THE FORMULAE-AS-TYPES NOTION OF CONSTRUCTION

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Dedicated to H. B. Curry on the occasion of his 80th birthday.

The following consists of notes which were privately circu-
lated in 1969. Since they have been referred to a few times in
the literature, it seems worth while to publish them. They have
been rearranged for easier reading, and some inessential correc-
tions have been made.

The ultimate goal was to develop a notion of construction
suitable for the interpretation of intuitionistic mathematics.
The notion of construction developed in the notes is certainly
too crude for that, so the use of the word *construction* is not
very appropriate. However, the terminology has been kept in
order to preserve the original title and also to preserve the
character of the notes. The title has a second defect; namely,
a *type* should be regarded as a abstract object whereas a *for-
mula* is the name of a type.

In Part I the ideas are illustrated for the intuitionistic
propositional calculus and in Part II they are applied to Heyt-
ing arithmetic.

I. INTUITIONISTIC PROPOSITIONAL CALCULUS

H. Curry (1958) has observed that there is a close correspon-
dence between *axioms* of positive implicational propositional
logic, on the one hand, and *basic combinators* on the other hand.
For example, the combinator $K = \lambda X.\lambda Y.X$ corresponds to the

axiom $\alpha \supset (\beta \supset \alpha)$.

The following notion of construction, for positive implica-
tional propositional logic, was motivated by Curry's observation.
More precisely, Curry's observation provided *half* the motivation.
The other half was provided by W. Tait's discovery of the close
correspondence between cut elimination and reduction of λ-terms
(W. W. Tait, 1965). It is convenient to use λ-terms rather than
combinators. This corresponds to the sequent formulation of pro-
positional logic.

1. *Formulation of the sequent calculus*

Let $P(\supset)$ denote positive implicational propositional logic.
The prime formulae of $P(\supset)$ are propositional variables. If
α and β are formulae, so is $\alpha \supset \beta$. A *sequent* has the form
$\Gamma \to \beta$, where Γ is a (possibly empty) finite sequence of formu-
lae and β is a formula. The axioms and rules of inference of
$P(\supset)$ are as follows.

(1.1)   Axioms: all sequents of the form
$$\alpha \to \alpha$$

(1.2) $$\frac{\Gamma, \alpha \to \beta}{\Gamma \to \alpha \supset \beta}$$

(1.3) $$\frac{\Gamma \to \alpha \qquad \Delta \to \alpha \supset \beta}{\Gamma, \Delta \to \beta}$$

(1.4)   Thinning, permutation and contraction
rules

2. *Type symbols, terms and constructions*

By a type symbol is meant a formula of $P(\supset)$. We will con-
sider a λ-formalism in which each term has a type symbol α as
a superscript (which we may not always write); the term is said
to be of type α. The rules of term formation are as follows.

(2.1)   Variables X^α, Y^β,... are terms

(2.2) λ-abstraction: from F^β get
$$(\lambda X^\alpha . F^\beta)^{\alpha \supset \beta} .$$

(2.3)   Application: from $G^{\alpha \supset \beta}$ and H^{α}

get $(G^{\alpha \supset \beta} H^{\alpha})^{\beta}$.

By a *construction* of a sequent $\Gamma \to \beta$ is meant a term F^{β}
of type β such that for every free variable X^{α} occurring in
F^{β} there is a corresponding occurrence of α in Γ (it being
understood that the existence of k distinct free variables of
the same type α in F^{β} is reflected by at least k occur-
rences of α in Γ) .   Thus X^{α} is a construction of $\alpha \to \alpha$
(but X^{α} is also a construction of $\beta, \alpha \to \alpha$).   Another example:
$\lambda X^{\alpha}.\lambda Y^{\beta}.X^{\alpha}$ is a construction of $\to \alpha \supset (\beta \supset \alpha)$.

3.   *Correspondence between derivation and terms*

Clearly the axioms and rules of inference 1.1-1.3 of $P(\supset)$
correspond exactly to the rules 2.1-2.3 of term formation.   A
construction of $\Gamma \to \beta$ is clearly also a construction of Γ,
$\alpha \to \beta$ (thinning); similarly for a permutation of Γ ; and the

contraction $\dfrac{\Gamma, \alpha, \alpha \to \beta}{\Gamma, \alpha \to \beta}$ corresponds to replacing two distinct

variables of type α by one variable of type α in the corre-
sponding construction.
Hence:

THEOREM 1.   Given any derivation of $\Gamma \to \beta$ in $P(\supset)$ we can
find a construction of $\Gamma \to \beta$ and conversely.

4.   *Interpretation of terms*

For an interpretation in ordinary set theory let each proposi-
tional variable (i.e., prime type symbol) denote a specific set
of basic objects.   Then every type symbol can be taken to denote
a set of things according to the rule: $\alpha \supset \beta$ denotes the set of
all functions whose domain is a superset of α and whose range
is a subset of β .   (According as to whether the superset de-
pends on the function in question, or whether it just depends
on α,   we get somewhat differing interpretations).   The vari-
ables of type α are interpreted as ranging over the set α .

It is now clear, by induction on the rules 2.1-2.3 of term for-
mation, how each term is to be interpreted as a function of the
objects over which its free variables range. Thus the closed
terms can be interpreted as a perfectly concrete set of function-
als of finite type over the basic objects. This interpretation
is used by H. Laüchli in the paper he read at the Summer Con-
ference at Buffalo in 1968: see (Laüchli, 1970) pp. 227-229.

Of course a constructivist would be interested in other inter-
pretations; for example, interpretations related to the *calcula-
tion* of terms (i.e., reduction to irreducible form). It is easy
to prove that the terms given above can be reduced to normal
(i.e., irreducible) form by λ-contractions. The relation be-
tween this and *cut elimination* will now be discussed briefly.

5. *Normalization of terms and cut elimination*

Clearly the cut rule for $P(\supset)$ corresponds to the following
rule of term formation: from F^{α} and G^{β} get $[F^{\alpha}/X^{\alpha}]G^{\beta}$
(the result of substituting F^{α} for the free variable X^{α} in
G^{β}, where no free variable in F^{α} becomes bound in $[F^{\alpha}/X^{\alpha}]G^{\beta}$).
Though we did not include substitution in our rules of term for-
mation, the rule 2.3 (application) is just about as bad--from
the viewpoint of obtaining irreducible terms. Professor Curry
is fond of pointing out how to get irreducible terms: simply
replace the rule 2.3 by:

(5.1)   from a variable X of type
           $\alpha_1 \supset (\alpha_2 \supset (\ldots \supset (\alpha_n \supset \beta)\ldots))$
           and terms $F_1,\ldots, F_n$ of types
           $\alpha_1,\ldots,\alpha_n$, respectively, get the
           term $XF_1\ldots F_n$ of type β .

Correspondingly, replace the rule 1.3 of $P(\supset)$ by the n-premise
rule

$$(5.2) \qquad \frac{\Gamma_1 \to \alpha_1 \quad \Gamma_2 \to \alpha_2 \ \ldots \ \Gamma_n \to \alpha_n}{\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ \alpha_1 \supset (\alpha_2 \supset (\ldots \supset (\alpha_n \supset \beta) \ldots)) \to \beta} \ .$$

Of course 5.2 can be obtained by n applications of one of the Gentzen rules

$$(5.3) \qquad \frac{\Gamma \to \alpha \quad \beta, \Delta \to \gamma}{\alpha \supset \beta, \ \Gamma, \ \Delta \to \gamma} \ ,$$

with γ equal to β, and use of $\beta, \Delta \to \beta$. We could replace 5.1 by a rule of term formation corresponding to 5.3, but 5.1 seems more natural. As a modification of Theorem 1 we have:

THEOREM 2. Let $P^*(\supset)$ be $P(\supset)$ with the rule 1.3 replaced by 5.2. Then! given any derivation of $\Gamma \to \beta$ in $P^*(\supset)$ we can find an irreducible construction of $\Gamma \to \beta$ and conversely.

Cut *elimination* can be taken to mean: the transformation of a proof of $\Gamma \to \beta$ in $P(\supset)$ into a proof of $\Gamma \to \beta$ in $P^*(\supset)$. Thus cut elimination can be obtained as a consequence of the reduction of terms to normal form. As mentioned in §4, such reduction is easy to prove for the terms under discussion. Results following from cut elemination in $P(\supset)$ (e.g.) the nonderivability of Peirce's Law $(\alpha \supset \beta . \supset \alpha) \supset \alpha)$ seem to be obtainable at least as easily from the normalizability of constructions.

6. *Addition of* $\neg$, $\wedge$ *and* $\vee$ *to* $P(\supset)$

Corresponding to each of these connectives we add certain closed prime terms to our supply of terms.

(i) For $\neg$: add a new prime formula f to $P(\supset)$. Then, for each formula α, introduce a term $A^{f \supset \alpha}$. As an exercise, the reader may wish to prove--for the resulting system--that there are no closed terms of type f . (By normalizability it is sufficient to prove this for irreducible terms). There *are* open terms of type f; for example, the variable X^f--which is a construction of $f \to f$.

(ii) For $\wedge$: add terms $B_1^{\alpha \supset (\beta \supset \alpha \wedge \beta)}$, $B_2^{\alpha \wedge \beta \supset \alpha}$ and $B_3^{\alpha \wedge \beta \supset \beta}$. These are just pairing and projection functionals ($\alpha \wedge \beta$ is the type of a pair of terms of types α and β). We do not need to add a term of type $\beta \supset (\alpha \supset \alpha \wedge \beta)$ because such a term can be defined as $\lambda Y^\beta . \lambda X^\alpha . B_1 X^\alpha Y^\beta$.

In connection with the theory of reducibility of constructions it is useful to postulate the contraction $B_2(B_1 FG)$ contr F and $B_3(B_1 FG)$ contr G . Then we get the following theorem.

THEOREM 3. Every closed irreducible term of type $\alpha \wedge \beta$ has the form $B_1 F^\alpha G^\beta$, where B_1 is as above.

[*Note added 1979* This treatment of $\neg$ and $\wedge$ does not seem to be appropriate for cut elimination in Gentzen's sequent calculus. As P. Martin-Löf soon pointed out to me, it *is* appropriate for D. Prawitz's theory of Gentzen's system of natural deduction. See Prawitz (1965). The terms $A^{f \supset \alpha}$ and $B_1^{\alpha \supset (\beta \supset \alpha \wedge \beta)}$ correspond to the inference rules $\dfrac{f}{\alpha}$ and $\dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$, respectively, while $B_2^{\alpha \wedge \beta \supset \alpha}$ and $B_3^{\alpha \wedge \beta \supset \beta}$ correspond to $\dfrac{\alpha \wedge \beta}{\alpha}$ and $\dfrac{\alpha \wedge \beta}{\beta}$, respectively.]

(iii) For $\vee$: there are two possibilities, corresponding to the discussion of weak existence and strong existence in §12, below. Corresponding to the case of weak existence we add terms $C_1^{\alpha \supset \alpha \vee \beta}$, $C_2^{\beta \supset \alpha \vee \beta}$ and $C_3^{\alpha \vee \beta \supset (\alpha \supset \gamma . \supset (\beta \supset \gamma . \supset \gamma))}$

It is useful to postulate the contractions $C_3(C_1 M)FG$ contr FM and $C_3(C_2 N)FG$ contr GN for all terms M, N, F, G of types α, β, $\alpha \supset \gamma$, $\beta \supset \gamma$, respectively. Then we get the following theorem.

THEOREM 4. Every closed irreducible term of type $\alpha \vee \beta$ has the form $C_1 F^\alpha$ or $C_2 G^\beta$, where C_1 and C_2 are as above.

II. HEYTING ARITHMETIC

We will be concerned mainly with the subsystem $H(\supset, \wedge, \forall)$ of Heyting arithmetic obtained by omitting $\vee$ and $\exists$. As is well-known, $\neg\, \alpha$ can be defined as $\alpha \supset 0 = 1$. In §12 we will make some remarks about the question of including $\exists$. Of course $\vee$ can be defined by use of $\exists$. The variables belonging to $H(\supset, \wedge, \forall)$ will be called *number variables*.

7. *Constructions*

Our constructions will be terms built up from prime terms by means of rules of term formation as indicated in (ii)-(iv), below. Every term is supplied with a unique type symbol. The numerical terms—namely, the terms belonging to $H(\supset, \wedge, \forall)$— have type 0 .

(i) *Type symbols*   The prime type symbols are: 0 and every equation of $H(\supset, \wedge, \forall)$. From these we generate all type symbols by the following two rules.

(a) From α and β get $\alpha \supset \beta$ and $\alpha \wedge \beta$.

(b) From α and a number variable x get $\forall\, x\alpha$.

(ii) *Prime terms*   These are:

(a) number variables $x, y, \ldots$; constants 0 and 1; function symbols for plus and times,

(b) variables $X^\alpha, Y^\beta, \ldots$,

(c) certain special terms, mentioned in §8, below, corresponding to axioms and rules of inference of $H(\supset, \wedge, \forall)$.

(iii) *λ-abstraction*:

(a) From F^β get $(\lambda X^\alpha . F^\beta)^{\alpha \supset \beta}$ as in §2.

(b) If x does not occur free in the type symbol of any free variable of F , form $(\lambda x F^\beta)^{\forall x\beta}$.

(iv) *Application*:

(a) From F^α and $G^{\alpha \supset \beta}$ form $(GF)^\beta$ as in §2.

(b) From $G^{\forall x\alpha(x)}$ and t of type 0 form $G(t)^{\alpha(t)}$.

8. *Special terms*

(i) terms of types $x + 0 = x$ and $x + (y + 1) = (x + y) + 1,$

(ii) terms of types $x \cdot 0 = 0$ and $x \cdot (y + 1) = x \cdot y + x$,

(iii) a term of type $x = x$,

(iv) a term of type $x = y \supset t(x) = t(y)$ for each term $t(x)$ of type 0,

(v) the terms B_1, B_2 and B_3 discussed in §6(ii),

(vi) a term $R^{\forall y \beta(y)}$ for each $\beta(y)$ of the form $\alpha(0) \supset [\forall x(\alpha(x) \supset \alpha(x + 1)) \supset \alpha(y)]$, also a term $R_n^{\beta(n)}$ for each numeral n.

9. *Constructions and derivations in* $H(\supset, \wedge, \forall)$

Define constructions as in §2. As in the case of $P(\supset)$--see §3-- the axioms of $H(\supset, \wedge, \forall)$ correspond to the existence of certain terms, and the rules of inference correspond to rules of term formation. In particular $\forall x \alpha(x) \to \alpha(t)$ has the construction $Y^{\forall x \alpha(x)} t$. If F is a construction of $\Gamma \to \alpha$, then $\lambda x.F$ is a construction of $\Gamma \to \forall x \alpha$. If $G(X^{\alpha(t)})$ is a construction of $\Gamma, \alpha(t) \to \beta$, then $G(Y^{\forall x \alpha(x)} t)$ is a construction of $\Gamma, \forall x \alpha(x) \to \beta$. Thus we obtain:

THEOREM 5. Given any derivation of $\Gamma \to \beta$ in $H(\supset, \wedge, \forall)$ we can find a construction of $\Gamma \to \beta$ and conversely.

10. *Interpretation of terms of* $H(\supset, \wedge, \forall)$

We extend the discussion of §4 as follows.

(i) We interpret each closed formula α as a set α^* in the following manner. If α is a closed equation, then α^* is the singleton set $\{1\}$ if the equation is true and the set $\{0\}$ if the equation is false. If $\alpha(n)^*$ has been defined for each numeral n, then define $(\forall x \alpha(x))^*$ as the set of all functions f such that $f(n) \in \alpha(n)^*$ for every n. Define $(\alpha \supset \beta)^*$ in terms of α^* and β^* as in §4. Define $(\alpha \wedge \beta)^*$ as the Cartesian product of α^* and β^*.

(ii) To each term we associate an object F^* by use of the following clauses. It should be noted that when F is closed, then the type symbol of F is a closed formula α; and F^* is

an element of α^* .

(a) Suppose F has the form $G(x_1,\ldots,x_k)$, where $x_1,\ldots,x_k$
are the free number variables of F . Assuming that $G(n_1,\ldots,n_k)^*$ has been defined for all numerals $n_1,\ldots,n_k$, we define
F^* to be the mapping which sends each k-tuple $n_1,\ldots,n_k$ into
$G(n_1,\ldots,n_k)^*$.

(b) Suppose F has no free number variables. If F has the
form $\forall x.G(x)$, then define F^* as the mapping which takes each
numeral n into $G(n)^*$. If F does not have this form, then
the free variables of F will have types $\beta_1,\ldots,\beta_k$ which have
been interpreted as sets $\beta_1^*,\ldots,\beta_k^*$ in (i), above, and F^* will
be a mapping from the Cartesian product of $\beta_1^*,\ldots,\beta_k^*$ into α^* .
In particular, if F has the form $\lambda Y^\beta.G$, then F^* is defined
in terms of G^* by the usual interpretation of λ-abstraction.
To be systematic we must allow free variables, here and in (a),
to occur vacuously.

(c) If F has the form $R_n^{\beta(n)}$ as in §8(vi), then F^* is de-
fined by primitive recursion on n . If F is $R^{\forall y\beta(y)}$, then
F^* is the mapping which takes each numeral n into $(R_n^{\beta(n)})^*$.

11. *Normalization of terms*

For the theory of reducibility of terms we postulate the fol-
lowing contraction schemes

(i) $(\lambda X.F(X))^{\alpha \supset \beta}G$ contr $F(G)^\beta$
            $(\lambda x.F(x))^{\forall x\alpha(x)}t$ contr $F(t)^{\alpha(t)}$

(ii) $B_2(B_1FG)$ contr   F
            $B_3(B_1FG)$ contr   G

(iii) $R_0^{\beta(0)}FG$ contr F

            $R_{n+1}^{\beta(n+1)}FG$ contr $Gn(R_n^{\beta(n)}FG)$

            $R^{\forall x\beta(x)}n$ contr $R_n^\beta(n)$.

Also we consider the steps in the calculation of closed numeri-
cal terms to be reduction steps.

By the method of Tait (1967) it is easy to prove:

THEOREM 6. Every term can be reduced to irreducible form.

Theorem 3 of §6 extends to the present terms. By use of
Theorem 3 it is easy to show that there is no irreducible con-
struction of a sequent of the form $\rightarrow m = n$ with $m \neq n$. Thus
by finitistic reasoning it can be proved that Theorem 6 implies
the consistency of $H(\supset, \wedge, \forall)$.

12. Introduction of $\exists$

How are we to handle the existential quantifier in our theory
of constructions? This appears to be a nontrivial question.
The following two alternatives suggest themselves.

(i) *Weak existence* In the formulation of $H(\supset, \wedge, \forall)$ take
the sequents

$$\rightarrow \alpha(t) \supset \exists y \alpha(y)$$

$$\rightarrow \exists y \alpha(y) \supset (\forall x(\alpha(x) \supset \beta) \supset \beta)$$

(x not free in β) to be axioms and introduce new prime terms C_1
C_2 of types $\alpha(t) \supset \exists y \alpha(y)$ and $\exists y \alpha(y) \supset (\forall x(\alpha(x) \supset \beta) \supset \beta)$,
respectively. Postulate the contraction $C_2(C_1 F)G$ contr GtF
for all terms F and G of types $\alpha(t)$ and $\forall x(\alpha(x) \supset \beta)$,
respectively. Theorems 3, 5 and 6 extend to $H(\supset, \wedge, \forall, \exists)$.
Corresponding to Theorem 4 we have:

THEOREM 7. Every closed irreducible term of type $\exists y \alpha(y)$
has the form $C_1^{\alpha(n) \supset \exists y \alpha(y)} F^{\alpha(n)}$.

Thus from an irreducible construction of $\rightarrow \exists y \alpha(y)$ we get
a numeral n and an irreducible construction of $\rightarrow \alpha(n)$ (assum-
ing $\exists y \alpha(y)$ closed).

(ii) *Strong existence (choice operators)* It is natural to in-
terpret an object of type $\exists y \alpha(y)$ as a pair $\langle t, F^{\alpha(t)} \rangle$. Thus,

in §10, $(\exists y\alpha(y))^*$ would be defined as the set of all pairs
$\langle n,Z\rangle$ such that $Z\varepsilon\alpha(n)^*$. Hence introduce projection opera-
tors P_1 and P_2 which give the required components t and
$F^{\alpha(t)}$ when applied to a pair $\langle t,F^{\alpha(t)}\rangle$ regarded as an object
of type $\exists y\alpha(y)$. The operators P_1 and P_2 can be considered
to have types $\exists y\alpha(y) \supset 0$ and $X^{\exists y\alpha(y)}\alpha(P_1 X^{\exists y\alpha(y)})$, respec-
tively. This takes us outside the formalism of $H(\supset,\wedge,\forall,\exists)$:
the type symbols of P_1 and P_2 are not formulae of
$H(\supset,\wedge,\forall,\exists)$. Nevertheless, it is clear what meaning we are to
attach to P_1 and P_2. Namely, P_1 applies to an object F
of type $\exists y\alpha(y)$ and yields an object $P_1 F$ of type 0; and P_2
when applied to F yields an object of type $\alpha(P_1 F)$.

   To illustrate the use of P_1 as a choice operator, let
$\alpha(x,y)$ be a formula of $H(\supset,\wedge,\forall,\exists)$ with free variables x and
y, and let F be a construction of type $\exists y\alpha(x,y)$. Then
$\lambda x.P_1 F$ is a term ϕ satisfying $\forall x\alpha(x,\phi(x))$.

   In general a type symbol of the form $\forall X^\alpha\beta$ arises by λ-
abstraction from a term F^β, where the variable X^α occurs in
β (as well as in other locations in F). A theory of such
terms would extend the theory developed in the preceding sec-
tions. In the latter theory the only variables occurring in
the type symbols have type 0 (they are, namely, numerical
variables of $H(\supset,\wedge,\forall,\exists)$).

   It might be of interest to know the answer to the following
question. Let G be a closed term obtained by extending our
supply of constructions in the manner just described. Suppose
G has type α where α is a formula of $H(\supset,\wedge,\forall,\exists)$. Must
there be a derivation of $\rightarrow \alpha$ in $H(\supset,\wedge,\forall,\exists)$?

13. Infinite constructions

   As is well-known, we do not have *cut elimination* for

$H(\supset,\wedge,\forall)$ unless the axiom of mathematical induction is replaced by an ω-rule. There is no difficulty in developing a theory of constructions for the ω-rule version of $H(\supset,\wedge,\forall)$. In fact, if one uses Tait's notion of infinite terms (Tait, 1967), one gets a very simple theory (modulo the question of handling infinite terms constructively).

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