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THE DETECTION OF MARKET ABUSE
ON FINANCIAL MARKETS:
A QUANTITATIVE APPROACH

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The detection of market abuse on financial markets: a quantitative approach.

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Key words: insider trading, market manipulation, abnormal return, detection of market abuse, alert, diffusion process.

Summary. In every country with legislation on market abuse, i.e. on market manipulation and insider trading, the repression of these offences is entrusted to supervisory and judicial authorities with powers that vary with the legislation in question. A procedure permitting cases of market abuse to be detected in real time is a need that is strongly felt by financial market supervisory authorities. Such a procedure consists basically in the analysis of the transactions carried out on the market by traders in order to detect anomalies that could be symptomatic of market abuse. The aim of this paper is to develop, through recourse to probability theory, a method for identifying cases of market abuse more effectively.

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Introduction

In each country that has legislation on market abuse, i.e. market manipulation and insider trading, the repression of these offences is entrusted to the supervisory and judicial authorities. Depending on the legislation in place, the powers of investigation and enforcement of the supervisory authority are quite broad. For an analysis of the different legal frameworks in Europe and elsewhere in the world, see the bibliography [Minenna, 2001].

The aim of the manipulation of a security on a financial market (so-called market manipulation) is to change its price or market participants' perception of its underlying value. This can be achieved in two ways, known respectively as market-based manipulation and information-based manipulation. The former is carried out directly on financial markets by means of transactions, including sham transactions, whereas the latter involves disclosing false or misleading information about issuing companies or securities traded on financial markets.

Insider trading consists in trading on financial markets with a view to exploiting information which is not yet publicly available (privileged or inside information) and which, if made public, would probably have a significant effect on the prices of the securities in question.

On the basis of these definitions, the elements that distinguish the two types of market abuse are discussed below.

A comparison of the aims of market manipulation with those of insider trading, from the perspective of profit maximization for the agents involved, reveals the first difference: a market manipulator may have an interest in the market knowing what he has done, whereas an insider trader seeks to hide his presence on the market. This difference implies that an insider, unlike a manipulator, is a price taker. Another difference is that insider trading is always based on exploiting privileged information, whereas this is not necessarily the case for market manipulation. A third difference is that insiders will always act in the direction in which the information in their possession will cause the price of the security to move, whereas for a manipulator the direction is indifferent and will depend on the type of manipulation undertaken.

The need to define a procedure that will permit financial market abuse to be detected is strongly felt by supervisory authorities. The question of market abuse detection has nonetheless been largely ignored in the financial literature, partly owing to the difficulty of accessing detailed data on trading, which is available instead to supervisory authorities.

The aim of this paper is to construct a procedure for identifying market abuses that can detect, for each security and on a daily basis, the possible presence of illicit behaviour.

On the basis of the information available to market supervisory authorities, the procedure is designed to verify, for each security, the presence of anomalies, known as market failures. To this end, the method used identifies a reference model for the various financial variables that make up the flow of information on trading in a given security that makes it possible to develop an indicator based on dynamic thresholds; the crossing of these thresholds indicates an anomalous

movement in the variable in question (known as an alert). The financial variables analyzed on the basis of the model thus become a series of alerts that signal the possible presence of market abuse.

The tripwires of the market abuse detection procedure were identified by analyzing what the theory of financial markets, supervisory experience and the empirical observation of the various cases of market abuse found by Consob suggested in relation to the various variables that comprise the flow of information on trading in securities on financial markets.

Once the tripwires have been chosen, the calibration of a market abuse detection procedure consists in calibrating the corresponding reference models, i.e. in their parametric specification for predictive purposes, and in identifying an algorithm that will permit the joint interpretation of the various alerts.

The observation of the cases of insider trading and market manipulation found by Consob permitted a valuable empirical verification for the calibration of the market abuse detection procedure.

The paper is divided into two parts: the first consists of a survey of the financial literature and of supervisory experience, in which elements useful for the construction of the tripwires are identified (see Section 1); the second outlines the procedure developed and contains a detailed description of the various tripwires and an explanation of how the procedure was calibrated (see Section 2).

1 Review of the literature and supervisory experience

Every procedure for detecting market abuses is based on an analysis of different traders' trades. The flows of elementary information on trades comprise: the prices, the quantities and the names of the traders who carried them out.

This section contains an analysis of the financial literature and Consob's supervisory experience designed to show how these flows of elementary information can be processed to construct the financial variables that serve to define the tripwires of the model. In particular, the following are addressed:

1. the theory of efficient markets with information that is homogeneous among traders (so-called classical asset-pricing theory);
2. the theory of financial markets with heterogeneous or asymmetric information;
3. the literature on the effects of insider trading and market manipulation on financial markets;

As regards the last of these points, three issues have been analyzed: the information content of trades by insiders for the purpose of inferring the future returns on securities [Minenna, 2001, Seyhun, 1998]; the estimation of the economic value of privileged information as a proxy for the calculation of the abnormal returns achieved by insiders [Minenna, 2002]; and the ability of the

market to read insider information and hence of prices to incorporate its value before it is made public [Meulbroek, 1992, Chackravarty, 1999, Cornell and Sirri, 1992, Bhattacharya et al., 2000].

The bibliography contains details on the literature referred to in this section [Barucci, 2002] and on supervisory experience [Tuccari, 1999].

1.1 Some theoretical aspects: classical asset-pricing theory

Some results of classical asset-pricing theory (in which agents have homogeneous information) show that the various flows of elementary information on trading in a security can be effectively summarized in two financial variables: the rate of return and the volume of trading. The analysis of this branch of theory with a view to identifying a set of tripwires serving to define a procedure for market abuse detection has therefore been carried out with reference to these two variables.

1.1.1 The rate of return

Classical asset-pricing theory is based on hypotheses that have already been widely applied in economic theory; those of perfectly competitive markets, which are complete (trades can be conditioned on all futures dates and events), agents who maximize an objective function (expected utility), and rational expectations (agents are risk neutral). When markets are in equilibrium and there are no opportunities for arbitrage, it has been shown that there exists a measure of equivalent probability, known as the risk-neutral probability, such that the expected conditioned single-period rates of return of securities portfolios are equal to the rate of return on the risk-free security. With respect to this measure of risk-neutral probability, it follows that:

- there does not exist an investment strategy capable of generating higher expected returns than that of the market portfolio or of the buy and hold strategy;
- the price of a security and the wealth generated by a discounted investment strategy, or a function thereof, are Martingale diffusion processes;
- excess returns (the return on a security net of the return on the risk-free security) are not autocorrelated and the information available at a given time does not permit future returns to be predicted.

These characteristics are summarized in the fact that the logarithm of the price of a security that does not distribute dividends follows a stochastic process known as *random walk*.

It can easily be seen that these hypotheses (risk neutrality and agents' preferences constant over time) are unrealistic and that the results they lead to are of little help in understanding the behaviour of the return on a security.

Dropping the hypothesis of risk-neutral agents, the implications of the theory of asset pricing become more complex. The measure of risk-neutral probability no longer coincides with the historical value (obtained from market data). Furthermore, the dynamic of the conditional expected return on a security no longer corresponds to that of the risk-free return. This dynamic is analyzed in the literature by comparing the return on a security with some state variables that describe the change in investment opportunities and agents' preferences. In particular, in the case of complete markets, in relation to the state variable used to describe the dynamic of the return on a security, the relevant theoretical models are the CCAPM (Consumption Capital Asset Pricing Model) [Rubinstein, 1976] and the ICAPM (Investment Capital Asset Pricing Model) [Breedon, 1979].

Thus, where agents are not risk neutral, there is no reason why the returns on securities should not be autocorrelated or why some currently available information should not be of value in forecasting future returns [Fama, 1991].

In the most recent literature it has been noted that the returns on portfolios and stock indices are characterized by mean reversion in the long run (negative autocorrelation of the returns) and by a trend (momentum effect) in the short run (positive autocorrelation of returns).

Lehman and Jegadeesh confirmed the presence of autocorrelation in the weekly returns on securities [Lehman, 1990, Jegadeesh, 1990].

Roll found evidence that the daily returns on individual securities are affected by the effects of the market's microstructure (e.g. transaction costs), which can lead to short-term mean reversion phenomena [Roll, 1984]. In particular, illiquid securities, characterized by a shallow market, are marked by a high price pressure, so that the returns may change sign in a short span of time.

Fama and French showed that an auto-regressive model based on the log of returns captures both the mean reversion and the momentum effect components [Fama and French, 1988].

In the light of these observations, the autocorrelation of returns, or their examination according to a mean-reverting dynamic, can provide an interesting indicator of the dynamic of a security. An anomaly in this dynamic signals a change in investment opportunities, and therefore in the riskiness of a security, the occurrence of a structural break or of a case of market abuse. Moreover, given the natural tendency for a security to be characterized by a mean-reversion component, and for this tendency to be inversely related to the security's liquidity, it can be seen that a joint analysis of returns and trading volumes is likely to be of help in discriminating between the different hypotheses of market failure.

1.1.2 Transactions: Trading volumes

In the literature agents are considered to trade securities on the financial market for the following reasons:

- risk sharing-hedging;
- speculation.

Trading that is connected with risk sharing-hedging serves to meet agents' need to hedge against the financial risk associated with their outstanding positions.

Speculative reasons are connected instead with the possibility of making profits, owing in part to the presence of heterogeneous information sets among agents.

In classical asset-pricing theory these reasons find scarce treatment; indeed, with complete markets, trading in connection with risk sharing-hedging can be explained only in an initial state of the market generated by an exogenous shock, which, given its nature, causes a structural break in the market. Where this occurs, such trading is necessary to reach a Pareto optimum allocation. Once this has been reached, under the hypothesis of rational expectations there will be no reason for any further trading. Speculative trading is excluded a priori given the hypothesis of agents with homogeneous information.

In this context, therefore, the literature fails to offer an effective explanation of the high turnover observed on financial markets and the large autoregressive component of such trading [Gallant et al., 1992].

Dropping the hypothesis of competitive markets, the absence of some markets and of specific financial instruments leads to an increase in trading volumes since agents, unable to hedge risk completely, tend to trade more frequently in the market.

All told, the theory of financial markets with operators possessing homogeneous information does not lead to a model that explains the empirical evidence of high turnovers and their large autoregressive components. On the other hand, it shows that a change in investment opportunities in agents' portfolio choices owing to a structural break will undoubtedly generate turnover with a certain autoregressive component.

The literature examined above thus does not identify a specific reference model that would explain the changes in trading volumes over time, although empirically it notes the presence of autoregressive components that should suggest the construction of a tripwire based on this variable.

The literature on financial markets in possession of non-homogeneous information surveyed in the following section offers greater insights into the dynamic of this financial variable.

1.2 Some theoretical insights: heterogeneous and asymmetric information

The hypothesis of heterogeneous and asymmetric information complicates the theoretical picture described above; indeed the existence of different information sets among agents leads to trading, not only for risk sharing-hedging reasons but also for speculative reasons.

The analysis of the behaviour of the returns on securities and trading volumes depends on the structure of the market (whether or not it is perfectly competitive) and on the nature of agents' information (heterogeneous or asymmetric).

A market is perfectly competitive when agents' behaviour does not influence the price (that is to say they are price takers); vice versa, if they are found to influence the price, the market is imperfectly competitive.

Information is heterogeneous when all the agents (or a large set of agents) receive private signals correlated with the fundamental value of the security; it is asymmetric when only a small subset of agents is in possession of privileged information regarding the fundamental value of the security.

The following presents the principal results in the literature on financial markets with heterogeneous and asymmetric information, with a view to identifying a set of useful tripwires for the definition of a Market Abuse Detection (M.A.D.) procedure.

1.2.1 Financial markets with heterogeneous information

Two important results found in the literature on financial markets in the presence of heterogeneous information are:

1. the existence of fully-revealing equilibrium prices;
2. the no-trade theorem.

The first result implies that, in the absence of noise (such as random demand on the part of noise traders), equilibrium prices promptly and accurately incorporate the value of the private information in agents' possession. This characteristic of prices was attributed by Grossman to the fact that, by trading on the market, agents transmit the value of the information in their possession to the others [Grossman, 1989]. In this context, the size of the exchange is an increasing function of the accuracy of the information (a closed-form relationship is established in the case of normal random variables and an exponential utility function).

The second result shows that, by itself, heterogeneous information does not give rise to turnover. In fact, if agents have achieved an ex ante Pareto optimal allocation in the absence of such information, differentiation of the information they possess will not generate trading [Milgrom and Stockey, 1982, Tirole, 1982].

Since these results lead the analysis of financial markets with heterogeneous information back to classical theory, they do not add anything to the analysis of the behaviour of the returns on securities and trading volumes. (see Section 1.1).

By adding a noise component, some later models make it possible to overcome the limitations connected with the existence of fully-revealing equilibrium prices and with the no-trade theorem under the hypothesis of perfectly competitive markets.

Kim and Verrecchia analyzed a model with private information obtained at a cost and a noise component in a two-period economy [Kim and Verrecchia, 1991]: in the first period agents trade on the basis of their heterogeneous private information regarding the fundamental value of the security; in the second a public announcement causes the expectations of the security's value to be homogeneous. The authors show that volumes in the second period are positively correlated with the absolute value of the change in price between the two periods considered and that the multiplier depends on the degree of heterogeneity of the agents' information. This static model highlights two important results:

1. volumes and absolute price changes are positively correlated;
2. the relationship between volumes and absolute price changes is positively related to the degree of heterogeneity of agents' information.

These results have also been obtained using models in which agents are characterized by opinions differentiated ex ante (and therefore not connected with private information) [Shalen, 1993] or by different interpretations of a public signal [Kandel and Pearson, 1995].

He and Wang developed a dynamic model that provides for the arrival on the market of private and public information together with a noise component. Agents can trade either to accommodate a shock on the supply side (non-informational trading) or to speculate on the performance of the security (informational trading). The authors show that the autocorrelation of volumes distinguishes private from public information. In particular, when a public announcement is made, large volumes are found only in a short interval around the announcement; when, instead, the information is private, the volumes tend to be autocorrelated, insofar as the information is transmitted via prices over time through trading, including that, subsequent to the arrival of the information, of uninformed agents who act as *followers* [He and Wang, 1995].

Harris and Raviv obtained a similar result by establishing a link between large volumes and autocorrelation among them with private information present in the market [Harris and Raviv, 1993].

Other models examine the relationship between volumes and returns, distinguishing between volumes that are information based and those that are not. Campbell et al. verified empirically that, if volumes are large for other than informational reasons (liquidity, preference shocks), then returns accompanied by high volumes will be followed by returns of the opposite sign (price reversal

or mean reversion) [Campbell et al., 1993, Conrad et al., 1994]. Llorente et al. showed, instead, that if volumes are generated by new private information arriving on the market (informational trading), the relationship may be of the opposite sign, but not necessarily so [Llorente, et al., 2001].

The latter models therefore show that there will be mean reversion if the volumes are based on non-informational reasons and a momentum effect (presence of a trend and positive autocorrelation among the returns) if they are based on private information.

Dropping the hypothesis of perfectly competitive markets, the fact that agents' transactions can influence the prices of securities on financial markets invalidates the results concerning the existence of fully-revealing equilibrium prices and the no-trade theorem.

The literature is scant, partly because imperfectly competitive markets are generally associated with the presence of asymmetric rather than heterogeneous information (see Section 1.2.2). An interesting study is that of Foster and Viswanathan, which shows that agents having different perceptions of the fundamental value of the security will fuel a chain reaction, since they will continue to trade owing to their different interpretations of the other agents' trading [Foster and Viswanathan, 1996].

The analysis of the literature on financial markets with heterogeneous information confirms that, for the purpose of defining a tripwire based on turnover, examining the time series according to an autocorrelated process can be of particular assistance. It also strongly suggests the desirability of defining a tripwire that examines returns in the light of a mean-reversion model and a calibration procedure that jointly evaluates the results of the tripwires based on volumes and returns.

1.2.2 Financial markets with asymmetric information

The literature on financial markets with asymmetric information shows how adverse selection phenomena due to the fear of trading with an agent in possession of privileged information can cause individual uninformed agents to decide not to carry out any trades and thus reduce the volume of trading on the market.

Assuming perfectly competitive markets, Wang analyzed a model with informed and uninformed agents, where the former can also trade off-market (private investment opportunities) for reasons not connected with the value of the security. The informed agents trade in response to private information or shocks related to private investment opportunities. The uninformed agents therefore face a problem of adverse selection linked to the risk of carrying out a trade with an informed agent and may be led not to trade on the market at all. Consequently, the volume of trading diminishes with the degree of information asymmetry, while it increases when information is disclosed because this reduces the problem of adverse selection. Even though the absolute effect is different from that found in the case of heterogeneous information (see Section 1.2.1),

the dynamic effect produced by the arrival of information in the public domain is the same.

Dropping the hypothesis of perfectly competitive markets, the picture becomes more variegated because an agent in possession of privileged information will define his behaviour taking into account the effect it will have on the price.

Kyle showed that in an imperfectly competitive market with a noise component the information transmitted by prices is inferior to that transmitted in a perfectly competitive market [Kyle, 1989].

In this context the financial literature presents some models that examine the behaviour of returns and trading volumes in relation to:

1. the microstructure of the market, and in particular to whether the market is order driven or quote driven;
2. the level of competitiveness from the point of view of dealers-market makers;
3. the effect of adverse selection on agents in the presence of informed agents.

Some interesting results are linked to the hypothesis of a dealer who sets bid and ask prices knowing that there are some informed and some uninformed agents in the market. In this case the dealer, fearing the effect of adverse selection (carrying out a transaction with an informed agent), tends to set a high bid-ask spread and, therefore, does not contribute to increasing the liquidity of the market. Easley and O'Hara showed that a dealer in a quote-driven market, knowing of the existence of informed traders, tends to set a higher bid-ask spread that produces inefficient conditions for trading [Easley and O'Hara, 1987]. In extreme situations, Glosten and Milgrom showed that this type of behaviour on the part of a dealer can lead to price levels that result in no transactions at all being carried out [Glosten and Milgrom, 1985].

Kyle analyzed a model in which an insider trader operates in a market with noise traders (whose demand is given by a random variable that does not depend on price) and a market maker who undertakes to clear the market at a price equal to the expected value of the dividend conditional on the flow of market orders. In an economy with only one opportunity to trade the insider trader, who knows the exact value of the dividend, tends to hide his information to prevent the market maker's price discovery process from reflecting the value of the private information exactly. This behaviour on the part of the insider translates operationally into an elasticity of his demand for private information that is inversely proportional to the depth of the market and directly proportional to the noise component. In particular, the insider, in order to maximize the profits obtained by exploiting the privileged information, may be led to trade modest quantities, so as to prevent his trading from revealing the value of the information to the market [Kyle, 1985].

Foster and Viswanathan, using reasoning analogous to Kyle's, showed that volumes are autocorrelated in an imperfectly competitive market in the presence of market abuse phenomena [Foster and Viswanathan, 1993].

The analysis of the literature on the financial markets in the presence of asymmetric information confirms the usefulness of observing the time series of volumes according to an autocorrelated process in order to construct a tripwire based on this financial variable. What is more, it suggests the desirability of examining factors related to the microstructure of markets and the trading methods used, with particular reference to the depth of the market and the presence of dominant traders. This analysis should be carried out through the construction of specific tripwires capable of revealing the presence of dominant operators in the market and the evolution of market concentration. Moreover, the consideration that market concentration should always be examined in relation to volumes and the depth of the market, suggests that the calibration of the procedure should be defined in a way that permits the joint evaluation of the results of the volume and market concentration tripwires.

1.3 The literature on the effects of insider trading and market manipulation.

The literature on market abuse is very limited and often connected to research carried out by supervisory authorities in support of their own activities.

In particular, the theme of manipulation is almost completely absent, partly owing to the unavailability of data on the trading of the various market participants, whereas insider trading has attracted the attention of several writers.

Among these, Meulbroek showed that higher volumes and abnormal returns are found on days marked by insider trading, since some uninformed agents “follow” the behaviour of the insiders (the so-called herd effect) and thus help the price of the security to rise more rapidly to the level it would reach if the information were disclosed [Meulbroek, 1992]. Similar results were obtained by Cornell and Sirri by examining the “Campbell and Taggart” case [Cornell and Sirri, 1992]. This analysis shows that insiders tend to hide their operations by carrying out rather small transactions, a fact already highlighted in the analysis of the financial literature 1.2.2.

Bhattacharya et al. showed that the disclosure of information on companies listed on the Mexican market does not have any effect on returns, volumes or volatility. This is due to the intense trading by insiders who anticipate the diffusion in the market of the value of the privileged information.(so-called pre-announcement information leakage) [Bhattacharya et al., 2000].

Chackravarty, analyzing the “Carnation” case, showed instead that a price effect does exist upon the disclosure of information, but also that it is not possible to distinguish the price effect due to the trading of insiders from that produced by the transactions of agents not in possession of privileged information [Chackravarty, 1999].

Bagliano et al. in an analysis of the Italian market showed that insider trading does not bring about a change in the autocorrelation of the time series of volumes and returns [Bagliano et al., 2001].

With reference to research carried out by supervisory authorities, Mitchell and Netter described the procedure adopted by the United States Securities and Exchange Commission (SEC) to estimate the abnormal returns connected with the disclosure of privileged information. This procedure uses the market model and, by means of an econometric approach of the event-studies type, estimates the abnormal return on a security in relation to the return on the market index. The results of the estimation form the basis for determining the value of the privileged information appropriated by the insider to the detriment of the market. On the basis of this value the SEC determines the sanctions to apply to the insider (so-called disgorgement) [Mitchell and Netter, 1994].

Minenna has recently proposed, as an alternative to the market model, the adoption of a diffusion process that infers future returns on the basis of a calibration based on the returns recorded by the security in a certain period. This innovative procedure detects the presence of abnormal returns in relation to the disclosure of privileged information. He has also shown that, by calibrating the diffusion process on the basis of an insider's trading strategy (information available to the supervisory authority), it is possible to calculate the value of the privileged information appropriated by the insider to the detriment of the market and hence the disgorgement [Minenna, 2001].

All told, these studies do not clarify whether the information possessed by insiders is incorporated in the price before the information is disclosed; however, they do confirm that volumes increase, that uninformed agents account for a substantial part of the increase, that insider traders tend to hide their presence in the market, that abnormal returns occur in the presence of privileged information and that they can be estimated using diffusion processes [Minenna, 2002].

In defining alerts based on trading volumes and returns, these considerations therefore confirm the need to analyze:

- the dynamic of trading volumes determined by the activity of the different market participants according to an autocorrelated process;
- the behaviour of the return on a security with a view to identifying the presence of abnormal returns using diffusion processes.

1.4 Consob's supervisory experience

Consob's supervisory experience has been acquired since 1991, the year in which Law 157/1991 provided a specific legal framework for countering insider trading and market manipulation. Since the law entered into force Consob has reported 140 cases of market abuse to the judicial authorities.

Consob carries out its supervisory action against market abuse by means of operational and analytical investigations. The former include not only analysis of intermediaries' market operations, the positions in cash and securities of individual clients, records of orders and transactions but also the collection of information on issuers, the analysis of their financial statements and research

reports, etc. These instruments permit the verification of the existence of the *fumus* of abusive conduct in the market and to ascertain the elements necessary to determine its scope.

The analytical instruments serve, instead, to evaluate the economic impact of abusive conduct on market integrity and any losses incurred by investors. In cases of insider trading, after identifying the privileged information, it is necessary to evaluate its price sensitivity and the value of the information that the insiders exploit at the expense of the market [Minenna, 2002]; in cases of market manipulation, after identifying the type of manipulation, it is necessary to examine the anomaly produced in the performance of the security and to quantify the damage caused to the market [Milia, 2001]. It needs to be remembered, in fact, that the sanction to be imposed on manipulators has to consider the economic and financial effects of their conduct on the market and investors.

The ex-post examination of the cases of market abuse found by Consob show that such conduct leaves a trace in the financial markets, in terms of both the price of the security and the volume of trading [Tuccari, 1999].

As for insider trading, considering the tendency of insiders to hide their presence in the market, it is the value of the privileged information and the moment of disclosure to the market, in relation among other things to the possible advance propagation of rumours, that determine anomalous movements in the security's price and the volume of trading. Hence the importance of the supervisory authority keeping a database of company information and setting up special enforcement units to monitor continuous and periodic information on companies.

In its experience with market manipulation Consob has found cases of both market-based manipulation and information-based manipulation.

The cases of market-based manipulation show that prices can be altered either with actual transactions (trade-based manipulation) or with sham transactions (wash sales/matched orders) and that some prices (specifically opening and closing prices) have a greater information value than others since they are referred to in rules on the functioning of the microstructure of markets. For example, manipulation of the opening price has been found in some cases because it determines the pay-off of the derivative component of structured products placed in the retail market.

It has also been found that market value, concentration of ownership and liquidity are factors that affect the probability of listed securities being manipulated. For example, in the case of thinly traded securities, agents may be tempted to carry out transactions so as to create the appearance of an active market, to drive the price above the level the market would otherwise express, and to boost the market value of a company that is about to be disposed of, etc. Manipulators are generally connected to the company's controlling shareholders and the organization of the market abuse requires agreements with intermediaries and institutional investors. These interventions often resemble those undertaken to stabilize the price of a security or in anticipation of the announcement of company news.

Turning to information-based manipulation, the cases found are connected

with the fundamental role that information plays in all financial investments. Two problems emerge in this connection: the first is the conflict of interest typical of financial intermediaries that publish research and investment advice and also operate directly in the market; the second is more typically Italian and regards the lack of “pure publishers”. That having been said, one type of manipulation found by Consob consists in the release of false information on company events or the situation of the company with the aim of influencing the prices of listed securities. This conduct is usually connected with the presence of controlling shareholders, who are often in financial difficulties. In some cases misleading or biased press reports have been found, by means of which the company communicates the existence of restructuring projects to the market, sometimes through channels not directly connected to the company. Another type of manipulation consists in the publication by intermediaries of research reports with exaggerated and/or false forecasts. In these cases the trading for own account of the intermediaries in question has often been found to be inconsistent with their recommendations.

The examples of market manipulation given above highlight the fact that both the prices of securities and the related returns generally undergo sharp changes (for example at the moment privileged information is disclosed), or show movements that cannot be attributed to a dynamic of the mean-reverting type (for example in the presence of manipulation). Furthermore, they show that trade volumes vary both in absolute terms, generally conserving an autoregressive component, and in terms of the composition of intermediaries and traders. With reference to the composition, it is necessary to consider two variables in particular:

1. the level of concentration of the intermediaries, that is the number of intermediaries and their shares of trading volumes (so-called static concentration);
2. the evolution of the concentration of the intermediaries, that is the change in each intermediary’s share of the volume of trading in a given security (so-called dynamic concentration).

These considerations, which summarize Consob’s supervisory experience, confirm the desirability of defining a tripwire based on trading volumes by examining the time series of volumes using an autoregressive model and on returns by calibrating a regression process of the mean-reverting type. They also suggest defining two tripwires that examine the composition of intermediaries-traders, distinguishing between static and dynamic concentration. Lastly, the empirical evidence shows how market abuse phenomena give rise to changes in the behaviour of several financial variables at the same time, which implies that the calibration of the procedure needs to be defined in a way that permits a summary evaluation of the results of all the tripwires.

2 The Market Abuse Detection procedure

2.1 Preamble

A M.A.D. procedure identifies, on a daily basis, the securities affected by illicit behaviour in the form of market manipulation or insider trading. It highlights the possible presence of market abuse phenomena by examining the behaviour of financial variables that correspond to the flows of elementary information on trading available to the supervisory authority.

The examination of the behaviour of the financial variables requires the definition of a reference model for each of them. The development of the reference models is aimed at the identification of dynamic thresholds, the crossing of which signals an anomalous movement in the variable in question (so-called alerts).

The financial variable and the related model thus become a tripwire of potential market abuse phenomena for the procedure. Having identified the tripwires, the calibration of a M.A.D. procedure consists in calibrating their models, that is in specifying their parameters and in identifying an algorithm permitting several alerts to be interpreted jointly.

As mentioned in the Introduction, the identification and calibration of the tripwires of the M.A.D. procedure have been identified and calibrated by analyzing:

- what the theory of financial markets and Consob's experience suggest;
- the cases of insider trading and market manipulation found by Consob.

2.2 The tripwires

The examination of the literature and supervisory experience have provided the following indications as to how transaction prices, the quantities traded and the names of the traders who carried out the transactions (i.e. the flows of elementary information on trading available to the supervisory authority) must be analyzed in order to construct financial variables whose behaviour can reveal market abuses:

- transaction prices are analyzed in terms of returns by studying the movements in the log of the price;
- returns generally undergo sharp changes (for example at the time privileged information is disclosed) or follow paths that are not of the mean-reverting type (for example in the presence of manipulative phenomena);
- the presence of abnormal returns is detected by estimating returns using diffusion processes;
- autoregressive models can capture both the discrete mean-reversion and momentum-effect components of returns;

- the quantities traded by individual agents are examined in terms of daily trading volumes according to an autocorrelated process;
- the agents are analyzed in relation to the quantities they have traded in a day, taking into account the depth of the market, the presence of dominant traders and the composition of the various intermediaries-traders;
- the composition of the market is evaluated using a two-stage process:
 - the level of concentration of the intermediaries, that is the number of intermediaries and their shares of trading volumes (so-called static concentration);
 - the evolution of the concentration of the intermediaries, that is the movement in each intermediary’s share of the volume of trading in a given security (so-called dynamic concentration).

On the basis of these indications four financial variables have been constructed that represent the behaviour of:

1. the volumes of trading in the security;
2. the returns on the security;
3. the static market concentration;
4. the dynamic market concentration.

The definition of a tripwire of a M.A.D. procedure requires the examination of these variables to be based on a reference model; appropriately calibrated, this defines the dynamic thresholds that trigger the procedure’s alerts. In particular, the construction of the tripwires must guarantee the detection, in real time, of the securities that may be the subject of market abuse.

The examination of the different cases of market abuse found by Consob has provided important support in defining the diffusion processes that describe the behaviour of the financial variables and thus characterize the reference models.

A description follows of the construction and functioning of the tripwires based on the four financial variables considered, with reference to the flows of elementary information on the trading in any security listed on the share market. To this end, P_t and Q_t are used to denote respectively the official price and the volume of trading observed in the financial markets for a generic security on day t .

2.2.1 The analysis of trading volumes

The examination of the time series of trading volumes Q_t was conducted according to an autocorrelated process, as suggested by the literature on financial markets and supervisory experience. It should be noted, however, that the literature does not indicate a particular reference model for this examination.

Considering the objective of defining a tripwire that is capable of revealing anomalies in the movement of this financial variable with a view to detecting market abuses, the analysis of the different cases found by Consob suggested assuming that volumes are governed in discrete time by the following autoregressive process, which, by construction, shows autocorrelation:

$$Q_k = \phi Q_{k-1} + \hat{\sigma} Z_k \quad (1)$$

where ϕ is a deterministic function of time, $Q_0 = q_0$ and $(Z_k)_{k \geq 0}$ is a sequence of random variables identically independently distributed as a normal with a zero mean and a unit variance on \mathbb{R}^1 . The variability of $\hat{\sigma}$ in relation to time is guaranteed by the fact that it multiplies a random variable whose variance is defined in relation to the unit of time.

For expository convenience, putting $\gamma = 1 - \phi$, (1) can be rewritten as follows:

$$Q_k - Q_{k-1} = -\gamma Q_{k-1} + \hat{\sigma} Z_k \quad (2)$$

A first consideration concerning (2) is that, since q_0 is a constant and Z_1, \dots, Z_k a sequence of independent random variables distributed as a normal with a zero mean and a unit variance, the solution $\{Q_k\}_{k \geq 0}$ is a Markov chain with respect to the filtration $\{\mathfrak{S}_k\}_{k \geq 0}$ generated by the sequence Z_1, \dots, Z_k , which assumes values on \mathbb{R}^1 and where k is the indicator of discrete time. The pair $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ defines the measurable space of $\{Q_k\}_{k \geq 0}$ where $\mathbb{B}(\mathbb{R}^1)$ is the Borel field on \mathbb{R}^1 . Every discrete Markov process defined in this way is identified by the initial distribution $v_0(\cdot)$ and by the probability of transition $\Pi_{1,k}(\cdot, \cdot)$ defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$.¹

The properties of the reference model sub (2) do not permit the behaviour of volumes to be forecast using a number of daily observations that refer to a time horizon of a month or less, if the statistical significance of the analysis is not to be lost or numerous procedural complications are not to be encountered.²

In order to construct a tripwire that responds to the objectives of the procedure, attention is focused on the distributive characteristics of the corresponding continuous-time version of (2) [Nelson, 1990].

The advantage of the continuous time approach is that by referring to a stochastic differential equation, if this has an integrated solution or if the distributive properties of the solution are known, it is possible to construct a confidence interval for the prediction of the variable described by the diffusion process. This interval defines the trading volume dynamic thresholds that identify the alerts of the M.A.D. procedure. The logic underlying the indicator to be constructed is taken over from that used by Minenna to predict the behaviour of the return on a security and detect the presence of abnormal returns [Minenna, 2001].

¹In particular $\forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$:

1. $P(Q_0 \in \Gamma) = v_0$;
2. $P(Q_{k+1} \in \Gamma | \mathfrak{S}_k) = P(Q_{k+1} \in \Gamma | Q_k) = \Pi_{1,k}(Q_k, \Gamma)$.

²For a discussion of the problems connected with the estimation of the parameters of the model in question, see the bibliography [Greene, 1993].

By rescaling the discrete Markov process $\{Q_k\}_{k \geq 0}$, it can be shown that (2) converges weakly to the diffusion process $\{Q_t\}$, given by the following stochastic differential equation:

$$dQ_t = -\theta Q_t dt + \sigma dW_t \quad (3)$$

where θ and σ are deterministic time functions and W_t is a standard unidimensional Brownian motion. The proof is given with reference to the generic diffusion process $\{X_t\}$ in appendix A.

(3) is the continuous-time version of (2). This stochastic differential equation is known in the literature as an Ornstein-Uhlenbeck arithmetic diffusion process, which has the following distributive properties with reference to any constant initial condition identified at time s , with $s < t$, equal to Q_s :

$$Q_t \sim N \left(Q_s e^{-\theta(t-s)}; \sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta(t-s)})} \right) \quad (4)$$

The proof of these distributive properties is given in appendix A with reference to the generic diffusion process $\{X_t\}$.

The relationship between (2) and (3) and the distributive properties of the latter stochastic differential equation (4) define the reference model to be used in examining the time series of trading volumes and thus uniquely define the tripwire with reference to this financial variable.

The dynamic thresholds that make it possible to identify the alerts for trading volumes have been constructed, in fact, by exploiting the properties of the reference model. In Section 2.3.1 the construction of these thresholds and their calibration are described in detail.

2.2.2 The analysis of returns

The theory of financial markets and supervisory experience indicated the desirability of using an autoregressive model to examine the time series of a security's returns. In particular, considering the objective of defining a tripwire capable of revealing anomalies in the behaviour of this variable with a view to detecting market abuses, the examination of the cases found by Consob showed that the following model appropriately describes the behaviour of this variable in discrete time:

$$R_k = \alpha + \lambda R_{k-1} + \hat{\sigma} Z_k \quad (5)$$

where $R = \ln(P)$, α and λ are deterministic functions of time,³ $R_0 = r_0$ and $(Z_k)_{k \geq 0}$ is a sequence of random variables identically independently distributed as a normal with a zero mean and a unit variance on \mathbb{R}^1 .

For expository convenience, putting $\lambda = 1 - \gamma$ and $\alpha = \gamma \cdot \eta$, (5) can be rewritten as follows:

³The variability of $\hat{\sigma}$ in relation to time is guaranteed by the fact that it multiplies a random variable whose variance is defined in relation to the unit of time.

$$R_k - R_{k-1} = \gamma \cdot (\eta - R_{k-1}) + \widehat{\sigma} Z_k \quad (6)$$

As in Section 2.2.1, in order to construct a tripwire that achieves the objectives of the procedure, we focus on the distributive characteristics of the corresponding continuous-time version of (6) [Nelson, 1990].

As already shown in the previous section, the advantage of the shift to continuous time consists in the identification *a priori* of the distributive properties of the process $\{R_k\}$ and thus in the possibility of easily defining a confidence interval for the prediction of the variable described by the diffusion process; by defining the dynamic thresholds that identify the alerts of the procedure, this process turns a financial variable into an alert.

By time rescaling, as in Section 2.2.1, the discrete Markov process $\{R_k\}_{k \geq 0}$ and showing (6) converges weakly towards the diffusion process $\{R_t\}$ given by the following stochastic differential equation:

$$dR_t = q(\mu - R_t) dt + \sigma dW_t \quad (7)$$

where q and μ are deterministic time functions and W_t is a standard unidimensional Brownian motion.⁴

(7) is the continuous-time version of (6). This is again an Ornstein-Uhlenbeck arithmetic diffusion process, which has the following distributive properties with reference to any constant initial condition identified at time s , with $s < t$, equal to R_s :⁵

$$R_t \sim N \left((R_s - \mu)e^{-q(t-s)} + \mu; \sqrt{\frac{\sigma^2}{2q} (1 - e^{-2q(t-s)})} \right) \quad (8)$$

The relationship between (6) and (7) and the distributive properties of the latter stochastic differential equation (8) define the reference model to be used in examining the time series of returns and thus uniquely define the tripwire with reference to this financial variable.

The dynamic thresholds that make it possible to identify the alerts for returns were in fact constructed by exploiting the properties of the reference model. In Section 2.3.2 we describe the construction of these thresholds and their calibration in detail.

2.2.3 The analysis of static concentration

Consob's supervisory experience and the financial literature showed the desirability, with a view to detecting market abuses, of examining the composition of the intermediaries-traders present on the market. A first analysis of this kind can be made by examining the so-called static concentration, that is the level of

⁴The proof is given with reference to the generic diffusion process $\{X_t\}$ in appendix A.

⁵The proof of these distributive properties is given in appendix A with reference to the generic diffusion process $\{X_t\}$.

concentration of the intermediaries, in the sense of the number of intermediaries and their shares of trading volumes.

The first step in constructing a tripwire on the basis of this aggregate is to identify the related financial variable. The examination of the various cases of market abuse found by Consob suggested that the best choice among the various indices of market concentration put forward in the literature would be an index of entropy, that is:

$$\Theta_t = \frac{1}{n_t} \sum_{i=1}^{n_t} \left(\frac{Q_t(i)}{\mu_t} \right)^\alpha \quad (9)$$

where

n_t is the number of traders on the market at time t

$Q_t(i)$, $i = 1, \dots, n_t$ are the quantities traded by the i -th intermediary at time t

$$\mu_t = \frac{\sum_{i=1}^{n_t} Q_t(i)}{n_t}$$

The reference model that gives the tripwire is based on a stochastic differential equation, that is:

$$d\Theta_t = -\zeta\Theta_t dt + \sigma dW_t \quad (10)$$

which we derived as for trading volumes, through the shift to continuous time of the corresponding discrete process (see Section 2.2.1), i.e.:

$$\Theta_k - \Theta_{k-1} = -\xi\Theta_{k-1} + \hat{\sigma}Z_k \quad (11)$$

This model has the following distributive properties with reference to any constant initial condition identified at time s , with $s < t$, equal to Θ_s :

$$\Theta_t \sim N \left(\Theta_s e^{-\zeta(t-s)}; \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta(t-s)})} \right) \quad (12)$$

The reason for this choice is that in normal trading conditions, that is in the absence of failures that could point to possible market abuse phenomena, the index for short-time horizons should be autoregressive in discrete time and mean-reverting in continuous time.

We examined the financial variable Θ_t with reference to the quantities bought, the quantities sold and the gross turnover, i.e. the sum of purchases and sales, of the various intermediaries-traders. Subsequently, we constructed three tripwires with the same mathematical formulation with reference to each of the above aggregates.

This choice was due to the need to capture not only the movement in the variable for the total turnover of the market but also the directions taken by individual intermediaries, and hence the market, since this could point to possible market abuse phenomena.

In Section 2.3.3 we give a detailed description of the calibration of the tripwires and of the construction of the dynamic thresholds, which, on the basis of the distributive properties of the reference model, make it possible to identify the alerts and the algorithm that, by interpreting the results of the three tripwires, gives the static concentration alert.

2.2.4 The analysis of dynamic concentration

Consob's supervisory experience and the financial literature showed that analysis of the composition of the market must also take into account the evolution of the concentration of the intermediaries, that is the behaviour of each intermediary's share of the volume of trading in a given security (so-called dynamic concentration).

This additional analysis of concentration can in fact reveal changes in the role played by a particular intermediary-trader in determining the total daily turnover in a security.

The first step in constructing a tripwire on the basis of this aggregate is to identify the related financial variable. The examination of the various cases of market abuse found by Consob suggested using an index of dissimilarity, that is:

$$\Psi_t = \sqrt{\frac{1}{\tilde{n}_t} \sum_{i=1}^{\tilde{n}_t} \tilde{Q}_t(i)^2} \quad (13)$$

where

$$\tilde{Q}_t(i) = Q_t(i) - Q_{t-k}(i)$$

$Q_t(i)$, $i = 1, \dots, n_t$ are the quantities traded by the i -th intermediary at time t

\tilde{n}_t is the number of traders with a value of $\tilde{Q}_t(i)$ other than zero

The reference model that gives the alert is based on a stochastic differential equation, that is:

$$d\Psi_t = -\omega\Psi_t dt + \sigma dW_t \quad (14)$$

which, as in the analysis of static concentration, was obtained through the shift to continuous time of the corresponding discrete process (see Section 2.2.3), i.e.:

$$\Psi_k - \Psi_{k-1} = -\nu\Psi_{k-1} + \hat{\sigma}Z_k \quad (15)$$

This model has the following distributive properties with reference to any constant initial condition identified at time s , with $s < t$, equal to Ψ_s :

$$\Psi_t \sim N \left(\Psi_s e^{-\omega(t-s)}; \sqrt{\frac{\sigma^2}{2\omega} (1 - e^{-2\omega(t-s)})} \right) \quad (16)$$

We examined the variable Ψ_t with reference to the quantities bought and sold and the total quantity traded by each intermediary, in the sense of the difference between purchases and sales (net turnover). Subsequently, three tripwires with the same mathematical formulation were constructed with reference to each of the above aggregates.

The decision to consider net turnover together with purchases and sales for this alert was due to the definition of this aggregate considered in relation to the characteristics of the variable Ψ_t . In fact, since the net turnover at a given time t is the synthesis of the operational behaviour of an intermediary-trader on the market, examining it through the variable Ψ_t - which by construction compares the quantitative trading data of the different intermediaries-traders with the corresponding figures for an earlier period - makes it possible to detect the changes in the operational behaviour of the different intermediaries-traders.

In Section 2.3.4 we give a detailed description of the calibration of the tripwires and of the construction of the dynamic thresholds, which, on the basis of the distributive properties of the reference model, makes it possible to identify the alerts and the algorithm that, by interpreting the results of the three tripwires, gives the dynamic concentration alert.

2.3 The calibration procedure

The calibration of a M.A.D. procedure consists in:

- deriving the prediction confidence intervals serving to identify the alerts of the various financial variables on the basis of the properties of the tripwires reference model;
- determining the time horizon needed for the specification of the parameters used in the prediction confidence intervals;
- defining the forecasting horizon of the tripwires;
- specifying the algorithm that, by jointly interpreting the various alerts, signals possible market abuse phenomena to be examined by the enforcement units (so-called warnings);
- defining the temporal validity of a warning generated by the M.A.D. procedure, that is the number of days the enforcement units must monitor the security following the warning.

Supervisory experience and the analysis of the cases of insider trading and market manipulation found by Consob permitted, through the *ex post* application of the M.A.D. procedure, the empirical verification for its calibration. In particular, this analysis showed that:

- the procedure must operate on a rolling basis with the daily updating of the estimation parameters of the tripwires; in other words, for the securities subject to monitoring, it must identify, on a daily basis, the

economic and financial phenomena potentially attributable to a case of market abuse;

- the calibration must be carried out, using a relatively short series of daily data (equal to or less than one month of trading); the resulting instability of the model makes it possible to capture the changes in the investment opportunities of the various agents operating on the market;
- the predictive capability must be equal to one trading day; the continuous updating of the predictions allows the model to identify promptly the cases that deserve to be analyzed in depth by the enforcement units.

These choices give rise to a large noise component, which was carefully evaluated in the construction of the tripwires.

The literature on financial markets and supervisory experience, corroborated by the above-mentioned empirical verification, have provided indications on how to construct the algorithm for the joint interpretation of the alerts produced by the various tripwires. In particular, it was found to be desirable to examine jointly the data on trading volumes, returns and market concentration.

2.3.1 The calibration of the volume-based tripwire⁶

Given the diffusion process of (3):

$$dQ_t = -\theta Q_t dt + \sigma dW_t \quad (3)$$

and the related distributive properties expressed with reference to an initial condition representing a daily time horizon, i.e.:

$$Q_t \sim N \left[Q_{t-1} e^{-\theta}; \sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta})} \right] \quad (17)$$

a prediction confidence interval can be constructed that, on the basis of the data on the trading volumes of the preceding days, infers the possible values for the following day.

In fact, since for any standard normal random variable Z :

$$P(-z_{\frac{\varkappa}{2}} \leq Z \leq z_{\frac{\varkappa}{2}}) = \varkappa$$

this last equation can be rewritten with reference to the diffusive process Q_t , i.e.:

$$P(-z_{\frac{\varkappa}{2}} \leq \frac{Q_t - Q_{t-1} e^{-\theta}}{\sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta})}} \leq z_{\frac{\varkappa}{2}}) = \varkappa$$

So that the desired prediction confidence interval can be defined as follows:

$$P \left(\begin{array}{c} -z_{\frac{\varkappa}{2}} \sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta})} + Q_{t-1} e^{-\theta} \leq \\ \leq Q_t \leq \\ \leq z_{\frac{\varkappa}{2}} \sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta})} + Q_{t-1} e^{-\theta} \end{array} \right) = \varkappa \quad (18)$$

⁶The intermediate steps of the mathematical proofs described in this section are available from the author upon request.

The extremes of the interval are the dynamic thresholds that define the alerts for the behaviour of trading volumes. Every time the observed behaviour of trading volumes falls outside the interval, there is an alert.

Having formally defined the fluctuation band, in order to calibrate the trading volume tripwire it is necessary to estimate the parameters of the stochastic differential equation (3) using the data observed in discrete time.

One effective way to do this, which makes it possible to arrive at an explicit formulation of the parameters and thus to avoid the use of numerical procedures, is to rewrite the discrete-time version of the diffusion process (3), i.e.:

$$Q_k - Q_{k-1} = -\gamma Q_{k-1} + \hat{\sigma} Z_k \quad (2)$$

so that it has the same characteristics as (3) in terms of conditional mean and variance.⁷

With reference to the conditional mean, the following must hold:

$$E(Q_k|Q_{k-1}) = Q_{k-1} - \gamma Q_{k-1} = Q_{k-1} e^{-\theta}$$

Hence we can rewrite the parameter γ of (2) in terms of the parameter θ of (3), i.e.:

$$\gamma = 1 - e^{-\theta} \quad (19)$$

For the standard deviation of (2) to be equal to that of (3), it is sufficient to impose the following equality:

$$\hat{\sigma} = \sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta})} \quad (20)$$

We can therefore rewrite (2), using (19) and (20), as:

$$Q_k - Q_{k-1} = (e^{-\theta} - 1) Q_{k-1} + \sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta})} Z_k \quad (21)$$

(21) becomes the discrete process that, on the basis of the daily observations of trading volumes, makes it possible to estimate the parameters of (3).

In fact a regression analysis based on the following model:

$$Q_k - Q_{k-1} = \hat{b} Q_{k-1} + e_k \quad (22)$$

where \hat{b} is the regression coefficient and e_k and $\hat{\sigma}$ are respectively the related error of fit and the mean squared error, i.e.:

$$\hat{\sigma} = \sqrt{\sum_k \frac{e_k^2}{n-2}} \quad (23)$$

⁷For more details on this aspect, see the bibliography [Dixit and Pindyck, 1994].

makes it possible to determine the parameter θ of (3) by resolving the following equation, which compares (21) with (22) :

$$\widehat{b} = -(1 - e^{-\theta}) \Rightarrow$$

Some simple mathematical manipulations give the value of the desired parameter:

$$\theta = \ln(\widehat{b} + 1)^{-1} \quad (24)$$

For the estimation of σ , using (20) and (24) and simplifying, we obtain the following equation:

$$\widehat{\sigma} = \sigma \sqrt{\frac{\ln(\widehat{b} + 1)^2}{\widehat{b}^2 + 2\widehat{b}}} \quad (25)$$

As mentioned earlier, the empirical verification of the tripwire was carried out by examining the various cases of market abuse found by Consob. This made it possible to identify the time horizon for estimating the regression of the data in discrete time, which we found to be 15 trading days. The value of the standardized normal random variable z , which defines the prediction confidence interval, is equal to 2.33 and therefore includes 99% of the possible forecasting scenarios of the financial variable.

2.3.2 The calibration of the return-based tripwire⁸

Given the diffusion process of (7):

$$dR_t = q(\mu - R_t)dt + \sigma dW_t \quad q, \sigma > 0 \quad (7)$$

and the related distributive properties expressed with reference to an initial condition representing a daily time horizon, i.e.:

$$R_t \sim N \left((R_{t-1} - \mu)e^{-q} + \mu; \sqrt{\frac{\sigma^2}{2q}(1 - e^{-2q})} \right) \quad (26)$$

we can construct, in the same way as in Section 2.3.1, a prediction confidence interval that, on the basis of the data on the returns of the preceding days, infers the possible values for the following day, i.e.:

$$P \left(\begin{array}{l} -z_{\frac{\varkappa}{2}} \sqrt{\frac{\sigma^2}{2q}(1 - e^{-2q})} + (R_{t-1} - \mu)e^{-q} + \mu \leq \\ \leq R_t \leq \\ \leq z_{\frac{\varkappa}{2}} \sqrt{\frac{\sigma^2}{2q}(1 - e^{-2q})} + (R_{t-1} - \mu)e^{-q} + \mu \end{array} \right) = \varkappa \quad (27)$$

⁸The intermediate steps of the mathematical proofs described in this section are available from the author upon request.

The extremes of the interval are the dynamic thresholds that define the alerts for the behaviour of returns. Every time the observed behaviour of returns falls outside the interval, there is an alert.

The estimation of the parameters follows similar steps to those described in Section 2.3.1, which makes it possible to arrive at a closed-form expression of the parameters and thus to avoid the use of numerical procedures.⁹

Thus, starting from (6), we obtain:

$$R_k - R_{k-1} = \gamma \cdot (\eta - R_{k-1}) + \hat{\sigma} Z_k \quad (6)$$

a discrete process that has the distributive characteristics of the stochastic differential equation (7) indicated in (26).

>From the equality of the conditional expected values we obtain the following relationships between the parameters of (6) and (7), i.e.:

$$\eta = \mu \quad (28)$$

$$\gamma = 1 - e^{-q} \quad (29)$$

>From the equality of the conditional variances we obtain:

$$\hat{\sigma} = \sqrt{\frac{\sigma^2}{2q} (1 - e^{-2q})} \quad (30)$$

We can therefore rewrite (6), using (28), (29) and (30), as:

$$R_k - R_{k-1} = (1 - e^{-q}) \cdot \mu + (e^{-q} - 1)R_{k-1} + \sqrt{\frac{\sigma^2}{2q} (1 - e^{-2q})} Z_k \quad (31)$$

(31) becomes the discrete process that, on the basis of the daily observations of returns, makes it possible to estimate the parameters of (7).

As in Section (2.3.1) a regression analysis on the basis of the following model:

$$R_k - R_{k-1} = \hat{a} + \hat{b}R_{k-1} + e_k \quad (32)$$

where \hat{a} and \hat{b} are the regression coefficients and e_k and $\hat{\sigma}$ are respectively the related error of fit and the mean squared error, i.e.:

$$\hat{\sigma} = \sqrt{\sum_k \frac{e_k^2}{n-2}} \quad (33)$$

makes it possible to determine the parameters q and μ of (7) by resolving the following set of equations, obtained by comparing (31) with (32):

$$\begin{cases} \hat{a} = (1 - e^{-q}) \cdot \mu \\ \hat{b} = (e^{-q} - 1) \end{cases} \Rightarrow$$

⁹For more details on this aspect, see the bibliography [Dixit and Pindyck, 1994].

Some simple mathematical manipulations give the value of the desired parameter:

$$\mu = -\frac{\widehat{a}}{\widehat{b}} \quad (34)$$

$$q = \ln(\widehat{b} + 1)^{-1} \quad (35)$$

For the estimation of σ , using (33) and simplifying, we obtain the following equation:

$$\sigma = \widehat{\sigma} \sqrt{\frac{\ln(\widehat{b} + 1)^2}{\widehat{b}^2 + 2\widehat{b}}} \quad (36)$$

As mentioned earlier, the empirical verification of the tripwire was carried out by examining the various cases of market abuse found by Consob. This made it possible to identify the time horizon for estimating the regression of the data in discrete time, which was found to be 15 trading days. The value of the standardized normal random variable z , which defines the prediction confidence interval, is equal to 2.33 and therefore includes 99% of the possible forecasting scenarios of the financial variable.

2.3.3 The calibration of the static market-concentration tripwire

The static concentration tripwire uses an entropy index as the financial variable to represent the level of concentration of the intermediaries-traders present on the market, i.e.

$$\Theta_t = \frac{1}{n_t} \sum_{i=1}^{n_t} \left(\frac{Q_t(i)}{\mu_t} \right)^\alpha \quad (9)$$

and a stochastic differential equation as the reference model, i.e.:

$$d\Theta_t = -\zeta\Theta_t dt + \sigma dW_t \quad (10)$$

the continuous-time version of the discrete-time process given below:

$$\Theta_k - \Theta_{k-1} = -\xi\Theta_{k-1} + \widehat{\sigma}Z_k \quad (11)$$

In calibrating this variable on a daily basis with a view to determining the dynamic thresholds that define the alerts, it should be noted that:

- by construction, as α increases, the expression (9) becomes more sensitive to the intermediaries that handle a large percentage of the trading volumes;
- intermediaries-traders do not necessarily carry out transactions involving a given security every day (so-called discontinuous trading). This implies that the use of daily data on the quantities traded by the i -th intermediary-trader to construct the index and hence the financial variable that defines

the tripwire risks introducing a large noise component that would make it difficult to interpret the values of the variable;

- the reference model is defined by the stochastic differential equation (10), the continuous-time version of the discrete-time autoregressive process (11).

Consequently, the same logical and computational approach was used as for the volume-based tripwire. In fact:

- the prediction confidence interval, i.e.:

$$P \left(\begin{array}{c} -z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta})} + \Theta_{t-1} e^{-\zeta} \leq \\ \leq \Theta_t \leq \\ \leq z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta})} + \Theta_{t-1} e^{-\zeta} \end{array} \right) = \alpha$$

was determined by exploiting the distributive properties of (10), expressed with reference to an initial condition representing a daily time horizon, i.e.:

$$\Theta_t \sim N \left[\Theta_{t-1} e^{-\zeta}; \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta})} \right] \quad (12)$$

- the estimation of the parameters of the stochastic differential equation (10), i.e.:

$$\begin{aligned} \zeta &= \ln(\widehat{b} + 1)^{-1} \\ \sigma &= \widehat{\sigma} \sqrt{\frac{\ln(\widehat{b} + 1)^2}{\widehat{b}^2 + 2\widehat{b}}} \end{aligned}$$

involved:

- a regression analysis applied to the following discrete process, redefined starting from (11) so as to have the same distributive characteristics as (10), i.e.:

$$\Theta_k - \Theta_{k-1} = (e^{-\zeta} - 1) \Theta_{k-1} + \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta})} Z_k$$

- some algebraic manipulations (see Section 2.3.1).

The analysis of the various cases of market abuse found by Consob, filtered through the characteristics of this tripwire, provided an effective empirical verification for the latter's calibration; in particular, it suggested:

- analyzing the quantities traded by each intermediary over a time horizon of 5 trading days, which overcomes the problem of the discontinuous trading of some intermediaries-traders;
- a value of 5 for the parameter α , which allows high concentrations to be highlighted;
- a reference window of 15 trading days for estimating the regression on the discrete data ;
- that the value of the standardized normal random variable z , which defines the prediction confidence interval, was equal to 2.33 and therefore included 99% of the possible forecasting scenarios of the financial variable.

The indicator was therefore specified as follows:

$$\Theta_t = \frac{1}{n_t} \sum_{i=1}^{n_t} \left(\frac{\hat{Q}_t(i)}{\mu_t} \right)^5 \quad (37)$$

where

$\hat{Q}_t(i) = \sum_{i=1}^5 Q_{t-5}(i)$ is the total quantity traded in the last 5 days by the i -th trader.

$$\mu_t = \frac{\sum_{i=1}^{n_t} \hat{Q}_t(i)}{n_t}$$

As indicated in Section 2.2.3, three tripwires are calculated for the static concentration (with reference to respectively the gross turnover, purchases and sales of the intermediaries-traders), since this is considered to represent the evolution of the static concentration, including with reference to the directions in which the market moves. The alert of this financial variable is triggered when at least one of the tripwires exceeds the corresponding threshold given by the prediction confidence interval.

2.3.4 The calibration of the dynamic market-concentration tripwire

The dynamic concentration tripwire uses an index of dissimilarity as the financial variable to represent the evolution of the trading of the intermediaries-traders present on the market, i.e.

$$\Psi_t = \sqrt{\frac{1}{\tilde{n}_t} \sum_{i=1}^{\tilde{n}_t} \tilde{Q}_t(i)^2} \quad (13)$$

and a stochastic differential equation as the reference model, i.e.:

$$d\Psi_t = -\omega\Psi_t dt + \sigma dW_t \quad (14)$$

the continuous-time version of the discrete-time autoregressive process given below:

$$\Psi_k - \Psi_{k-1} = -\nu\Psi_{k-1} + \widehat{\sigma}Z_k \quad (15)$$

In calibrating this variable on a daily basis with a view to determining the dynamic thresholds that define the alerts, given the characteristics of the construction of the tripwire, that is to say that:

- the variable $\widetilde{Q}_t(i)$ compares a quantity at time t with the corresponding value at an earlier time, i.e. $Q_t(i) - Q_{t-k}(i)$,
- the reference model is defined by the stochastic differential equation (14), the continuous version of the discrete autoregressive process (15),

it is necessary to define the lag of $\widetilde{Q}_t(i)$ and the reference window of the estimation of the regression on the discrete data needed for the parametric specification of (14).

The same logical and computational approach was therefore used as for the static market-concentration tripwire. In fact:

- the prediction confidence interval, i.e.:

$$P \left(\begin{array}{c} -z_{\frac{\varkappa}{2}} \sqrt{\frac{\sigma^2}{2\omega} (1 - e^{-2\omega})} + \Psi_{t-1}e^{-\omega} \leq \\ \leq \Psi_t \leq \\ \leq z_{\frac{\varkappa}{2}} \sqrt{\frac{\sigma^2}{2\omega} (1 - e^{-2\omega})} + \Psi_{t-1}e^{-\omega} \end{array} \right) = \varkappa$$

was determined by exploiting the distributive properties of (13) expressed with reference to an initial condition representing a daily time horizon, i.e.:

$$\Psi_t \sim N \left[\Psi_{t-1}e^{-\omega}; \sqrt{\frac{\sigma^2}{2\omega}(1 - e^{-2\omega})} \right] \quad (16)$$

- the estimation of the parameters of the stochastic differential equation (13), i.e.:

$$\begin{aligned} \omega &= \ln(\widehat{b} + 1)^{-1} \\ \sigma &= \widehat{\sigma} \sqrt{\frac{\ln(\widehat{b} + 1)^2}{\widehat{b}^2 + 2\widehat{b}}} \end{aligned}$$

involved:

- a regression analysis applied to the following discrete process, redefined starting from (15) so as to have the same distributive characteristics as (13), i.e.:

$$\Psi_k - \Psi_{k-1} = (e^{-\omega} - 1) \Psi_{k-1} + \sqrt{\frac{\sigma^2}{2\omega}(1 - e^{-2\omega})} Z_k$$

- some algebraic manipulations (see Section 2.3.3).

The analysis of the various cases of market abuse found by Consob, filtered through the characteristics of this tripwire, provided an effective empirical verification for the latter’s calibration; in particular, it suggested:

- constructing the variable $\tilde{Q}_t(i)$ by comparing the quantities traded by each intermediary at time t with the corresponding value observed at time $t - 5$ (in other words the lag is equal to the trading week);
- a reference window of 15 trading days for estimating the regression on the discrete data.

The indicator was therefore specified as follows:

$$\Psi_t = \sqrt{\frac{1}{\tilde{n}_t} \sum_{i=1}^{\tilde{n}_t} [Q_t(i) - Q_{t-5}(i)]^2} \quad (38)$$

As explained in Section 2.2.4, three tripwires are calculated for the dynamic concentration (with reference to respectively the net turnover, purchases and sales of the intermediaries-traders), since this is considered to represent the evolution of the dynamic concentration, including with reference to the directions in which the market moves. The alert of this financial variable is triggered when at least one of the tripwires exceeds the corresponding threshold given by the prediction confidence interval.

2.3.5 The algorithm for reading the alerts

In order to complete the calibration of the M.A.D. procedure, it is necessary:

- to specify the algorithm that, by jointly interpreting the various alerts, signals possible phenomena of market abuse to be examined by the enforcement units (so-called warnings);
- to define the temporal validity of a warning generated by the M.A.D. procedure; known as the critical period, this is the number of days the enforcement units must monitor the security following the warning.

To this end, we list the working mechanism of a generic tripwire and the dynamic thresholds that, for each of the tripwires constructed, define anomalous behaviour on the part of the various financial variables considered.

The tripwire identifies the dynamic thresholds and thus the anomalies in the behaviour of the financial variable analyzed, on a daily basis, with reference to a rolling set of observations. In particular, the dynamic thresholds that define the prediction confidence interval for the financial variable at time t are calculated with reference to the interval $[t - k, t]$ where $k = 15$. The system for identifying the alerts is thus of the rolling type with a reference window of 15 trading days. The crossing of the dynamic thresholds corresponding to the extremes of the prediction confidence interval defined by the generic tripwire signals an anomaly in the behaviour of the financial variable in question; the extremes of the prediction confidence intervals that define the alerts for each financial variable are listed below:

1. $Q_t \notin (Q_{\text{inf}}; Q_{\text{sup}})$

$$Q_{\text{inf}} = -z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta})} + Q_{t-1} e^{-\theta}$$

$$Q_{\text{sup}} = +z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\theta} (1 - e^{-2\theta})} + Q_{t-1} e^{-\theta}$$
2. $R_t \notin (R_{\text{inf}}; R_{\text{sup}})$

$$R_{\text{inf}} = \mu - z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2q} (1 - e^{-2q})} + (R_{t-1} - \mu) e^{-q}$$

$$R_{\text{sup}} = \mu + z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2q} (1 - e^{-2q})} + (R_{t-1} - \mu) e^{-q}$$
3. $\Theta_t \notin (\Theta_{\text{inf}}; \Theta_{\text{sup}})$ ¹⁰

$$\Theta_{\text{inf}} = -z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta})} + \Theta_{t-1} e^{-\zeta}$$

$$\Theta_{\text{sup}} = +z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\zeta} (1 - e^{-2\zeta})} + \Theta_{t-1} e^{-\zeta}$$
4. $\Psi_t \notin (\Psi_{\text{inf}}; \Psi_{\text{sup}})$ ¹¹

$$\Psi_{\text{inf}} = -z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\omega} (1 - e^{-2\omega})} + \Psi_{t-1} e^{-\omega}$$

$$\Psi_{\text{sup}} = +z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma^2}{2\omega} (1 - e^{-2\omega})} + \Psi_{t-1} e^{-\omega}$$

The analysis of the literature and supervisory experience showed that the data on trading volumes, the behaviour of returns and the evolution of market concentration needed to be analyzed jointly. In this sense the examination of

¹⁰This expression refers to the generic tripwire of the static concentration. It should be remembered that three tripwires are calculated for this financial variable with reference to respectively the gross turnover, purchases and sales of the intermediaries-traders. The alert is triggered when at least one of the tripwires crosses one of the thresholds indicated by the predictive confidence interval.

¹¹This expression refers to the generic tripwire of the dynamic concentration. It should be remembered that three tripwires are calculated for this financial variable with reference to respectively the net turnover, purchases and sales of the intermediaries-traders. The alert is triggered when at least one of the tripwires crosses one of the thresholds indicated by the predictive confidence interval.

the various cases of market abuse found by Consob, filtered by means of the *ex post* application of the procedure, has shown that an effective algorithm for the joint reading of the various alerts produced by the tripwires aimed at identifying a warning for a security is to consider a trading day to be anomalous if at least three of the four tripwires have signaled an alert.

The decision to require at least three tripwires to signal an alert to have a warning for a security was therefore taken with the help of the empirical verification carried out on the basis of the cases of market abuse analyzed by Consob, but also with a view to selecting the most important phenomena.

The critical period (i.e. the period of validity of the warning) was taken to be the five subsequent trading days. In the event of more than three alerts on successive days, the critical period starts from the last day on which there was a warning.

The specification of the algorithm that produces warnings for the joint interpretation of the alerts produced by the various financial variables completes the description of the M.A.D. procedure developed in this paper.

3 Conclusions

The definition of a Market Abuse Detection procedure is a need strongly felt by supervisory authorities, for which it is an effective instrument for preventing and repressing criminal behaviour in the forms of market manipulation and insider trading.

This paper presents a M.A.D. procedure that detects, for each security and on a daily basis, the presence of market abuse phenomena by means of a set of tripwires that analyzes the flows of elementary information on trading in securities on financial markets available to supervisory authorities. These flows have been transformed, on the basis of the literature, supervisory experience and the cases of market abuse found by Consob into four diffusion processes that represent the behaviour of:

1. the trading volumes of the security;
2. the returns on the security;
3. the static concentration of the market;
4. the dynamic concentration of the market.

In a M.A.D. procedure a tripwire produces an alert every time the observed behaviour of a financial variable is not consistent with the predictive hypotheses of the underlying reference model, hypotheses identified by means of dynamic thresholds. The tripwires were constructed with a view to ensuring the procedure will permit the detection in real time of securities possibly subject to market abuse.

The reference models that define the tripwires were specified by calibrating a number of stochastic differential equations.

After identifying the individual tripwires, the algorithm that permitted the joint interpretation of the various alerts was calibrated to generate warnings about securities, i.e. to signal securities possibly subject to market abuse for investigation by the competent enforcement authorities.

The M.A.D. procedure is a first step in a quantity-based approach to the supervision of financial markets. The procedure developed can undoubtedly be improved by introducing new tripwires on other financial variables, such as volatility, and examining data on intra-day trading, such as the inter-arrival time. This is a task that is left for future research, in the belief that quantitative analysis is an effective way to enhance the ability of supervisory authorities to carry out the tasks entrusted to them.

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Appendix A

A.1 The theorem of convergence on \mathbb{R}^1

Let (Ω, \mathfrak{S}) and $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ be measurable spaces where $\mathbb{B}(\mathbb{R}^1)$ is the Borel field on \mathbb{R}^1 . A function mapping Ω into \mathbb{R}^1 , $X : \Omega \rightarrow \mathbb{R}^1$, i.e. that assigns $\forall \omega \in \Omega$ a value $X(\omega)$ of \mathbb{R}^1 , is called $(\mathfrak{S}, \mathbb{B}(\mathbb{R}^1))$ measurable, and is also defined as a random variable on (Ω, \mathfrak{S}) if the pre-images¹² of the measurable sets on \mathbb{R}^1 are measurable sets on Ω , that is if $\forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$:

$$\{\omega \in \Omega | X(\omega) \in \Gamma\} = X^{-1}(\Gamma) \in \mathfrak{S}$$

Taking a measure P such that $P(\Omega) = 1$, we call the triple $(\Omega, \mathfrak{S}, P)$ the probability space and the function P_X defined on $\mathbb{B}(\mathbb{R}^1)$ such that $\forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$:

$$P_F(\Gamma) = P(X^{-1}(\Gamma)) = P\{\omega \in \Omega | X(\omega) \in \Gamma\}$$

(IS) the probability distribution of X .

For a given X , as defined above, we can always choose a probability space $(\Omega, \mathfrak{S}, P) = (\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1), P_F)$ where $X^{-1}(\Gamma) = \omega$ and P_F is the measure of probability for X .

Given the triple $(\Omega, \mathfrak{S}, P)$, let $\{X_k\}_{k \geq 0}$ be a discrete Markov process (or chain) with respect to the filtration $\{\mathfrak{S}_k\}_{k \geq 0}$, generated by the sequence of random variables X_0, X_1, \dots, X_k for $k \in \mathbb{N}$ where $X : \Omega \rightarrow \mathbb{R}^1$. Accordingly, $\{X_k\}_{k \geq 0}$ takes values on \mathbb{R}^1 where k is an indicator of discrete time. As previously stated with reference to the generic random variable X , the pair $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ defines the measurable space of $\{X_k\}_{k \geq 0}$. Every discrete Markov process defined in this way is identified by the initial distribution $v_0(\cdot)$ and the transition probability $\Pi_{1,k}(\cdot, \cdot)$ ¹³ defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$,¹⁴ i.e.:

- $P(X_0 \in \Gamma) = v_0$;
- $P(X_{k+1} \in \Gamma | \mathfrak{S}_k) = P(X_{k+1} \in \Gamma | X_k) = \Pi_{1,k}(X_k, \Gamma)$.

This implies that $\Pi_{1,k}(\cdot, \cdot)$ is such that:

- $\Pi_{1,k}(x, \cdot)$ is a measure of probability on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ for every $x \in \mathbb{R}^1$;
- $\Pi_{1,k}(\cdot, \Gamma)$ is $\mathbb{B}(\mathbb{R}^1)$ measurable for all $\Gamma \in \mathbb{B}(\mathbb{R}^1)$.

We then rescale the discrete Markov process $\{X_k\}_{k \geq 0}$ by defining for every $h > 0$ a new discrete Markov process $\{X_{kh}\}_{kh \geq 0}$ with respect to the filtration $\{\mathfrak{S}_{kh}\}_{kh \geq 0}$, generated by the sequence of random variables $X_0, X_h, X_{2h}, \dots, X_{kh}$ for $k \in \mathbb{N}$ which assumes values on \mathbb{R}^1 , where kh is the new indicator of discrete time. In other words, we divide the k time intervals into $\frac{1}{h}$ subintervals with length h .

¹²By pre-images is meant the inverse of the function: $X^{-1}(\Gamma)$.

¹³The subscripts of Π means that the movement is from k with a time interval equal to 1.

¹⁴ $\Pi_{1,k}(\cdot, \cdot)$ is such that:

1. $\Pi_{1,k}(x, \cdot)$ is a measure of probability on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ for every $x \in \mathbb{R}^1$;
2. $\Pi_{1,k}(\cdot, \Gamma)$ is $\mathbb{B}(\mathbb{R}^1)$ measurable for all the $\Gamma \in \mathbb{B}(\mathbb{R}^1)$.

It follows evidently that for $h \downarrow 0$ the k time intervals are subdivided into an infinite number of subintervals of the same size.

This process is also identified by an initial probability $v_0(\cdot)$ and a probability of transition $\Pi_{h,kh}(\cdot, \cdot)$,¹⁵ both defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$.¹⁶ It follows that:

- $P(X_0 \in \Gamma) = v_0 \quad \forall \Gamma \in \mathbb{B}(\mathbb{R}^1);$
- $P(X_{(k+1)h} \in \Gamma | \mathfrak{S}_{kh}) = P(X_{(k+1)h} \in \Gamma | X_{kh}) = \Pi_{h,kh}(X_{kh}, \Gamma).$

Let $D([0, \infty), \mathbb{R}^1) \stackrel{def}{=} \{f : [0; \infty) \rightarrow \mathbb{R}^1 : \forall t, f(t^+) = f(t) \text{ and } f(t^-) \text{ exists}\}$ be the Skorokhod space, i.e. the space that goes from $[0; \infty)$ to \mathbb{R}^1 with continuous paths to the right and finite paths to the left.

Let $\{X_t^h\}$ be a continuous process generated on the basis of $\{X_{kh}\}_{kh \geq 0}$ for $kh \leq t < (k+1)h$ where t is the indicator of continuous time. By construction $\{X_t^h\}$ assumes values on D and the pair $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ defines the measurable space of $\{X_t^h\}$.¹⁷ In particular, it is evident that $\{X_t^h\}$ is a jump chain defined by the moment of the jump time, which occurs at the times $J_{kh} = kh \quad \forall k \geq 0$, i.e.:

$$J_0 = 0, \quad J_{(k+1)h} = \inf(t \geq J_{kh} : X_t^h \neq X_{kh}) \quad \forall k \geq 0^{18}$$

or:

$$J_0 = 0, \quad (k+1)h = \inf(t \geq kh : X_t^h \neq X_{kh}) \quad \forall k \geq 0$$

or:

$$J_0 = 0, \quad (k+1)h = \sup(k \leq \frac{t}{h} : X_t^h \neq X_{kh}) \quad \forall k \geq 0^{19}$$

or, defining $[\frac{t}{h}] \stackrel{def}{=} \sup(k \leq \frac{t}{h} : X_t^h \neq X_{kh}) :$

$$J_0 = 0, \quad J_{(k+1)h} = [\frac{t}{h}] \quad \forall k \geq 0$$

and by the holding time with a length of $(k+1)h - kh$ for $k \geq 0$ in which $\{X_t^h\} = \{X_{kh}\}$ for $kh \leq t < (k+1)h$.

By construction $\{X_t^h\}$ for $kh \leq t < (k+1)h$ is a continuous Markov function (or chain) with respect to the filtration $\{\mathfrak{S}_t^h\}_{t \geq 0}$, with an initial distribution P_h^0 and probability of transition P_h defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$.

It follows that:

- $P(X_0^h = y) = P_h^0 \quad \forall y \in \mathbb{R}^1;$
- $P(X_t^h = y | \mathfrak{S}_s^h) = P(X_t^h = y | X_s^h = x) = P_h(x, y) \quad \forall x, y \in \mathbb{B}(\mathbb{R}^1) \text{ and } t = s + h$

Moreover, by construction, the following relationships exist between $\{X_t^h\}$ and $\{X_{kh}\} :$

¹⁵The subscripts of Π means that the movement is from kh with a time interval equal to h .

¹⁶ $\Pi_{h,kh}(\cdot, \cdot)$ is such that:

1. $\Pi_{h,kh}(x, \cdot)$ is a measure of probability on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ for every $x \in \mathbb{R}^1;$
2. $\Pi_{h,kh}(\cdot, \Gamma)$ is $\mathbb{B}(\mathbb{R}^1)$ measurable for all the $\Gamma \in \mathbb{B}(\mathbb{R}^1)$.

¹⁷It should be remembered that this measurable space hypothesis is a sufficient condition for the existence of a measure of probability.

¹⁸i.e. the smallest value of t for $kh \leq t < (k+1)h$ such that $X_t^h \neq X_{kh}^h$

¹⁹i.e. the largest value of k for $1 < k \leq \frac{t}{h}$ such that $X_t^h \neq X_{kh}^h$

- $P_h(X_0^h \in \Gamma) = v_0 \quad \forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$
- $P_h(X_t^h = X_{kh}, \quad \forall kh \leq t < (k+1)h) = 1$
- $P_h(X_{(k+1)h} \in \Gamma | \mathfrak{S}_{kh}) = P_h(X_{(k+1)h} \in \Gamma | X_{kh}) = P(X_{(k+1)h} \in \Gamma | X_{kh}) = \Pi_{h,kh}(X_{kh}, \Gamma)$ almost certainly under $P_h, \forall k \geq 0, \forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$.
- $P_h(X_t^h = X_{(k+1)h} | X_s^h = X_{kh}) = P_h(X_t^h = X_{(k+1)h} | X_s^h = X_{[\frac{t}{h}]h}) = \Pi_{h, [\frac{t}{h}]h}(X_{[\frac{t}{h}]h}, X_{(k+1)h})$ where $t = s + h^{20}$

Lastly, we define $\{X_t\}$ as the diffusion process characterized by the following stochastic differential equation:

$$dX_t = b(x, t)dt + \sigma(x, t)dW_t$$

In what follows we identify the conditions under which it is true that for $h \downarrow 0$:

$$\{X_t^h\} \xrightarrow{d} \{X_t\}$$

where the symbol \xrightarrow{d} indicates weak convergence.

The conditional first moment of the discrete process $\{X_k\}_{k \geq 0}$ is defined as:

$$E(X_{k+1} \in \Gamma | X_k = x) = \sum_k (X_{k+1} - X_k) \Pi_{1,k}(x, \Gamma)$$

It follows that the conditional first moment of the discrete process $\{X_k\}_{k \geq 0}$ expressed in terms of the process $\{X_{kh}\}_{kh \geq 0}$ is:²¹

$$\begin{aligned} E(X_{k+1} \in \Gamma | X_k = x) &= \frac{1}{h} E(X_{(k+1)h} \in \Gamma | X_{kh} = x) \\ &= \frac{1}{h} \sum_k (X_{(k+1)h} - X_{kh}) \Pi_{h,kh}(x, \Gamma) \end{aligned}$$

and, thus, expressed in terms of the continuous process $\{X_t^h\}$:²²

$$\begin{aligned} E(X_{k+1} \in \Gamma | X_k = x) &= \frac{1}{h} E(X_t^h = y | X_s^h = x) \\ &= \frac{1}{h} \int_{\mathbb{R}} (y - x) P_h(x, dy) \\ &= \frac{1}{h} \int_{\mathbb{R}} (y - x) \Pi_{h, [\frac{t}{h}]h}(x, dy) \end{aligned}$$

>From now we focus on the conditional moments of the discrete process $\{X_k\}_{k \geq 0}$ expressed in terms of $\{X_t^h\}$. In particular, the conditional first moment just defined becomes:

²⁰In less formal terms the first equality, by exploiting the Markov property of the X_t^h , only states that of all the values of the holding time the one taken is that which is closest to the jump. This is so because at the jump time, denoted by $J_{(k+1)h} = [\frac{t}{h}]$, $X_t^h \neq X_{kh}$ and therefore $X_s^h = X_{[\frac{t}{h}]h}$.

²¹The multiplication by $\frac{1}{h}$ is necessary given the construction of the process $\{X_{kh}\}$

²²The last equality is guaranteed by the relationship [4] existing between $\{X_t^h\}$ and $\{X_{kh}\}$.

$$b_h(x, t) = \frac{1}{h} \int_{\mathbb{R}} (y - x) \Pi_{h, [\frac{t}{h}]_h}(x, dy) \quad (39)$$

the conditional second moment is therefore defined as:

$$a_h(x, t) = \frac{1}{h} \int_{\mathbb{R}} (y - x)^2 \Pi_{h, [\frac{t}{h}]_h}(x, dy) \quad (40)$$

The higher moments of the absolute value $\forall \delta > 0$ are defined analogously:²³

$$c_{h, \delta}(x, t) = \frac{1}{h} \int_{\mathbb{R}} |(y - x)|^{2+\delta} \Pi_{h, [\frac{t}{h}]_h}(x, dy) \quad (41)$$

It is not certain that the Lebesgue integrals²⁴ of (39) and (40) are finite; consequently, since the integrating function is always limited on finite intervals, it follows that the related truncated moments, $\forall \varepsilon > 0$:

$$b_h^*(x, t) = \frac{1}{h} \int_{|y-x| \leq \varepsilon} (y - x) \Pi_{h, [\frac{t}{h}]_h}(x, dy) \quad (42)$$

$$a_h^*(x, t) = \frac{1}{h} \int_{|y-x| \leq \varepsilon} (y - x)^2 \Pi_{h, [\frac{t}{h}]_h}(x, dy) \quad (43)$$

will certainly be finite.

In other words, (39) and (40) are integrated on $|y - x| \leq \varepsilon$ instead of \mathbb{R} . It is evident that where:

$$\Delta_{h, \varepsilon}(x, t) = \frac{1}{h} \int_{|y-x| > \varepsilon} \Pi_{h, [\frac{t}{h}]_h}(x, dy) = 0 \quad \forall \varepsilon > 0 \quad (44)$$

i.e. the probability of a shift on \mathbb{R} of the Markov chain $\{X_t^h\}$ greater than ε is null, the truncated and untruncated moments will coincide.

In the light of the above we prove the following theorem:

Theorem 1 *Given the conditions 1-3, or 1a-3, presented below, the sequence $\{X_t^h\}$ converges weakly for $h \downarrow 0$ to the process $\{X_t\}$; it has a unique distribution and is characterized by the following stochastic differential equation:*

$$dX_t = b(x, t)dt + \sigma(x, t)dW_t \quad (45)$$

where W_t is a standard unidimensional Brownian motion. The distribution is independent from the choice of $\sigma(x, t)$ and $\{X_t\}$ remains finite within finite time intervals $\forall t \in [0, T]$, i.e.:

$$P[|X_t| < \infty] = 1$$

²³It is not necessary that $\delta \in \mathbb{N}$. If $\delta \notin \mathbb{N}$, the expression does not represent a higher order absolute conditional moment.

²⁴It should be remembered that the the Lebesgue and Riemann integrals coincide when they exist.

Condition 1 There exists $a(x, t)$, a continuous measure mapping from $\mathbb{R}^1 \times [0, \infty)^{25}$ into the space \mathbb{R}^{+26} and there exists $b(x, t)$, a continuous measure mapping from $\mathbb{R}^1 \times [0, \infty)$ on \mathbb{R}^1 such that $\forall x \in \mathbb{R}^1, \forall T > 0, \forall t \in [0, T]$:

$$\lim_{h \downarrow 0} |b_h^*(x, t) - b(x, t)| = 0 \quad (46)$$

$$\lim_{h \downarrow 0} |a_h^*(x, t) - a(x, t)| = 0 \quad (47)$$

$$\lim_{h \downarrow 0} \Delta_{h, \varepsilon}(x, t) = 0 \quad (48)$$

In less formal terms (46) requires the truncated conditional first moment to converge, for $h \downarrow 0$, to a function $b(x, t)$ that will therefore become the conditional first moment of the process $\{X_t\}$. A similar requirement is imposed by (47). (48) is nothing but the limit of the convergence condition (44) of the truncated moments.

In what follows we show that condition 1 implies condition 1a, which is computationally simpler:

Condition 1a If there exists a $\delta > 0$ such that $\forall x \in \mathbb{R}^1, \forall T > 0, \forall t \in [0, T]$:

$$\lim_{h \downarrow 0} c_{h, \delta}(x, t) = 0 \quad (49)$$

then there exists $a(x, t)$, a continuous measure mapping from $\mathbb{R}^1 \times [0, \infty)$ into the space \mathbb{R}^+ and there exists $b(x, t)$, a continuous measure mapping from $\mathbb{R}^1 \times [0, \infty)$ on \mathbb{R}^1 such that $\forall x \in \mathbb{R}^1, \forall T > 0, \forall t \in [0, T]$:

$$\lim_{h \downarrow 0} |b_h(x, t) - b(x, t)| = 0$$

$$\lim_{h \downarrow 0} |a_h(x, t) - a(x, t)| = 0$$

Proof:

To verify that condition 1 is equivalent to condition 1a, it is sufficient to show that:

1. the convergence to zero of hypothesis (49) causes the convergence to zero of hypothesis (48) of condition 1, i.e.:

$$\lim_{h \downarrow 0} c_{h, \delta}(x, t) = 0 \Rightarrow \lim_{h \downarrow 0} \Delta_{h, \varepsilon}(x, t) = 0$$

2. the conditional first and second moments, $b_h(x, t)$ and $a_h(x, t)$, indicated in (39) and (40) exist and are finite.

²⁵The real-valued interval $[0, \infty)$.

²⁶It should be remembered that the space \mathbb{R}^+ characterizes the second moments of any random variable.

As regards the first point, in the first place we note the Markov inequality, which holds if X takes on non-negative values,²⁷ i.e.:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Since $|y - x|$ is a variable with values that are always positive, the Markov inequality can be specified for $a = \varepsilon$ and with reference to (44). The left-hand term becomes:

$$P(|y - x| \geq \varepsilon) = \frac{1}{h} \int_{|y-x|>\varepsilon} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) = \Delta_{h,\varepsilon}(x, t)$$

while the expected value is:

$$E(|y - x|) = \frac{1}{h} \int_{-\infty}^{\infty} |y - x| \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy)$$

Hence, it follows from Markov's inequality that:

$$\frac{1}{h} \int_{|y-x|>\varepsilon} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) \leq \frac{1}{h} \varepsilon^{-1} \int_{-\infty}^{\infty} |y - x| \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) \quad (50)$$

A similar procedure is followed for the stochastic function $|y - x|^{2+\delta}$, giving:

$$\frac{1}{h} \int_{|y-x|^{2+\delta}>\varepsilon^{2+\delta}} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) \leq \frac{1}{h} \varepsilon^{-(2+\delta)} \int_{-\infty}^{\infty} |y - x|^{2+\delta} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) \quad (51)$$

The right-hand term can be seen to be the conditional n-th moment, $c_{h,\delta}(x, t)$, referred to in (41), up to a factor $\varepsilon^{-(2+\delta)}$. Hence, using (51) we obtain:

$$\frac{1}{h} \int_{|y-x|^{2+\delta}>\varepsilon^{2+\delta}} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) \leq \varepsilon^{-(2+\delta)} c_{h,\delta}(x, t) \quad (52)$$

and since the left-hand terms of (50) and (51) are equal, i.e.:

$$\frac{1}{h} \int_{|y-x|^{2+\delta}>\varepsilon^{2+\delta}} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) = \frac{1}{h} \int_{|y-x|>\varepsilon} \Pi_{h, [\frac{\varepsilon}{h}]h}(x, dy) = \Delta_{h,\varepsilon}(x, t) \quad (53)$$

combining (52) and (53) gives the following inequality:

$$\Delta_{h,\varepsilon}(x, t) \leq \varepsilon^{-(2+\delta)} c_{h,\delta}(x, t) \quad (54)$$

²⁷In fact, if X is a positive stochastic variable ($X \geq 0$) and $\phi(x)$ is its probability density, then, for every a , the expected value of X is:

$$E(X) = \int_0^a x\phi(x)dx + \int_a^\infty x\phi(x)dx \geq \int_a^\infty x\phi(x)dx \geq \int_a^\infty a\phi(x)dx = aP(X \geq a)$$

which implies: $P(X \geq a) \leq \frac{E(X)}{a}$

Q.E.D.

Since $\varepsilon^{-(2+\delta)}$ is a positive constant, if, under hypothesis (49) of condition 1a:

$$\lim_{h \downarrow 0} c_{h,\delta}(x, t) = 0 \quad (49)$$

then:

$$\varepsilon^{-(2+\delta)} \lim_{h \downarrow 0} c_{h,\delta}(x, t) = 0$$

and, therefore, using (54):

$$\lim_{h \downarrow 0} c_{h,\delta}(x, t) = 0 \Rightarrow \lim_{h \downarrow 0} \Delta_{h,\varepsilon}(x, t) = 0$$

This is equivalent to saying that the convergence to zero under hypothesis (49) of condition 1a causes hypothesis (48) of condition 1 to converge to zero.

Q.E.D.

As regards the second point, for expository convenience, we introduce the following expression:

$$\frac{1}{h} \int_{\mathbb{R}} (y-x)^k \Pi_{h, [\frac{t}{h}]_h}(x, dy) \quad (55)$$

which, for $k = 1$, is equivalent to $b_h(x, t)$ and, for $k = 2$, is equivalent to $a_h(x, t)$.

It is sufficient to verify for $k = 1, 2$ that:

- a) $\lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x| \in [0,1]} (y-x)^k \Pi_{h, [\frac{t}{h}]_h}(x, dy)$ is finite;
- b) $\lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x| > 1} (y-x)^k \Pi_{h, [\frac{t}{h}]_h}(x, dy) = 0$.

Recalling what was stated earlier with reference to the existence of truncated moments $\forall \varepsilon > 0$, (expressions (42) and (43)), setting $\varepsilon = 1$, the fact that the limit referred to in point a) is finite is immediately verified.

As regards the convergence to zero of the limit referred to in point b), for $k = 1, 2$ and for $(y-x) > 1$ the following inequality is always satisfied:

$$(y-x)^k \Pi_{h, [\frac{t}{h}]_h}(x, dy) \leq |y-x|^{2+\delta} \Pi_{h, [\frac{t}{h}]_h}(x, dy)$$

It follows that:

$$\frac{1}{h} \int_{|y-x| > 1} (y-x)^k \Pi_{h, [\frac{t}{h}]_h}(x, dy) \leq \frac{1}{h} \int_{|y-x| > 1} |y-x|^{2+\delta} \Pi_{h, [\frac{t}{h}]_h}(x, dy)$$

and thus that:

$$\lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|>1} (y-x)^k \Pi_{h, [\frac{t}{h}]_h}(x, dy) \leq \lim_{h \downarrow 0} \frac{1}{h} \int_{|y-x|>1} |y-x|^{2+\delta} \Pi_{h, [\frac{t}{h}]_h}(x, dy)$$

since under hypothesis (49) of condition 1a the right-hand limit is null, the left-hand limit is also certainly null.

Q.E.D.

Condition 2 For $h \downarrow 0$ the initial probability of the process $\{X_t\}$ is identical to that of the process $\{X_t^h\}$

$$\lim_{h \downarrow 0} P_h(X_0^h \in \Gamma) = v_0 \quad i.e. \quad P(X_0 \in \Gamma) = v_0 \quad \forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$$

Condition 2 is evidently satisfied by construction.

Conditions 1 and 2 do not necessarily guarantee that the process $\{X_t\}$ exists. Hence the need for an additional assumption:

Condition 3 v_0 , $a(x, t)$ and $b(x, t)$ uniquely specify the distribution of the process $\{X_t\}$, characterized by an initial distribution v_0 , a conditional second moment $a(x, t)$ and a conditional first moment $b(x, t)$.

The conditions for verifying the latter statement are many and belong to the standard results of stochastic theory. See the bibliography for more details [Ethier and Kurtz, 1986, Stroock and Varadhan, 1979].

Since (45) is a generic Ito unidimensional process with parameters $b(x, t)$ and $\sigma(x, t)$,²⁸ from what has been said above, the theorem is proved.

A.2 An application: the convergence of an AR(1) process

The proof given below can easily be extended to the following processes: (2), (6) (11) and (15).²⁹

In particular, we will show, by applying the theorem referred to in Section A1, that the following AR(1) process

$$X_k - X_{k-1} = \alpha - \gamma X_{k-1} + \widehat{\sigma} Z_k \quad (56)$$

converges weakly to the diffusion process $\{X_t\}$ characterized by the following stochastic differential equation:

$$dX_t = (\alpha - \theta X_t) dt + \sigma dW_t \quad (57)$$

where α and θ are deterministic functions of time and W_t is a standard unidimensional Brownian motion.

²⁸It should be noted that given the characteristics of the process no further assumptions are needed to be able to affirm that a function $\sigma(x, t)$ always exists such that the following equality is always satisfied:

$$a(x, t) = \sqrt{\sigma(x, t)}$$

²⁹The intermediate steps of the mathematical proofs described in this section are available from the author upon request.

A first consideration regarding (56) is that since x_0 is a constant and Z_1, \dots, Z_k a sequence of independent random variables distributed normally with a zero mean and a unit variance, the solution $\{X_k\}_{k \geq 0}$ is a Markov chain with respect to the filtration $\{\mathfrak{S}_k\}_{k \geq 0}$ generated by the sequence Z_1, \dots, Z_k that assumes values on \mathbb{R}^1 and where k is the indicator of discrete time. The pair $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ defines the measurable space of $\{X_k\}_{k \geq 0}$ where $\mathbb{B}(\mathbb{R}^1)$ is the Borel field on \mathbb{R}^1 . Every discrete Markov process defined in this way is identified by the initial distribution $v_0(\cdot)$ and the probability of transition $\Pi_{1,k}(\cdot, \cdot)$ defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ ³⁰.

We rescale the discrete Markov process $\{X_k\}_{k \geq 0}$ by defining for every $h > 0$ a new discrete Markov process $\{X_{kh}\}_{kh \geq 0}$, with respect to the filtration $\{\mathfrak{S}_{kh}\}_{kh \geq 0}$, generated by the sequence of random variables $Z_0, Z_h, Z_{2h}, \dots, Z_{kh}$, which are identically independently distributed as a normal with zero mean and variance equal to h , for $k \in \mathbb{N}$, that assumes values on \mathbb{R}^1 , where kh is the new indicator of discrete time. In other words, the k time intervals are divided into $\frac{1}{h}$ subintervals with a length h , i.e.

$$X_{kh} - X_{(k-1)h} = \alpha_h - \gamma_h X_{(k-1)h} + \hat{\sigma} \sqrt{h} Z_k$$

or

$$X_{kh} - X_{(k-1)h} = \alpha_h - \gamma_h X_{(k-1)h} + \hat{\sigma} Z_{kh}$$

where, by construction, $\alpha_h = \alpha \cdot h$ and γ_h must be chosen to guarantee the following equivalence:³¹

$$\begin{aligned} X_k - X_{k-1} &= \sum_{j=1}^{\frac{1}{h}} X_{(k-1)+jh} - X_{(k-1)+j h-h} \\ \alpha - \gamma X_{k-1} + \hat{\sigma} Z_k &= \sum_{j=1}^{\frac{1}{h}} \alpha \cdot h - \gamma_h X_{(k-1)+j h-h} + \hat{\sigma} Z_{kh} \end{aligned}$$

where $Z_{kh} \sim N(0, \sqrt{h})$

It is evident that for $h \downarrow 0$ the k time intervals are divided into infinite subintervals of the same size.

This process is also defined by an initial probability $v_0(\cdot)$ and the probability of transition is $\Pi_{h,kh}(\cdot, \cdot)$, both defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$.³²

Defining the Skorokhod space (i.e. the space mapping $[0; \infty)$ into \mathbb{R}^1 with continuous paths to the right and finite paths to the left) as $D([0, \infty), \mathbb{R}^1)$,

³⁰ $\forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$:

1. $P(X_0 \in \Gamma) = v_0$;
2. $P(X_{k+1} \in \Gamma | \mathfrak{S}_k) = P(X_{k+1} \in \Gamma | X_k) = \Pi_{1,k}(X_k, \Gamma)$.

³¹ It is well known that the temporal rescaling of a random variable in $\frac{1}{h}$ parts measuring h takes place by means of the following transformation:

$$\sqrt{h}Z - \sqrt{h}E(Z) + E(Z)$$

³² So that:

1. $P(X_0 \in \Gamma) = v_0 \quad \forall \Gamma \in \mathbb{B}(\mathbb{R}^1)$;
2. $P(X_{(k+1)h} \in \Gamma | \mathfrak{S}_{kh}) = P(X_{(k+1)h} \in \Gamma | X_{kh}) = \Pi_{h,kh}(X_{kh}, \Gamma)$.

let $\{X_t^h\}$ be a continuous process generated on the basis of $\{X_{kh}\}_{kh \geq 0}$ for $kh \leq t < (k+1)h$ where t is the indicator of continuous time:³³

$$X_t^h - X_{t-1}^h = \alpha_h - \gamma_h X_{t-1}^h + \widehat{\sigma} Z_t^h \quad (58)$$

By construction $\{X_t^h\}$ for $kh \leq t < (k+1)h$ is a continuous Markov process (or chain) with respect to the filtration $\{\mathfrak{S}_t^h\}_{t \geq 0}$, with an initial distribution P_h^0 , equal by construction to v_0 , and a probability of transition P_h defined on $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$.³⁴

Given the construction of process (58), with respect to process (56), to prove the weak convergence of the latter to (57), it is sufficient to verify the convergence of (58) to (57). It can then be shown, by applying the theorem referred to in A1, that (58) converges weakly to (57), i.e.:

$$dX_t = (\alpha - \theta X_t) dt + \sigma dW_t \quad (57)$$

In what follows we verify the conditions required for the application of the theorem.

To verify condition 1a, the first step is to examine the convergence to zero of the conditional moment of the absolute value of a moment higher than the second for $h \downarrow 0$; to simplify the calculations, it is better to consider a conditional even-order moment, so as to avoid the computational problems connected with the absolute value.

We define the conditional fourth moment of the discrete process $\{X_k\}_{k \geq 0}$, expressed in terms of the process $\{X_t^h\}$, as:

$$c_{h,1}(x, t) = \frac{1}{h} \mathbf{E} \left[(X_{t+1}^h - X_t^h)^4 \mid \mathfrak{S}_t^h \right] =$$

Given (58):

$$= \frac{1}{h} \mathbf{E} \left[(\alpha_h - \gamma_h X_t^h + \widehat{\sigma} Z_{t+1}^h)^4 \mid \mathfrak{S}_t^h \right] =$$

by means of some mathematical manipulations and using the properties of the expected value and those of the sequence $Z_0^h, Z_1^h, Z_2^h, \dots, Z_t^h$,³⁵ we obtain:

³³By construction $\{X_t^h\}$ assumes values on D and the pair $(\mathbb{R}^1, \mathbb{B}(\mathbb{R}^1))$ defines the measurable space of $\{X_t^h\}$. In particular, it is evident that $\{X_t^h\}$ is a jump chain defined by the jump time, which occurs at times $J_{kh} = kh \forall k \geq 0$, and by the holding time, which measures $(k+1)h - kh$ for $k \geq 0$ in which $\{X_t^h\} = \{X_{kh}\}$ for $kh \leq t < (k+1)h$.

In view of the above, the deterministic functions α and γ remain unchanged.

³⁴So that:

1. $P(X_0^h = y) = P_h^0 \quad \forall y \in \mathbb{R}^1$;
2. $P(X_t^h = y \mid \mathfrak{S}_s^h) = P(X_t^h = y \mid X_s^h = x) = P_h(x, y) \quad \forall x, y \in \mathbb{B}(\mathbb{R}^1)$ and $t = s + h$

On the relationships between $\{X_t^h\}$ and $\{X_{kh}\}$, see section A1.

³⁵Since the sequence $Z_0^h, Z_1^h, Z_2^h, \dots, Z_t^h$, consists of random variables identically independently distributed as a normal with zero mean and unit variance, the first moment is equal to zero, the odd moments are null, the second moment is equal to h and the fourth to $3h^2$.

$$= 3\hat{\sigma}^4 h + 6\alpha^2 h^2 \hat{\sigma}^2 - 12\alpha h \gamma_h X_t^h \hat{\sigma}^2 + 6\gamma_h^2 (X_t^h)^2 \hat{\sigma}^2 + \alpha^4 h^3 - 4\alpha^3 h^2 \gamma_h X_t^h + 6\gamma_h^2 \alpha^2 (X_t^h)^2 h - 4\alpha \gamma_h^3 (X_t^h)^3 + \frac{\gamma_h^4 (X_t^h)^4}{h}$$

We calculate the limit for $h \downarrow 0$:

$$\lim_{h \downarrow 0} c_{h,1}(x, t) = \lim_{h \downarrow 0} 6\gamma_h^2 (X_t^h)^2 \hat{\sigma}^2 - 4\alpha \gamma_h^3 (X_t^h)^3 + \frac{\gamma_h^4 (X_t^h)^4}{h} \quad (59)$$

To satisfy condition 1a and ensure the existence of $a(x, t)$ and $b(x, t)$, the limit (59) must be equal to 0. This is equivalent to saying that the function γ_h must be defined in a way to ensure that result.

Once the first and second moments have also been calculated, the function γ_h will be defined so as to guarantee, by applying theorem 1, that (58) converges weakly (57).

We calculate the conditional first moment, $b_h(x, t)$:

$$b_h(x, t) = \frac{1}{h} \mathbf{E} [X_{t+1}^h - X_t^h | \mathfrak{S}_t^h] =$$

Given (58):

$$= \frac{1}{h} \mathbf{E} [\alpha_h - \gamma_h X_t^h + \hat{\sigma} Z_{t+1}^h | \mathfrak{S}_t^h] =$$

proceeding in the same way as for the conditional fourth moment, after some mathematical manipulations , we obtain the following result:

$$= \left(\alpha - \frac{\gamma_h}{h} X_t^h \right)$$

We calculate the limit for $h \downarrow 0$:

$$\lim_{h \downarrow 0} b_h(x, t) = \lim_{h \downarrow 0} \alpha - \frac{\gamma_h}{h} X_t^h \quad (60)$$

To satisfy condition 1a, the function γ_h must be chosen so that the limit (60) is equal to $\alpha - \theta X_t$.

We calculate the conditional second moment, $a_h(x, t)$:

$$a_h(x, t) = \frac{1}{h} \mathbf{E} \left[(X_{t+1}^h - X_t^h)^2 | \mathfrak{S}_t^h \right] =$$

Given (58):

$$= \frac{1}{h} \mathbf{E} \left[(\alpha_h - \gamma_h X_t^h + \hat{\sigma} Z_{t+1}^h)^2 | \mathfrak{S}_t^h \right] =$$

proceeding in the same way as for the conditional fourth moment, after some mathematical manipulations, the following result is obtained:

$$= \alpha^2 h + \frac{\gamma_h^2}{h} (X_t^h)^2 - 2\alpha\gamma_h X_t^h + \hat{\sigma}^2$$

We calculate the limit for $h \downarrow 0$:

$$\lim_{h \downarrow 0} a_h(x, t) = \lim_{h \downarrow 0} \frac{\gamma_h^2}{h} (X_t^h)^2 - 2\alpha\gamma_h X_t^h + \hat{\sigma}^2 \quad (61)$$

To satisfy condition 1a, the function γ_h must be chosen so as to guarantee that the limit (61) is equal to σ^2 .³⁶

To satisfy condition 1a, it is necessary to find the function γ_h for which the limits (59), (60) and (61) converge to the values previously indicated and listed below, i.e.:

$$\begin{cases} \lim_{h \downarrow 0} 6\gamma_h^2 (X_t^h)^2 \hat{\sigma}^2 - 4\alpha\gamma_h^3 (X_t^h)^3 + \frac{\gamma_h^4 (X_t^h)^4}{h} \stackrel{?}{=} 0 \\ \lim_{h \downarrow 0} (\alpha - \frac{\gamma_h}{h} X_t^h) \stackrel{?}{=} \alpha - \theta X_t \\ \lim_{h \downarrow 0} \frac{\gamma_h^2}{h} (X_t^h)^2 - 2\alpha\gamma_h X_t^h + \hat{\sigma}^2 \stackrel{?}{=} \sigma^2 \end{cases}$$

Let us adopt the definition $\gamma_h \stackrel{def}{=} \theta \cdot h$ and verify the system by calculating the limits:

$$\begin{cases} \lim_{h \downarrow 0} 6\theta^2 h^2 (X_t^h)^2 \hat{\sigma}^2 - 4\theta^3 h^3 \alpha (X_t^h)^3 + \theta^4 h^3 (X_t^h)^4 = 0 \\ \lim_{h \downarrow 0} (\alpha - \theta X_t^h) = \alpha - \theta X_t \\ \lim_{h \downarrow 0} \theta^2 h (X_t^h)^2 - 2\alpha\theta h X_t^h + \sigma^2 = \sigma^2 \end{cases}$$

It follows that the limits (59), (60) and (61) converge to the quantities required to satisfy condition 1a. In particular, it is found that:

$$b(x, t) = \alpha - \theta X_t \quad (62)$$

$$a(x, t) = \sigma^2 \quad (63)$$

Condition 2 is evidently satisfied by the construction of the process $\{X_t^h\}$, so that condition 3 is also satisfied. Theorem 1 can therefore be applied and, using the results (62) and (63), we can conclude that (58) converges weakly to 57.

Q.E.D.

³⁶It should be noted that the construction of the process X_t^h means that on finding the value of γ_h , $\hat{\sigma} = \sigma$.

A.3 The distributive properties of an Ornstein-Uhlenbeck diffusion process

The proof given below can easily be extended to the following processes: (3), (7), (10) and (14).³⁷

Given the process:

$$dX_t = -qX_t dt + \sigma dW_t \quad q, \sigma > 0 \quad (64)$$

where $dW_t = \varepsilon dt$ and $\varepsilon \sim N(0, 1)$ is white noise.

It will be shown that:

$$X_t \sim N\left(e^{-qt} X_0; \sqrt{\sigma^2 \frac{1 - e^{-2qt}}{2q}}\right)$$

The stochastic differential equation is linear since the coefficients of dt and dW are linear functions of X . It is also autonomous since the coefficients are independent from t . A theorem concerning stochastic differential equations states that such an equation with the characteristics described above has a solution and that this solution is unique, given an initial condition $X_0 = x$ with x independent from dW .

The solution of the stochastic differential equation (64) corresponds to the search for the solution of the following Cauchy problem:

$$\begin{aligned} dX_t &= -qX_t dt + \sigma dW_t & q, \sigma > 0 \\ X_0 &= x \end{aligned} \quad (64)$$

Defining $Y_t = X_t e^{qt}$ and applying ITO's lemma to (64):

$d(X_t e^{qt}) = \frac{d}{dt}(X_t e^{qt})dt + \frac{d}{dX}(X_t e^{qt})dX_t + \frac{1}{2} \frac{d^2}{dX^2}(X_t e^{qt})dX_t^2$
so that:

$$d(X_t e^{qt}) = qX_t e^{qt} dt + e^{qt} dX_t$$

and using (64):

$$d(X_t e^{qt}) = qX_t e^{qt} dt + e^{qt} (-qX_t dt + \sigma dW_t)$$

simplifying and integrating, we obtain:

$$X_t = \int_0^t \sigma e^{-q(t-s)} dW_s + X_0 e^{-qt} \quad (65)$$

(65) is the strong solution of (64).

³⁷The intermediate steps of the mathematical demonstrations described in this section are available upon request.

The aim now is to derive, with reference to (65), the distribution of X_t , and the related conditional expected value and variance: $E(X_t|\mathfrak{F})$ and $Var(X_t|\mathfrak{F})$, by means of the derivation of the moment generating function of X_t , $M_{X_t}(\gamma) = E(e^{\gamma X_t})$.

To this end we adopt the following definition $Q_t = e^{\alpha(t)X_t - \beta(t)}$ and derive the related stochastic differential equation starting from (64) by means of ITO's formula:

$$d(e^{\alpha(t)X_t - \beta(t)}) = \left[\frac{d}{dt} (\alpha(t)X_t - \beta(t)) \right] (e^{\alpha(t)X_t - \beta(t)})dt + \alpha(t)(e^{\alpha(t)X_t - \beta(t)})dX_t + \frac{1}{2}\alpha^2(t)(e^{\alpha(t)X_t - \beta(t)})dX_t^2$$

Putting $\frac{d}{dt}\alpha(t) = \alpha'(t)$, $\frac{d}{dt}\beta(t) = \beta'(t)$, in order to derive $M_{X_t}(\gamma)$, it is necessary to identify the functional forms of $\alpha(t)$ and $\beta(t)$ such that the following expression is null:

$$[\alpha'(t)X_t e^{\alpha(t)X_t - \beta(t)} - q\alpha(t)X_t e^{\alpha(t)X_t - \beta(t)} + \frac{1}{2}\sigma^2\alpha^2(t)e^{\alpha(t)X_t - \beta(t)} - \beta'(t) \cdot e^{\alpha(t)X_t - \beta(t)}]dt = 0$$

The values of these functions are found to be:

$$\alpha(t) = \gamma e^{-q(T-t)} \quad (66)$$

$$\beta(t) = \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-t)}}{2q} \quad (67)$$

So that:

$$d(e^{\alpha(t)X_t - \beta(t)}) = \alpha(t)(e^{\alpha(t)X_t - \beta(t)})\sigma dW_t$$

and, using (66) and (67) and integrating, we obtain:

$$e^{\gamma e^{-q(T-t)}X_t - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-t)}}{2q}} = \int \gamma e^{-q(T-t)} (e^{\gamma e^{-q(T-t)}X_t - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-t)}}{2q}}) \sigma dW_t$$

The expression $e^{\gamma e^{-q(T-t)}X_t - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-t)}}{2q}}$ is a Martingale, hence at the generic time t :

$$E \left[(e^{\gamma e^{-q(T-t)}X_t - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-t)}}{2q}}) | \mathfrak{F}_0 \right] = (e^{\gamma e^{-q(T-0)}X_0 - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-0)}}{2q}})$$

and at time T :

$$E \left[(e^{\gamma e^{-q(T-T)}X_T - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-T)}}{2q}}) | \mathfrak{F}_0 \right] = (e^{\gamma e^{-q(T-0)}X_0 - \frac{1}{2}\sigma^2\gamma^2 \frac{e^{-2q(T-0)}}{2q}})$$

simplifying and exploiting the properties of the exponential function and the expected value and specifying the expression for $T = t$, we obtain:

$$E [e^{\gamma X_t} | \mathfrak{F}_0] = e^{\gamma e^{-qt}X_0 + \frac{1}{2}\sigma^2\gamma^2 \frac{1-e^{-2qt}}{2q}}$$

In order to obtain the moment generating function of X_t , $E(e^{\gamma X_t})$, the expected value operator is applied again and, exploiting the related properties, it follows that:

$$M_{X_t}(\gamma) = E(e^{\gamma X_t}) = e^{\gamma e^{-qt} X_0 + \frac{1}{2} \sigma^2 \gamma^2 \frac{1 - e^{-2qt}}{2q}} \quad (68)$$

(68) can be seen to have the form of the moment generating function of a normal distribution.³⁸ So that:

$$X_t \sim N\left(e^{-qt} X_0; \sqrt{\sigma^2 \frac{1 - e^{-2qt}}{2q}}\right)$$

Q.E.D.

³⁸In fact, given $X_t \sim N(\mu, \sigma^2)$
 $M_{X_t}(\gamma) = E(e^{\gamma X(t)}) = e^{\mu \gamma + \frac{1}{2} \sigma^2 \gamma^2}$

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