THE DIFFERENCE BETWEEN CONSECUTIVE PRIMES, II

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1. *Introduction*

Beginning with Hoheisel [**8**], many authors have found shorter and shorter intervals $[x - x^{\theta}, x]$ that must contain a prime number. The most recent result is $\theta = 0.535$: see Baker and Harman [**1**], where the history of the problem is discussed. In the present paper we prove:

THEOREM 1. *For all $x > x_0$, the interval $[x - x^{0.525}, x]$ contains prime numbers.*

With enough effort, the value of x_0 could be determined effectively.

The paper has much in common with [**1**]; in particular we use the sieve method of Harman [**4**, **5**]. We no longer use zero density estimates, however, but rather mean value results on Dirichlet polynomials similar to those that give rise to such estimates. Compare, for example, work of Iwaniec and Pintz [**9**] and Baker, Harman and Pintz [**2**]. Much of the improvement over [**1**] arises from the use of Watt's theorem [**11**] on a particular kind of mean value. More accurate estimates for six-dimensional integrals are also used to good effect. There is in addition a device which uses a two-dimensional sieve to get an asymptotic formula for a 'one-dimensionally sieved' set; see Lemmas 16, 17. Unfortunately, these lemmas, which would be of great significance for $\theta = 0.53$, are not very numerically significant when θ drops to 0.525; the same applies to the 'rôle reversals' discussed below.

Let us introduce enough notation to permit an outline of the proof. When $\mathscr{E}$ is a finite sequence of positive integers, counted with multiplicity, we write $|\mathscr{E}|$ for the number of terms of $\mathscr{E}$, and

$$\mathscr{E}_d = \{m\colon dm \in \mathscr{E}\}.$$

Let

$$P(z) = \prod_{p<z} p,$$

where the symbol p is reserved for a prime variable; and let

$$S(\mathscr{E}, z) = |\{m \in \mathscr{E}\colon (m, P(z)) = 1\}|.$$

Let θ be a positive number,

$$0.524 < \theta \leqslant 0.535. \tag{1.1}$$

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Let $\mathscr{L} = \log x$, $y_1 = x\exp(-3\mathscr{L}^{1/3})$, $y = x^{\theta+\varepsilon}$,

$$\mathscr{A} = [x-y, x) \cap \mathbb{Z} \quad \text{and} \quad \mathscr{B} = [x-y_1, x) \cap \mathbb{Z},$$

where ε is a sufficiently small positive number.

Buchstab's identity is the equation

$$S(\mathscr{E}, z) = S(\mathscr{E}, w) - \sum_{w \leqslant p < z} S(\mathscr{E}_p, p),$$

where $2 \leqslant w < z$; $S(\mathscr{A}, x^{1/2})$ counts the primes we are looking for. Our philosophy is to use Buchstab's identity to produce parallel decompositions of $S(\mathscr{A}, x^{1/2})$ and $S(\mathscr{B}, x^{1/2})$:

$$S(\mathscr{A}, x^{1/2}) = \sum_{j=1}^{k} S_j - \sum_{j=k+1}^{l} S_j,$$

$$S(\mathscr{B}, x^{1/2}) = \sum_{j=1}^{k} S_j^* - \sum_{j=k+1}^{l} S_j^*.$$

Here $S_j \geqslant 0$, $S_j^* \geqslant 0$ and for $j \leqslant t < k$ or $j > k$ we have

$$S_j = \frac{y}{y_1} S_j^*(1 + o(1))$$

as $x \to \infty$. Thus

$$S(\mathscr{A}, x^{1/2}) \geqslant \frac{y}{y_1}\left(S(\mathscr{B}, x^{1/2}) - \sum_{j=t+1}^{k} S_j^*\right)(1 + o(1)).$$

We must thus ensure that not too many sums are discarded, that is, fall into the category $t < j \leqslant k$.

Just as in [**1**] we use Buchstab's identity twice to reach the decomposition

$$\begin{aligned} S(\mathscr{A}, x^{1/2}) &= S(\mathscr{A}, x^{\nu(0)}) - \sum_{\nu(0) \leqslant \alpha_1 < 1/2} S(\mathscr{A}_{p_1}, x^{\nu(\alpha_1)}) \\ &\quad + \sum_{\substack{\nu(0) \leqslant \alpha_1 < 1/2 \\ \nu(\alpha_1) \leqslant \alpha_2 < \min(\alpha_1, (1-\alpha_1)/2)}} S(\mathscr{A}_{p_1 p_2}, p_2) \\ &= \Sigma_1 - \Sigma_2 + \Sigma_3, \quad \text{say.} \end{aligned} \tag{1.2}$$

(Here $p_j = x^{\alpha_j}$.) We give asymptotic formulae for Σ_1 and Σ_2. The piecewise linear function $\nu(\dots)$ is larger (for given θ) than its counterpart in [**1**]. From this point on, rôle reversals are employed. To illustrate this, note that

$$S(\mathscr{A}_{p_1 p_2}, p_2) = |\{p_1 p_2 h \in \mathscr{A} : p \mid h \Rightarrow p \geqslant p_2\}|.$$

If K is a region in which $\alpha_1 > 1 - \alpha_1 - \alpha_2$, we note that

$$\begin{aligned} \sum_{(\alpha_1, \alpha_2) \in K} S(\mathscr{A}_{p_1 p_2}, p_2) = \{hp_2 h_1 \in \mathscr{A} : &(\mathscr{L}^{-1}\log h_1, \mathscr{L}^{-1}\log p_2) \in K, \\ &p \mid h_1 \Rightarrow p > h_1^{1/2}, p \mid h \Rightarrow p \geqslant p_2\}, \end{aligned}$$

leading readily to the formula (in which $h = x^{\beta_3}$)

$$\sum_{(\alpha_1,\alpha_2)\in K} S(\mathscr{A}_{p_1p_2}, p_2) = (1+o(1)) \sum_{\substack{(1-\alpha_2-\beta_3,\alpha_2)\in K\\ p\mid h\Rightarrow p>p_2}} S\left(\mathscr{A}_{hp_2}, \left(\frac{x}{hp_2}\right)^{1/2}\right)$$

which we term a rôle-reversal. The point here is that our asymptotic formulae

$$\sum_{m\sim M}\sum_{n\sim N} a_m b_m S(\mathscr{A}_{mn}, x^{\nu}) = \frac{y}{y_1}(1+o(1)) \sum_{m\sim M}\sum_{n\sim N} a_m b_n S(\mathscr{B}_{mn}, x^{\nu}) \qquad (1.3)$$

require certain upper bounds on M and N; see Lemmas 12 and 13. Here $m\sim M$ means $M\leqslant m<2M$; $m\asymp M$ means $B^{-1}M<m<BM$; B is a positive absolute constant, which need not have the same value at each occurrence.

It will generally be beneficial to attempt as many decompositions as possible. There are two reasons for this. First, if there are several variables, there should often be a combination of variables which satisfy one of our criteria for obtaining an asymptotic formula. Second, if there are many variables, the contribution is already quite small. To see this, note that if $*$ represents $x^{\nu}<p_n<p_{n-1}<\ldots<p_1<x^{\lambda}$, then

$$\sum{}^{*} S(\mathscr{B}_{p_1\ldots p_n}, p_n)$$
$$= \frac{y_1}{\mathscr{L}}(1+o(1)) \int_{\alpha_1=\nu}^{\lambda}\int_{\alpha_2=\nu}^{\alpha_1}\cdots\int_{\alpha_n=\nu}^{\alpha_{n-1}} \omega\left(\frac{1-\alpha_1-\ldots-\alpha_n}{\alpha_n}\right)\frac{d\alpha_1\ldots d\alpha_n}{\alpha_1\ldots\alpha_n^2}$$

(compare [**1**]). Moreover,

$$\omega\left(\frac{1-\alpha_1-\ldots-\alpha_n}{\alpha_n}\right)\leqslant 1 \quad\text{and}\quad \int_{\alpha_1}\cdots\int_{\alpha_n}\frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_n}{\alpha_n^2}\leqslant\frac{(\log(\lambda/\nu))^n}{n!\,\nu}.$$

For $\theta=0.525$ we shall have $\nu\geqslant 0.05$. Hence the contribution from $p_1\leqslant x^{1/10}$ (for which one can take $n=8$) is at most

$$\frac{y}{\mathscr{L}}\frac{(\log 2)^8}{8!0.05}(1+o(1)) < 0.000002y\mathscr{L}^{-1}.$$

(If ‘asymptotic formula regions’, in the sense of (1.4) below, are not discarded, we get a better estimate still.)

However, when rôle-reversals are used it may not always be beneficial to perform as many decompositions as possible. The reason for this is that with rôle-reversals, a sum may be replaced by the difference of two sums, each substantially larger than the original one. If not enough combinations of variables lie in ‘asymptotic formula regions’, we have made matters worse. For example, when decomposing in straightforward fashion we count

$$p_1\ldots p_n m,\ p\mid m\Rightarrow p>p_n.$$

When rôle-reversals are used we may have

$$p_1\ldots p_n klm,\ p\mid k\Rightarrow p>p_r,\quad p\mid l\Rightarrow p>p_s,\quad p\mid m\Rightarrow p>p_n.$$

The first expression gives rise to a term

$$\omega\left(\frac{1-\alpha_1-\ldots-\alpha_n}{\alpha_n}\right)\frac{1}{\alpha_1\ldots\alpha_{n-1}\alpha_n^2},$$

while the second leads to a term

$$\omega\left(\frac{f_1}{\alpha_r}\right)\omega\left(\frac{f_2}{\alpha_s}\right)\omega\left(\frac{f_3}{\alpha_n}\right)\frac{1}{(\alpha_1\ldots\alpha_n)\alpha_r\alpha_s\alpha_n}$$

for certain expressions f_1, f_2 and f_3. The corresponding integral can then be larger than the original term under consideration.

The final decomposition of Σ_2, given in § 6, arises from Lemmas 12 and 13, together with formulae of the type

$$\sum_{(\alpha_1,\ldots,\alpha_r)\in K} S(\mathscr{A}_{p_1\ldots p_r}, p_r) = \frac{y}{y_1}(1+o(1)) \sum_{(\alpha_1,\ldots,\alpha_r)\in K} S(\mathscr{B}_{p_1\ldots p_r}, p_r) \tag{1.4}$$

discussed in § 5.

2. *Application of Watt's theorem*

Let $T = x^{1-\theta-\varepsilon/2}$ and $T_0 = \exp(\mathscr{L}^{1/3})$. In this section we seek a result of the type

$$\int_{1/2+iT_0}^{1/2+iT} |M(s)N(s)K(s)|\,|ds| \ll x^{1/2}\mathscr{L}^{-A} \tag{2.1}$$

where $M(s)$ and $N(s)$ are Dirichlet polynomials,

$$M(s) = \sum_{m\sim M} a_m m^{-s}, \quad N(s) = \sum_{n\sim N} b_n n^{-s},$$

and $K(s)$ is a 'zeta factor', that is,

$$K(s) = \sum_{k\sim K} k^{-s} \text{ or } \sum_{k\sim K} (\log k)k^{-s}.$$

Note the convention of the same symbol for the polynomial and its 'length'. Of course, 1 is a Dirichlet polynomial of length 1. We shall assume without comment that each Dirichlet polynomial that appears has length at most x and coefficients bounded by a power of the divisor function τ: thus, whenever a sequence $(a_m)_{m\sim M}$ is mentioned, we assume that

$$|a_m| \leqslant \tau(m)^B.$$

(This property may be readily verified for the particular polynomials employed later.) The bound (2.1), and any bound in which A appears, is intended to hold for every positive A; the constant implied by the '$\ll$' or 'O' notation may depend on A, B and ε.

It is not a long step from (2.1) to a 'fundamental lemma' of the type

$$\sum_{m\sim M} a_m \sum_{n\sim N} b_n S(\mathscr{A}_{mn}, w) = \frac{y}{y_1}(1+o(1)) \sum_{m\sim M} a_m \sum_{n\sim N} b_n S(\mathscr{B}_{mn}, w) \tag{2.2}$$

with

$$w = \exp\left(\frac{\mathscr{L}}{\log \mathscr{L}}\right). \tag{2.3}$$

This will be demonstrated in § 3.

LEMMA 1. *Let*

$$N(s) = \sum_{p_i \sim P_i} (p_1 \dots p_u)^{-s} \tag{2.4}$$

where $u \leqslant B$, $P_i \geqslant w$ *and* $P_1 \dots P_u \leqslant x$. *Then, for* $\operatorname{Re} s = \frac{1}{2}$,

$$|N(s)| \leqslant g_1(s) + \dots + g_r(s), \quad \text{with } r \leqslant \mathscr{L}^B, \tag{2.5}$$

where each g_i *is of the form*

$$\mathscr{L}^B \prod_{i=1}^{h} |N_i(s)|, \quad \text{with } h \leqslant B, N_1 \dots N_h \leqslant x, \tag{2.6}$$

and among the Dirichlet polynomials $N_1, \dots, N_h$ *the only polynomials of length greater than* $T^{1/2}$ *are zeta factors.*

Proof. It clearly suffices to prove (2.5) for

$$N(s) = \sum_{n \sim N} \Lambda(n) n^{-s}$$

where Λ is von Mangoldt's function. We now obtain the desired result by the identity of Heath-Brown [**6**].

We shall refer to polynomials $N(s)$ 'of type (2.4)' to indicate that the hypothesis of Lemma 1 holds for $N(s)$.

LEMMA 2. *If* $K(s)$ *is a zeta factor,* $1 \leqslant U \leqslant T$, $K \leqslant 4U$ *and* $M < T$, *then*

$$\int_{1/2+iU/2}^{1/2+iU} |M(s)|^2 |K(s)|^4 |ds| \ll U^{1+\varepsilon}(1 + M^2 U^{-1/2}). \tag{2.7}$$

Proof. For $K \leqslant U^{1/2}$ and $M \leqslant U^{1/2}$ this is proved in all essentials by Watt [**11**] in the course of the proof of his main theorem. For $K \leqslant U^{1/2}$ and $M > U^{1/2}$ we have

$$\int_{1/2+iU/2}^{1/2+iU} |M(s)|^2 |K(s)|^4 |ds| \ll \|M\|_\infty^2 \int |K(s)|^4 |ds|$$
$$\ll M^{1+\varepsilon} U \ll M^2 U^{1/2+\varepsilon}.$$

Now suppose that $U^{1/2} < K \leqslant 4U$. Using a reflection principle based on [**10**, Theorem 4.13], we may replace K by a zeta factor of length $K' \leqslant U^{1/2}$ with error $E = O(1)$. Thus $|K|^4 \ll |K'|^4 + |E|^4$. Since

$$\int_{1/2+iU/2}^{1/2+iU} |M(s)|^2 |E|^4 |ds| \ll \int_{1/2+iU/2}^{1/2+iU} |M(s)|^2 |ds|$$
$$\ll (M + U) U^{\varepsilon},$$

the general case of Lemma 2 now follows.

LEMMA 3. *Let $MN_1N_2K = x$. Suppose that M, N_1 and N_2 are of type* (2.4) *and $K(s)$ is a zeta factor, $K \ll x^{3/4}$. Let $M = x^{\alpha}$ and $N_j = x^{\beta_j}$ and suppose that*

$$\alpha \leqslant \theta, \tag{2.8}$$

$$\beta_1 + \tfrac{1}{2}\beta_2 \leqslant \tfrac{1}{2}(1+\theta) - \alpha'. \tag{2.9}$$

Here and subsequently $\alpha' = \max(\alpha, 1-\theta)$. *Suppose further that*

$$\beta_2 \leqslant \tfrac{1}{4}(1+3\theta) - \alpha', \tag{2.10}$$

$$\beta_1 + \tfrac{3}{2}\beta_2 \leqslant \tfrac{1}{4}(3+\theta) - \alpha'. \tag{2.11}$$

Then for $1 \leqslant U \leqslant T$,

$$\int_{U/2}^{U} |(MN_1N_2K)(\tfrac{1}{2}+it)|\, dt \ll x^{1/2}\mathscr{L}^{-A}. \tag{2.12}$$

Proof. Suppose first that $4U < K$ and write $N = MN_1N_2$. Lemma 5 of [**2**] yields $\|M\|_\infty \ll M^{1/2}\mathscr{L}^{-A}$ if $M > x^{\varepsilon}$ and similar results for N_1 and N_2. By an application of Lemmas 4.2 and 4.8 of [**11**] we obtain

$$\|K\|_\infty \ll \frac{K^{1/2}}{U}, \quad \|KN\|_\infty \ll \frac{K^{1/2}}{U} M^{1/2}(N_1N_2)^{1/2}\mathscr{L}^{-A} = \frac{x^{1/2}}{U}\mathscr{L}^{-A}.$$

Hence the integral in (2.12) is

$$\ll U\|KN\|_\infty \ll x^{1/2}\mathscr{L}^{-A}.$$

Now suppose that $K \leqslant 4U$. The integral in (2.12) is

$$\begin{aligned}
&\left(\int |M|^2\right)^{1/2}\left(\int |N_1^2N_2|^2\right)^{1/4}\left(\int |K^2N_2|^2\right)^{1/4} \\
&\qquad \ll x^{\varepsilon/50}(M+T)^{1/2}(N_1^2N_2+T)^{1/4}T^{1/4}(1+N_2^2T^{-1/2})^{1/4} \\
&\qquad \lll x^{\gamma}
\end{aligned}$$

by Lemma 2 and the mean value theorem [**2**, (3.3)]. Here

$$\begin{aligned}
\gamma = \tfrac{1}{2}\alpha' + \tfrac{1}{4}\max(2\beta_1+\beta_2, 1-\theta) + \tfrac{1}{4}(1-\theta) \\
+ \max(0, \tfrac{1}{2}\beta_2 - \tfrac{1}{8}(1-\theta)) - \tfrac{1}{25}\varepsilon.
\end{aligned}$$

The conditions (2.8)–(2.11) guarantee that $\gamma \leqslant \frac{1}{2} - \frac{1}{25}\varepsilon$.

LEMMA 4. *The conclusion of Lemma* 3 *holds if the hypotheses* (2.9)–(2.11) *are replaced by:*

$$\text{either } \beta_1 \leqslant \tfrac{1}{2}(1-\theta) \text{ or } N_1 \text{ is a zeta factor;} \tag{2.13}$$

$$\beta_2 \leqslant \tfrac{1}{8}(1+3\theta) - \tfrac{1}{2}\alpha'. \tag{2.14}$$

Proof. If either $K > 4U$, or $N_1 > 4U$ and N_1 is a zeta factor, we may proceed as at the beginning of the proof of Lemma 3. Thus we may suppose that these cases are excluded. The integral in (2.12) is at most

$$\left(\int |M|^2\right)^{1/2}\left(\int |N_1|^4\right)^{1/4}\left(\int |KN_2|^4\right)^{1/4} \ll x^{\delta},$$

where

$$\delta = \tfrac{1}{2}\alpha' + \tfrac{1}{2}(1-\theta) - \tfrac{1}{10}\varepsilon + \tfrac{1}{4}\max(0, 4\beta_2 - \tfrac{1}{2}(1-\theta)).$$

(If $\beta_1 \leqslant \frac{1}{2}(1-\theta)$, the mean value theorem yields $\int |N_1|^4 \ll T x^{\varepsilon/4}$; if N_1 is a zeta factor and $N_1 \leqslant 4U$, the same bound follows from (2.7).) The result now follows, in view of (2.14).

LEMMA 5. *Let $K(s)$ be a zeta factor, $K \ll x^{3/4}$. Suppose that $M = x^{\alpha}$, $N = x^{\beta}$, $\alpha \leqslant \theta$ and*

$$\beta \leqslant \min(\tfrac{1}{2}(3\theta + 1 - 4\alpha'), \tfrac{1}{5}(3 + \theta - 4\alpha')). \tag{2.15}$$

Suppose further that $M(s)$ and $N(s)$ are Dirichlet polynomials of the type (2.4). *Then*

$$\int_{T_0}^{T} |(MNK)(\tfrac{1}{2} + it)|\, dt \ll x^{1/2}\mathscr{L}^{-A}. \tag{2.16}$$

Proof. Let $[\frac{1}{2}U, U] \subset [T_0, T]$. It suffices to get the above mean value bound over $[\frac{1}{2}U, U]$.

Let

$$a = \min(2\theta - 2\alpha', \tfrac{1}{5}(1 - 3\theta + 2\alpha')).$$

We may suppose that

$$\beta > \tfrac{1}{2}(1-\theta),$$

since otherwise the result follows from Lemma 4 with $\beta_2 = 0$.

In view of Lemma 1, we may suppose that

$$N = N_1 \ldots N_t,$$

where $N_j = x^{\delta_j}$, $\delta_1 \leqslant \ldots \leqslant \delta_t$ and any N_j with $\delta_j > \frac{1}{2}(1-\theta)$ is a zeta factor.

We now give two cases in which (2.16) is valid.

Case 1. There is a subproduct x^{δ} of $N_1 \ldots N_t$ which is either a zeta factor or has $\delta \leqslant \frac{1}{2}(1-\theta)$. Moreover,

$$\beta - \delta \leqslant a.$$

If $\alpha' \geqslant \frac{1}{12}(13\theta - 1)$, then

$$a \leqslant 2\theta - 2\alpha' \leqslant \tfrac{1}{8}(1 + 3\theta) - \tfrac{1}{2}\alpha',$$

while if $\alpha' < \frac{1}{12}(13\theta - 1)$, then

$$a \leqslant \tfrac{1}{5}(1 - 3\theta + 2\alpha') \leqslant \tfrac{1}{8}(1 + 3\theta) - \tfrac{1}{2}\alpha'.$$

Now (2.16) follows on applying Lemma 4.

Case 2. There is a subproduct x^{δ} of $N_1 \ldots N_t$ such that

$$a \leqslant \delta \leqslant \beta - a.$$

Let $\beta_2 = \min(\delta, \beta - \delta)$; then $\beta_2 \in [a, \frac{1}{2}\beta]$. Let $\beta_1 = \beta - \beta_2$. Then

$$\beta_1 + \tfrac{1}{2}\beta_2 = \beta - \tfrac{1}{2}\beta_2 \leqslant \beta - \tfrac{1}{2}a \leqslant \tfrac{1}{2}(\theta + 1) - \alpha'.$$

Moreover,

$$\beta_2 \leqslant \tfrac{1}{2}(\tfrac{1}{2}(3\theta + 1 - 4\alpha')) = \tfrac{1}{4}(3\theta + 1) - \alpha'$$

and

$$\beta_1 + \tfrac{3}{2}\beta_2 \leqslant \tfrac{5}{4}\beta \leqslant \tfrac{1}{4}(3+\theta-4\alpha').$$

Now (2.16) follows from Lemma 3.

We may now complete the proof of the lemma. If $\delta_t \leqslant a$, there is evidently a subsum of $\delta_1 + \ldots + \delta_t$ in $[a, 2a]$. Now

$$2a \leqslant \beta - a,$$

since if $\alpha' \geqslant \frac{1}{12}(13\theta - 1)$, then

$$3a \leqslant 6\theta - 6\alpha' \leqslant \tfrac{1}{2}(1-\theta) < \beta,$$

while if $\alpha' < \frac{1}{12}(13\theta - 1)$, then

$$3a \leqslant \tfrac{1}{5}(3 - 9\theta + 6\alpha') \leqslant \tfrac{1}{2}(1-\theta) < \beta.$$

Thus Case 2 holds when $\delta_t \leqslant a$, and of course Case 2 also holds when $a < \delta_t \leqslant \beta - a$.

Finally suppose that $\delta_t > \beta - a$; then we are in Case 1 with $\delta = \delta_t$. This completes the proof of Lemma 5.

3. *Sieve asymptotic formulae*

In this section we establish formulae of the type (2.2) and use them as a stepping stone to formulae of type (1.3). In order to link (2.2) or (1.3) to the behaviour of Dirichlet polynomials we use the following variant of [**2**, Lemma 11].

LEMMA 6. *Let* $F(s) = \sum_{k \asymp x} c_k k^{-s}$. *If*

$$\int_{T_0}^{T} |F(\tfrac{1}{2}+it)|\,dt \ll x^{1/2}\mathscr{L}^{-A}, \tag{3.1}$$

then

$$\sum_{k\in\mathscr{A}} c_k = \frac{y}{y_1}\sum_{k\in\mathscr{B}} c_k + O(y\mathscr{L}^{-A}). \tag{3.2}$$

LEMMA 7. *Let a and u be positive numbers, $w = x^{1/u}$ and $D = x^a$. Suppose that*

$$1/a < u < (\log x)^{1-\varepsilon}. \tag{3.3}$$

Then

$$\sum_{\substack{d \mid P(w)\\ d > D}} \frac{1}{d} \ll \exp(\log\log w + 2ua - ua\log ua). \tag{3.4}$$

The implied constant is absolute.

Proof. Let $\rho = (u \log ua)/\mathscr{L}$. We use the simple inequality

$$\exp(cy) - 1 \leqslant (\exp(c) - 1)y \tag{3.5}$$

for $c > 0$ and $0 \leqslant y \leqslant 1$. Now

$$\sum_{\substack{d \mid P(w) \\ d > D}} \frac{1}{d} \leqslant \frac{1}{D^{\rho}} \sum_{d \mid P(w)} \frac{d^{\rho}}{d} = \frac{1}{D^{\rho}} \prod_{p < w} (1 + p^{\rho - 1})$$

$$= \frac{1}{D^{\rho}} \exp\left(\sum_{p < w} \log(1 + p^{\rho - 1}) \right)$$

$$\leqslant \frac{1}{D^{\rho}} \exp\left(\sum_{p < w} \frac{p^{\rho} - 1}{p} + \sum_{p < w} \frac{1}{p} \right). \tag{3.6}$$

We apply (3.5) with $y = \log p / \log w$ and $c = \log ua$. The last expression in (3.6) is at most

$$\frac{1}{D^{\rho}} \exp\left(\frac{\exp(\log ua) - 1}{\log w} \sum_{p < w} \frac{\log p}{p} + \sum_{p < w} \frac{1}{p} \right)$$

$$= \exp\left(-ua \log ua + ua\left(1 + O\left(\frac{1}{\log w}\right)\right) + \log\log w + O(1) \right)$$

by Mertens' theorems for $\sum p^{-1} \log p$ and $\sum p^{-1}$. This completes the proof.

LEMMA 8. *Let* $M(s) = \sum_{m \sim M} a_m m^{-s}$, $N(s) = \sum_{n \sim N} b_n n^{-s}$, $M = x^{\alpha}$ *and* $N = x^{\beta}$, *with* $\alpha \leqslant \theta - \varepsilon$ *and*

$$\beta \leqslant \min\left(\tfrac{1}{2}(3\theta + 1 - 4\alpha'), \tfrac{1}{5}(3 + \theta - 4\alpha')\right) - 2\varepsilon. \tag{3.7}$$

Suppose further that $M(s)$ *and* $N(s)$ *are of type* (2.4). *Then* (2.2) *holds.*

Proof. We follow the proof of [**2**, Lemma 12]. We must prove (3.2) with

$$c_k = \sum_{m,n} \sum_{\substack{d \mid l, d \mid P(w) \\ mnl = k}} a_m b_n \mu(d).$$

According to Lemma 15 of Heath-Brown [**7**],

$$\sum_{d \mid l, d \mid P(w)} \mu(d) = \sum_{\substack{d \mid l, d \mid P(w) \\ d \leqslant \gamma}} \mu(d) + O\left(\sum_{\substack{d \mid l, d \mid P(w) \\ \gamma \leqslant d < \gamma w}} 1 \right)$$

where $\gamma = x^{\varepsilon/2}$. Let

$$c_k' = \sum_{\substack{m,n; d \mid P(w), d \leqslant \gamma \\ l \equiv 0 \, (\mathrm{mod}\, d), mnl = k}} a_m b_n \mu(d), \quad c_k'' = \sum_{\substack{m,n; d \mid P(w) \\ \gamma \leqslant d < w\gamma, l \equiv 0 \, (\mathrm{mod}\, d) \\ mnl = k}} |a_m b_n|.$$

Then

$$\sum_{k \in \mathscr{A}} c_k = \sum_{k \in \mathscr{A}} c_k' + O\left(\sum_{k \in \mathscr{A}} c_k'' \right).$$

Suppose for the moment that $MND > x^{1/4}$. We now apply Lemma 5 to $M_1(s)N(s)K(s)$, where

$$M_1(s) = \sum_{m; d \sim D} \frac{a_m \mu(d)}{(md)^s}, \quad N(s) = \sum_n \frac{b_n}{n^s}, \quad K(s) = \sum_{kMND \asymp x} k^{-s},$$

and then sum over $D=\gamma 2^{-j}\in(\frac{1}{2},\frac{1}{2}\gamma]$. In each case M_1 has length at most Mx^{ε} and Lemma 5 is applicable. We conclude that

$$\sum_{k\in\mathscr{A}} c_k' = \frac{y}{y_1}\sum_{k\in\mathscr{B}} c_k' + O(y\mathscr{L}^{-A}). \tag{3.8}$$

We reach the same conclusion with c_k'' in place of c_k' by modifying $M_1(s)$ in an obvious fashion. Finally we bound $\sum_{k\in\mathscr{A}} c_k''$ by

$$\begin{aligned}\ll \frac{y}{y_1}\sum_{k\in\mathscr{B}} c_k'' &\ll y\sum_{m,n}\frac{|a_n|\,|b_n|}{mn}\sum_{\substack{d\mid P(w)\\ \gamma\leqslant d<w\gamma}}\frac{1}{d}\\ &\ll y\sum_{m,n}\frac{|a_m|}{m}\frac{|b_n|}{n}\exp(-\tfrac{1}{2}u\varepsilon\log u\varepsilon)\ll y\mathscr{L}^{-A},\end{aligned}$$

from Lemma 7. Here $w=x^{1/u}$, so that $u=\log\mathscr{L}$. Now (3.2) follows on assembling this together with (3.8).

Now suppose that $MND\leqslant x^{1/4}$, so that (3.8) and its analogue for c_k'' take the form

$$\sum_{\substack{nl\in\mathscr{A}\\ n\ll x^{1/4}}} a_n - \frac{y}{y_1}\sum_{\substack{nl\in\mathscr{B}\\ n\ll x^{1/4}}} a_n \ll y\mathscr{L}^{-A}.$$

This bound is easily established, because the left-hand side is

$$\sum_n a_n\left\{\frac{y}{n}+O(1)\right\} - \frac{y}{y_1}\sum_{n\sim N} a_n\left\{\frac{y_1}{n}+O(1)\right\} \ll \sum_n |a_n| \ll x^{1/4+\varepsilon}.$$

The proof may now be carried through as in the case $MND>x^{1/4}$.

LEMMA 9. *Let $LMN=x$. Let g be a natural number, $g\leqslant B$. Suppose that*

$$M=x^{\sigma_1},\quad N=x^{\sigma_2},$$

$$|\sigma_1-\sigma_2|<2\theta-1+\tfrac{1}{8}\varepsilon, \tag{3.9}$$

$$1-(\sigma_1+\sigma_2)<\min\left(4\theta-2,\ \frac{(8g-4)\theta-(4g-3)}{4g-1},\ \frac{24g\theta-(12g+1)}{4g-1}\right). \tag{3.10}$$

Suppose further that the Dirichlet polynomial $L(s)$ satisfies

$$\sup_{t\in[T_o,T]}|L(\tfrac{1}{2}+it)|\ll L^{1/2}\mathscr{L}^{-A}. \tag{3.11}$$

Then $F(s)=L(s)M(s)N(s)$ satisfies (3.1).

Proof. This is a variant of Theorem 4 of [**2**]. The only modification needed to the argument in [**2**] comes in Case 2(ii), where the expressions I_1 and I_2 must be replaced by

$$\begin{aligned}I_1' = {}&(TM^{f(\sigma_1)})^{1/2-1/4g}(TN^{f(\sigma_2)})^{1/2-1/4g}\\ &\times(L^{2g-2g\sigma_3})^{1/2g}M^{\sigma_1-1/2}N^{\sigma_2-1/2}L^{\sigma_3-1/2},\end{aligned}$$

and

$$I_2' = (TM^{f(\sigma_1)})^{1/2-1/12g}(TN^{f(\sigma_2)})^{1/2-1/12g} \times (TL^{4g-6g\sigma_3})^{1/6g} M^{\sigma_1-1/2} N^{\sigma_2-1/2} L^{\sigma_3-1/2}.$$

Here

$$f(\sigma) = \min(1-2\sigma, 4-6\sigma), \tag{3.12}$$

a function which has the simple property

$$\alpha f(\sigma) + \sigma - \tfrac{1}{2} \leqslant \alpha f(\tfrac{3}{4}) + \tfrac{1}{4} = \tfrac{1}{4}(1-2\alpha) \tag{3.13}$$

for any $\alpha \in [\frac{1}{6}, \frac{1}{2}]$. Thus

$$I_1' \leqslant T^{1-1/2g}(MN)^{1/8g} L^{1/2} = T^{1-1/2g} x^{1/8g} L^{1/2-1/8g} \ll x^{1/2}\mathscr{L}^{-A}$$

by (3.10). Similarly

$$I_2' \leqslant T(MN)^{1/24g} L^{1/6} = T x^{1/24g} L^{1/6-1/24g} \ll x^{1/2}\mathscr{L}^{-A}.$$

The desired result follows just as in [**2**].

We now require combinatorial lemmas designed to bring Lemma 9 into play after a number of 'Buchstab decompositions' of the left-hand side of (2.2).

LEMMA 10. *Let* $0 \leqslant \alpha \leqslant \frac{1}{2} + \varepsilon$ *and let* h *be the least positive integer with*

$$\alpha \geqslant \tfrac{1}{2} - 2h(\theta - \tfrac{1}{2}).$$

Let $k \geqslant 0$,

$$\frac{2(\theta-\alpha)}{2h-1} \geqslant \alpha_1 \geqslant \ldots \geqslant \alpha_k > 0, \tag{3.14}$$

and suppose that

$$\alpha + \alpha_1 + \ldots + \alpha_{k-1} + \tfrac{1}{2}\alpha_k \leqslant 1 - \theta \quad \text{if } k > 0. \tag{3.15}$$

Then, writing

$$\alpha^* = \max\left(\frac{2h(1-\theta)-\alpha}{2h-1}, \frac{2(h-1)\theta+\alpha}{2h-1}\right) \tag{3.16}$$

we have

$$\alpha + \alpha_1 + \ldots + \alpha_k \leqslant \alpha^*. \tag{3.17}$$

Proof. Suppose first that $k \geqslant h$. Since the α_j are decreasing, (3.15) yields

$$\alpha + (h - \tfrac{1}{2})\alpha_k \leqslant 1 - \theta, \tag{3.18}$$

$$\alpha_k \leqslant (1-\theta-\alpha)/(h-\tfrac{1}{2}), \tag{3.19}$$

so that

$$\alpha + \alpha_1 + \ldots + \alpha_k \leqslant 1 - \theta + \tfrac{1}{2}\alpha_k \leqslant 1 - \theta + \frac{1-\theta-\alpha}{2h-1} = \frac{2h(1-\theta)-\alpha}{2h-1}.$$

Now suppose that $k < h$. We apply (3.14) to obtain

$$\alpha + \alpha_1 + \ldots + \alpha_k \leqslant \alpha + (h-1)\frac{2(\theta-\alpha)}{2h-1} = \frac{2(h-1)\theta+\alpha}{2h-1}.$$

This completes the proof of Lemma 10.

Note that the function α^* of α satisfies

$$1-\theta \leqslant \frac{h-\theta}{2h-1} \leqslant \alpha^* \leqslant \tfrac{1}{2}+\varepsilon. \tag{3.20}$$

LEMMA 11. *Make the hypotheses of Lemma* 10, *and suppose further that*

$$\alpha+\alpha_1+\ldots+\alpha_k+\tfrac{1}{2}\alpha_{k+1} > 1-\theta. \tag{3.21}$$

Then the numbers

$$\gamma_1 = \alpha+\alpha_1+\ldots+\alpha_k$$

and

$$\gamma_2 = 1-(\alpha+\alpha_1+\ldots+\alpha_{k+1})$$

satisfy

$$|\gamma_1-\gamma_2| \leqslant 2\theta-1. \tag{3.22}$$

Proof. We have

$$\begin{aligned}\gamma_1-\gamma_2 &= 2(\alpha+\alpha_1+\ldots+\alpha_k)+\alpha_{k+1}-1\\ &\geqslant 2(1-\theta)-1 = -(2\theta-1)\end{aligned} \tag{3.23}$$

from (3.21). If $k \geqslant h$, then we have (3.19) from the previous proof. Since $\alpha \geqslant \frac{1}{2}-2h(\theta-\frac{1}{2})$,

$$\alpha_k \leqslant \frac{1-\theta-\alpha}{h-\frac{1}{2}} \leqslant \frac{\frac{1}{2}-\theta+2h(\theta-\frac{1}{2})}{h-\frac{1}{2}} = 2\theta-1.$$

Since the α_j are decreasing,

$$\begin{aligned}\gamma_1-\gamma_2 &= 2(\alpha+\alpha_1+\ldots+\alpha_{k-1}+\tfrac{1}{2}\alpha_k)+\alpha_k+\alpha_{k+1}-1\\ &\leqslant 2(1-\theta)+2(2\theta-1)-1 = 2\theta-1.\end{aligned}$$

If $k<h$, then (3.14) yields

$$\gamma_1-\gamma_2 \leqslant 2\alpha-1+(2h-1)\frac{2(\theta-\alpha)}{2h-1} = 2\theta-1.$$

This completes the proof of Lemma 11.

LEMMA 12. *Let* $\alpha \in [0, \frac{1}{2}]$,

$$0 \leqslant \beta \leqslant \min(\tfrac{1}{2}(3\theta+1-4\alpha^*), \tfrac{1}{5}(3+\theta-4\alpha^*)) - 2\varepsilon.$$

Let $M(s)=\sum_{m\sim M} a_m m^{-s}$, $N(s)=\sum_{n\sim N} b_n n^{-s}$, $2M=x^\alpha$ *and* $N=x^\beta$, *where* $M(s)$ *and* $N(s)$ *are of type* (2.4). *Let*

$$I_h = [\tfrac{1}{2}-2h(\theta-\tfrac{1}{2}), \tfrac{1}{2}-(2h-2)(\theta-\tfrac{1}{2})),$$

and write

$$\nu(\alpha) = \min\left(\frac{2}{2h-1}(\theta-\alpha), \frac{36\theta-17}{19}\right) \quad (\alpha \in I_h, h \geqslant 1).$$

Then (1.3) *holds for every* $\nu \leqslant \nu(\alpha)$.

This result sharpens Lemmas 5 and 6 of [**1**]. It is clear that $\nu(\alpha) \geqslant 2\theta-1$. The upper bound on β never falls below $\frac{1}{2}(3\theta-1)-2\varepsilon$.

Proof of Lemma 12. The summation conditions $m \sim M$ and $n \sim N$ will be omitted. Let

$$\psi(l, z) = \begin{cases} 1 & \text{if } (l, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

From Buchstab's identity,

$$\psi(l, z) = \psi(l, w) - \sum_{\substack{ph = l \\ w \leqslant p < z}} \psi(h, p). \tag{3.24}$$

Here w is given by (2.3).

We must prove that, taking $z = x^{\nu}$,

$$c_k(0) = \sum_{mnl = k} a_m b_n \psi(l, z)$$

satisfies (3.2). From (3.24),

$$c_k(0) = c_k'(0) - c_k''(0) - c_k(1)$$

where

$$c_k'(0) = \sum_{mnl = k} a_m b_n \psi(l, w),$$

$$c_k''(0) = \sum_{\substack{mnp_1 h_1 = k \\ w \leqslant p_1 < z \\ mp_1^{1/2} > x^{1-\theta}}} a_m b_n \psi(h_1, p_1),$$

$$c_k(1) = \sum_{\substack{mnp_1 h_1 = k \\ w \leqslant p_1 < z \\ mp_1^{1/2} \leqslant x^{1-\theta}}} a_m b_n \psi(h_1, p_1).$$

We continue the process by applying (3.24) to $c_k(1)$. In general, let

$$c_k(j) = \sum_{\substack{mp_1 \dots p_j h_j = k \\ w \leqslant p_j < \dots < p_1 < z \\ mp_1 \dots p_{j-1} p_j^{1/2} \leqslant x^{1-\theta}}} a_m b_n \psi(h_j, p_j);$$

then (3.24) gives

$$c_k(j) = c_k'(j) - c_k''(j) - c_k(j+1)$$

where $c_k'(j)$ is obtained from $c_k(j)$ by replacing $\psi(h_j, p_j)$ by $\psi(h_j, w)$, and

$$c_k''(j) = \sum_{\substack{mp_1 \dots p_j p_{j+1} h_{j+1} = k \\ w \leqslant p_{j+1} < \dots < p_1 < z \\ mp_1 \dots p_{j-1} p_j^{1/2} \leqslant x^{1-\theta} < mp_1 \dots p_j p_{j+1}^{1/2}}} a_m b_n \psi(h_{j+1}, p_{j+1}).$$

For $j \geqslant \mathscr{L} / \log w = \log \mathscr{L}$, the sum $c_k(j)$ is empty and decomposition ceases.

Each $c_k'(j)$ satisfies (3.2). To see this, we write $p_i = x^{\alpha_i}$. Then since $m \leqslant x^{\alpha}$, where $\alpha \in I_h$, we have

$$mp_1 \dots p_j \leqslant x^{\alpha^*} \tag{3.25}$$

in the sum $c_k'(j)$, by Lemma 10. We now obtain (3.2) by an appeal to Lemmas 6 and 8, taking

$$M(s) = \sum_{\substack{mp_1 \dots p_{j-1} p_j^{1/2} \leqslant x^{1-\theta} \\ w \leqslant p_j < \dots < p_1 < z}} a_m (mp_1 \dots p_j)^{-s}.$$

Each $c''_k(j)$ satisfies (3.2). For in $c''_k(j)$, we may write

$$h_{j+1} = p'_1 \dots p'_u$$

where $u \leqslant \log \mathscr{L}$. It suffices to prove (3.2) for the portion $c''_k(j,u)$ of $c''_k(j)$ arising from a fixed value of u.

As in the proof of Lemma 1 of [**3**], we may remove the conditions of summation, $x^{1-\theta} < mp_1 \dots p_j p_{j+1}^{1/2}$, $p_{j+1} < p_j$ and $p_{j+1} \leqslant p'_1$, by the use of Perron's formula, which modifies the coefficients of the Dirichlet polynomials in an acceptable way. (The same process is implicit several times in the rest of the paper.) We apply Lemmas 6 and 9 with $g = 5$, with $M(s)$ produced by grouping $m, p_1, \dots, p_j$, $L(s)$ corresponding to p_{j+1}, and $N(s)$ produced by grouping the remaining variables. The restriction to dyadic ranges presents no problem, and we have only to verify (3.9)–(3.11). For (3.9) we appeal to Lemma 11; the condition (3.21) derives from one of the summation conditions for $c''_k(j)$. For (3.10) we note that

$$1 - (\sigma_1 + \sigma_2) \leqslant \nu(\alpha),$$

and $\nu(\alpha)$ is defined in such a way that (3.10) holds. We complete the proof of (3.2) for $c''_k(j)$ by noting that (3.11) follows from [**2**, Lemma 5]. We now obtain (3.4) for $c_k(0)$ on combining the results for the $O(\log \mathscr{L})$ expressions into which we have decomposed it.

LEMMA 13. *Let $M = x^{\alpha}$, $N_1 = x^{\beta}$ and $N_2 = x^{\gamma}$, where $M(s)$, $N_1(s)$ and $N_2(s)$ are of type* (2.4) *and suppose that $\alpha \leqslant \frac{1}{2}$ and either*

(i)
$$2\beta + \gamma \leqslant 1 + \theta - 2\alpha^* - 2\varepsilon,$$
$$\gamma \leqslant \tfrac{1}{4}(1 + 3\theta) - \alpha^* - \varepsilon,$$
$$2\beta + 3\gamma \leqslant \tfrac{1}{2}(3 + \theta) - 2\alpha^* - 2\varepsilon,$$

or

(ii)
$$\beta \leqslant \tfrac{1}{2}(1 - \theta), \quad \gamma \leqslant \tfrac{1}{8}(1 + 3\theta - 4\alpha^*) - \varepsilon.$$

Let

$$b_n = \sum_{\substack{n_1 n_2 = n \\ n_1 \sim N_1, n_2 \sim N_2}} A_{n_1} B_{n_2}.$$

Then (1.3) *holds whenever $\nu \leqslant \nu(\alpha)$.*

Proof. We follow the proof of Lemma 12, altering only the discussion of $c'_k(j)$, where $N(s)$ no longer satisfies the requirements of Lemma 8. However, $N(s) = N_1(s)N_2(s)$, and at the point in the proof of Lemma 8 where we appeal to Lemma 5, we may substitute an appeal to Lemma 3 in Case (i), or to Lemma 4 in Case (ii). Modified in this way, the proof of the necessary variant of Lemma 8 goes through, and we obtain the desired result.

4. *The two-dimensional sieve*

For a given positive integer m let us write, suppressing dependence on R,

$$\mathscr{E}^m = \{rl\colon mrl \in \mathscr{A}, r \sim R\}$$

and define $\mathscr{F}^m$ similarly with $\mathscr{A}$ replaced by $\mathscr{B}$.

LEMMA 14. *Let $x^{1/4} < M \leqslant x^{1/2}$ and $MN^2R < x^{1-2\varepsilon}$, and suppose $M(s)$ is of type* (2.4). *Then*

$$\sum_{m\sim M}\sum_{n\sim N} a_m b_n S(\mathscr{E}^{mn^2}, w) = \frac{y}{y_1}\sum_{m\sim M}\sum_{n\sim N} a_m b_n S(\mathscr{F}^{mn^2}, w) + O(y\mathscr{L}^{-A}). \quad (4.1)$$

Proof. As in the proof of Lemma 8, the left-hand side of (4.1) is

$$\sum_{m,n} a_m b_n \sum_{\substack{d\mid P(w)\\ d\leqslant\gamma}} \mu(d)|\mathscr{E}_d^{mn^2}| + O\Bigg(\sum_{m,n}|a_m b_n| \sum_{\gamma\leqslant d<\gamma w}|\mathscr{E}_d^{mn^2}|\Bigg). \quad (4.2)$$

Let $L = x/MN^2R$. For given $d \mid P(w)$,

$$\begin{aligned}|\mathscr{E}_d^{mn^2}| &= \sum_{eh=d}\ \sum_{\substack{r\sim R\\ e\mid r\\ (r,h)=1}}\ \sum_{\substack{l\asymp L\\ h\mid l\\ mn^2rl\in\mathscr{A}}} 1\\ &= \sum_{eh=d}\ \sum_{r'\sim R/e}\Bigg(\sum_{\substack{f\mid h\\ f\mid r'}}\mu(f)\Bigg)\sum_{\substack{l'\asymp Lh^{-1}\\ mn^2er'hl'\in\mathscr{A}}} 1\\ &= \sum_{efg=d}\mu(f)\sum_{\substack{r''\sim R/ef\\ mn^2ef^2gr''l'\in\mathscr{A}}}\ \sum_{l'\asymp L/fg} 1. \end{aligned} \quad (4.3)$$

Thus, given any coefficients λ_d, with $|\lambda_d| \leqslant 1$,

$$\sum_{m,n} a_m b_n \sum_{\substack{d<\gamma w\\ d\mid P(w)}} \lambda_d |\mathscr{E}_d^{mn^2}| = \sum_{k\in\mathscr{A}} c_k \quad (4.4)$$

with

$$c_k = \sum_{\substack{efg<\gamma w\\ efg\mid P(w)}} \lambda_{efg}\,\mu(f) \sum_{\substack{m,n\\ mn^2ef^2grl=k}} a_m b_n \sum_{r\sim R/ef}\ \sum_{l\asymp L/fg} 1.$$

Let $E \geqslant 1$, $F \geqslant 1$, $G \geqslant 1$, $EFG \ll \gamma w$, $R_1 = R/EF$, $L_1 = L/FG$,

$$M(s) = \sum_{m\sim M} a_m m^{-s} \sum_{\substack{e\sim E,\, f\sim F\\ g\sim G}} \lambda_{efg}\,\mu(f)(ef^2g)^{-s},$$

$$L_1(s) = \sum_{l\asymp L_1} l^{-s} \quad \text{and} \quad R_1(s) = \sum_{r\asymp R_1} r^{-s}.$$

Lemma 2 and the mean value theorem yield, for $T_0 < U \leqslant T$ and $\max(R_1, L_1) \leqslant 4U$,

$$\begin{aligned}\int_{U/2}^{U} |MR_1L_1(\tfrac12 + it)|\,dt &\leqslant \Big(\int |M|^2\Big)^{1/2}\Big(\int |R_1|^4\Big)^{1/4}\Big(\int |L_1|^4\Big)^{1/4}\\ &\ll x^{1/4+(1-\theta)/2+\varepsilon} \ll x^{1/2-\varepsilon}\mathscr{L}^{-A}.\end{aligned}$$

If R_1 or L_1 is greater than $4U$, we get the bound

$$\int_{U/2}^{U} |MR_1L_1(\tfrac{1}{2}+it)|\,dt \ll x^{1/2}\mathscr{L}^{-A}$$

by arguing as at the start of the proof of Lemma 3. Consequently,

$$F(s) = \sum_{k \asymp x} c_k k^{-s} = M(s)R_1(s)L_1(s)\sum_n \frac{b_n}{n^{2s}}$$

satisfies

$$\int_{T_0}^{T} |F(\tfrac{1}{2}+it)|\,dt \ll x^{1/2}\mathscr{L}^{-2A}\sum_n \frac{|b_n|}{n}$$
$$\ll x^{1/2}\mathscr{L}^{-A}$$

and (3.2) holds. In particular, the first summand in (4.2) is

$$\frac{y}{y_1}\sum_{m,n} a_m b_n \sum_{\substack{d \mid P(w) \\ d \leqslant \gamma}} \mu(d)|\mathscr{F}_d^{mn^2}| + O(y\mathscr{L}^{-A}) \tag{4.5}$$

and the second is

$$O\left(\frac{y}{y_1}\sum_{m,n} |a_m|\,|b_n| \sum_{\gamma \leqslant d < \gamma w} |\mathscr{F}_d^{mn^2}|\right) + O(y\mathscr{L}^{-A}). \tag{4.6}$$

Moreover,

$$\sum_{m,n} |a_m b_n| \sum_{\gamma \leqslant d < \gamma w} |\mathscr{F}_d^{mn^2}| \ll \sum_{m,n} |a_m b_n| \sum_{\substack{r \sim R/ef \\ mn^2ef^2grl \in \mathscr{B}}} \sum_{l \asymp L/fg} 1$$
$$\ll \sum_{m,n} |a_m b_n| \sum_{\gamma \leqslant efg < \gamma w} \sum_{r \sim R/ef} \frac{y_1}{mn^2ef^2gr}.$$

Consider, for example, the part of the last sum with $e \geqslant \gamma^{1/3}$; this is

$$\ll y_1\mathscr{L}^B \sum_{\substack{\gamma^{1/3} \leqslant e < \gamma w \\ e \mid P(w)}} \frac{1}{e} \ll y_1\mathscr{L}^{-A}$$

by Lemma 7. It is now easy to complete the proof on combining this estimate with the formulas (4.5) and (4.6).

LEMMA 15. *Let $M \asymp x^{\alpha}$ with $\frac{1}{4} < \alpha \leqslant \frac{1}{2}$; let $R \ll x^{1/2-2\varepsilon}$. Suppose that $M(s)$ is of type* (2.4). *Then*

$$\sum_{m \sim M} a_m S(\mathscr{E}^m, x^{\nu}) = \frac{y}{y_1}\sum_{m \sim M} a_m S(\mathscr{F}^m, x^{\nu}) + O(y\mathscr{L}^{-A})$$

for all $\nu \leqslant \nu(\alpha)$.

Proof. We must show that

$$c_k(0) = \sum_{\substack{m \sim M \\ mrl = k}} a_m \sum_{r \sim R} \sum_l \psi(rl, z)$$

satisfies (3.2), where $z = x^{\nu}$. Imitating the proof of Lemma 12, we find that this reduces to proving (3.2) for $c'_k(j)$, $c''_k(j)$, where

$$c'_k(j) = \sum_{mrl=k} a_m \sum_{\substack{w \leqslant p_j < \dots < p_1 < z \\ p_1 \dots p_j \mid rl \\ mp_1 \dots p_{j-1} p_j^{1/2} \leqslant x^{1-\theta}}} \psi\left(\frac{rl}{p_1 \dots p_j}, w\right),$$

$$c''_k(j) = \sum_{mrl=k} a_m \sum_{\substack{w \leqslant p_j < \dots < p_1 < z \\ p_1 \dots p_{j+1} \mid rl \\ mp_1 \dots p_{j-1} p_j^{1/2} \leqslant x^{1-\theta} < mp_1 \dots p_j p_{j+1}^{1/2}}} \psi\left(\frac{rl}{p_1 \dots p_{j+1}}, p_{j+1}\right)$$

(with obvious modifications when $j = 0$).

Let $\sum_H$ denote a sum over all subsets H of $\{1, \dots, j\}$; write $u(H) = \prod_{i \in H} p_i$ and $v(H) = \prod_{i \notin H, i \leqslant j} p_i$. For $p_1 \dots p_j$ counted in $c'_k(j)$ there is an H such that $u(H) \mid r$ and $(r, v(H)) = 1$. Thus, writing $r' = r/u(H)$ and $l' = l/v(H)$, we have

$$c'_k(j) = \sum_H \sum_m a_m \sum_{p_1, \dots, p_j} \sum_{\substack{r' \sim R/u(H), (r', v(H)) = 1 \\ mr' p_1 \dots p_j l' = k}} \sum_{l'} \psi(r'l', w).$$

Inserting the factor $\sum_{K \cap H = \emptyset} (-1)^{|K|} \sum_{u(K) \mid r'} 1$ in place of the condition $(r', v(H)) = 1$, we arrive at

$$c'_k(j) = \sum_H \sum_{K \cap H = \emptyset} (-1)^{|K|} \sum_m a_m \sum_{\substack{p_1, \dots, p_j \\ mr'' u(K) p_1 \dots p_j l' = k}} \sum_{r'' \sim R/u(H)u(K)} \sum_{l'} \psi(r''l', w)$$

$$= \sum_H \sum_{K \cap H = \emptyset} (-1)^{|K|} \sum_{m'} A_{m'} \sum_{\substack{n \\ m'n^2 r'' l' = k}} b_n \sum_{r''} \sum_{l'} \psi(r''l', w).$$

Here

$$A_{m'} = \sum_{\substack{m \\ m \prod_{i \notin K} p_i = m'}} \sum_{p_i \ (i \notin K)} a_m, \quad b_n = \sum_{\substack{p_i \ (i \in K) \\ u(K) = n}} 1. \tag{4.7}$$

Note that, recalling (3.25),

$$m'n \leqslant x^{1/2}, \quad nr'' \ll R \ll x^{1/2 - 2\varepsilon}; \tag{4.8}$$

hence $m' \leqslant x^{1/2}$ and $m'n^2 r'' \ll x^{1-2\varepsilon}$ in the last expression for $c'_k(j)$. Since there are fewer than $2^{2j} \leqslant \mathcal{L}^2$ possibilities for H and K, Lemma 14 yields (3.2) for $c'_k(j)$.

We may proceed similarly to obtain

$$c''_k(j-1) = \sum_H \sum_{K \cap H = \emptyset} (-1)^{|K|}$$
$$\sum_m \sum_{\substack{w \leqslant p_j < \dots < p_1 < z \\ mp_1 \dots p_{j-1}^{1/2} \leqslant x^{1-\theta} < mp_1 \dots p_j^{1/2} \\ mr'' u(K) p_1 \dots p_j l' = k}} \sum_{r'' \sim R/u(H)u(K)} \sum_{l'} \psi(r''l', p_j).$$

We approximate this expression by the contribution from $K = \emptyset$,

$$c_k = \sum_H \sum_m \sum_{p_1,\ldots,p_j} \sum_{r''} \sum_{l'} \psi(r''l', p_j)$$
$$mr''p_1\ldots p_j l' = k$$

where the summation conditions for m, $p_1, \ldots, p_j$, r'', l' are as in the preceding sum. We have

$$|c''_k(j-1) - c_k| \leqslant \sum_H \sum_{\substack{K \cap H = \emptyset \\ K \neq \emptyset}} \sum_{m'} |A_{m'}| \sum_n b_n \sum_{r'' \sim R/u(H)u(K)} \sum_{l'} \psi(r''l', w) = C_k,$$
$$m'n^2r''l' = k$$

say. Here $A_{m'}$ and b_n are as in (4.7) and we see that

$$\sum_{k \in \mathscr{A}} C_k = \frac{y}{y_1} \sum_{k \in \mathscr{B}} C_k + O(y\mathscr{L}^{-A}),$$

while, for some H and K,

$$\sum_{k \in \mathscr{B}} C_k \ll y_1 \mathscr{L}^2 \sum_{m'} \frac{|A_{m'}|}{m'} \sum_{r'' < x} \frac{1}{r''} \sum_{l' < x} \frac{1}{l'} \sum_{n > w} \frac{1}{n^2} \ll y_1 \mathscr{L}^{-A}.$$

Obviously, then, it suffices to prove (3.2) for c_k in place of $c''_k(j-1)$. The argument is now essentially the same as the discussion of $c''_k(j)$ at the end of the proof of Lemma 12 and we omit it. This completes the proof of Lemma 15.

In the next lemma we use Lemma 15 to obtain a formula of the shape

$$\sum_{p_1 \sim M} \sum_{p_2 \sim R} S(\mathscr{A}_{p_1 p_2}, x^\nu) = \frac{y}{y_1} \sum_{p_1 \sim M} \sum_{p_2 \sim R} S(\mathscr{B}_{p_1 p_2}, x^\nu) + O(y\mathscr{L}^{-A}) \tag{4.9}$$

that would be inaccessible by the method of § 3.

LEMMA 16. *Suppose that* $M = x^\alpha$,

$$R \leqslant M \quad \textit{and} \quad M^2 R < x.$$

Then (4.9) *holds for* $\nu = 2\theta - 1$.

Proof. In view of Lemma 12 we may suppose that $R \geqslant x^{(3\theta - 1)/2 - 2\varepsilon}$. Lemma 15 yields

$$\sum_{p_1 \sim M} S(\mathscr{E}^{p_1}, x^\nu) = \frac{y}{y_1} \sum_{p_1 \sim M} S(\mathscr{F}^{p_1}, x^\nu) + O(y\mathscr{L}^{-A}).$$

Here

$$\mathscr{E}^{p_1} = \{rl \colon r \sim R,\ p_1 rl \in \mathscr{A}\}$$

and $\mathscr{F}^{p_1}$ is defined similarly with $\mathscr{B}$ in place of $\mathscr{A}$.

We have

$$\begin{aligned}\sum_{p_1\sim M} S(\mathscr{E}^{p_1}, x^\nu) &= \sum_{p_1\sim M} |\{p'_1\ldots p'_u l\colon p'_1\ldots p'_u \sim R,\ p_1p'_1\ldots p'_u l\in\mathscr{A},\\ &\qquad\qquad x^\nu\leqslant p'_1\leqslant\ldots\leqslant p'_u,\ (l, P(x^\nu))=1\}|\\ &= \sum_{p_1\sim M}\ \sum_{\substack{x^\nu\leqslant p'_1\leqslant\ldots\leqslant p'_u,\\ p'_1\ldots p'_u\sim R}} S(\mathscr{A}_{p_1p'_1\ldots p'_u}, x^\nu)\\ &= \sum_{p_1\sim M}\sum_{p'_1\sim R} S(\mathscr{A}_{p_1p'_1}, x^\nu)\\ &\quad+\sum_{\substack{p_1\sim M\\ x^\nu\leqslant p'_1\leqslant(2R)^{1/2}}}\ \sum_{\substack{p'_1\leqslant p'_2\leqslant\ldots\leqslant p'_u\\ p'_2\ldots p'_u\sim R/p'_1}} S(\mathscr{A}_{p_1p'_1p'_2\ldots p'_u}, x^\nu). \qquad (4.10)\end{aligned}$$

The second sum in the last expression is

$$S=\frac{y}{y_1}\sum_{\substack{p_1\sim M\\ x^\nu\leqslant p'_1\leqslant(2R)^{1/2}}}\ \sum_{\substack{p'_1\leqslant p'_2\leqslant\ldots\leqslant p'_u\\ p'_2\ldots p'_u\sim R/p'_1}} S(\mathscr{B}_{p_1p'_1\ldots p'_u}, x^\nu)+O(y\mathscr{L}^{-A})$$

by an application of Lemma 12. For this it suffices to note that $MR^{1/2}\ll x^{1/2}$, and

$$R/p_1\ll x^{1/3}p_1^{-1}\ll x^{1/3-(2\theta-1)}\ll x^{(3\theta-1)/2-2\varepsilon}$$

since $\theta>0.524$. (Removal of the condition $p'_1\leqslant p'_2$ is covered in [**3**], as pointed out earlier.) Now

$$\begin{aligned}\sum_{p_1\sim M}\sum_{p'_1\sim R} S(\mathscr{A}_{p_1p'_1}, x^\nu) &= \frac{y}{y_1}\left\{\sum_{p_1\sim M} S(\mathscr{F}^{p_1}, x^\nu)-S\right\}+O(y\mathscr{L}^{-A})\\ &= \frac{y}{y_1}\sum_{p_1\sim M}\sum_{p'_1\sim R} S(\mathscr{B}_{p_1p'_1}, x^\nu)+O(y\mathscr{L}^{-A})\end{aligned}$$

by an obvious variant of (4.10). This completes the proof of Lemma 16.

Lemma 17. *Let $M_1\leqslant M_2\leqslant M_3$, $M_1M_2M_3^2\leqslant x$ and $M_1\geqslant x^{2\theta-1}$, and suppose $M_1(s)$ and $M_3(s)$ are of type* (2.4). *Then, for $0<\nu<2\theta-1$,*

$$\begin{aligned}&\sum_{m_1\sim M_1}\ \sum_{p_2\sim M_2}\ \sum_{m_3\sim M_3} a_{m_1}b_{m_3}S(\mathscr{A}_{m_1p_2m_3}, x^\nu)\\ &\quad=\frac{y}{y_1}\sum_{m_1\sim M_1}\ \sum_{p_2\sim M_2}\ \sum_{m_3\sim M_3} a_{m_1}b_{m_3}S(\mathscr{B}_{m_1p_2m_3}, x^\nu)+O(y\mathscr{L}^{-A}).\end{aligned}$$

Proof. We have

$$M_1M_3\leqslant(M_1M_2M_3^2)^{1/2}\leqslant x^{1/2},\quad M_2^3\ll xM_1^{-1}\ll x^{2-2\theta}.$$

If $M_2\leqslant x^{(3\theta-1)/2-2\varepsilon}$, the result follows from Lemma 12. Suppose now that

$$M_2>x^{(3\theta-1)/2-2\varepsilon}.$$

Then $M_1 M_3 > M_2 > x^{1/4}$. Lemma 15 yields an asymptotic formula for

$$\sum_{m_1 \sim M_1} \sum_{m_3 \sim M_3} a_{m_1} b_{m_3} S(\mathscr{E}^{m_1 m_3}, x^\nu).$$

(Here $R = M_2$.)

In analogy with (4.10),

$$\begin{aligned}
&\sum_{m_1, m_3} a_{m_1} b_{m_3} S(\mathscr{E}^{m_1 m_3}, x^\nu) \\
&\qquad = \sum_{m_1, m_3} a_{m_1} b_{m_3} \sum_{\substack{x^\nu \leqslant p_1' \leqslant \ldots \leqslant p_u' \\ p_1' \ldots p_u' \sim M_2}} S(\mathscr{A}_{m_1 m_2 p_1' \ldots p_u'}, x^\nu) \\
&\qquad = \sum_{m_1, m_3} a_{m_1} b_{m_3} \sum_{p_1' \sim M_2} S(\mathscr{A}_{m_1 m_3 p_1'}, x^\nu) \\
&\qquad\quad + \sum_{\substack{m_1, m_3 \\ x^\nu \leqslant p_1' \leqslant (2M_2)^{1/2}}} \sum_{\substack{p_1' \leqslant p_2' \leqslant \ldots \leqslant p_u' \\ p_2' \ldots p_u' \sim M_2/p_1'}} S(\mathscr{A}_{m_1 m_3 p_1' \ldots p_u'}, x^\nu).
\end{aligned}$$

The second sum in the last expression is

$$\frac{y}{y_1} \sum_{\substack{m_1, m_3 \\ x^\nu \leqslant p_1' \leqslant (2M_2)^{1/2}}} \sum_{\substack{p_1' \leqslant p_2' \leqslant \ldots \leqslant p_u' \\ p_2' \ldots p_u' \sim M_2/p_1'}} S(\mathscr{B}_{m_1 m_3 p_1' \ldots p_u'}, x^\nu) + O(y\mathscr{L}^{-A}).$$

To see this we divide it into two subsums defined by

(i) $m_1 m_3 p_1' \leqslant x^{1/2}$,

(ii) $m_1 m_3 p_1' > x^{1/2}$.

If condition (i) holds, then

$$p_2' \ldots p_u' \ll M_2 x^{-(2\theta-1)} \ll x^{(2-2\theta)/3-(2\theta-1)} \ll x^{(3\theta-1)/2-2\varepsilon}$$

since $\theta > 0.524$. We now get the desired result from Lemma 12, with variables regrouped as $m = m_1 m_3 p_1'$ and $n = p_2' \ldots p_u'$.

If condition (ii) holds, then we regroup the variables differently, taking

$$m = m_3 p_2' \ldots p_u' \ll m_3 M_2 / p_1' \ll m_3^2 m_1 M_2 x^{-1/2} \ll x^{1/2},$$

and

$$n = m_1 p_1' \ll m_1 M_2^{1/2} \ll x M_2^{-5/2} \ll x^{1-5(3\theta-1)/4+6\varepsilon} \ll x^{(3\theta-1)/2-2\varepsilon}.$$

Once again, the desired result follows from Lemma 12. We may now complete the proof in exactly the same way as the previous lemma.

5. *Further asymptotic formulae*

Let $L_1 \ldots L_l = x$, $l \geqslant 3$, $L_j = x^{\alpha_j}$ and $\alpha_j \geqslant \varepsilon$. We shall find a region of $(\alpha_1, \ldots, \alpha_l)$ in which

$$\int_{T_0}^{T} |L_1(\tfrac{1}{2}+it) \ldots L_l(\tfrac{1}{2}+it)|^h \, dt \ll x^{h/2} \mathscr{L}^{-A} \tag{5.1}$$

for every $A > 0$. Here $h = 1$ or 2. The case $h = 2$ is needed for application to primes in almost all short intervals, which we shall consider in another paper. For $h = 1$, (5.1) permits us to evaluate

$$\sum_{p_1 \sim L_1} \cdots \sum_{p_{l-1} \sim L_{l-1}} S(\mathscr{A}_{p_1 \ldots p_{l-1}}, p_{l-1}).$$

This is essentially an application of Lemma 6. We have already discussed the removal of the condition $(b_l, P(p_{l-1})) = 1$ in counting $p_1 \ldots p_{l-1} b_l$ in the last sum.

In order to prove (5.1) we need only prove

$$\sum_j |L_1(\tfrac{1}{2} + it_j) \ldots L_l(\tfrac{1}{2} + it_j)|^h \ll x^{h/2} \mathscr{L}^{-A} \tag{5.2}$$

for any set $\mathscr{S} = \{t_1, t_2, \ldots\}$ in $[T_0, T]$ with $|t_i - t_j| \geqslant 1$ $(i \neq j)$. By a simple dyadic decomposition we may assume that

$$L_j^{\sigma_j - 1/2} < |L_j(\tfrac{1}{2} + it)| \leqslant 2 L_j^{\sigma_j - 1/2}$$

where $\frac{1}{2} \leqslant \sigma_j \leqslant 1 + \varepsilon$ and the left-hand inequality is to be deleted when $\sigma_j = \frac{1}{2}$. We recall that the Dirichlet polynomials L_j have $L_j^{\sigma_j - 1} \leqslant \mathscr{L}^B$. For $j = 3$ we need to hypothesize the stronger inequality

$$L_3^{\sigma_3 - 1} \ll \mathscr{L}^{-A} \tag{5.3}$$

for every $A > 0$.

Now (5.2) will follow if we show that

$$S := |\mathscr{S}| \mathscr{L}^{-B} \prod_j L_j^{h(\sigma_j - 1)} \ll \mathscr{L}^{-A} \tag{5.4}$$

(the product $\prod_j$ runs over $j = 1, \ldots, l$ unless otherwise stated). We have at our disposal the bounds (f as in (3.12)):

$$|\mathscr{S}| \mathscr{L}^{-B} \ll \max(L_j^{g_j(2 - 2\sigma_j)} L_3^{k_j(2 - 2\sigma_3)}, T L_j^{g_j f(\sigma_j)} L_3^{k_j f(\sigma_3)})$$

where k_1 and k_2 are 0 or 1, and k_j is 0 for $j > 2$. To see this, apply Lemma 1 of [**2**] to $L_j^{g_j} L_3^{k_j}$. We write $(I)_j$ as an abbreviation for

$$|\mathscr{S}| \mathscr{L}^{-B} \ll L_j^{g_j(2 - 2\sigma_j)} L_3^{k_j(2 - 2\sigma_3)}$$

and $(II)_j$ for

$$|\mathscr{S}| \mathscr{L}^{-B} \ll T L_j^{g_j f(\sigma_j)} L_3^{k_j f(\sigma_3)}.$$

LEMMA 18. *With the above notation, suppose that*

$$\frac{h}{2g_1} + \frac{h}{2g_2} \leqslant 1, \quad v := 1 - \left(\frac{k_1}{g_1} + \frac{k_2}{g_2}\right) > 0, \quad \sum_{i \neq 3} \frac{h}{2g_i} + \frac{hv}{2g_3} > 1. \tag{5.5}$$

Let $u = 1 - \sum_{i > 3} h/2g_i$ *and suppose that*

$$\tfrac{1}{6} hv \leqslant g_3 \left(u - \frac{h}{2g_1} - \frac{h}{2g_2}\right) \leqslant \tfrac{1}{2} hv.$$

Let $b_1, c_1, a_2, c_2, b_3, c_3, a_4, c_4, a_5, b_5, c_5, a_6, b_6, c_6$ *be non-negative numbers,*

$a_j \in [h/6g_1, h/2g_1]$, $b_j \in [h/6g_2, h/2g_2]$,

$$u = \frac{h}{2g_1} + b_1 + c_1 = a_2 + \frac{h}{2g_2} + c_2 = \frac{h}{2g_1} + b_3 + c_3$$

$$= a_4 + \frac{h}{2g_2} + c_4 = a_r + b_r + c_r \quad (r \geqslant 5),$$

$$\tfrac{1}{6}\left(-2g_3c_1 + h - \frac{hk_1}{g_1}\right) \leqslant k_2b_1 \leqslant \tfrac{1}{2}\left(-2g_3c_1 + h - \frac{hk_1}{g_1}\right),$$

$$\tfrac{1}{6}\left(-2g_3c_2 + h - \frac{hk_2}{g_2}\right) \leqslant k_1a_2 \leqslant \tfrac{1}{2}\left(-2g_3c_2 + h - \frac{hk_2}{g_2}\right),$$

$$\tfrac{1}{6}\left(h - \frac{hk_1}{g_1}\right) \leqslant k_2b_3 + g_3c_3 \leqslant \tfrac{1}{2}\left(h - \frac{hk_1}{g_1}\right),$$

$$\tfrac{1}{6}\left(h - \frac{hk_2}{g_2}\right) \leqslant k_1a_4 + g_3c_4 \leqslant \tfrac{1}{2}\left(h - \frac{hk_2}{g_2}\right),$$

$$\tfrac{1}{6}(-2g_3c_5 + h) \leqslant k_1a_5 + k_2b_5 \leqslant \tfrac{1}{2}(-2g_3c_5 + h),$$

$$\tfrac{1}{6}h \leqslant k_1a_6 + k_2b_6 + g_3c_6 \leqslant \tfrac{1}{2}h.$$

Then (5.1) *holds whenever* $\alpha_j \geqslant g_j^{-1}(1-\theta)$ $(j > 3)$,

$$\alpha_2(\tfrac{1}{4}h + \tfrac{1}{2}g_2b_1) + \alpha_3\left(-\tfrac{1}{2}g_3c_1 + \tfrac{1}{4}h - \frac{hk_1}{4g_1} + \tfrac{1}{2}k_2b_1\right) \geqslant (b_1 + \varepsilon)(1-\theta), \quad (5.6)$$

$$\alpha_1(\tfrac{1}{4}h + \tfrac{1}{2}g_1a_2) + \alpha_3\left(-\tfrac{1}{2}g_3c_2 + \tfrac{1}{4}h - \frac{hk_2}{4g_2} + \tfrac{1}{2}k_1a_2\right) \geqslant (a_2 + \varepsilon)(1-\theta), \quad (5.7)$$

$$\alpha_2(\tfrac{1}{4}h + \tfrac{1}{2}g_2b_3) + \alpha_3\left(\tfrac{1}{2}g_3c_3 + \tfrac{1}{4}h - \frac{hk_1}{4g_1} + \tfrac{1}{2}k_2b_3\right) \geqslant \left(u - \frac{h}{2g_1} + \varepsilon\right)(1-\theta), \quad (5.8)$$

$$\alpha_1(\tfrac{1}{4}h + \tfrac{1}{2}g_1a_4) + \alpha_3\left(\tfrac{1}{2}g_3c_4 + \tfrac{1}{4}h - \frac{hk_2}{4g_2} + \tfrac{1}{2}k_1a_4\right) \geqslant \left(u - \frac{h}{2g_2} + \varepsilon\right)(1-\theta), \quad (5.9)$$

$$\alpha_1(\tfrac{1}{4}h + \tfrac{1}{2}g_1a_5) + \alpha_2(\tfrac{1}{4}h + \tfrac{1}{2}g_2b_5) + \alpha_3(-\tfrac{1}{2}g_3c_5 + \tfrac{1}{4}h + \tfrac{1}{2}k_1a_5 + \tfrac{1}{2}k_2b_5)$$
$$\geqslant (a_5 + b_5 + \varepsilon)(1-\theta), \quad (5.10)$$

$$\alpha_1(\tfrac{1}{4}h + \tfrac{1}{2}g_1a_6) + \alpha_2(\tfrac{1}{4}h + \tfrac{1}{2}g_2b_6) + \alpha_3(\tfrac{1}{2}g_3c_6 + \tfrac{1}{4}h + \tfrac{1}{2}k_1a_6 + \tfrac{1}{2}k_2b_6)$$
$$\geqslant (u + \varepsilon)(1-\theta) \quad (5.11)$$

and

$$\alpha_3\left(\tfrac{1}{2}g_3\left(u - \frac{h}{2g_1} - \frac{h}{2g_2}\right) + \tfrac{1}{4}hv\right) \geqslant \left(u - \frac{h}{2g_1} - \frac{h}{2g_2} + \varepsilon\right)(1-\theta). \quad (5.12)$$

Proof. Since $\alpha_j \geqslant g_j^{-1}(1-\theta)$ for $j>3$, we have $(I)_j$ for $j>3$. There are thus eight cases to consider.

Case $(I)_1$, $(I)_2$, $(I)_3$. Define λ by

$$\frac{h}{2g_1}+\frac{h}{2g_2}+\lambda\left(\frac{v}{2g_3}+\sum_{j>3}\frac{1}{2g_j}\right)=1$$

so that $0 \leqslant \lambda < h$ from (5.5). Then

$$S \ll \prod_{j=1}^{2}(L_j^{2g_j-2g_j\sigma_j}L_3^{2k_j-2k_j\sigma_3})^{h/2g_j}(L_3^{2g_3-2g_3\sigma_3})^{\lambda v/2g_3}$$

$$\times\prod_{j>3}(L_j^{2g_j-2g_j\sigma_j})^{\lambda/2g_j}\prod_j L_j^{h(\sigma_j-1)}.$$

Every $L_j^{\sigma_j-1}$ in the above product has non-negative exponent: the exponent is 0 for $j \leqslant 2$;

$$h\left(-\frac{k_1}{g_1}-\frac{k_2}{g_2}+1\right)-\lambda v=(h-\lambda)v$$

for $j=3$; and $h-\lambda$ for $j>3$. Since $(h-\lambda)v>0$, (5.4) now follows from (5.3).

Case $(I)_1$, $(II)_2$, $(I)_3$. Since $b_1+c_1=u-h/2g_1$, we have

$$S \ll (L_1^{2g_1(1-\sigma_1)}L_3^{2k_1(1-\sigma_3)})^{h/2g_1}(TL_2^{g_2f(\sigma_2)}L_3^{k_2f(\sigma_3)})^{b_1}$$

$$\times(L_3^{2g_3-2g_3\sigma_3})^{c_1}\prod_{j>3}(L_j^{2g_j-2g_j\sigma_j})^{h/2g_j}\prod_j L_j^{h(\sigma_j-1)}.$$

The monomials in $L_j^{\sigma_j-1}$ $(j\neq 2,\ j\neq 3)$ have product 1 and can be omitted; the corresponding step will be implicit in subsequent cases. Thus $S \ll S_1(\sigma_2,\sigma_3)$, where

$$S_1(\sigma_2,\sigma_3)=T^{b_1}L_2^{h(\sigma_2-1)+g_2b_1f(\sigma_2)}L_3^{(-2g_3c_1+h-hk_1/g_1)(\sigma_3-1)+k_2b_1f(\sigma_3)}$$

$$\leqslant S_1(\tfrac{3}{4},\tfrac{3}{4}).$$

For the last inequality, we appeal to (3.13), using

$$\tfrac{1}{6}h \leqslant g_2b_1 \leqslant \tfrac{1}{2}h \tag{5.13}$$

and

$$\tfrac{1}{6}\left(-2g_3c_1+h-\frac{hk_1}{g_1}\right)\leqslant k_2b_1 \leqslant \tfrac{1}{2}\left(-2g_3c_1+h-\frac{hk_1}{g_1}\right). \tag{5.14}$$

In subsequent cases, we leave the appeal to (3.13) implicit but mention the inequalities corresponding to (5.13) and (5.14).

Finally, (5.6) yields

$$S_1(\tfrac{3}{4},\tfrac{3}{4})=T^{b_1}L_2^{-h/4-g_2b_1/2}L_3^{g_3c_1/2-h/4+hk_1/4g_1-k_2b_1/2} \ll \mathscr{L}^{-A}.$$

Case $(II)_1$, $(I)_2$, $(I)_3$. This is similar to the previous case, with L_1 and L_2 interchanged.

Case $(I)_1$, $(II)_2$, $(II)_3$. Since $b_3 + c_3 = u - h/2g_1$, we have

$$S \ll (L_1^{2g_1(1-\sigma_1)} L_3^{2k_1(1-\sigma_3)})^{h/2g_1} (TL_2^{g_2 f(\sigma_2)} L_3^{k_2 f(\sigma_3)})^{b_3} (TL_3^{g_3 f(\sigma_3)})^{c_3}$$
$$\times \prod_{j>3} (L_j^{2g_j - 2g_j\sigma_j})^{h/2g_j} \prod_j L_j^{h(\sigma_j - 1)}.$$

Thus $S \ll S_2(\sigma_2, \sigma_3)$ where

$$S_2(\sigma_2, \sigma_3) = T^{u-h/2g_1} L_2^{h(\sigma_2-1)+g_2 b_3 f(\sigma_2)} L_3^{(h-hk_1/g_1)(\sigma_3-1)+(k_2 b_3 + g_3 c_3) f(\sigma_3)}$$
$$\leqslant S_2(\tfrac{3}{4}, \tfrac{3}{4}),$$

since

$$\tfrac{1}{6} h \leqslant g_2 b_3 \leqslant \tfrac{1}{2} h,$$
$$\tfrac{1}{6}\left(h - \frac{hk_1}{g_1}\right) \leqslant k_2 b_3 + g_3 c_3 \leqslant \tfrac{1}{2}\left(h - \frac{hk_1}{g_1}\right).$$

Finally, (5.8) yields

$$S_2(\tfrac{3}{4}, \tfrac{3}{4}) = T^{u-h/2g_1} L_2^{-h/4 - g_2 b_3/2} L_3^{-h/4 + hk_1/4g_1 - k_2 b_3/2 - g_3 c_3/2} \ll \mathscr{L}^{-A}.$$

Case $(II)_1$, $(I)_2$, $(II)_3$. This is similar to the previous case, with L_1 and L_2 interchanged.

Case $(II)_1$, $(II)_2$, $(I)_3$. Since $a_5 + b_5 + c_5 = u$, we have

$$S \ll (TL_1^{g_1 f(\sigma_1)} L_3^{k_1 f(\sigma_3)})^{a_5} (TL_2^{g_2 f(\sigma_2)} L_3^{k_2 f(\sigma_3)})^{b_5}$$
$$\times (L_3^{2g_3 - 2g_3\sigma_3})^{c_5} \prod_{j>3} (L_j^{2g_j - 2g_j\sigma_j})^{h/2g_j} \prod_j L_j^{h(\sigma_j-1)}.$$

Thus $S \ll S_3(\sigma_1, \sigma_2, \sigma_3)$, where

$$S_3(\sigma_1, \sigma_2, \sigma_3) = T^{a_5+b_5} L_1^{h(\sigma_1-1)+g_1 a_5 f(\sigma_1)} L_2^{h(\sigma_2-1)+g_2 b_5 f(\sigma_2)}$$
$$\times L_3^{(h-2g_3 c_5)(\sigma_3-1)+(k_1 a_5 + k_2 b_5) f(\sigma_3)}$$
$$\leqslant S_3(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}).$$

Here we use

$$\tfrac{1}{6} h \leqslant g_1 a_5 \leqslant \tfrac{1}{2} h, \quad \tfrac{1}{6} h \leqslant g_2 b_5 \leqslant \tfrac{1}{2} h,$$
$$\tfrac{1}{6}(h - 2g_3 c_5) \leqslant k_1 a_5 + k_2 b_5 \leqslant \tfrac{1}{2}(h - 2g_3 c_5).$$

Finally, (5.10) yields

$$S_3(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}) = T^{a_5+b_5} L_1^{-h/4 - g_1 a_5/2} L_2^{-h/4 - g_2 b_5/2} L_3^{-h/4 + g_3 c_5/2 - k_1 a_5/2 - k_2 b_5/2}$$
$$\ll \mathscr{L}^{-A}.$$

Case $(II)_1$, $(II)_2$, $(II)_3$. Since $a_6 + b_6 + c_6 = u$, we have

$$S \ll (TL_1^{g_1 f(\sigma_1)} L_3^{k_1 f(\sigma_3)})^{a_6} (TL_2^{g_2 f(\sigma_2)} L_3^{k_2 f(\sigma_3)})^{b_6} (TL_3^{g_3 f(\sigma_3)})^{c_6}$$
$$\times \prod_{j>3} (L_j^{2g_j - 2g_j\sigma_j})^{h/2g_j} \prod_j L_j^{h(\sigma_j-1)}.$$

Thus $S \ll S_4(\sigma_1, \sigma_2, \sigma_3)$, where

$$S_4(\sigma_1, \sigma_2, \sigma_3) = T^u L_1^{h(\sigma_1-1)+g_1 a_6 f(\sigma_1)} L_2^{h(\sigma_2-1)+g_2 b_6 f(\sigma_2)} \times L_3^{h(\sigma_3-1)+(k_1 a_6 + k_2 b_6 + g_3 c_6) f(\sigma_3)} \leqslant S_4(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}).$$

Here we use

$$\tfrac{1}{6}h \leqslant g_1 a_6 \leqslant \tfrac{1}{2}h, \quad \tfrac{1}{6}h \leqslant g_2 b_6 \leqslant \tfrac{1}{2}h,$$

$$\tfrac{1}{6}h \leqslant k_1 a_6 + k_2 b_6 + g_3 c_6 \leqslant \tfrac{1}{2}h.$$

Finally, (5.11) yields

$$S_4(\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{3}{4}) = T^u L_1^{-h/4 - g_1 a_6/2} L_2^{-h/4 - g_2 b_6/2} L_3^{-h/4 - (k_1 a_6 + k_2 b_6 + g_3 c_6)/2} \ll \mathscr{L}^{-A}.$$

Case $(I)_1$, $(I)_2$, $(II)_3$. Let $c = u - h/2g_1 - h/2g_2$. Then

$$S \ll (L_1^{2g_1 - 2g_1\sigma_1} L_3^{2k_1 - 2k_1\sigma_3})^{h/2g_1} (L_2^{2g_2 - 2g_2\sigma_2} L_3^{2k_2 - 2k_2\sigma_3})^{h/2g_2} \times (T L_3^{g_3 f(\sigma_3)})^c \prod_j L_j^{h(\sigma_j - 1)}.$$

Thus $S \ll S_5(\sigma_3)$, where

$$S_5(\sigma_3) = T^c L_3^{hv(\sigma_3-1)+cg_3 f(\sigma_3)} \leqslant S_5(\tfrac{3}{4}).$$

Here we use

$$\tfrac{1}{6}hv \leqslant g_3 c \leqslant \tfrac{1}{2}hv.$$

Finally, (5.12) yields

$$S_5(\tfrac{3}{4}) = T^c L^{-hv/4 - cg_3/2} \ll \mathscr{L}^{-A}.$$

This completes the proof of Lemma 18.

The cases that are helpful in the present paper, where $h = 1$, are $(g_1, g_2, g_3, g_4) = (1, 2, 3, d)$ where $d = 4$ or 5. We take $k_1 = k_2 = 0$. Thus

$$u = 1 - \tfrac{1}{2d}, \quad b_1 + c_1 = \tfrac{1}{2} - \tfrac{1}{2d}, \quad c_1 = \tfrac{1}{6}, \quad \tfrac{1}{12} \leqslant b_1 \leqslant \tfrac{1}{4}.$$

This is satisfied for $(b_1, c_1) = (\tfrac{1}{3} - \tfrac{1}{2d}, \tfrac{1}{6})$. Similarly $(a_2, c_2) = (\tfrac{7}{12} - \tfrac{1}{2d}, \tfrac{1}{6})$. Next,

$$b_3 + c_3 = \tfrac{1}{2} - \tfrac{1}{2d}, \quad \tfrac{1}{12} \leqslant b_3 \leqslant \tfrac{1}{4}, \quad \tfrac{1}{18} \leqslant c_3 \leqslant \tfrac{1}{6}.$$

This is satisfied for (b_3, c_3) equal to either $(\tfrac{1}{3} - \tfrac{1}{2d}, \tfrac{1}{6})$ or $(\tfrac{1}{4}, \tfrac{1}{4} - \tfrac{1}{2d})$ (it is obvious that b_3 or c_3 should be chosen as an endpoint of its permitted interval). Next,

$$a_4 + c_4 = \tfrac{3}{4} - \tfrac{1}{2d}, \quad \tfrac{1}{6} \leqslant a_4 \leqslant \tfrac{1}{2}, \quad \tfrac{1}{18} \leqslant c_4 \leqslant \tfrac{1}{6}$$

is satisfied for (a_4, c_4) equal to either $(\tfrac{1}{2}, \tfrac{1}{4} - \tfrac{1}{2d})$ or $(\tfrac{7}{12} - \tfrac{1}{2d}, \tfrac{1}{6})$. Next,

$$c_5 = \tfrac{1}{6}, \quad a_5 + b_5 = \tfrac{5}{6} - \tfrac{1}{2d}, \quad \tfrac{1}{6} \leqslant a_5 \leqslant \tfrac{1}{2}, \quad \tfrac{1}{12} \leqslant b_5 \leqslant \tfrac{1}{4}$$

is satisfied for (a_5, b_5) equal to either $(\frac{1}{2}, \frac{1}{3}-\frac{1}{2d})$ or $(\frac{7}{12}-\frac{1}{2d}, \frac{1}{4})$. Finally,

$$a_6+b_6+c_6=1-\tfrac{1}{2d}, \quad \tfrac{1}{6}\leqslant a_6\leqslant\tfrac{1}{2}, \quad \tfrac{1}{12}\leqslant b_6\leqslant\tfrac{1}{4}, \quad \tfrac{1}{18}\leqslant c_6\leqslant\tfrac{1}{6}.$$

This is satisfied for (a_6, b_6, c_6) equal to $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}-\frac{1}{2d})$ or $(\frac{7}{12}-\frac{1}{2d}, \frac{1}{4}, \frac{1}{6})$ or $(\frac{1}{2}, \frac{1}{3}-\frac{1}{2d}, \frac{1}{6})$.

The region of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for which (5.1) holds via Lemma 18 is thus a union of polytopes obtained from various choices of (b_3, c_3), (a_4, c_4), (a_5, b_5) and (a_6, b_6, c_6).

6. *The final decomposition*

In what follows, we ignore the presence of ε for brevity. Let $\theta=0.525$. We begin with some further notation needed to describe the further decomposition of Σ_3 in (1.2). Write

$$U_n=\{(\alpha_1,\ldots,\alpha_n)\colon 0<\alpha_n<\alpha_{n-1}<\ldots<\alpha_1, 2\alpha_n<1-\alpha_1-\ldots-\alpha_{n-1}\}.$$

Let

$$G=\bigcup_{n=2}^{\infty} G_n$$

where

$$G_n=\{(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n\colon \text{an asymptotic formula can be obtained for } p_1\ldots p_n r\in\mathscr{A}, p_j\sim x^{\alpha_j}, (r, P(p_n))=1\}.$$

(The means of obtaining the asymptotic formula is, of course, Lemma 9 or Lemma 18.) Put

$$D_0=\{(\alpha,\beta)\colon 0\leqslant\alpha\leqslant\tfrac{1}{2}, 0\leqslant\beta\leqslant\min(\tfrac{1}{2}(3\theta+1-4\alpha^*), \tfrac{1}{2}(3+\theta-4\alpha^*))\}$$

with α^* as in § 3,

$$D_1=\{(\alpha,\beta,\gamma)\colon 0\leqslant\alpha\leqslant\tfrac{1}{2}, \gamma\leqslant\tfrac{1}{4}(1+3\theta)-\alpha^*, \beta+\tfrac{1}{2}\gamma\leqslant\tfrac{1}{2}(1+\theta)-\alpha^*, \beta+\tfrac{3}{2}\gamma\leqslant\tfrac{1}{4}(3+\theta)-\alpha^*\},$$

$$D_2=\{(\alpha,\beta,\gamma)\colon 0\leqslant\alpha\leqslant\tfrac{1}{2}, \beta\leqslant\tfrac{1}{2}(1-\theta), \gamma\leqslant\tfrac{1}{8}(1+3\theta)-\tfrac{1}{2}\alpha^*\},$$

$$D^*=\{(\alpha,\beta,\gamma,\delta)\colon (\alpha,\beta,\gamma,\delta,\delta) \text{ can be partitioned into } (\eta,\zeta)\in D_0 \text{ or } (\eta,\zeta,\lambda)\in D_1\cup D_2\},$$

$$R=\{(\alpha,\beta,\gamma)\colon (\alpha,\beta,\gamma)\notin D_1\cup D_2, (\alpha,\beta,\gamma) \text{ cannot be partitioned into } (\eta,\zeta)\in D_0\}.$$

In case the language is unclear, $(\alpha,\beta,\gamma,\delta,\delta)$ can be partitioned into $(\eta,\zeta)\in D_0$ if, for example, $(\alpha+2\delta, \beta+\gamma)$ or $(\alpha+\delta+\gamma, \beta+\delta)$ is in D_0.

Note that D_0, D_1 and D_2 correspond to conditions on variables which allow a further decomposition via Lemma 12 or 13; while D^* allows two further decompositions. In regions corresponding to R, rôle-reversals will be needed to perform further decompositions.

Presented with a sum such as

$$\sum_{p,q} S(\mathscr{A}_{pq}, q),$$

we may be able to give an asymptotic formula for some of the almost-primes counted. We can make these visible by writing, for example,

$$\sum_{p,q} S(\mathscr{A}_{pq}, q) = \sum_{p,q} S\left(\mathscr{A}_{pq}, \left(\frac{x}{pq}\right)^{1/2}\right) + \sum_{\substack{p,q \\ q<r<(x/pq)^{1/2}}} S(\mathscr{A}_{pqr}, r)$$

(the Buchstab identity in reverse). We define a new function to take into account the possible savings introduced by this technique.

Given $\boldsymbol{\alpha} \in U_n$, write

$$u = \left[\frac{1 - \alpha_1 - \ldots - \alpha_n}{\alpha_n}\right].$$

Then $u \geqslant 1$ by definition of U_n, and $S(\mathscr{A}_{p_1 \ldots p_n}, p_n)$ counts numbers with up to u prime factors. Now write

$$w(\boldsymbol{\alpha}, 1) = \frac{1}{\alpha_{n+1}} \quad \text{where } \alpha_{n+1} = 1 - \alpha_1 - \ldots - \alpha_n.$$

Define $w(\boldsymbol{\alpha}, k)$ inductively by

$$w(\boldsymbol{\alpha}, k+1) = w(\boldsymbol{\alpha}, k) + \int^{*} \frac{d\beta_1 \ldots d\beta_k}{\beta_1 \beta_2 \ldots \beta_k (\alpha_{k+1} - \beta_1 - \ldots - \beta_k)}$$

where $*$ denotes the region

$$\alpha_n \leqslant \beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_k \leqslant \tfrac{1}{2}(\alpha_{n+1} - \beta_1 - \ldots - \beta_{k-1}),$$

$$(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_k) \notin G.$$

Finally we write

$$w(\boldsymbol{\alpha}) = w(\boldsymbol{\alpha}, u).$$

We then have

$$w(\boldsymbol{\alpha}) \leqslant \frac{\omega(\alpha_{n+1}/\alpha_n)}{\alpha_n}$$

and, for $1 \leqslant k \leqslant u$,

$$w(\boldsymbol{\alpha}) \leqslant w(\boldsymbol{\alpha}, k) + \int^{*} \omega\left(\frac{\alpha_{n+1} - \beta_1 - \ldots - \beta_k}{\beta_k}\right) \frac{d\beta_1}{\beta_1} \ldots \frac{d\beta_{k-1}}{\beta_{k-1}} \frac{d\beta_k}{\beta_k^2}. \tag{6.1}$$

This is a translation into integrals of the following fact. If we apply the Buchstab identity in reverse u times to

$$\sum_{p_1, \ldots, p_n} S(\mathscr{A}_{p_1, \ldots, p_n}, p_n),$$

the loss from regions for which we cannot give an asymptotic formula is less than the corresponding loss, if we only apply the identity k times and discard all $p_1 \ldots p_n q_1 \ldots q_k h_{k+1}$ with $(h_{k+1}, P(q_k)) = 1$ for which an asymptotic formula cannot be given. We use (6.1) in some numerical calculations with $k = 2$ or 3. We shall also use

$$\omega(u) = \begin{cases} 1/u & \text{for } 1 \leqslant u \leqslant 2, \\ (1+\log(u-1))/u & \text{for } 2 \leqslant u \leqslant 3, \end{cases}$$

$$\omega(u) \leqslant \tfrac{1}{3}(1+\log 2) \quad \text{if } u > 3.$$

We require a development of the above notation to take into account rôle-reversals. Let $\boldsymbol{\alpha}_3 \in U_3$. Put $\boldsymbol{\alpha}_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $\nu \leqslant \alpha_4 \leqslant \frac{1}{2}\alpha_1$. We write

$$u' = \left[\frac{\alpha_1 - \alpha_4}{\alpha_4}\right].$$

Define $w^*(\alpha_4, j)$ and $Y(\alpha_4, j)$ by

$$w^*(\boldsymbol{\alpha}_4, 1) = w(\boldsymbol{\alpha}_3)\,\frac{1}{\alpha_1 - \alpha_4}, \quad w^*(\alpha_4, j) = w(\boldsymbol{\alpha}_3) Y(\boldsymbol{\alpha}_4, j),$$

$$w^*(\boldsymbol{\alpha}_4, j+1) = w(\boldsymbol{\alpha}_3)\left(Y(\boldsymbol{\alpha}_4, j) + \int^{\dagger} \frac{d\gamma_1 \ldots d\gamma_j}{\gamma_1 \ldots \gamma_j(\alpha_1 - \alpha_4 - \gamma_1 - \ldots - \gamma_j)}\right).$$

Here the last expression is to be interpreted as a sum of multiple integrals (including those counted by $w(\boldsymbol{\alpha}_3)$) with the integration condition † dependent on which multiple integral from $w(\boldsymbol{\alpha}_3)$ is multiplying it. If, for example, one takes the term

$$\int \frac{d\beta_1 \ldots d\beta_k}{\beta_1 \ldots \beta_k(1 - \alpha_1 - \alpha_2 - \alpha_3 - \beta_1 - \ldots - \beta_k)}$$

from $w(\boldsymbol{\alpha}_3)$, then the conditions on $\gamma_1, \ldots, \gamma_j$ are

$$\alpha_4 \leqslant \gamma_1 \leqslant \ldots \leqslant \gamma_j \leqslant \tfrac{1}{2}(\alpha_1 - \alpha_4 - \gamma_1 - \ldots - \gamma_{j-1}),$$

$$(\alpha_2, \alpha_3, \alpha_4, \gamma_1, \ldots, \gamma_j, \beta_1, \ldots, \beta_k, 1 - \alpha_1 - \alpha_2 - \alpha_3 - \beta_1 - \ldots - \beta_k) \notin G.$$

Let

$$w^*(\alpha_4) = w^*(\boldsymbol{\alpha}_4, u').$$

Of course, we have

$$w^*(\boldsymbol{\alpha}_4) \leqslant \frac{w(\boldsymbol{\alpha}_3)\omega((\alpha_1 - \alpha_4)/\alpha_4)}{\alpha_4},$$

and various other upper bounds could be derived using small numbers of integration variables.

Now define non-overlapping polygons A, B, C, D, E, F, whose union is $\{(\alpha_1, \alpha_2) \in U_2 \colon \nu(0) \leqslant \alpha_1 \leqslant \frac{1}{2}, \nu(\alpha_1) \leqslant \alpha_2\}$, by the following sets of inequalities:

A: $\frac{1}{4} \leqslant \alpha_1 \leqslant \frac{2}{5}$, $\quad \frac{1}{3}(1 - \alpha_1) \leqslant \alpha_2 \leqslant \min(\alpha_1, \frac{1}{2}(3\theta - 1), 1 - 2\alpha_1)$;

B: $\frac{1}{4}(3 - 3\theta) \leqslant \alpha_1 \leqslant \frac{1}{2}$,

$\max(\frac{1}{2}\alpha_1, 1 - 2\alpha_1) \leqslant \alpha_2 \leqslant \min(\frac{1}{2}(3\theta - 1), \frac{1}{2}(1 - \alpha_1))$;

C: $\nu(0) \leqslant \alpha_1 \leqslant \frac{1}{3}$, $\quad \nu(\alpha_1) \leqslant \alpha_2 \leqslant \min(\alpha_1, \frac{1}{3}(1 - \alpha_1))$;

D: $\frac{1}{3} \leqslant \alpha_1 \leqslant \frac{1}{2}$, $\quad \nu(\alpha_1) \leqslant \alpha_2 \leqslant \max(\frac{1}{3}(1 - \alpha_1), \frac{1}{2}\alpha_1)$;

E: $\frac{1}{2}(3\theta-1)\leqslant\alpha_1\leqslant\frac{1}{4}(3-3\theta)$, $\frac{1}{2}(3\theta-1)\leqslant\alpha_2\leqslant\min(\alpha_1,1-2\alpha_1)$;

F: $\frac{1}{3}\leqslant\alpha_1\leqslant 2-3\theta$, $\max(1-2\alpha_1,\frac{1}{2}(3\theta-1))\leqslant\alpha_2\leqslant\frac{1}{2}(1-\alpha_1)$.

Note that

$$(\alpha_1,\alpha_2)\in A \iff (1-\alpha_1-\alpha_2,\alpha_2)\in B$$

and a similar relation holds between E and F. Moreover, in $A\cup B\cup E\cup F$ only products of three primes are counted. So

$$\sum_{(\alpha_1,\alpha_2)\in B} S(\mathscr{A}_{p_1p_2},p_2)=\sum_{(\alpha_1,\alpha_2)\in A} S(\mathscr{A}_{p_1p_2},p_2),$$

$$\sum_{(\alpha_1,\alpha_2)\in F} S(\mathscr{A}_{p_1p_2},p_2)=\sum_{(\alpha_1,\alpha_2)\in E} S(\mathscr{A}_{p_1p_2},p_2),$$

and

$$\Sigma_3=2\sum_{(\alpha_1,\alpha_2)\in A} S(\mathscr{A}_{p_1p_2},p_2)+2\sum_{(\alpha_1,\alpha_2)\in E} S(\mathscr{A}_{p_1p_2},p_2)$$
$$+\sum_{(\alpha_1,\alpha_2)\in C} S(\mathscr{A}_{p_1p_2},p_2)+\sum_{(\alpha_1,\alpha_2)\in D} S(\mathscr{A}_{p_1p_2},p_2).$$

Now we consider A in more detail. If we discarded the sum over A, our 'loss' would be ≈ 0.1971. In fact we shall make only a small saving when the exponent is 0.525; we would have much greater success when $\theta=0.53$. We apply Buchstab's identity to get

$$\sum_{(\alpha_1,\alpha_2)\in A} S(\mathscr{A}_{p_1p_2},p_2)=\sum_{(\alpha_1,\alpha_2)\in A} S(\mathscr{A}_{p_1p_2},x^{\nu(\alpha_1)})-\sum_{(\alpha_1,\alpha_2,\alpha_3)\in A'} S(\mathscr{A}_{p_1p_2p_3},p_3).$$

We can give an asymptotic formula for the first sum on the right-hand side. If $(\alpha_1,\alpha_2,\alpha_3)\in D_1\cup D_2$ or $(\alpha_1+\alpha_3,\alpha_2)\in D_0$, or $(\alpha_2+\alpha_3,\alpha_1)\in D_0$, then a further straightforward decomposition of the final sum is possible. In the remaining part of A' we note that $\alpha_1+\alpha_2\geqslant\frac{1}{2}$. Writing h for a number counted by $S(\mathscr{A}_{p_1p_2p_3},p_3)$, we have $h\sim x^{\alpha_4}$ with

$$\alpha_4+\alpha_3\leqslant\tfrac{1}{2},\quad \alpha_2\leqslant\tfrac{1}{2}(3\theta-1).$$

A rôle-reversal yields

$$\sum_{p_1,p_2,p_3} S(\mathscr{A}_{p_1p_2p_3},p_3)=\sum_{h,p_2,p_3} S\left(\mathscr{A}_{hp_2p_3},\left(\frac{x}{hp_2p_3}\right)^{1/2}\right)$$
$$=\sum_{h,p_2,p_3} S(\mathscr{A}_{hp_2p_3},x^{\nu(\alpha_4+\alpha_3)})-\sum_{h,p_2,p_3,q} S(\mathscr{A}_{hp_2p_3q},q).$$

We omit the conditions of summation for brevity. On the left-hand side we had to count numbers $hp_2p_3p_1$, and in the last sum on the right we count numbers hp_2p_3qr, so we speak of this step as 'decomposition of p_1'. Further decompositions may be possible in either the straightforward decomposition or the decomposition of p_1.

Altogether we get a 'loss' from region A of

$$\int_{(\alpha_1,\alpha_2)\in A}\min\left(\frac{1}{\alpha_1\alpha_2(1-\alpha_1-\alpha_2)},\frac{1}{\alpha_1\alpha_2}(I_1+I_2)+\frac{1}{\alpha_2}I_3\right)d\alpha_1\,d\alpha_2$$

with

$$I_1 = \int_{\substack{\boldsymbol{\alpha}_3 \in U_3 \setminus R \\ \boldsymbol{\alpha}_4 \notin D^* \cup G}} \frac{w(\boldsymbol{\alpha}_4)}{\alpha_3 \alpha_4} \, d\alpha_3 \, d\alpha_4,$$

$$I_2 = \int_{\substack{\boldsymbol{\alpha}_3 \in U_3 \setminus R \\ \boldsymbol{\alpha}_4 \in D^* \setminus G}} \frac{1}{\alpha_3 \alpha_4} \min\left(w(\boldsymbol{\alpha}_4), \int_{\substack{\boldsymbol{\alpha}_6 \in U_6 \\ \boldsymbol{\alpha}_6 \notin G}} \frac{w(\boldsymbol{\alpha}_6)}{\alpha_5 \alpha_6} \, d\alpha_5 \, d\alpha_6 \right) d\alpha_3 \, d\alpha_4,$$

$$I_3 = \int_{\boldsymbol{\alpha}_3 \in R} \frac{1}{\alpha_3} \omega\left(\frac{1 - \alpha_1 - \alpha_2 - \alpha_3}{\alpha_3} \right) \int_{\substack{\nu \\ \boldsymbol{\alpha}_4 \notin G}}^{\alpha_1/2} \frac{w^*(\alpha_4)}{\alpha_4} \, d\alpha_4 \, d\alpha_3.$$

Here, for the sake of clarity, we have omitted further decompositions after a rôle-reversal, and not considered the six-dimensional region where two further decompositions are possible. In this way we obtain a loss less than 0.15, and so a loss from regions A and B less than 0.3.

For region E we perform two further decompositions; Lemma 16 covers

$$\sum_{p_1, p_2} S(\mathscr{A}_{p_1 p_2}, x^\nu).$$

There may now be a rôle-reversal preceding the next decomposition; whether or not this is the case, it is easy to see that Lemma 17 covers

$$\sum_{h, p_2, p_3} S(\mathscr{A}_{h p_2 p_3}, x^\nu)$$

where h runs either over primes or over integers coprime to $P(p_3)$. The loss from this region is less than 0.03 (and so less than 0.06 from $E \cup F$). Without using the two-dimensional sieve we would have had to discard all of this region with a loss from $E \cup F$ of ≈ 0.0864, so the saving with $\theta = 0.525$ is quite small.

For region C it is only necessary to reverse the rôles of variables for a small part of the sum

$$\sum_{\substack{(\alpha_1, \alpha_2) \in C \\ \boldsymbol{\alpha}_3 \in U_3}} S(\mathscr{A}_{p_1 p_2 p_3}, p_3).$$

For example, we can perform a further decomposition in a straightforward manner whenever $\alpha_1 \leqslant 0.2875$ since

$$\alpha_2 + \alpha_3 \leqslant \tfrac{1}{2}(\alpha_1 + \alpha_2 + 2\alpha_3) \leqslant \tfrac{1}{2}.$$

Again, if $\alpha_1 + \alpha_3 \leqslant \frac{1}{2}$, we have $(\alpha_1 + \alpha_3, \alpha_2) \in D_0$. Otherwise,

$$(1 - \alpha_1 - \alpha_2 - \alpha_3) + \alpha_2 < \tfrac{1}{2}$$

and we can reverse rôles to decompose α_1. Altogether, the loss from region C is less than 0.21, while if C were discarded, we would lose

$$\int_C \omega\left(\frac{1 - \alpha_1 - \alpha_2}{\alpha_2} \right) \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2^2} > 1.$$

Finally, region D can be tackled by analysing when straightforward decompositions are possible and when a rôle-reversal is essential. In this region p_1 is often the largest variable. The loss from region D is less than 0.34, again a great saving on the trivial estimate.

Combining all our estimates we conclude that, for all large x,

$$\pi(x + x^{0.525}) - \pi(x) \geqslant \frac{x^{0.525}}{\log x}(1 - 0.3 - 0.06 - 0.21 - 0.34)$$
$$= \frac{9}{100}\frac{x^{0.525}}{\log x}.$$

As the exponent decreases further, the savings over the trivial bounds from regions A, B, E and F become negligible and the contributions from regions C and D rise fairly rapidly, leading to a trivial lower bound.

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