(Quantum) Fields on Causal Sets

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Outline

1. Causal Sets: discrete gravity
2. Continuum-Discrete correspondence: sprinklings
3. Relativistic fields on manifolds & their propagators
4. Propagation on causal sets: hop-and-stop
5. Classical Field Theory on causal sets
6. Quantum Field Theory on causal sets

Spacetime

In Einstein's theory of general relativity, spacetime is a continuous 4-dimensional manifold M endowed with a metric $g_{\mu\nu}$ of Lorentzian signature $(-+++)$.

The manifold M represents a continuous collection of idealised events, *points* in spacetime. The metric encodes the geometry of M, which manifests itself physically in gravitational effects such as tidal forces and gravitational lensing.

Nine of the 10 components of $g_{\mu\nu}$ encode the lightcone structure of spacetime (the "causal order" $\prec_g$). The 10th component sets the local physical scale (the volume factor $\sqrt{-g}$).

"Causal Order + Volume = Geometry"

Zeeman 1964; Hawking, King & McCarthy 1976; Malament 1976

Atomic Spacetime

Our two fundamental theories of Nature, general relativity and quantum theory, are plagued by self-contradictions that arise around the Planck scale $L_p = \sqrt{\hbar G/c^3} \approx 10^{-33} cm$. This has led many researchers to doubt the persistence of a spacetime continuum down to arbitrarily small sizes.

Causal Set theory is an approach to quantum gravity in which the deep structure of spacetime is postulated to be atomic and in which causal order is a primary concept.

’t Hooft 1979; Myrheim 1979; Bombelli, Lee, Meyer, Sorkin 1987

The mathematical structure that encapsulates these two principles is called a causal set, which is thought to replace the continuum (manifold) description of spacetime at small scales.

A causal set $(C, \prec)$ is a locally finite partially ordered set, meaning that $\prec$ is

1. Irreflexive: $x \nprec x$,
2. Transitive: $x \prec y$ and $y \prec z$ implies $x \prec z$,
3. Locally finite: $|I(x, z)| := \mathrm{card}(\{y : x \prec y \prec z\}) < \infty$.

You can also view a causal set as a directed acyclic graph.

The elements of C are thought of as "atoms of spacetime". The order relation $\prec$ represents the causal structure and the number of elements encodes the spacetime volume.

"Order + Number = Geometry"

Let us first look a the correspondence between causal sets and continuous manifolds.

Sprinklings

Given a Lorentzian manifold (M, g) we can generate a causal set C_M by performing a "sprinkling", placing points at random into the manifold according to a Poisson process such that if $R \subset M$:

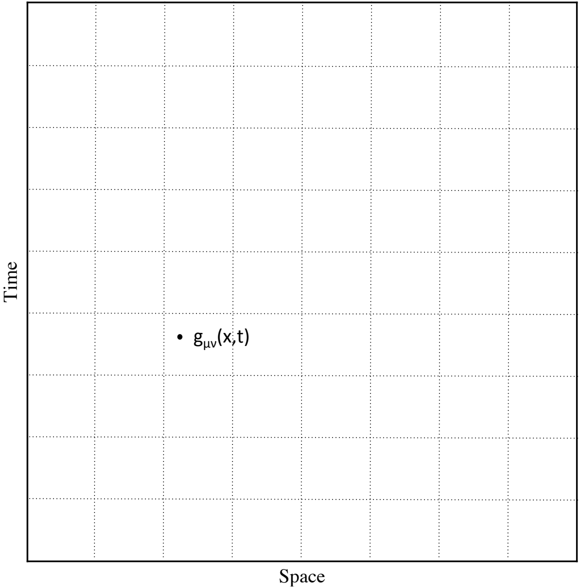
$$\mathbb{P}(N(R) = k) = \frac{(\rho V_R)^k e^{-\rho V_R}}{k!}.$$

For a sprinkling of density ρ the expected number of points in a region $R \subset M$ will be $\langle N(R) \rangle = \rho V_R$.

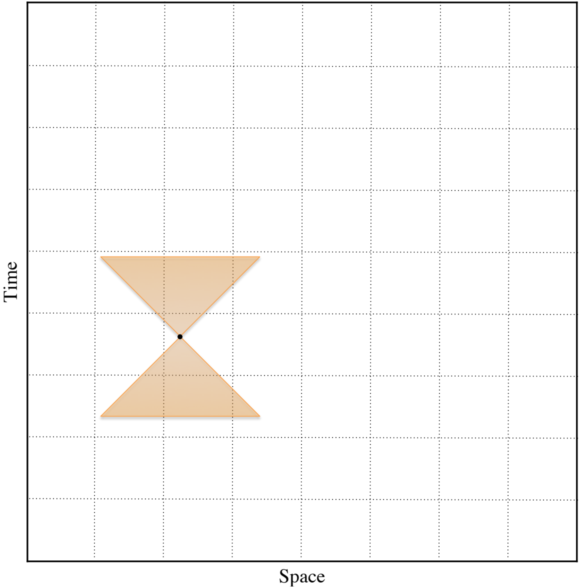
The causal set C_M is defined as the set whose elements are the sprinkled points and whose order relation is inherited from the causal relation $\prec_g$ on (M, g).

The simplest example is $1+1$ dimensional Minkowski space $\mathbb{M}^2$ with Cartesian coordinates (x, t) and metric $ds^2 = -dt^2 + dx^2$.

Sprinkling into $\mathbb{M}^2$ $(c = 1)$



Sprinkling into $\mathbb{M}^2$



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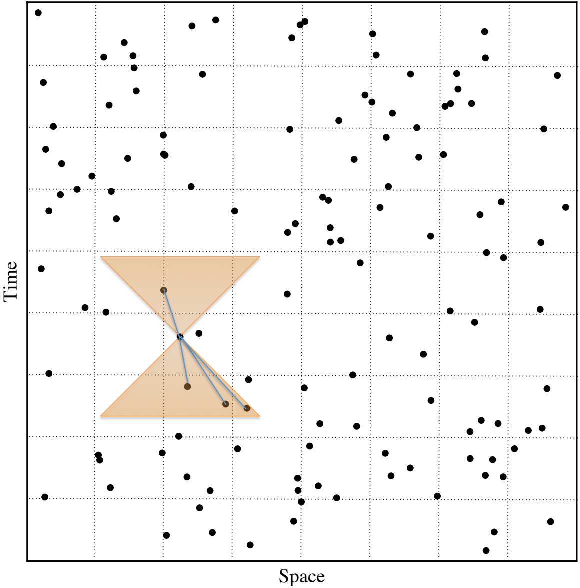
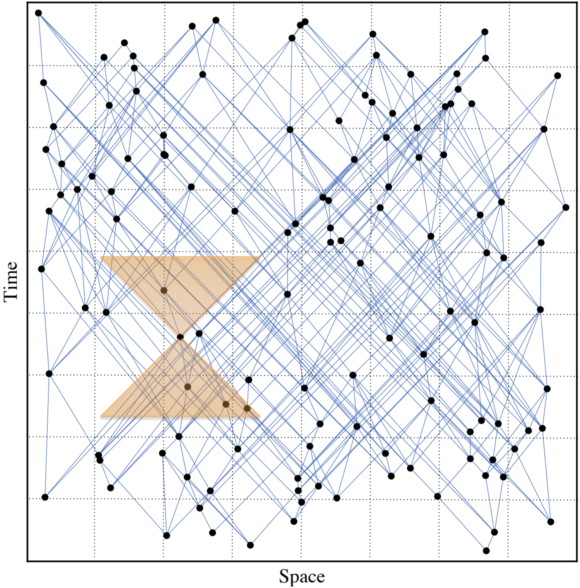
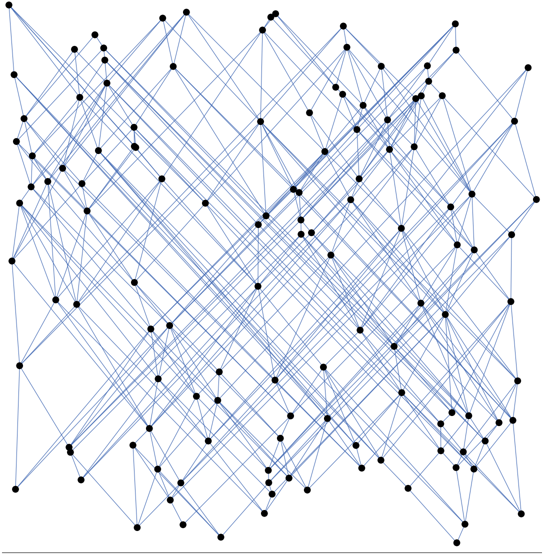


Sprinkling into $\mathbb{M}^2$



Sprinkling into $\mathbb{M}^2$



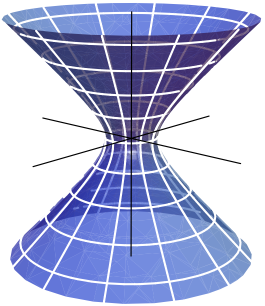
Sprinkling into dS^2

De Sitter space is the maximally symmetric spacetime of constant positive curvature. In $1+1$ dimensions it can be viewed as a hyperboloid embedded in 3-dimensional Minkowski space:

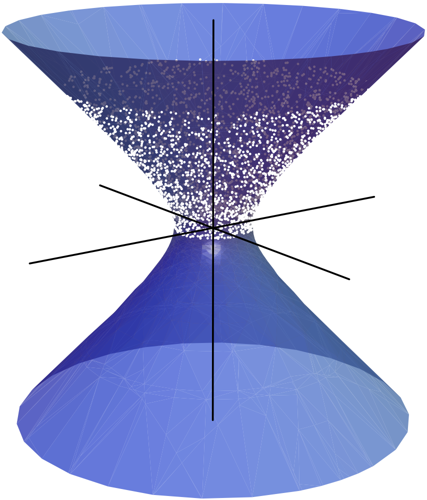
$$-(X^0)^2 + (X^1)^2 + (X^2)^2 = \ell^2.$$

In closed global coordinates:

$$\begin{aligned} X^0 &= \ell \sinh t \\ X^1 &= \ell \cosh t \cos\theta \\ X^2 &= \ell \cosh t \, \sin\theta \\ ds^2 &= -\ell^2 dt^2 + \ell^2 \cosh^2 t \, d\theta^2 \end{aligned}$$



Sprinkling into dS^2



Geodesic Distance

The geodesic distance between two elements is not explicitly contained in the sprinkling. However, metric information is intrinsically encoded in the causal set.

If $(C, \prec)$ is a sprinkling into an n–dimensional spacetime (M, g) and $L_{ij}^{\max}$ denotes the longest chain between $\nu_i, \nu_j \in C$ then

$$\lim_{\rho \to \infty} \mathbb{E}[\rho^{-\frac{1}{n}} L_{ij}^{\max}] = c_n d_{ij}$$

where d_{ij} is the geodesic distance between ν_i and ν_j in (M, g) and c_n depends only on the dimension.

Myrheim 1978, Ilie et.al. 2005, Bachmat 2012

For $\rho < \infty$ this correspondence still holds up to small corrections.

Relativistic fields on spacetime manifolds

Our current best models of the fundamental laws governing the dynamics of matter are based on the theory of relativistic fields living on continuous spacetime manifolds.

The simplest example of a relativistic field is a non-interacting scalar field $\phi : M \to \mathbb{C}$ satisfying the Klein–Gordon equation

$$(\Box - m^2)\phi(\mathbf{x}, t) = 0$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and ∇_μ is the covariant derivative.

The Klein-Gordon equation encodes the dynamics of the field: given Cauchy data "$\phi(\mathbf{x}, t_0)$ and $\dot{\phi}(\mathbf{x}, t_0)$" it fully predicts its evolution.

Propagators

Equivalently, the dynamics can be encoded in the propagators (Green functions) of the field, which are solutions to

$$(\Box - m^2)G(X, Y) = \delta(X - Y)$$

satisfying certain boundary conditions.

For example, once we have the retarded Green function $G_R(X, Y)$ defined by the boundary condition

$$G_R(X, Y) = 0 \text{ unless } X \prec Y$$

we can use it to “propagate forward” initial data.

The retarded propagator G_R in $\mathbb{M}^2$ and dS^2

In $1+1$ dimensional Minkowski space, $\Box = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}$ and

$$G_R(X_i, X_j) = \frac{1}{2}\chi(X_i \prec X_j) J_0(m d_{ij})$$

where d_{ij} is the geodesic distance in $\mathbb{M}^2$ between X_i and X_j and J_0 is a Bessel function of the first kind.

In dS^2, $\ell^2 \Box = -\frac{\partial^2}{\partial t^2} - \tanh t \frac{\partial}{\partial t} + \mathrm{sech}^2 t \frac{\partial^2}{\partial \theta^2}$ and $G_R(X_i, X_j) =$

$$\frac{\mathrm{sech} \pi\mu}{4}\chi(X_i \prec X_j) \mathrm{Im}\left[{}_2F_1\left(\frac{1}{2}-\mu, \frac{1}{2}+\mu, 1; \frac{1+\cosh\frac{d_{ij}}{\ell}}{2}\right)\right]$$

where $\mu = \frac{1}{2}\sqrt{1-4m^2\ell^2}$, d_{ij} is the geodesic distance in dS^2 between X_i and X_j and ${}_2F_1$ is a hypergeometric function.

Models of matter propagation on causal sets

Let us look at some simple propagation models from one element/node to another on a causal set.

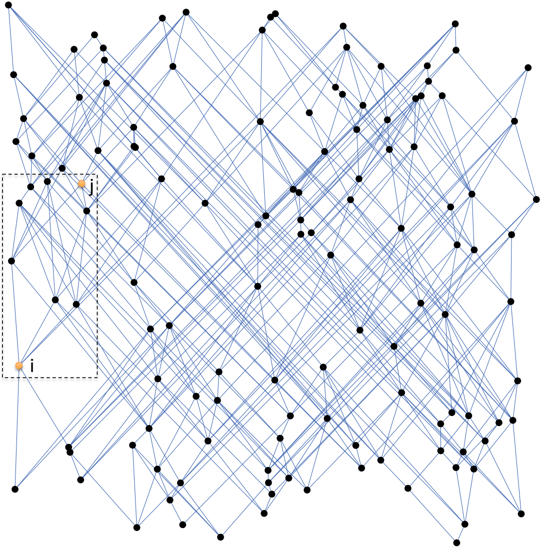
Such models are automatically "relativistic" when thought of as ocurring on sprinklings into continuum spacetimes due to the relativistic invariance of the causal order relation.

It turns out that a very simple model (Johnston 2010) already has rich structure, reproducing the propagators of scalar fields.

If time permits, I will show you that this model also encodes the structure of the *quantum* field, defining a preferred quantum state for the field solely in terms of causal structure.

N.B. In the following I will always assume a natural labelling of $(C, \prec)$: $\nu_i \prec \nu_j \implies i < j$.

A simple hop-and-stop model for a propagator



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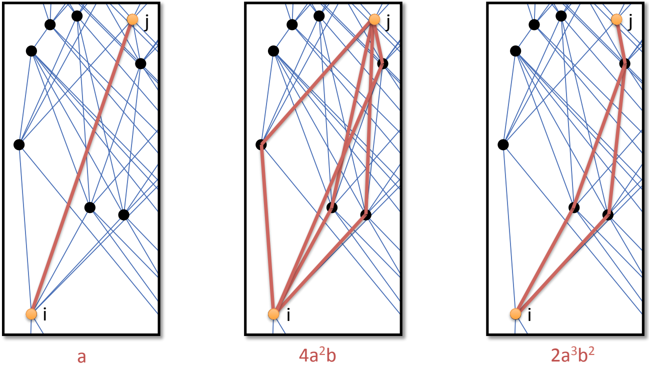


Summing over paths: $P_{ij} = a + 4a^2b + 2a^3b^2$

Summing over maximal paths: $P_{ij} = 2a^3b^2.$

The causal set propagator

For a general causal set, the number of paths of length k from node ν_i to node ν_j is just $[\mathbf{C}^k]_{ij}$ where $\mathbf{C}$ is the causal/adjacency matrix of the causal set. The propagation amplitude is then:

$$\mathbf{P} = a\mathbf{C} + a^2 b\mathbf{C}^2 + a^3 b^2 \mathbf{C}^3 + \ldots = a\mathbf{C}(\mathbf{1} - ab\mathbf{C})^{-1}$$

The amplitude $\tilde{\mathbf{P}}$ corresponding to the sum over *maximal* paths is obtained by replacing $\mathbf{C}$ by its transitive reduction $\tilde{\mathbf{C}}$.

In $1+1$ dimensions, $\mathbf{P}$ matches the continuum *retarded causal propagator* G_R for $a = \frac{1}{2}$ and $b = m^2/\rho$ for sprinklings in flat space (Johnston 2010, Afshordi et. al. 2012). There is good evidence that this is also true in curved spacetimes (Aslanbeigi & MB 2013).

In $3+1$ dimensions, there is some evidence that $\tilde{\mathbf{P}}$ matches the continuum retarded propagator for $a = \frac{\sqrt{\rho}}{\sqrt{24}\pi}$ and $b = -m^2/\rho$.

Comparison of causal set and continuum propagators

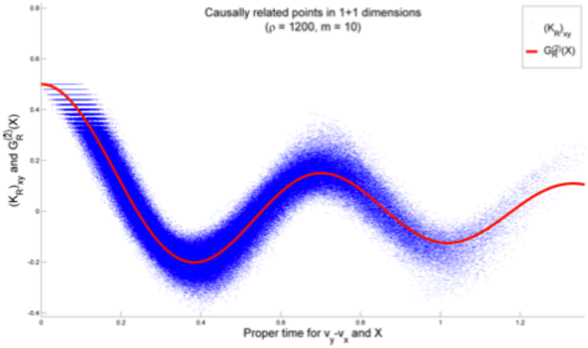
The matrix $\mathbf{P}$ (as well as $\tilde{\mathbf{P}}$) is retarded since by definition $[\mathbf{C}^k]_{ij} = 0$ unless $\nu_i \prec \nu_j$. This suggests an analogy with the continuum retarded propagator G_R.

To compare the discrete propagator $\mathbf{P}$ with G_R, fix $\rho = 1$ and

1. Sprinkle N points into a region of Minkowski or de Sitter space in $1+1$ dimensions
2. Compute the causal/adjacency matrix $\mathbf{C}$
3. Evaluate $\mathbf{P} = \frac{1}{2}\mathbf{C}(\mathbf{1} - \frac{m^2}{2}\mathbf{C})^{-1}$
4. Plot the values $\mathbf{P}_{ij}$ against geodesic distance d_{ij}
5. Compare with the continuum propagator G_R

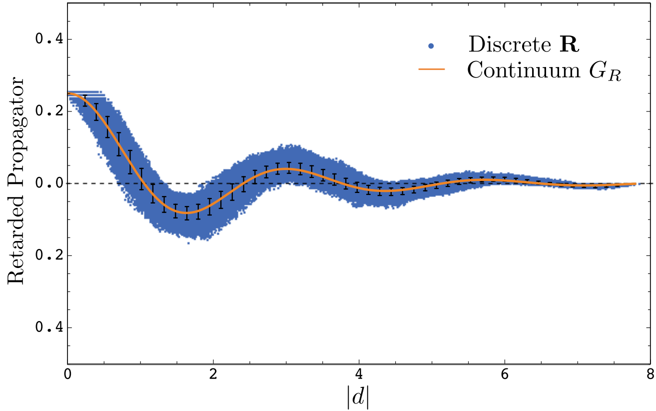
Comparison in $\mathbb{M}^2$

A plot of P_{ij} for a field of mass $m = 10$ against d_{ij} for causally related elements in a $\rho = 1000$ sprinkling into Minkowski spacetime. The red line is the continuum function $G_R(X_i, X_j)$.



Comparison in dS^2

A plot of P_{ij} for a field of mass $m = 2.4$ against d_{ij} for causally related elements in a $\rho = 76$ sprinkling into de Sitter spacetime with $\ell = 1$. The orange line is the continuum function $G_R(X_i, X_j)$.



An analytic proof for $\mathbb{M}^2$ sprinklings

For a sprinkling into $\mathbb{M}^2$ the expected number of paths from a node ν_i to a node ν_j can be calculated analytically (Meyer 1988):

$$\langle \mathbf{C}_{ij}^k \rangle = \chi(X_i \prec X_j) \frac{(\rho\, d_{ij}^2)^k}{2^k \Gamma(k+1)^2}$$

where $X_i := X(\nu_i)$.

The expected value of $\mathbf{P}_{ij}$ for a sprinkling is then

$$\begin{aligned} \langle \mathbf{P}_{ij} \rangle &= \sum_{k=1}^{\infty} a^k b^{k-1} \langle \mathbf{C}_{ij}^k \rangle = \chi(X_i \prec X_j) \sum_{k=1}^{\infty} 2^{-k} \Gamma(k+1)^{-2} a^k b^{k-1} d_{ij}^{2k} \\ &= \chi(X_i \prec X_j) a J_0(-i\sqrt{2ab\rho} d_{ij}). \end{aligned}$$

This agrees with the retarded propagator $G_R(X_i, X_j)$ for $a = 1/2$ and $b = -m^2/\rho$. (Johnston 2010)

Katz Centrality

The Katz centrality or status index K_i (Katz 1953) of a node ν_i in a network is a generalisation of the degree of ν_i, measuring the "relative influence of the node within the network". It is defined for an arbitrary "attenuation factor" $\alpha \in \mathbb{R}$ as

$$\begin{aligned} K_i &:= \sum_n \alpha^n (\text{number of } n^{th}\text{-nearest neighbours of } \nu_i) \\ &= \sum_n \alpha^n \sum_j [\mathbf{C}^n]_{ij} = \sum_j [\alpha \mathbf{C}(1 - \alpha \mathbf{C})]_{ij} = \sum_j \mathbf{P}_{ij} \end{aligned}$$

where $\mathbf{P}_{ij}$ is the discrete propagator with $a = \alpha$ and $b = 1$.

For a causal set obtained by sprinkling into a spacetime (M, g) (alternatively: that can be faithfully embedded into (M, g)), $\langle \mathbf{P}_{ij} \rangle$ is equal to 2α times the retarded propagator $G_R(X_i, X_j)$ of a scalar field on (M, g) with mass $m = \sqrt{\rho}$.

Quantum Fields

Our models of fundamental particle physics are *quantum* field theories. The field ϕ becomes an operator acting on a Hilbert space $\hat{\phi} : \mathcal{H} \to \mathcal{H}$ subject to the two conditions

$$(\Box - m^2)\hat{\phi}(X) = 0 \qquad \text{and} \qquad [\hat{\phi}(X), \hat{\phi}(Y)] = i\Delta(X, Y)$$

where $i\Delta(X, Y) := G_R(X, Y) - G_R(Y, X)$.

Physical predictions are made by choosing a quantum state $\omega : \mathcal{H} \to \mathbb{C}$, i.e. a positive linear functional on $\mathcal{H}$, that defines expectation values for operators $\langle \hat{\mathcal{O}} \rangle_\omega := \omega(\hat{\mathcal{O}})$.

For a free field, knowledge of the two-point (Wightman) function $W(X, Y) = \langle \hat{\phi}(X)\hat{\phi}(Y) \rangle$ is enough: all expectation values $\langle \hat{\mathcal{O}} \rangle$ are polynomials in W.

Quantum Fields: The usual approach

How is the vacuum state for a scalar field on a spacetime manifold (M, g) generally found? The usual (canonical) approach relies on special symmetries of (M, g).

When (M, g) is time translation invariant, or asymptotically time translation invariant (say in the infinite past), then we can define an energy operator (Hamiltonian) $\hat{H}$ and a "ground state" ω_0 in which the expectation value of $\hat{H}$ is minimised.

On the causal set there is no meaningful analog of time tranlation invariance. We need another approach, more appropriate to the intrinsic structure of the causal set.

Everything from causal structure?

On the causal set, we just have $\mathbf{P}$, which gives us $\mathbf{\Delta} = \mathbf{P} - \mathbf{P}^T$. Can the commutator alone yield the full algebra of operators (and the "vacuum state")? (Noldus)

For that, $\mathbf{\Delta}$ alone would need to give us the discrete counterpart of $W(X,Y)$ — without additional input.

The antisymmetric part of W is given by $\Delta/2$:

$$\langle\hat{\phi}(X)\hat{\phi}(Y)\rangle - \langle\hat{\phi}(Y)\hat{\phi}(X)\rangle = \langle[\hat{\phi}(X),\hat{\phi}(Y)]\rangle = i\Delta(X,Y)$$

The real part of W is what's missing.

However, noting that $i\mathbf{\Delta}$ is a Hermitian matrix, we can use the polar composition to define its "positive part" $\frac{1}{2}(i\mathbf{\Delta} + \sqrt{-\mathbf{\Delta}^2})$: a positive matrix whose imaginary part is $\mathbf{\Delta}/2$. (Johnston, Sorkin)

The causal set two-point function

Hence, identify

$$\mathbf{W} = \frac{1}{2}\left(i\boldsymbol{\Delta} + \sqrt{-\boldsymbol{\Delta}^2}\right)$$

as the "discrete two-point function" on the causal set.

We thus have a procedure at hand that specifies the quantum theory of a (free) scalar field uniquely from the adjacency matrix:

$$\mathbf{C} \to \mathbf{R} \to \boldsymbol{\Delta} \to \mathbf{W}.$$

There was no mention of specifying a "minimum energy state", no additional input. So, in sprinklings into spacetimes with time translation invariance, where a physical ground state is uniquely prescribed by the requirement of "minimum energy", does $\mathbf{W}$ agree with its continuum counterpart $W(X, Y)$?

$\mathbf{W}$ in a $\mathbb{M}^2$ sprinkling with $m > 0$

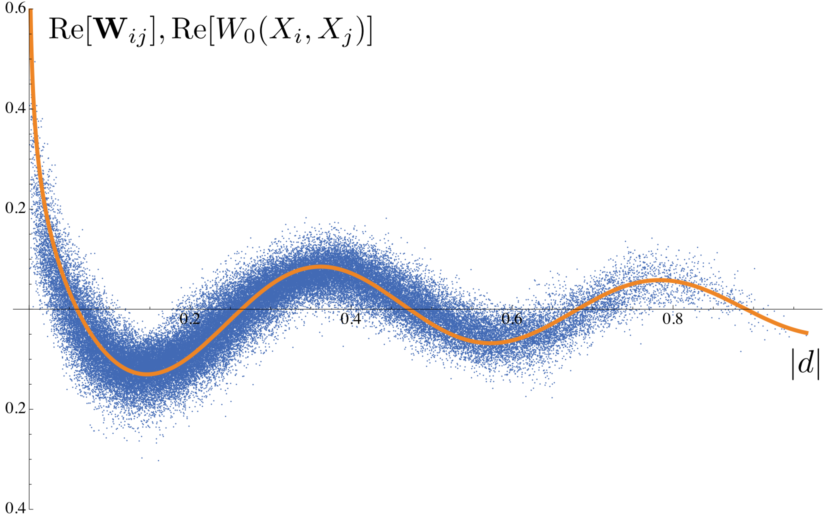
The two–point function of the *unique* Poincaré–invariant ground state ω_0 of a massive scalar field in $\mathbb{M}^2$ is:

$$W_0(X_i, X_j) = \langle \hat{\phi}(X_i)\hat{\phi}(X_j)\rangle_0 = \frac{1}{4} H_0^{(1)}\left[imd_{ij}\right].$$

where $H_0^{(1)}$ is a Hankel function of the first kind.

To compare the causal set two–point function with $W(X_i, X_j)$, we generate a causal set $(C, \prec)$ by a sprinkling into $\mathbb{M}^2$.

We then plot $\mathrm{Re}[\mathbf{W}_{ij}]$ (recall that the imaginary part of $\mathbf{W}$ is proportional to $i\mathbf{\Delta}$) against d_{ij} for all pairs of causally related points $\nu_i, \nu_j \in C$ and compare it to $\mathrm{Re}[W_0(X_i, X_j)]$.



A scatter plot of $\mathrm{Re}[\mathbf{W}_{ij}]$ against d_{ij} for causally related elements in a sprinkling with $\rho = 1000$ into a causal diamond in $\mathbb{M}^2$ for $m = 15$. The orange line is the continuum $\mathrm{Re}[W_0(x, y)]$.