *show* takes a value of any type that is a member of *Show*, and returns its representation as a string. For example, *show* $(2 + 2)$ yields the string `"4"`, as does *show* $(6-2)$ and *show* applied to any other expression whose value is 4.

Furthermore, we can construct strings such as:

```
"The sum of " ++ show x ++ " and " ++ show y ++ " is "
  ++ show (x + y) ++ "."
```

with no difficulty. In particular, because $(+\!\!+)$ is right associative, the number of steps to construct this string is directly proportional to its total length, and we can't expect to do any better than that. (Since $(+\!\!+)$ needs to reconstruct its left argument, if it were left associative the above expression would repeatedly reconstruct the same sub-string on each application of $(+\!\!+)$. If the total string length were n, then in the worst case the number of steps needed to do this would be proportional to n^2, instead of proportional to n in the case where $(+\!\!+)$ is right associative.)

Unfortunately, this strategy breaks down when construction of the list is nested. A particularly nasty version of this problem arises for tree-shaped data structures. Consider a function *showTree* that converts a value of type *Tree* into a string, as in:

```
showTree (Branch (Branch (Leaf 2) (Leaf 3)) (Leaf 4))
==> "< <2|3>|4>"
```

We can define this behavior straightforwardly as follows:

$$showTree :: (Show\ a) \Rightarrow Tree\ a \rightarrow String$$

```
showTree (Leaf x)
  = show x
showTree (Branch l r)
  = "<" ++ showTree l ++ "|" ++ showTree r ++ ">"
```

Each of the recursive calls to *showTree* introduces more applications of (++), but since they are nested, a large amount of list reconstruction takes place (similar to the problem that would arise if (++) were left associative). If the tree being converted has size n, then in the worst case the number of steps needed to perform this conversion is proportional to n^2. This is no good!

To restore linear complexity, suppose we had a function *shows*:

```
shows :: (Show a) => a -> String -> String
```

which takes a showable value and a string and returns that string with the value's representation concatenated at the front. For example, we would expect *shows* $(2 + 2)$ `"hello"` to return the string `"4hello"`. The string argument should be thought of as an "accumulator" for the final result.

Using *shows* we can define a more efficient version of *showTree* which, like *shows*, has a string accumulator argument. Let's call this function *showsTree*:

```
showsTree :: (Show a) => Tree a -> String -> String
showsTree (Leaf x) s
  = shows x s
showsTree (Branch l r) s
  = "<" ++ showsTree l ("|" ++ showsTree r (">" ++ s))
```

This function requires a number of steps directly proportional to the size of the tree, thus solving our efficiency problem. To see why this is so, note that the accumulator argument s is never reconstructed. It is simply passed as an argument in one recursive call to *shows* or *showsTree*, and is incrementally extended to its left using (++).

showTree can now be re-defined in terms of *showsTree* using an empty accumulator:

```
showTree t = showsTree t ""
```

Exercise B.1 Prove that this version of *showTree* is equivalent to the old.

Although this solves our efficiency problem, the presentation of this function (and others like it) can be improved somewhat. First, let's create a type synonym (part of the Standard Prelude):

```
type ShowS = String → String
```

Second, we can avoid carrying accumulators around, and also avoid amassing parentheses at the right end of long sequences of concatenations, by using functional composition:

```
showsTree :: (Show a) ⇒ Tree a → ShowS
showsTree (Leaf x)
   = shows x
showsTree (Branch l r)
   = ("<"++) ∘ showsTree l ∘ ("|"++) ∘ showsTree r ∘ (">"++)
```

Details: This can be simplified slightly more by noting that ("c"++) is equivalent to ('c':) for any character c.

Something more important than just tidying up the code has come about by this transformation: We have raised the presentation from an *object level* (in this case, strings) to a *function level*. You can read the type signature of *showsTree* as saying that *showsTree* maps a tree into a *showing function*. Functions like ("<"++) and ("a string"++) are primitive showing functions, and we build up more complex ones by function composition.

The actual *Show* class in Haskell has two additional levels of complexity (and functionality): (1) the ability to specify the *precedence* of a string being generated, which is important when *show*ing a data type that has infix constructors, since it determines when parentheses are needed, and (2) a function for *show*ing a *list* of values of the type under consideration, since lists have special syntax in Haskell and are so commonly used that they deserve special treatment. The full definition of the *Show* class is given by:

```
class Show a where
  showsPrec :: Int → a → ShowS
  showList  :: [a] → ShowS

  showList []       = showString "[]"
  showList (x : xs) = showChar '[' ∘ shows x ∘ showl xs
    where showl []       = showChar ']'
          showl (x : xs) = showString ", " ∘ shows x ∘ showl xs
```

Note the default method for *showList*, and its "function level" style of definition.

In addition to this class declaration the Standard Prelude defines the following functions, which return us to where we started our journey in this section:

```
shows :: (Show a) ⇒ a → ShowS
shows = showsPrec 0
show  :: (Show a) ⇒ a → String
show x = shows x ""
```

Some details about *showsPrec* can be found in the Haskell Report, but if you are not displaying constructors in infix notation, the precedence can be ignored. Furthermore, the default method for *showList* is perfectly good for most uses of lists that you will encounter. Thus, for example, we can finish our *Tree* example by declaring it to be an instance of the class *Show* very simply as:

```
instance (Show a) ⇒ Show (Tree a) where
  showsPrec n = showsTree
```

B.5 The Read Class

Now that we can convert trees into strings, let's turn to the inverse problem: converting strings into trees. The basic idea is to define a *parser* for a type a, which at first glance seems as if it should be a function of type $String \to a$. This simple approach has two problems, however: (1) it's possible that the string is ambiguous, leading to more than one way to interpret it as a value of type a, and (2) it's possible that only a prefix of the string will parse correctly. Thus we choose instead to return a list of $(a, String)$ pairs as the result of a parse. If all goes well we will always get a singleton list such as $[(v, "")]$ as the result of a parse, but we cannot count on it (in fact, when recursively parsing sub-strings, we will expect a singleton list with a *non-empty* trailing string).

The Standard Prelude provides a type synonym for parsers of the kind just described:

```
type ReadS a = String → [(a, String)]
```

and also defines a function *reads* that by analogy is similar to *shows*:

```
reads :: (Read a) ⇒ ReadS a
```

We will return later to the precise definition of this function, but for now let's use it to define a parser for the *Tree* data type, whose string represen-

tation is as described in the previous section. List comprehensions give us a convenient idiom for constructing such parsers:[1]

$$\begin{aligned}
&\mathit{readsTree} :: (\mathit{Read}\ a) \Rightarrow \mathit{ReadS}\ (\mathit{Tree}\ a)\\
&\mathit{readsTree}\ (\texttt{'<'} : s) = [(\mathit{Branch}\ l\ r, u) \mid (l, \texttt{'|'} : t) \leftarrow \mathit{readsTree}\ s,\\
&\qquad\qquad\qquad\qquad (r, \texttt{'>'} : u) \leftarrow \mathit{readsTree}\ t]\\
&\mathit{readsTree}\ s = \qquad [(\mathit{Leaf}\ x, t) \mid (x, t) \leftarrow \mathit{reads}\ s]
\end{aligned}$$

Let's take a moment to examine this function definition in detail. There are two main cases to consider: If the string has the form $\texttt{'<'} : s$ we should have the representation of a branch, in which case parsing s as a tree should yield a left branch l followed by a string of the form $\texttt{'|'} : t$; parsing t as a tree should then yield the right branch r followed by a string of the form $\texttt{'>'} : u$. The resulting tree $\mathit{Branch}\ l\ r$ is then returned, along with the trailing string u. Note the expressive power we get from the combination of pattern matching and list comprehension.

If the initial string is not of the form $\texttt{'<'} : s$, then we must have a leaf, in which case the string is parsed using the generic *reads* function, and the result is directly returned.

If we accept on faith for the moment that there is a *Read* instance for *Int* that behaves as one would expect, e.g.:

$$\begin{aligned}
&(\mathit{reads}\ \texttt{"5 golden rings"}) :: [(\mathit{Int}, \mathit{String})]\\
&\Longrightarrow [(5, \texttt{" golden rings"})]
\end{aligned}$$

then you should be able to verify the following calculations:

$$\begin{aligned}
&\mathit{readsTree}\ \texttt{"< <1|2>|3>"}\\
&\Longrightarrow
\end{aligned}$$

There are a couple of shortcomings, however, in our definition of *readsTree*. One is that the parser is quite rigid in that it allows no "white space" (such as extra spaces, tabs, or line feeds) before or between the elements of the tree representation. The other is that the way we parse our punctuation symbols (`'<'`, `'|'`, and `'>'`) is quite different from the way we parse leaf values and sub-trees. This lack of uniformity makes the function definition harder to read.

We can address both of these problems by using a *lexical analyzer*, which parses a string into primitive "lexemes" defined by some rules about the string construction. The Standard Prelude defines a lexical analyzer:

[1] An even more elegant approach to parsing uses monads and parser combinators. These are part of a standard parsing library distributed with most Haskell systems.

$lex :: ReadS\ String$

whose lexical rules are those of the Haskell language, which can be found in the Haskell Report. For our purposes, an informal explanation is sufficient:

lex normally returns a singleton list containing a pair of strings: the first string is the first lexeme in the input string, and the second string is the remainder of the input. White space – including Haskell comments – is completely ignored. If the input string is empty or contains only white-space and comments, *lex* returns [("", "")]; if the input is not empty in this sense, but also does not begin with a valid lexeme after any leading white-space, *lex* returns [].

Using this lexical analyzer, our tree parser can be rewritten as:

```
readsTree :: (Read a) ⇒ ReadS (Tree a)
readsTree s = [(Branch l r, x) | ("<", t) ← lex s,
                                 (l,       u) ← readsTree t,
                                 ("|", v) ← lex u,
                                 (r,       w) ← readsTree v,
                                 (">", x) ← lex w          ]
              ++
              [(Leaf x, t) | (x, t) ← reads s]
```

This definition solves both problems mentioned earlier: white-space is suitably ignored, and parsing of sub-strings has a more uniform structure.

To tie all of this together, let's first look at the definition of the class *Read* in the Standard Prelude:

```
class Read a where
  readsPrec :: Int → ReadS a
  readList  :: ReadS [a]
  readList   = readParen False (λr → [pr | ("[", s) ← lex r,
                                           pr       ← readl s])
    where readl s = [([], t) | ("]", t)   ← lex s] ++
                    [(x : xs, u) | (x, t)   ← reads s,
                                   (xs, u)  ← readl' t]
          readl' s = [([], t) | ("]", t)   ← lex s] ++
                     [(x : xs, v) | (",", t) ← lex s,
                                    (x, u)   ← reads t,
                                    (xs, v)  ← readl' u]
  readParen      :: Bool → ReadS a → ReadS a
  readParen b g = if b then mandatory else optional
```

```
                  where optional r = g r ++ mandatory r
                        mandatory r = [(x, u) | ("(", s) ← lex r,
                        sc                      (x, t) ← optional s,
                                                (")", u) ← lex t]
```

The default method for *readList* is rather tedious, but otherwise straightforward.

reads can now be defined, along with an even higher-level function, *read*:

```
reads :: (Read a) ⇒ ReadS a
reads = readsPrec 0

read  :: (Read a) ⇒ String → a
read s = case [x | (x, t) ← reads s, ("", "") ← lex t] of
           [x] → x
           [] → error "PreludeText.read: no parse"
           _ → error "PreludeText.read: ambiguous parse"
```

The definition of *reads* (like *shows*) should not be surprising. The definition of *read* assumes that exactly one parse is expected, and thus causes a run-time error if there is no unique parse or if the input contains anything more than a representation of exactly one value of type a (and possibly comments and white-space).

You can test that the *Read* and *Show* instances for a particular type are working correctly by applying $(read \circ show)$ to a value in that type, which in most situations should be the identity function.

B.6 The Index Class

The Standard Prelude defines a type class of array indices:

```
class (Ord a) ⇒ Ix a where
  range   :: (a, a) → [a]
  index   :: (a, a) → a → Int
  inRange :: (a, a) → a → Bool
```

Arrays are defined elsewhere, but the index class is useful for other things besides arrays, so I will describe it here.

Instance declarations are provided for *Int*, *Integer*, *Char*, *Bool*, and tuples of *Ix* types; in addition, instances may be automatically derived for enumerated and tuple types. You should think of the primitive types as vector indices, and tuple types as indices of multidimensional rectangular

arrays. Note that the first argument of each of the operations of class *Ix* is a pair of indices; these are typically the *bounds* (first and last indices) of an array. For example, the bounds of a 10-element, zero-origin vector with *Int* indices would be $(0, 9)$, while a 100 by 100 1-origin matrix might have the bounds $((1, 1), (100, 100))$. (In many other languages, such bounds would be written in a form like $1 : 100, 1 : 100$, but the present form fits the type system better, since each bound is of the same type as a general index.)

The *range* operation takes a bounds pair and produces the list of indices lying between those bounds, in index order. For example,

$$\mathit{range}\ (0, 4) \Longrightarrow [0, 1, 2, 3, 4]$$
$$\mathit{range}\ ((0, 0), (1, 2)) \Longrightarrow [(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)]$$

The *inRange* predicate determines whether an index lies between a given pair of bounds. (For a tuple type, this test is performed componentwise, and then combined with $(\wedge)$.) Finally, the *index* operation determines the (zero-based) position of an index within a bounded range; for example:

$$\mathit{index}\ (1, 9)\ 2 \Longrightarrow 1$$
$$\mathit{index}\ ((0, 0), (1, 2))\ (1, 1) \Longrightarrow 4$$

B.7 The Numeric Classes

The *Num* class and the numeric class hierarchy were briefly described in Section 7.5. Figure B.1 gives the full class declarations.

```
class (Eq a, Show a) ⇒ Num a where
  (+), (−), (∗) :: a → a → a
  negate :: a → a
  abs, signum :: a → a
  fromInteger :: Integer → a
class (Num a, Ord a) ⇒ Real a where
  toRational :: a → Rational
class (Real a, Enum a) ⇒ Integral a where
  quot, rem, div, mod :: a → a → a
  quotRem, divMod :: a → a → (a, a)
  toInteger :: a → Integer
class (Num a) ⇒ Fractional a  where
  (/) :: a → a → a
  recip :: a → a
  fromRational :: Rational → a
class (Fractional a) ⇒ Floating a where
  pi :: a
  exp, log, sqrt :: a → a
  (∗∗), logBase :: a → a → a
  sin, cos, tan :: a → a
  asin, acos, atan :: a → a
  sinh, cosh, tanh :: a → a
  asinh, acosh, atanh :: a → a
class (Real a, Fractional a) ⇒ RealFrac a where
  properFraction :: (Integral b) ⇒ a → (b, a)
  truncate, round :: (Integral b) ⇒ a → b
  ceiling, floor :: (Integral b) ⇒ a → b
class (RealFrac a, Floating a) ⇒ RealFloat a where
  floatRadix :: a → Integer
  floatDigits :: a → Int
  floatRange :: a → (Int, Int)
  decodeFloat :: a → (Integer, Int)
  encodeFloat :: Integer → Int → a
  exponent :: a → Int
  significand :: a → a
  scaleFloat :: Int → a → a
  isNaN, isInfinite, isDenormalized, isNegativeZero, isIEEE
    :: a → Bool
```

Figure B.1: Standard Numeric Classes

Appendix C

Built-in Types Are Not Special

Throughout this text we have introduced many "built-in" types such as lists, tuples, integers, and characters. We have also shown how new user-defined types can be defined. Aside from special syntax, you might be wondering if the built-in types are in any way more special than the user-defined ones. The answer is *no*. The special syntax is for convenience and for consistency with historical convention, but has no semantic consequence.

We can emphasize this point by considering what the type declarations would look like for these built-in types if in fact we were allowed to use the special syntax in defining them. For example, the *Char* type might be written as:

```
data Char = 'a' | 'b' | 'c' | ...   -- This is not valid
          | 'A' | 'B' | 'C' | ...   -- Haskell code!
          | '1' | '2' | '3' | ...
```

These constructor names are not syntactically valid; to fix them we would have to write something like:

```
data Char = Ca | Cb | Cc | ...
          | CA | CB | CC | ...
          | C1 | C2 | C3 | ...
```

Even though these constructors are actually more concise, they are quite unconventional for representing characters, and thus the special syntax is used instead.

In any case, writing "pseudo-Haskell" code in this way helps us to see

through the special syntax. We see now that *Char* is just a data type consisting of a large number of nullary (meaning they take no arguments) constructors. Thinking of *Char* in this way makes it clear why, for example, we can pattern-match against characters; i.e., we would expect to be able to do so for any of a data type's constructors.

Similarly, using pseudo-Haskell, we could define *Int* and *Integer* by:

```
-- more pseudo-code:
data Int = (-2^29) | ... | -1 | 0 | 1 | ... | (2^29 - 1)
data Integer = ... -2 | -1 | 0 | 1 | 2 ...
```

(Recall that -2^{29} to 2^{29-1} is the minimum range for the *Int* data type.) *Int* is clearly a much larger enumeration than *Char*, but it's still finite! In contrast, the pseudo-code for *Integer* (the type of arbitrary precision integers) is intended to convey an *infinite* enumeration (and in that sense only, the *Integer* data type *is* somewhat special).

Haskell has a data type called *unit* which has exactly one value: (). The name of this data type is also written (). This is trivially expressed in Haskell pseudo-code:

```
data () = ()   -- more pseudo-code
```

Tuples are also easy to define playing this game:

```
data (a, b)       = (a, b)   -- more pseudo-code
data (a, b, c)    = (a, b, c)
data (a, b, c, d) = (a, b, c, d)
```

and so on. Each declaration above defines a tuple type of a particular length, with parentheses playing a role in both the expression syntax (as data constructor) and type-expression syntax (as type constructor). By "and so on" we mean that there are an infinite number of such declarations, reflecting the fact that tuples of all finite lengths are allowed in Haskell.

The list data type is also easily handled in pseudo-Haskell, and more interestingly, it is recursive:

```
data [a] = [] | a : [a]   -- more pseudo-code
infixr 5 :
```

We can now see clearly what we described about lists earlier: [] is the empty list, and (:) is the infix list constructor; thus $[1, 2, 3]$ must be equivalent to the list $1 : 2 : 3 : []$. (Note that (:) is right associative.) The type of [] is $[a]$, and the type of (:) is $a \to [a] \to [a]$.

Details: The way (:) is defined here is actually legal syntax—infix constructors are permitted in **data** declarations, and are distinguished from infix operators (for pattern-matching purposes) by the fact that they must begin with a colon (a property trivially satisfied by ":").

At this point the reader should note carefully the differences between tuples and lists, which the above definitions make abundantly clear. In particular, note the recursive nature of the list type whose elements are homogeneous and of arbitrary length, and the non-recursive nature of a (particular) tuple type whose elements are heterogeneous and of fixed length. The typing rules for tuples and lists should now also be clear:

For $(e_1, e_2, ..., en)$, $n \geqslant 2$, if Ti is the type of ei, then the type of the tuple is $(T_1, T_2, ..., Tn)$.

For $[e_1, e_2, ..., en]$,$n \geqslant 0$, each ei must have the same type T, and the type of the list is $[T]$.

Appendix D

Pattern-Matching Details

In this chapter we will look at Haskell's pattern-matching process in greater detail.

Haskell defines a fixed set of patterns for use in case expressions and function definitions. Pattern matching is permitted using the constructors of any type, whether user-defined or pre-defined in Haskell. This includes tuples, strings, numbers, characters, etc. For example, here's a contrived function that matches against a tuple of "constants:"

$$\begin{aligned} &\mathit{contrived} :: ([a], \mathit{Char}, (\mathit{Int}, \mathit{Float}), \mathit{String}, \mathit{Bool}) \rightarrow \mathit{Bool} \\ &\mathit{contrived}\ ([\,], \texttt{'b'}, (1, 2.0), \texttt{"hi"}, \mathit{True}) = \mathit{False} \end{aligned}$$

This example also demonstrates that *nesting* of patterns is permitted (to arbitrary depth).

Technically speaking, *formal parameters* to functions are also patterns—it's just that they *never fail to match a value*. As a "side effect" of a successful match, the formal parameter is bound to the value it is being matched against. For this reason patterns in any one equation are not allowed to have more than one occurrence of the same formal parameter.

A pattern that may fail to match is said to be *refutable*; for example, the empty list [] is refutable. Patterns such as formal parameters that never fail to match are said to be *irrefutable*. There are three other kinds of irrefutable patterns, which are summarized below.

As-Patterns Sometimes it is convenient to name a pattern for use on the right-hand side of an equation. For example, a function that duplicates the first element in a list might be written as:

$$f\ (x : xs) = x : x : xs$$

Note that $x : xs$ appears both as a pattern on the left-hand side, and as an expression on the right-hand side. To improve readability, we might prefer to write $x{:}xs$ just once, which we can achieve using an *as-pattern* as follows:[1]

$$f\ s@(x : xs) = x : s$$

Technically speaking, as-patterns always result in a successful match, although the sub-pattern (in this case $x : xs$) could, of course, fail.

Wildcards Another common situation is matching against a value we really care nothing about. For example, the functions *head* and *tail* can be written as:

$$\begin{aligned} head\ (x{:}\ \_) &= x \\ tail\ (\_ : xs) &= xs \end{aligned}$$

in which we have "advertised" the fact that we don't care what a certain part of the input is. Each wildcard will independently match anything, but in contrast to a formal parameter, each will bind nothing; for this reason more than one are allowed in an equation.

Lazy Patterns There is one other kind of pattern allowed in Haskell. It is called a *lazy pattern*, and has the form $\sim pat$. Lazy patterns are *irrefutable*: matching a value v against $\sim pat$ always succeeds, regardless of pat. Operationally speaking, if an identifier in pat is later "used" on the right-hand-side, it will be bound to that portion of the value that would result if v were to successfully match pat, and $\perp$ otherwise.

Lazy patterns are useful in contexts where infinite data structures are being defined recursively. For example, infinite lists are an excellent vehicle for writing *simulation* programs, and in this context the infinite lists are often called *streams*.

Pattern-Matching Semantics

So far we have discussed how individual patterns are matched, how some are refutable, some are irrefutable, etc. But what drives the overall process? In

[1] Another advantage to doing this is that a naive implementation might otherwise completely reconstruct $x : xs$ rather than re-use the value being matched against.

what order are the matches attempted? What if none succeed? This section addresses these questions.

Pattern matching can either *fail*, *succeed* or *diverge*. A successful match binds the formal parameters in the pattern. Divergence occurs when a value needed by the pattern diverges (i.e. is non-terminating) or results in an error ($\perp$). The matching process itself occurs "top-down, left-to-right." Failure of a pattern anywhere in one equation results in failure of the whole equation, and the next equation is then tried. If all equations fail, the value of the function application is $\perp$, and results in a run-time error.

For example, if *bot* is a divergent or erroneous computation, and if $[1, 2]$ is matched against $[0, bot]$, then 1 fails to match 0, so the result is a failed match. But if $[1, 2]$ is matched against $[bot, 0]$, then matching 1 against *bot* causes divergence (i.e. $\perp$).

The only other twist to this set of rules is that top-level patterns may also have a boolean *guard*, as in this definition of a function that forms an abstract version of a number's sign:

$$\begin{array}{lll} sign\ x \mid x > 0 & = 1 \\ \quad \mid & x == 0 = 0 \\ \quad \mid & x < 0 & = -1 \end{array}$$

Note here that a sequence of guards is given for a single pattern; as with patterns, these guards are evaluated top-down, and the first that evaluates to *True* results in a successful match.

An Example The pattern-matching rules can have subtle effects on the meaning of functions. For example, consider this definition of *take*:

$$\begin{array}{ll} take\ 0\ \_ & = [\,] \\ take\ \_\ [\,] & = [\,] \\ take\ n\ (x : xs) & = x : take\ (n-1)\ xs \end{array}$$

and this slightly different version (the first 2 equations have been reversed):

$$\begin{array}{ll} take1\ \_\ [\,] & = [\,] \\ take1\ 0\ \_ & = [\,] \\ take1\ n\ (x : xs) & = x : take1\ (n-1)\ xs \end{array}$$

Now note the following:

$$\begin{array}{lcl} \mathit{take}\ 0\ \mathit{bot} & \Longrightarrow & [\,] \\ \mathit{take1}\ 0\ \mathit{bot} & \Longrightarrow & \bot \\ \\ \mathit{take}\ \mathit{bot}\ [\,] & \Longrightarrow & \bot \\ \mathit{take1}\ \mathit{bot}\ [\,] & \Longrightarrow & [\,] \end{array}$$

We see that *take* is "more defined" with respect to its second argument, whereas *take1* is more defined with respect to its first. It is difficult to say in this case which definition is better. Just remember that in certain applications, it may make a difference. (The Standard Prelude includes a definition corresponding to *take*.)

Case Expressions

Pattern matching provides a way to "dispatch control" based on structural properties of a value. However, in many circumstances we don't wish to define a *function* every time we need to do this. Haskell's *case expression* provides a way to solve this problem. Indeed, the meaning of pattern matching in function definitions is specified in the Haskell Report in terms of case expressions, which are considered more primitive. In particular, a function definition of the form:

$$\begin{array}{l} f p_{11} ... p_{1k} = e_1 \\ ... \\ f p_{n1} ... p_{nk} = e_n \end{array}$$

where each p_{ij} is a pattern, is semantically equivalent to:

$$\begin{array}{lll} f\ x_1\ x_2\ ...\ xk = \textbf{case}\ (x_1, ..., xk)\ \textbf{of} & (p_{11}, ..., p_{1k}) & \rightarrow e_1 \\ & ... & \\ & (p_{n1}, ..., p_{nk}) & \rightarrow e_n \end{array}$$

where the *xi* are new identifiers. For example, the definition of *take* given earlier is equivalent to:

$$\begin{array}{l} \mathit{take}\ m\ ys = \textbf{case}\ (m, ys)\ \textbf{of} \\ \quad (0, \_) \quad\ \rightarrow [\,] \\ \quad (\_, [\,]) \quad\ \rightarrow [\,] \\ \quad (n, x : xs) \rightarrow x : \mathit{take}\ (n-1)\ xs \end{array}$$

For type correctness, the types of the right-hand sides of a case expression or set of equations comprising a function definition must all be the same; more precisely, they must all share a common principal type.

The pattern-matching rules for case expressions are the same as we have given for function definitions.

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