Chapter 2

Stellar Dynamics in Galaxies

2.1 Introduction

A system of stars behaves like a fluid, but one with unusual properties. In a normal fluid two-body interactions are crucial in the dynamics, but in contrast star-star encounters are very rare. Instead stellar dynamics is mostly governed by the interaction of individual stars with the mean gravitational field of all the other stars combined. This has profound consequences for how the dynamics of the stars within galaxies are described mathematically, allowing for some considerable simplifications.

This chapter establishes some basic results relating to the motions of stars within galaxies. The virial theorem provides a very simple relation between the total potential and kinetic energies of stars within a galaxy, or other system of stars, that has settled down into a steady state. The virial theorem is derived formally here. The timescale for stars to cross a system of stars, known as the crossing time, is a simple but important measure of the motions of stars. The relaxation time measures how long it takes for two-body encounters to influence the dynamics of a galaxy, or other system of stars. An expression for the relaxation time is derived here, which is then used to show that encounters between stars are so rare within galaxies that they have had little effect over the lifetime of the Universe.

The motions of stars within galaxies can be described by the collisionless Boltzmann equation, which allows the numbers of stars to be calculated as a function of position and velocity in the galaxy. The equation is derived from first principles here. Similarly, the Jeans Equations relate the densities of stars to position, velocity, velocity dispersion and gravitational potential.

2.2 The Virial Theorem

2.2.1 The basic result

Before going into the main material on stellar dynamics, it is worth stating – and deriving – a basic principle known as the *virial theorem.* It states that for any system of particles bound by an inverse-square force law, the time-averaged kinetic energy $\langle T \rangle$ and the time-averaged potential energy $\langle U \rangle$ satisfy

$$2 \langle T \rangle + \langle U \rangle = 0 , \tag{2.1}$$

for a steady equilibrium state. $\langle T \rangle$ will be a very large positive quantity and $\langle U \rangle$ a very large negative quantity. Of course, for a galaxy to hold together, the total energy $\langle T \rangle + \langle U \rangle < 0$; the virial theorem provides a much tighter constraint than this alone. Typically, $\langle T \rangle$ and $-\langle U \rangle \sim 10^{50}$ to 10^{54} J for galaxies.

In practice, many systems of stars are not in a perfect final steady state and the virial theorem does not apply exactly. Despite this, it does give important, approximate results for many astronomical systems.

The virial theorem was first devised by Rudolf Clausius (1822–1888) to describe motions of particles in thermodynamics. The term *virial* comes from the Latin word for force, *vis*.

2.2.2 Deriving the virial theorem from first principles

To prove the virial theorem, consider a system of N stars. Let the ith star have a mass m_i and a position vector $\mathbf{x}_i$. The position vector will be measured from the centre of mass of the system, and we shall assume that this centre of mass moves with uniform motion. The velocity of the ith star is $\dot{\mathbf{x}}_i \equiv \mathrm{d}\mathbf{x}_i/\mathrm{d}t$, where t is the time.
Consider the moment of inertia I of this system of stars defined here as

$$I \equiv \sum_{i=1}^{N} m_i \, \mathbf{x}_i \cdot \mathbf{x}_i = \sum_{i=1}^{N} m_i \, x_i^2 \quad . \tag{2.2}$$

(Note that this is a different definition of moment of inertia to the moment of inertia about a particular axis that is used to study the rotation of bodies about an axis.)
Differentiating with respect to time t,

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i \, \mathbf{x}_i \cdot \mathbf{x}_i\right) = \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(m_i \, \mathbf{x}_i \cdot \mathbf{x}_i \Big) \\
&= \sum_i m_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{x}_i \cdot \mathbf{x}_i \Big) \qquad \text{assuming that the masses } m_i \text{ do not change} \\
&= \sum_i m_i \Big(\dot{\mathbf{x}}_i \cdot \mathbf{x}_i + \mathbf{x}_i \cdot \dot{\mathbf{x}}_i \Big) \qquad \text{from the product rule} \\
&= 2 \sum_i m_i \, \dot{\mathbf{x}}_i \cdot \mathbf{x}_i
\end{aligned}$$

Differentiating again,

$$\begin{aligned}
\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} &= 2 \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \, \dot{\mathbf{x}}_i \cdot \mathbf{x}_i = 2 \sum_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(m_i \, \dot{\mathbf{x}}_i \cdot \mathbf{x}_i \Big) = 2 \sum_i m_i \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{\mathbf{x}}_i \cdot \mathbf{x}_i \Big) \\
&= 2 \sum_i m_i \Big(\ddot{\mathbf{x}}_i \cdot \mathbf{x}_i + \dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_i \Big) \\
&= 2 \sum_i m_i \, \ddot{\mathbf{x}}_i \cdot \mathbf{x}_i + 2 \sum_i m_i \, \dot{x}_i{}^2
\end{aligned} \tag{2.3}$$

The kinetic energy of the ith particle is $\frac{1}{2} m_i \dot{x}_i{}^2$. Therefore the total kinetic energy of the entire system of stars is

$$T = \sum_i \tfrac{1}{2} \, m_i \, \dot{x}_i^2 \qquad\qquad \therefore \sum_i m_i \, \dot{x}_i^2 = 2\,T$$

Substituting this into Equation 2.3,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \;=\; 4\,T \;+\; 2\sum_i m_i\,\ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i \quad , \tag{2.4}$$

at any time t.
We now need to remember that the average of any parameter $y(t)$ over time $t = 0$ to τ is

$$\langle y \rangle \;=\; \frac{1}{\tau}\int_0^\tau y(t)\;\mathrm{d}t$$

Consider the average value of $\mathrm{d}^2 I/\mathrm{d}t^2$ over a time interval $t = 0$ to τ.

$$\begin{aligned}
\left\langle \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} \right\rangle &= \frac{1}{\tau}\int_0^\tau \left(4\,T \;+\; 2\sum_i m_i\,\ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i \right)\;\mathrm{d}t \\
&= \frac{4}{\tau}\int_0^\tau T\;\mathrm{d}t \;+\; \frac{2}{\tau}\int_0^\tau \sum_i m_i\,\ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i\;\;\mathrm{d}t \\
&= 4\,\langle T\rangle \;+\; 2\sum_i \frac{m_i}{\tau}\int_0^\tau \ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i\;\;\mathrm{d}t \quad \text{assuming } m_i \text{ is constant over time} \\
&= 4\,\langle T\rangle \;+\; 2\sum_i m_i\,\langle \ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i\rangle
\end{aligned} \tag{2.5}$$

When the system of stars eventually reaches equilibrium, the moment of inertia I will be constant. So, $\langle \mathrm{d}^2 I/\mathrm{d}t^2\rangle = 0$. An alternative way of visualising this is by considering that I will be bounded in any physical system and $\mathrm{d}^2 I/\mathrm{d}t^2$ will also be finite. Therefore the long-time average $\langle \frac{\mathrm{d}^2 I}{\mathrm{d}t^2}\rangle$ will vanish as τ becomes large, i.e. $\lim_{\tau\to\infty}\langle \frac{\mathrm{d}^2 I}{\mathrm{d}t^2}\rangle = \lim_{\tau\to\infty}(\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}^2 I}{\mathrm{d}t^2}\,\mathrm{d}t) \;\to\; 0$ because $\mathrm{d}^2 I/\mathrm{d}t^2$ remains finite.
Substituting for $\langle \mathrm{d}^2 I/\mathrm{d}t^2\rangle = 0$ into Equation 2.5,

$$4\,\langle T\rangle \;+\; 2\sum_i m_i\,\langle \ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i\rangle \;=\; 0 \;\; .$$

$$\therefore \quad 2\,\langle T\rangle \;+\; \sum_i m_i\,\langle \ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i\rangle \;=\; 0 \;\; . \tag{2.6}$$

The term $\sum_i m_i\,\langle \ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i\rangle$ is related to the gravitational potential. We next need to show how.

Newton's Second Law of Motion gives for the ith star,

$$m_i\,\ddot{\mathbf{x}}_i \;=\; \sum_{\substack{j\\ j\neq i}} \mathbf{F}_{ij}$$

where $\mathbf{F}_{ij}$ is the force exerted on the ith star by the jth star. Using the law of universal gravitation,

$$m_i\,\ddot{\mathbf{x}}_i \;=\; \sum_{\substack{j\\ j\neq i}} -\frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\;(\mathbf{x}_i - \mathbf{x}_j) \quad .$$

Taking the scalar product (dot product) with $\mathbf{x}_i$,

$$m_i\,\ddot{\mathbf{x}}_i\,\boldsymbol{.}\,\mathbf{x}_i \;=\; \Bigg(\sum_{\substack{j\\ j\neq i}} -\frac{G\,m_i\,m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\;(\mathbf{x}_i - \mathbf{x}_j)\Bigg)\boldsymbol{.}\,\mathbf{x}_i$$

Summing over all i,

$$\sum_i m_i \ddot{\mathbf{x}}_i \boldsymbol{.} \mathbf{x}_i \;=\; -\sum_i \sum_{\substack{j \\ j\neq i}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} (\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_i \;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} (\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_i \tag{2.7}$$

Switching i and j, we have

$$\sum_j m_j\, \ddot{\mathbf{x}}_j \boldsymbol{.} \mathbf{x}_j \;=\; -\sum_{\substack{j,i \\ i\neq j}} \frac{G\, m_j\, m_i}{|\mathbf{x}_j - \mathbf{x}_i|^3}\; (\mathbf{x}_j - \mathbf{x}_i) \boldsymbol{.} \mathbf{x}_j \tag{2.8}$$

Adding Equations 2.7 and 2.8,

$$\sum_i m_i \ddot{\mathbf{x}}_i \boldsymbol{.} \mathbf{x}_i + \sum_j m_j \ddot{\mathbf{x}}_j \boldsymbol{.} \mathbf{x}_j \;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} (\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_i \;-\; \sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_j\, m_i}{|\mathbf{x}_j - \mathbf{x}_i|^3} (\mathbf{x}_j - \mathbf{x}_i) \boldsymbol{.} \mathbf{x}_j$$

$$\therefore \quad 2 \sum_i m_i\, \ddot{\mathbf{x}}_i \boldsymbol{.} \mathbf{x}_i \;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \Big((\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_i \;+\; (\mathbf{x}_j - \mathbf{x}_i) \boldsymbol{.} \mathbf{x}_j \Big)$$

But

$$\begin{aligned}
(\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_i \;+\; (\mathbf{x}_j - \mathbf{x}_i) \boldsymbol{.} \mathbf{x}_j \;&=\; (\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_i \;-\; (\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} \mathbf{x}_j \\
&=\; (\mathbf{x}_i - \mathbf{x}_j) \boldsymbol{.} (\mathbf{x}_i - \mathbf{x}_j) \qquad \text{(factorising)} \\
&=\; |\mathbf{x}_i - \mathbf{x}_j|^2
\end{aligned}$$

$$\therefore \quad 2 \sum_i m_i\, \ddot{\mathbf{x}}_i \boldsymbol{.} \mathbf{x}_i \;=\; -\sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}\; |\mathbf{x}_i - \mathbf{x}_j|^2$$

$$\therefore \quad \sum_i m_i\, \ddot{\mathbf{x}}_i \boldsymbol{.} \mathbf{x}_i \;=\; -\frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \tag{2.9}$$

We now need to find the total potential energy of the system.
The gravitational potential at star i due to star j is

$$\Phi_{i\,j} \;=\; -\frac{G\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Therefore the gravitational potential at star i due to all other stars is

$$\Phi_i \;=\; \sum_{\substack{j \\ j\neq i}} \Phi_{i\,j} \;=\; \sum_{\substack{j \\ j\neq i}} -\frac{G\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Therefore the gravitational potential energy of star i due to all the other stars is

$$U_i \;=\; m_i\, \Phi_i \;=\; -\, m_i \sum_{\substack{j \\ j\neq i}} \frac{G\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

The total potential energy of the system is therefore

$$U \;=\; \sum_i U_i \;=\; \frac{1}{2} \sum_i \left(- m_i \sum_{\substack{j \\ j \neq i}} \frac{G\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|} \right)$$

The factor $\frac{1}{2}$ ensures that we only count each pair of stars once (otherwise we would count each pair twice and would get a result twice as large as we should). Therefore,

$$U \;=\; -\frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{G\, m_i\, m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Substituting for the total potential energy into Equation 2.9,

$$\sum_i m_i\, \ddot{\mathbf{x}}_i \,\boldsymbol{.}\, \mathbf{x}_i \;=\; U$$

Equation 2.6 uses time-averaged quantities. So, averaging over time $t = 0$ to τ,

$$\frac{1}{\tau} \int_0^\tau \sum_i m_i\, \ddot{\mathbf{x}}_i \,\boldsymbol{.}\, \mathbf{x}_i \,\mathrm{d}t \;=\; \langle U \rangle$$

$$\therefore \quad \sum_i m_i\, \frac{1}{\tau} \int_0^\tau \ddot{\mathbf{x}}_i \,\boldsymbol{.}\, \mathbf{x}_i \,\mathrm{d}t \;=\; \langle U \rangle$$

$$\therefore \quad \sum_i m_i\, \langle\, \ddot{\mathbf{x}}_i \,\boldsymbol{.}\, \mathbf{x}_i \,\rangle \;=\; \langle U \rangle$$

Substituting this into Equation 2.6,

$$2\,\langle T \rangle \;+\; \langle U \rangle \;=\; 0$$

This is Equation 2.1, the Virial Theorem.

It is also possible to rederive the virial theorem using tensors. This *tensor virial theorem* uses a tensor moment of inertia and tensor representations of the kinetic and potential energies. This is beyond the scope of this course.

2.2.3 Using the Virial Theorem

The virial theorem applies to systems of stars that have reached a steady equilibrium state. It can be used for many galaxies, but can also be used for other systems such as some star clusters. However, we need to be careful that we use the theorem only for equilibrium systems.

The theorem can be applied, for example, to:

- elliptical galaxies
- evolved star clusters, e.g. globular clusters
- evolved clusters of galaxies (with the galaxies acting as the particles, not the individual stars)

Examples of places where the virial theorem cannot be used are:

- merging galaxies
- newly formed star clusters
- clusters of galaxies that are still forming/still have infalling galaxies

(Note that the virial theorem does also apply to stars or planets in circular orbits, but we do not normally use it for these simple cases because a direct analysis based on the acceleration is more straightforward.)

The virial theorem provides an easy way to make rough estimates of masses, because velocity measurements can give $\langle T \rangle$. To do this we need to measure the observed velocity dispersion of stars (the dispersion along the line of sight using radial velocities obtained from spectroscopy). The theorem then gives the total gravitational potential energy, which can provide the total mass. This mass, of course, is important because it includes dark matter. Virial masses are particularly important for some galaxy clusters (using galaxies, or atoms in X-ray emitting gas, as the particles).

But it is prudent to consider virial mass estimates as order-of-magnitude only, because (i) generally one can measure only line-of-sight velocities, and getting $T = \frac{1}{2}\sum_i m_i \dot{\mathbf{x}}_i^2$ from these requires more assumptions (e.g. isotropy of the velocity distribution); and (ii) the systems involved may not be in a steady state, in which case of course the virial theorem does not apply — some clusters of galaxies are may be quite far from a steady state.

Note that for galaxies beyond our own, we cannot measure three-dimensional velocities of stars directly (although some projects are now achieving this for some Local Group galaxies). We have to use radial velocities (the component of the velocity along the line of sight to the galaxy) only, obtained from spectroscopy through the Doppler shift of spectral lines. Beyond nearby galaxies, radial velocities of individual stars become difficult to obtain. It becomes necessary to measure velocity dispersions along the line of sight from the observed widths of spectral lines in the combined light of millions of stars.

2.2.4 Deriving masses from the Virial Theorem: a naive example

Consider a spherical elliptical galaxy of radius R that has uniform density and which consists of N stars each of mass m having typical velocities v.
From the virial theorem,

$$2\langle T \rangle + \langle U \rangle = 0$$

where $\langle T \rangle$ is the time-averaged total kinetic energy and $\langle U \rangle$ is the average total potential energy.
We have

$$T = \sum_{i=1}^{N} \frac{1}{2} m v^2 = \frac{1}{2} N m v^2$$

and averaging over time, $\langle T \rangle = \frac{1}{2} N m v^2$ also. (Note that strictly speaking we are taking the typical velocity to mean the root mean square velocity.)
The total gravitational potential energy of a uniform sphere of mass M and radius R (a standard result) is

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

where G is the universal gravitational constant. So the time-averaged potential energy of the galaxy is

$$\langle U \rangle = -\frac{3}{5}\frac{GM^2}{R}$$

where M is the total mass. Substituting this into the virial theorem equation,

$$2\left(\frac{1}{2}Nmv^2\right) - \frac{3}{5}\frac{GM^2}{R} = 0$$

But the total mass is $M = Nm$.

$$\therefore \quad v^2 = \frac{3}{5}\frac{NGm}{R} = \frac{3}{5}\frac{GM}{R}$$

The calculation is only approximate, so we shall use

$$v^2 \simeq \frac{NGm}{R} \simeq \frac{GM}{R} . \tag{2.10}$$

This gives the mass to be

$$M \simeq \frac{v^2 R}{G} . \tag{2.11}$$

So an elliptical galaxy having a typical velocity $v = 350\ \mathrm{km\,s^{-1}} = 3.5 \times 10^5\ \mathrm{m\,s^{-1}}$, and a radius $R = 10$ kpc $= 3.1 \times 10^{20}$ m, will have a mass $M \sim 6 \times 10^{41}$ kg $\sim 3 \times 10^{11} M_\odot$.

2.2.5 Example: the fundamental plane for elliptical galaxies

We can derive a relationship between scale size, central surface brightness and central velocity dispersion for elliptical galaxies that is rather similar to the fundamental plane, using only assumptions about a constant mass-to-light ratio and a constant functional form for the surface brightness profile.

We shall assume here that:

- the mass-to-light ratio is constant for ellipticals (all E galaxies have the same M/L regardless of their size or mass), and
- elliptical galaxies have the same functional form for the mass distribution, only scalable.

Let I_0 be the central surface brightness and R_0 be a scale size of a galaxy (in this case, different galaxies will have different values of I_0 and R_0). The total luminosity will be

$$L \propto I_0 R_0^2 ,$$

because I_0 is the light per unit projected area. Since the mass-to-light ratio is a constant for all galaxies, the mass of the galaxy is $M \propto L$.

$$\therefore \ M \propto I_0 R_0^2 .$$

From the virial theorem, if v is a typical velocity of the stars in the galaxy

$$v^2 \simeq \frac{GM}{R_0} .$$

The observed velocity dispersion along the line sight, σ_0, will be related to the typical velocity v by $\sigma_0 \propto v$ (because v is a three-dimensional space velocity). So

$$\sigma_0^2 \propto \frac{M}{R_0} . \qquad \therefore \; M \propto \sigma_0^2 \, R_0 .$$

Equating this with $M \propto I_0 \, R_0^2$ from above, $\quad \sigma_0^2 \, R_0 \propto I_0 \, R_0^2$.

$$\therefore \; R_0 \, I_0 \, \sigma_0^{-2} \simeq \text{constant} .$$

This is close to, but not the same, as the observed fundamental plane result $R_0 \, I_0^{0.8} \, \sigma_0^{-1.3} \simeq$ constant. The deviation from this virial prediction probably has something to do with a varying mass-to-light ratio, and may be caused by differences in ages between galaxies causing differences in luminosity.

2.3 The Crossing Time, $T_{\mathbf{cross}}$

The *crossing time* is a simple, but important, parameter that measures the timescale for stars to move significantly within a system of stars. It is sometimes called the *dynamical timescale.*

It is defined as

$$T_{cross} \equiv \frac{R}{v} , \tag{2.12}$$

where R is the size of the system and v is a typical velocity of the stars.

As a simple example, consider a stellar system of radius R (and therefore an overall size $2R$), having N stars each of mass m; the stars are distributed roughly homogeneously, with v being a typical velocity, and the system is in dynamical equilibrium. Then from the virial theorem,

$$v^2 \simeq \frac{NGm}{R} .$$

The crossing time is then

$$T_{cross} \equiv \frac{2R}{v} \simeq \frac{2R}{\sqrt{\frac{NGm}{R}}} \simeq 2\sqrt{\frac{R^3}{NGm}} . \tag{2.13}$$

But the mass density is

$$\rho = \frac{Nm}{\frac{4}{3}\pi R^3} = \frac{3Nm}{4\pi R^3} .$$

$$\therefore \; \frac{R^3}{Nm} = \frac{3}{4\pi\rho} .$$

$$\therefore \; T_{cross} = 2\sqrt{\frac{3}{4\pi G\rho}}$$

So approximately,

$$T_{cross} \sim \frac{1}{\sqrt{G\rho}} . \tag{2.14}$$

Although this equation has been derived for a particular case, that of a homogeneous sphere, it is an important result and can be used for order of magnitude estimates in other situations. (Note that ρ here is the mass density of the system, averaged over a volume of space, and not the density of individual stars.)

Example: an elliptical galaxy of 10^{11} stars, radius 10 kpc.

$$\begin{aligned} R &\simeq 10\,\text{kpc} \simeq 3.1 \times 10^{20}\,\text{m} \\ N &= 10^{11} \\ m &\simeq 1\,M_\odot \simeq 2 \times 10^{30}\,\text{kg} \end{aligned}$$

$$T_{cross} \simeq 2\sqrt{\frac{R^3}{NGm}} \quad \text{gives} \quad T_{cross} \simeq 10^{15}\,\text{s} \simeq 10^{8}\,\text{yr}.$$

The Universe is 14 Gyr old. So if a galaxy is $\simeq$ 14 Gyr old, there are $\simeq$ few $\times$ 100 crossing times in a galaxy's lifetime so far.

2.4 The Relaxation Time, $T_{\textbf{relax}}$

The relaxation time is the time taken for a star's velocity v to be changed significantly by two-body interactions. It is defined as the time needed for a change Δv^2 in v^2 to be the same as v^2, i.e. the time for

$$\Delta v^2 = v^2 \quad . \tag{2.15}$$

To estimate the relaxation time we need to consider the nature of encounters between stars in some detail.

2.5 Star-Star Encounters

2.5.1 Types of encounters

We might expect that stars, as they move around inside a galaxy or other system of stars, will experience close encounters with other stars. The gravitational effects of one star on another would change their velocities and these velocity perturbations would have a profound effect on the overall dynamics of the galaxy. The dynamics of the galaxy might evolve with time, as a result only of the internal encounters between stars.

The truth, however, is rather different. Close star-star encounters are extremely rare and even the effects of distant encounters are so slight that it takes an extremely long time for the dynamics of galaxies to change substantially.

We can consider two different types of star-star encounters:

- *strong encounters* – a close encounter that strongly changes a star's velocity – these are *very* rare in practice
- *weak encounters* – occur at a distance – they produce only very small changes in a star's velocity, but are much more common

2.5.2 Strong encounters

A strong encounter between two stars is defined so that we have a strong encounter if, at the closest approach, the change in the potential energy is larger than or equal to the initial kinetic energy.

For two stars of mass m that approach to a distance r_0, if the change in potential energy is larger than than initial kinetic energy,

$$\frac{Gm^2}{r_0} \geq \frac{1}{2}mv^2 ,$$

where v is the initial velocity of one star relative to the other.

$$\therefore \quad r_0 \leq \frac{2Gm}{v^2} .$$

So we define a strong encounter radius

$$r_S \equiv \frac{2Gm}{v^2} . \tag{2.16}$$

A strong encounter occurs if two stars approach to within a distance $r_S \equiv 2Gm/v^2$.

For an elliptical galaxy, $v \simeq 300\ \text{kms}^{-1}$. Using $m = 1M_\odot$, we find that $r_S \simeq 3 \times 10^9$ m $\simeq$ 0.02 AU. This is a very small figure on the scale of a galaxy. The typical separation between stars is ~ 1 pc $\simeq$ 200 000 AU.

For stars in the Galactic disc in the solar neighbourhood, we can use a velocity dispersion of $v = 30\ \text{kms}^{-1}$ and $m = 1M_\odot$. This gives $r_S \simeq 3 \times 10^{11}$ m $\simeq$ 2 AU. This again is very small on the scale of the Galaxy.

So strong encounters are very rare. The mean time between them in the Galactic disc is $\sim 10^{15}$ yr, while the age of the Galaxy is $\simeq 13 \times 10^9$ yr. In practice, we can ignore their effect on the dynamics of stars.

2.5.3 Distant weak encounters between stars

A star experiences a weak encounter if it approaches another to a minimum distance r_0 when

$$r_0 > r_S \equiv \frac{2Gm}{v^2} \tag{2.17}$$

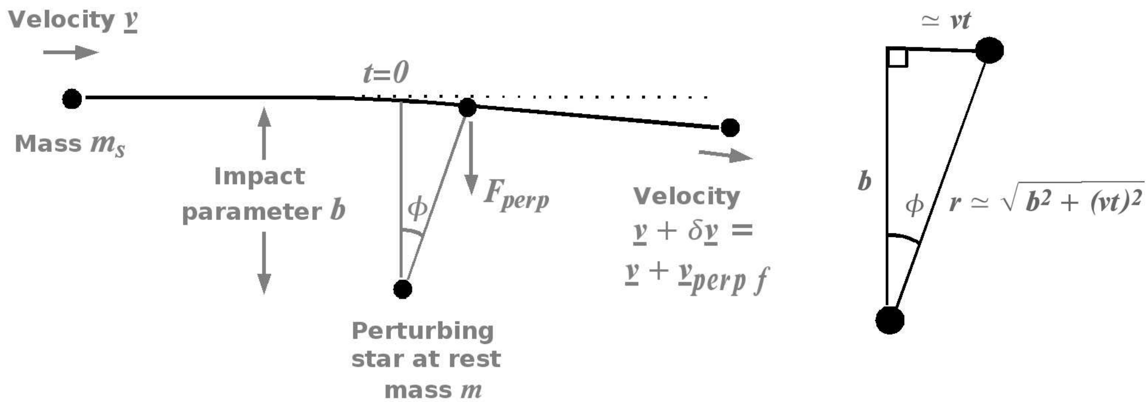
where v is the relative velocity before the encounter and m is the mass of the perturbing star. Weak encounters in general provide only a tiny perturbation to the motions of stars in a stellar system, but they are so much more numerous than strong encounters that they are more important than strong encounters in practice.

We shall now derive a formula that expresses the change δv in the velocity v during a weak encounter (Equation 2.19 below). This result will later be used to derive an expression for the square of the velocity change caused by a large number of weak encounters, which will then be used to obtain an estimate of the relaxation time in a system of stars.

Consider a star of mass m_s approaching a perturbing star of mass m with an impact parameter b. Because the encounter is weak, the change in the direction of motion will be small and the change in velocity will be perpendicular to the initial direction of motion. At any time t when the separation is r, the component of the gravitational force perpendicular to the direction of motion will be

$$F_{perp} = \frac{G m_s m}{r^2} \cos\phi \ ,$$

where ϕ is the angle at the perturbing mass between the point of closest approach and the perturbed star. Let the component of velocity perpendicular to the initial direction of motion be v_{perp} and let the final value be $v_{perp\,f}$.



Making the approximation that the speed along the trajectory is constant, $r \simeq \sqrt{b^2 + v^2 t^2}$ at time t if $t = 0$ at the point of closest approach. Using $\cos\phi = b/r \simeq b/\sqrt{b^2 + v^2 t^2}$ and applying $F = ma$ perpendicular to the direction of motion we obtain

$$\frac{\mathrm{d}v_{perp}}{\mathrm{d}t} = \frac{G\,m\,b}{(b^2 + v^2 t^2)^{3/2}} \ ,$$

where v_{perp} is the component at time t of the velocity perpendicular to the initial direction of motion. Integrating from time $t = -\infty$ to ∞,

$$\Big[v_{perp} \Big]_0^{v_{perp\,f}} = G\,m\,b \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(b^2 + v^2 t^2)^{3/2}} \ .$$

We have the standard integral $\int_{-\infty}^{\infty} (1 + s^2)^{-3/2}\, \mathrm{d}s = 2$ (which can be shown using the substitution $s = \tan x$). Using this standard integral, the final component of the velocity perpendicular to the initial direction of motion is

$$v_{perp\,f} = \frac{2Gm}{bv} \ . \tag{2.18}$$

Because the deflection is small, the change of velocity is $\delta v \equiv |\delta \mathbf{v}| = v_{perp\,f}$. Therefore the change in the velocity v is given by

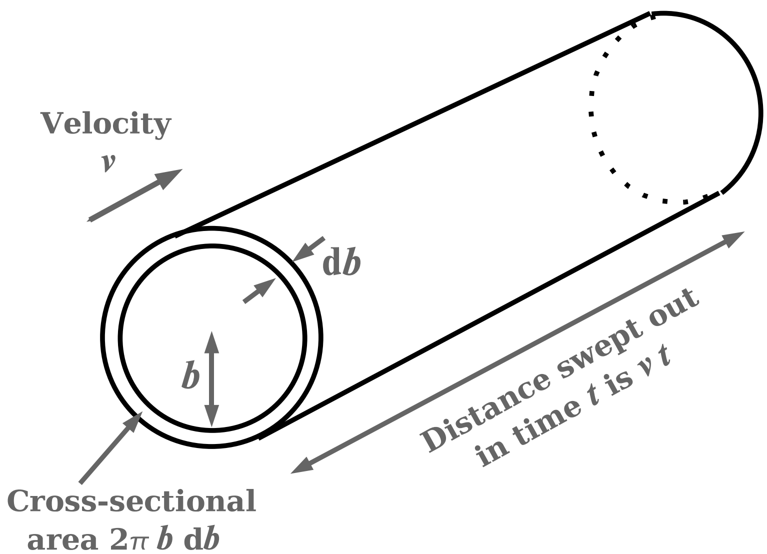
$$\delta v = \frac{2Gm}{bv} \ , \tag{2.19}$$

where G is the constant of gravitation, b is the impact parameter and m is the mass of the perturbing star.

As a star moves through space, it will experience a number of perturbations caused by weak encounters. Many of these velocity changes will cancel, but some net change will occur over time. As a result, the sum over all δv will remain small, but the sum of the squares δv^2 will build up with time. It is this change in v^2 that we need to consider in the definition of the relaxation time (Equation 2.15). Because the change in velocity $\delta \mathbf{v}$ is perpendicular to the initial velocity $\mathbf{v}$ in a weak encounter, the change in v^2 is therefore $\delta v^2 \equiv v_f^2 - v^2 = |\mathbf{v} + \delta\mathbf{v}|^2 - v^2 = (\mathbf{v}+\delta\mathbf{v}) \boldsymbol{.} (\mathbf{v}+\delta\mathbf{v}) - v^2 = \mathbf{v} \boldsymbol{.} \mathbf{v} + 2\mathbf{v} \boldsymbol{.} \delta\mathbf{v} + \delta\mathbf{v} \boldsymbol{.} \delta\mathbf{v} - v^2 = 2\mathbf{v} \boldsymbol{.} \delta\mathbf{v} + (\delta v)^2 = (\delta v)^2$, where $\mathbf{v}_f$ is the final velocity of the star. The change in v^2 resulting from a single encounter that we need to consider is

$$\delta v^2 = \left(\frac{2Gm}{bv}\right)^2 . \qquad (2.20)$$

Consider all weak encounters occurring in a time period t that have impact parameters in the range b to $b + \mathrm{d}b$ within a uniform spherical system of N stars and radius R.



The volume swept out by impact parameters b to $b + \mathrm{d}b$ in time t is $2\,\pi\,b\,\mathrm{d}b\,v\,t$. Therefore the number of stars encountered with impact parameters between b and $b + \mathrm{d}b$ in time t is

$$\text{(volume swept out) (number density of stars)} = \left(2\,\pi\,b\,\mathrm{d}b\,v\,t\right) \frac{N}{\frac{4}{3}\pi R^3} = \frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2R^3}$$

The total change in v^2 caused by all encounters in time t with impact parameters in the range b to $b + \mathrm{d}b$ will be

$$\Delta v^2 = \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2\,R^3}\right)$$

Integrating over b, the total change in a time t from all impact parameters from b_{min} to b_{max} is

$$\Delta v^2(t) = \int_{b_{min}}^{b_{max}} \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3\,b\,v\,t\,N\,\mathrm{d}b}{2\,R^3}\right) = \frac{3}{2}\left(\frac{2Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \int_{b_{min}}^{b_{max}} \frac{\mathrm{d}b}{b}$$

$$\therefore \quad \Delta v^2(t) = 6 \left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right) \quad . \tag{2.21}$$

It is sometimes useful to have an expression for the change in v^2 that occurs in one crossing time. In one crossing time $T_{cross} = 2R/v$, the change in v^2 is

$$\begin{aligned} \Delta v^2(T_{cross}) &= 6 \left(\frac{Gm}{v}\right)^2 \frac{v}{R^3} \left(\frac{2R}{v}\right) N \ln\left(\frac{b_{max}}{b_{min}}\right) \\ &= 12\,N \left(\frac{Gm}{Rv}\right)^2 \ln\left(\frac{b_{max}}{b_{min}}\right) \quad . \end{aligned} \tag{2.22}$$

The maximum scale over which weak encounters will occur corresponds to the size of the system of stars. So we shall use $b_{max} \simeq R$.

$$\Delta v^2(T_{cross}) = 12\,N \left(\frac{Gm}{Rv}\right)^2 \ln\left(\frac{R}{b_{min}}\right) \quad . \tag{2.23}$$

We are more interested here in the relaxation time T_{relax}. The relaxation time is defined as the time taken for $\Delta v^2 = v^2$. Substituting for Δv^2 from Equation 2.21 we get,

$$6 \left(\frac{Gm}{v}\right)^2 \frac{v\,T_{relax}\,N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right) = v^2 \quad .$$

$$\therefore \quad T_{relax} = \frac{1}{6N \ln\left(\frac{b_{max}}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2} \quad , \tag{2.24}$$

or putting $b_{max} \simeq R$,

$$T_{relax} = \frac{1}{6N \ln\left(\frac{R}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2} \quad . \tag{2.25}$$

Equation 2.25 enables us to estimate the relaxation time for a system of stars, such as a galaxy or a globular cluster. Different derivations can have slightly different numerical constants because of the different assumptions made.

In practice, b_{min} is often set to the scale on which strong encounters begin to operate, so $b_{min} \simeq 1$ AU. The precise values of b_{max} and b_{min} have relatively little effect on the estimation of the relaxation time because of the log dependence.

As an example of the calculation of the relaxation time, consider an elliptical galaxy. This has: $v \simeq 300$ kms^{-1} $= 3.0 \times 10^5$ ms^{-1}, $N \simeq 10^{11}$, $R \simeq 10$ kpc $\simeq 3.1 \times 10^{20}$ m and $m \simeq 1\ M_\odot \simeq 2.0 \times 10^{30}$ kg. So, $\ln(R/b_{min}) \simeq 21$ and $T_{relax} \sim 10^{24}$ s $\sim 10^{17}$ yr. The Universe is 14×10^9 yr old, which means that the relaxation time is $\sim 10^8$ times the age of the Universe. So star-star encounters are of no significance for galaxies.

For a large globular cluster, we have: $v \simeq 10$ kms^{-1} $= 10^4$ ms^{-1}, $N \simeq 500\,000$, $R \simeq 5$ pc $\simeq 1.6 \times 10^{17}$ m and $m \simeq 1\ M_\odot \simeq 2.0 \times 10^{30}$ kg. So, $\ln(R/b_{min}) \simeq 15$ and $T_{relax} \sim 5 \times 10^{15}$ s $\sim 10^7$ yr. This is a small fraction (10^{-3}) of the age of the Galaxy. Two body interactions are therefore significant in globular clusters.

The importance of the relaxation time calculation is that it enables us to decide whether we need to allow for star-star interactions when modelling the dynamics of a system of stars. This is discussed further in Section 2.7 below.

2.6 The Ratio of the Relaxation Time to the Crossing Time

An approximate expression for the ratio of the relaxation time to the crossing time can be calculated easily. Dividing the expressions for the relaxation and crossing times (Equations 2.25 and 2.12),

$$\frac{T_{relax}}{T_{cross}} = \frac{1}{12N \ln\left(\frac{R}{b_{min}}\right)} \frac{R^2 v^4}{(Gm)^2} \ .$$

For a uniform sphere, from the virial theorem (Equation 2.10),

$$v^2 \simeq \frac{NGm}{R}$$

and setting b_{min} equal to the strong encounter radius $r_S = 2GM/v^2$ (Equation 2.16), we get,

$$\frac{T_{relax}}{T_{cross}} = \frac{1}{12N \ln\left(\frac{Rv^2}{2GM}\right)} \frac{R^2 v^4}{(Gm)^2} \simeq \frac{N^2}{12N \ln(N)}$$

$$\therefore \frac{T_{relax}}{T_{cross}} \simeq \frac{N}{12 \ln N} \ . \tag{2.26}$$

For a galaxy, $N \sim 10^{11}$. Therefore $T_{relax}/T_{cross} \sim 10^9$. For a globular cluster, $N \sim 10^5$ and $T_{relax}/T_{cross} \sim 10^3$.

2.7 Collisional and Collisionless Systems

It is possible to classify the dynamics of systems of matter according to whether the interactions of individual particles in those systems are important or not. Such systems are said to be either *collisional* or *collisionless*. The dynamics are of these systems are:

- **collisional** if interactions between individual particles substantially affect their motions;
- **collisionless** if interactions between individual particles do **not** substantially affect their motions.

Note that this definition was encountered in Chapter 1 relating to the encounters between different systems. Here it applies *within* a single system of mass: the effects are all internal to the system.

The relaxation time calculations showed that galaxies are in general *collisionless* systems. (But an exception to this might be the region around the central nuclei of galaxies where the density of stars is very large.) Globular clusters are *collisional* over the lifetime of the Universe. Gas, whether in galaxies or in the laboratory, is *collisional*.

Modelling becomes much easier if two-body encounters can be ignored. Fortunately, we can ignore these star-star interactions when modelling galaxies and this makes possible the use of a result called the collisionless Boltzmann equation later.

2.8 Violent Relaxation

Stars in galaxies are collisionless systems, as we have seen. Therefore, the stars in a steady state galaxy will continue in steady state orbits without perturbing each other. The average distribution of stars will not change with time.

However, the situation can be very different in a system that is not in equilibrium. A changing gravitational potential will cause the orbits of the stars to change. Because the stars determine the overall potential, the change in their orbits will change the potential. This process of changes in the dynamics of stars caused by changes in their net potential is called *violent relaxation.*

Galaxies experienced violent relaxation during their formation, and this was a process that brought them to the equilibrium state that we see many of them in today. Interactions between galaxies can also bring about violent relaxation. The process takes place relatively quickly ($\sim 10^8$ yr) and redistributes the motions of stars.

2.9 The Nature of the Gravitational Potential in a Galaxy

The gravitational potential in a galaxy can be represented as essentially having two components. The first of these is the broad, smooth, underlying potential due to the entire galaxy. This is the sum of the potentials of all the stars, and also of the dark matter and the interstellar medium. The second component is the localised deeper potentials due to individual stars.

We can effectively regard the potential as being made of a smooth component with very localised deep potentials superimposed on it. This is illustrated figuratively in Figure 2.1.

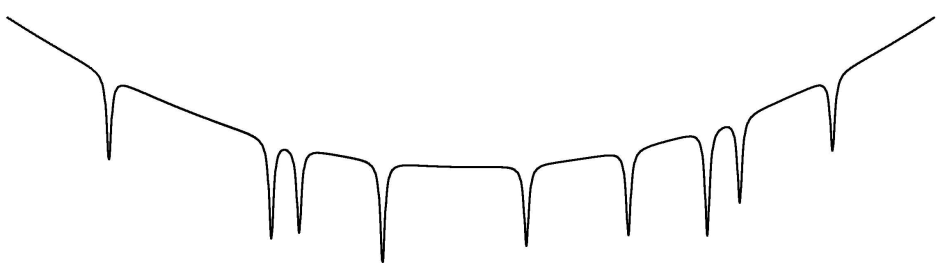


Figure 2.1: A sketch of the gravitational potential of a galaxy, showing the broad potential of the galaxy as a whole, and the deeper, localised potentials of individual stars.

Interactions between individual stars are rare, as we have seen, and therefore it is the broad distribution that determines the motions of stars. Therefore, we can represent the dynamics of a system of stars using only the smooth underlying component of the gravitational potential $\Phi(\mathbf{x}, t)$, where $\mathbf{x}$ is the position vector of a point and t is the time. If the galaxy has reached a steady state, Φ is $\Phi(\mathbf{x})$ only. We shall neglect the effect of the localised potentials of stars in the following sections, which is an acceptable approximation as we have shown.

2.10 Gravitational potentials, density distributions and masses

2.10.1 General principles

The distribution of mass in a galaxy – including both the visible and dark matter – determines the gravitational potential. The potential Φ at any point is related to the local density ρ by Poisson's Equation,

$$\nabla^2\Phi \; (\equiv \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{\nabla}\Phi) \;=\; 4\pi\, G\rho \;. \tag{2.27}$$

This means that if we know the density $\rho(\mathbf{x})$ as a function of position across a galaxy, we can calculate the potential Φ, either analytically or numerically, by integration. Alternatively, if we know $\Phi(\mathbf{x})$, we can calculate the density profile $\rho(\mathbf{x})$ by differentiation. In addition, because the acceleration due to gravity $\mathbf{g}$ is related to the potential by $\mathbf{g} = -\boldsymbol{\nabla}\Phi$, we can compute $\mathbf{g}(\mathbf{x})$ from $\Phi(\mathbf{x})$ and vice-versa. Similarly, substituting for $\mathbf{g} = -\boldsymbol{\nabla}\Phi$ in the Poisson Equation gives $\boldsymbol{\nabla}\boldsymbol{.}\mathbf{g} = -\,4\pi G\rho\,$.

These computations are often done for some example theoretical representations of the potential or density. A number of convenient analytical functions are encountered in the literature, depending on the type of galaxy being modelled and particular circumstances.

The issue of determining actual density profiles and potentials from observations of galaxies is much more challenging, however. Observations readily give the projected density distributions of stars on the sky, and we can attempt to derive the three-dimensional distribution of stars from this; this in turn can give the density of visible matter $\rho_{_{VIS}}(\mathbf{x})$ across the galaxy. However, it is the total density $\rho(\mathbf{x})$, including dark matter $\rho_{_{DM}}(\mathbf{x})$, that is relevant gravitationally, with $\rho(\mathbf{x}) = \rho_{_{DM}}(\mathbf{x}) + \rho_{_{VIS}}(\mathbf{x})$. The dark matter distribution can only be inferred from the dynamics of visible matter (or to a limited extent from gravitational lensing of background objects). In practice, therefore, the three-dimension density distribution $\rho(\mathbf{x})$ and the gravitational potential $\Phi(\mathbf{x})$ are poorly known, particularly where dark matter dominates far from the central regions.

2.10.2 Spherical symmetry

Calculating the relationship between density and potential is much simpler if we are dealing with spherically symmetric distributions, which are appropriate in some circumstances such as spherical elliptical galaxies. Under spherical symmetry, ρ and Φ are functions only of the radial distance r from the centre of the distribution. Therefore,

$$\nabla^2\Phi \;=\; \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) \;=\; 4\,\pi\,G\,\rho \tag{2.28}$$

because Φ is independent of the angles θ and ϕ in a spherical coordinate system (see Appendix C).

Another useful parameter for spherically symmetric distributions is the mass $M(r)$ that lies inside a radius r. We can relate this to the density $\rho(r)$ by considering a thin spherical shell of radius r and thickness dr centred on the distribution. The mass of this shell is $\mathrm{d}M(r) = \rho(r) \times \text{surface area} \times \text{thickness} = 4\pi r^2 \rho(r)\,\mathrm{d}r$. This gives us the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi\, r^2 \,\rho \ , \tag{2.29}$$

often known as the equation of continuity of mass. The total mass is $M_{tot} = \lim_{r\to\infty} M(r)$.

The gravitational acceleration $\mathbf{g}$ in a spherical distribution has an absolute value $|\mathbf{g}|$ of

$$g = \frac{GM(r)}{r^2} \ , \tag{2.30}$$

at a distance r from the centre, where G is the constant of gravitation (derived in Appendix B), and is directed towards the centre of the distribution.

If we know how one of these functions (ρ, Φ or $M(r)$) depends on radial distance r, we can calculate the others relatively easily when we have spherical symmetry. For example, if know the potential $\Phi(r)$ as a function of r, we can differentiate it to get the mass $M(r)$ interior to r, and by differentiating it again we can get the density $\rho(r)$. On the other hand, if we know $\rho(r)$ as a function of r, we can integrate it to get $M(r)$, and integrating it again gives $\Phi(r)$.

Comparing equations 2.28 and 2.29, we find that

$$M(r) = \frac{r^2}{G}\frac{\mathrm{d}\Phi}{\mathrm{d}r} \ , \tag{2.31}$$

when we have spherical symmetry. This allows us to convert between $M(r)$ and $\Phi(r)$ directly for this spherically symmetric case.

2.10.3 Two examples of spherical potentials

The Plummer Potential

A function that is often used for the theoretical modelling of spherically-symmetric galaxies is the Plummer potential. This has a gravitational potential Φ at a radial distance r from the centre that is given by

$$\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2 + a^2}} \ , \tag{2.32}$$

where M_{tot} is the total mass of the galaxy and a is a constant. The constant a serves to flatten the potential in the core.

For this potential the density ρ at a radial distance r is

$$\rho(r) = \frac{3M_{tot}}{4\pi}\frac{a^2}{(r^2 + a^2)^{5/2}} \ , \tag{2.33}$$

which can be derived from the expression for Φ using the Poisson equation $\nabla^2\Phi = 4\pi G\rho$. This density scales with radius as $\rho \sim r^{-5}$ at large radii.

The mass interior to a point $M(r)$ can be computed from the density ρ using $\mathrm{d}M/\mathrm{d}r = 4\pi r^2\rho$, or from the potential Φ using Gauss's Law in the form $\int_S \boldsymbol{\nabla}\Phi \boldsymbol{.} \mathrm{d}\boldsymbol{S} = 4\pi GM(r)$ for a spherical surface of radius r. The result is

$$M(r) = \frac{M_{tot}\, r^3}{(r^2 + a^2)^{3/2}} \ . \tag{2.34}$$

The Plummer potential was first used in 1911 by H. C. K. Plummer (1875–1946) to describe globular clusters. Because of the simple functional forms, the Plummer model is sometimes useful for approximate analytical modelling of galaxies, but the r^{-5} density profile is much steeper than elliptical galaxies are observed to have.

The Dark Matter Profile

A density distribution that is often used in modelling galaxies is one that is sometimes called the dark matter profile. The total density is given by

$$\rho(r) \;=\; \frac{\rho_0}{1\,+\,(r/a)^2} \;=\; \frac{\rho_0\, a^2}{r^2\,+\,a^2} \;, \tag{2.35}$$

where ρ_0 is the central density ($\rho(r)$ at $r = 0$) and a is a constant. The mass interior to a radius r is

$$M(r) \;=\; 4\pi\rho_0 \int_0^r \frac{r'^2}{1 + r'^2/a^2}\,\mathrm{d}r' \;=\; 4\pi\rho_0\, a^2\,(\,r \,-\, a\tan^{-1}(r/a)\,)\;.$$

Spiral galaxies with this profile would have rotation curves that are flat for $r \gg a$, which is exactly what is observed. This profile therefore represents successfully the large amount of dark matter that is observed at large distances r from the centres of galaxies. One weakness is that the mass interior to a radius tends to infinity as r increases: $\lim_{r\to\infty} M(r) \to \infty$. In practice, therefore, the density profiles of real galaxies must fall below the dark matter profile at some very large distances. These issues are discussed further in Chapter 5.

The Isothermal Sphere

The density distribution known as the isothermal sphere is a spherical model of a galaxy that is identical to the distribution that would be followed by a stable cloud of gas having the same temperature everywhere. A spherically-symmetric cloud of gas having a single temperature T throughout would have a gas pressure $P(r)$ at a radius r from its centre that is related to T by the ideal gas law as $P(r) = n_p k_B T$, where $n_p(r)$ is the number density of gas particles (atoms or molecules) at radius r and k_B is the Boltzmann constant. This can also be expressed in terms of the density ρ as $P(r) = k_B\rho(r)T/m_p$, where m_p is the mean mass of each particle in the gas.

The cloud will be supported by hydrostatic equilibrium, so therefore

$$\frac{\mathrm{d}P}{\mathrm{d}r} \;=\; -\,\frac{GM(r)}{r^2}\,\rho(r)\;, \tag{2.36}$$

where $M(r)$ is the mass enclosed within a radius r. The gradient in the mass is $\mathrm{d}M/\mathrm{d}r = 4\pi r^2\rho(r)$.

These equations have a solution

$$\rho(r) \;=\; \frac{\sigma^2}{2\pi G\, r^2}\;, \quad \text{and} \quad M(r) \;=\; \frac{2\sigma^2}{G}\, r\;, \quad \text{where} \quad \sigma^2 \;\equiv\; \frac{k_B T}{m_p}\;, \tag{2.37}$$

where m_p is the mass of each gas particle. The parameter σ is the root-mean-square velocity in any direction. This is only one of a number of solutions and it is called the *singular isothermal sphere*.

The isothermal sphere model for a system of stars is defined to be a model that has the same density distribution as the isothermal gas cloud. Therefore, an isothermal galaxy would also have a density $\rho(r)$ and mass $M(r)$ interior to a radius r given by

$$\rho(r) \quad = \quad \frac{\sigma^2}{2\pi G\, r^2} \ , \qquad \text{and} \quad M(r) \quad = \quad \frac{2\sigma^2}{G}\, r \ , \tag{2.38}$$

for a singular isothermal sphere, where σ is root-mean-square velocity of the stars along any direction.

The singular isothermal sphere model is sometimes used for the analytical modelling of galaxies. While it has some advantages of simplicity, it does suffer from the disadvantage of being unrealistic in some important respects. Most significantly, the model fails totally at large radii: formally the limit of $M(r)$ as $r \longrightarrow \infty$ is infinite.

2.11 Phase Space and the Distribution Function $f(x, v, t)$

To describe the dynamics of a galaxy, we could use:

- the positions of each star, $\mathbf{x}_i$
- the velocities of each star, $\mathbf{v}_i$

where $i = 1$ to N, with $N \sim 10^6$ to 10^{12}. However, this would be impractical numerically.

If we tried to store these data on a computer as 4-byte numbers for every star in a galaxy having $N \sim 10^{12}$ stars, we would need $6 \times 4 \times 10^{12}$ bytes $\sim 2 \times 10^{13}$ bytes $\sim$ 20 000 Gbyte. This is such a large data size that the storage requirements are prohibitive. If we needed to simulate a galaxy theoretically, we would need to follow the galaxy over time using a large number of time steps. Storing the complete set of data for, say, $10^3 - 10^6$ time steps would be impossible. Observationally, meanwhile, it is impossible to determine the positions and motions of every star in any galaxy, even our own.

In practice, therefore, people represent the stars in a galaxy using the *distribution function* $f(\mathbf{x}, \mathbf{v}, t)$ over position $\mathbf{x}$ and velocity $\mathbf{v}$, at a time t. This is the probability density in the 6-dimensional phase space of position and velocity at a given time. It is also known as the "phase space density". It requires only modest data resources to store the function numerically for a model of a galaxy, while f can also be modelled analytically.

The number of stars in a rectangular box between x and $x + \mathrm{d}x$, y and $y + \mathrm{d}y$, z and $z + \mathrm{d}z$, with velocity components between v_x and $v_x + \mathrm{d}v_x$, v_y and $v_y + \mathrm{d}v_y$, v_z and $v_z + \mathrm{d}v_z$, is $f(\mathbf{x}, \mathbf{v}, t)\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z\ \mathrm{d}v_x\ \mathrm{d}v_y\ \mathrm{d}v_z \ \equiv \ f(\mathbf{x}, \mathbf{v}, t)\ \mathrm{d}^3\mathbf{x}\ \mathrm{d}^3\mathbf{v}$. The number density $n(\mathbf{x}, \mathbf{v}, t)$ of stars in space can be obtained from the distribution function f by integrating over the velocity components,

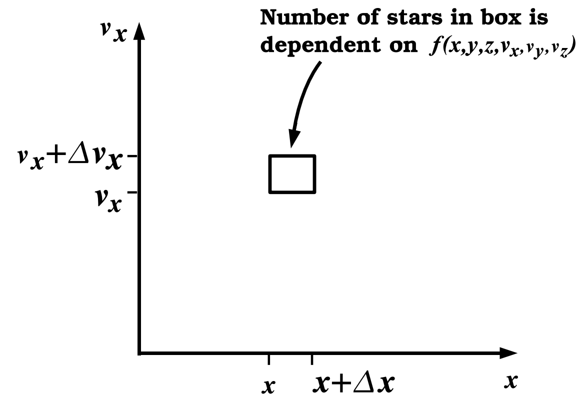
$$n(\mathbf{x}, \mathbf{v}, t) \quad = \quad \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t)\ \mathrm{d}v_x\ \mathrm{d}v_y\ \mathrm{d}v_z \quad = \quad \int_{-\infty}^{\infty} f(\mathbf{x}, \mathbf{v}, t)\ \mathrm{d}^3\mathbf{v} \ . \tag{2.39}$$

2.12 The Continuity Equation

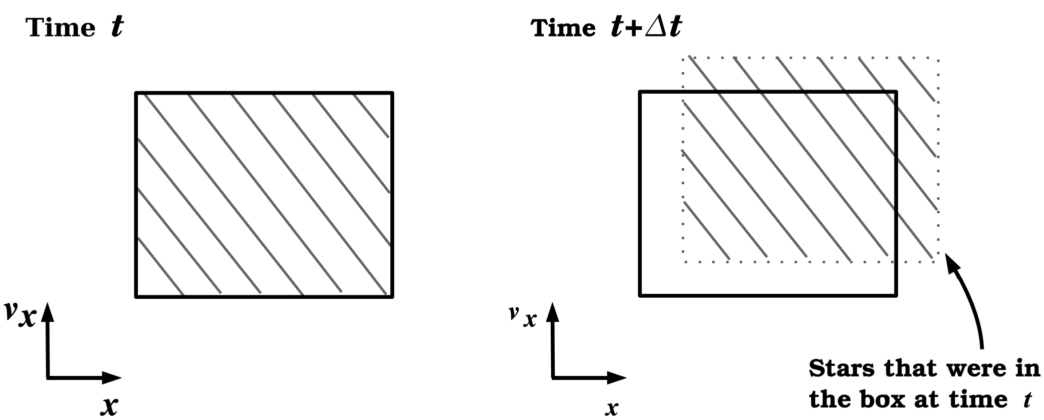
We shall assume here that stars are conserved: for the purpose of modelling galaxies we shall assume that the number of stars does not change. This means ignoring star formation and the deaths of stars, but it is acceptable for the present purposes.

The assumption that stars are conserved results in the *continuity equation*. This expresses the rate of change in the distribution function f as a function of time to the rates of change with position and velocity. The equation becomes an important starting point in deriving other equations that relate f to the gravitational potential and to observational quantities.

Consider the $x - v_x$ plane within the 6-dimensional phase space (x, y, z, v_x, v_y, v_z) in Cartesian coordinates. Consider a rectangular box in the plane extending from x to $x + \Delta x$ and v_x to $v_x + \Delta v_x$.



But the velocity v_x means that stars move in x ($v_x \equiv \mathrm{d}x/\mathrm{d}t$). So there is a flow of stars through the box in both the x and the v_x directions.



We can represent the flow of stars by the continuity equation:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\left(f\frac{\mathrm{d}x}{\mathrm{d}t}\right) + \frac{\partial}{\partial y}\left(f\frac{\mathrm{d}y}{\mathrm{d}t}\right) + \frac{\partial}{\partial z}\left(f\frac{\mathrm{d}z}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_x}\left(f\frac{\mathrm{d}v_x}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_y}\left(f\frac{\mathrm{d}v_y}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_z}\left(f\frac{\mathrm{d}v_z}{\mathrm{d}t}\right) = 0 \ . \qquad (2.40)$$

This can be abbreviated as

$$\boxed{\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\left(f\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_i}\left(f\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)\right) = 0\,,} \tag{2.41}$$

where $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $v_1 \equiv v_x$, $v_2 \equiv v_y$, and $v_3 \equiv v_z$. It is sometimes also abbreviated as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\boldsymbol{.}\left(f\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right) + \frac{\partial}{\partial \mathbf{v}}\boldsymbol{.}\left(f\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) = 0\ , \tag{2.42}$$

where, in this notation, for any vectors $\mathbf{a}$ and $\mathbf{b}$ with components (a_1, a_2, a_3) and (b_1, b_2, b_3),

$$\frac{\partial}{\partial \mathbf{a}}\boldsymbol{.}\mathbf{b} \equiv \sum_{i=1}^{3}\frac{\partial b_i}{\partial a_i}\ . \tag{2.43}$$

(Note that it does not mean a direct differentiation by a vector).

It is also possible to simplify the notation further by introducing a combined phase space coordinate system $\mathbf{w} = (\mathbf{x}, \mathbf{v})$ with components $(w_1, w_2, w_3, w_4, w_5, w_6) = (x, y, z, v_x, v_y, v_z)$. In this case the continuity equation becomes

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6}\frac{\partial}{\partial w_i}(f\dot{w}_i) = 0\ . \tag{2.44}$$

The equation of continuity can also be expressed in terms of the momentum $\mathbf{p} = m\mathbf{v}$, where m is mass of an element of gas, as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\boldsymbol{.}\left(f\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right) + \frac{\partial}{\partial \mathbf{p}}\boldsymbol{.}\left(f\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right) = 0\ . \tag{2.45}$$

2.13 The Collisionless Boltzmann Equation

2.13.1 The importance of the Collisionless Boltzmann Equation

Equation 2.25 showed that the relaxation time for galaxies is very long, significantly longer than the age of the Universe: galaxies are collisionless systems. This, fortunately, simplifies the analysis of the dynamics of stars in galaxies.

It is possible to derive an equation from the continuity equation that more explicitly states the relation between the distribution function f, position $\mathbf{x}$, velocity $\mathbf{v}$ and time t. This is the collisionless Boltzmann equation (C.B.E.), which takes its name from a similar equation in statistical physics derived by Boltzmann to describe particles in a gas. It states that

$$\boxed{\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i}\right) \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = 0\,.} \tag{2.46}$$

The collisionless Boltzmann equation therefore provides a relationship between the density of stars in phase space for a galaxy with position $\mathbf{x}$, stellar velocity $\mathbf{v}$ and time t.

2.13.2 A derivation of the Collisionless Boltzmann Equation

The continuity equation (2.41) states that

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}\left(f\,\frac{\mathrm{d}x_i}{\mathrm{d}t}\right) + \frac{\partial}{\partial v_i}\left(f\,\frac{\mathrm{d}v_i}{\mathrm{d}t}\right)\right) = 0\ ,$$

where f is the distribution function in the Cartesian phase space $(x_1, x_2, x_3, v_1, v_2, v_3)$. But the acceleration of a star is given by the gradient of the gravitational potential Φ:

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial \Phi}{\partial x_i}$$

in each direction (i.e. for each value of i for $i = 1, 2, 3$). (This is simply $\mathrm{d}\mathbf{v}/\mathrm{d}t = \mathbf{g} = -\boldsymbol{\nabla}\Phi$ resolved into each dimension.)

We also have $\dfrac{\mathrm{d}x_i}{\mathrm{d}t} = v_i$, so,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\partial}{\partial x_i}(f v_i) + \frac{\partial}{\partial v_i}\left(-f\frac{\partial \Phi}{\partial x_i}\right)\right) = 0\ .$$

But v_i is a coordinate, not a value associated with a particular star: we are using the continuous function f rather than considering individual stars. Therefore v_i is independent of x_i. So,

$$\frac{\partial}{\partial x_i}(f v_i) = v_i\,\frac{\partial f}{\partial x_i}\ .$$

The potential $\Phi \equiv \Phi(\mathbf{x}, t)$ does not depend on v_i: Φ is independent of velocity.

$$\therefore\quad \frac{\partial}{\partial v_i}\left(f\frac{\mathrm{d}\Phi}{\mathrm{d}x_i}\right) = \frac{\partial \Phi}{\partial x_i}\,\frac{\partial f}{\partial v_i}$$

$$\therefore\quad \frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i\,\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\,\frac{\partial f}{\partial v_i}\right) = 0\ .$$

But $\dfrac{\mathrm{d}v_i}{\mathrm{d}t} = -\dfrac{\partial \Phi}{\partial x_i}$, so,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(v_i\,\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\,\frac{\partial f}{\partial v_i}\right) = 0\ . \tag{2.47}$$

This is the collisionless Boltzmann equation. It can also be written as

$$\boxed{\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\,\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\,\frac{\partial f}{\partial v_i}\right) = 0\,.} \tag{2.46}$$

Alternatively it can expressed as,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6}\dot{w}_i\,\frac{\partial f}{\partial w_i} = 0\ , \tag{2.48}$$

where $\mathbf{w} = (\mathbf{x}, \mathbf{v})$ is a 6-dimensional coordinate system, and also as

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{x}} + \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{v}} = 0 \ , \tag{2.49}$$

and as

$$\frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{x}} + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{p}} = 0 \ . \tag{2.50}$$

Note the use here of the notation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\boldsymbol{\cdot}\frac{\partial f}{\partial \mathbf{x}} \equiv \sum_{i=1}^{3}\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} \ , \qquad \text{etc.} \tag{2.51}$$

2.13.3 Deriving the Collisionless Boltzmann Equation using Hamiltonian Mechanics

[This section is not examinable.]

The collisionless Boltzmann equation can also be derived from the continuity equation using Hamiltonian mechanics. This derivation is given here. It has the advantage of being neat. However, do not worry if you are not familiar with Hamiltonian mechanics: this is given as an alternative to Section 2.13.2.

Hamilton's Equations relate the differentials of the position vector $\mathbf{x}$ and of the (generalised) momentum $\mathbf{p}$ to the differential of the Hamiltonian H:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}} \ , \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial \mathbf{x}} \ . \tag{2.52}$$

(In this notation this means

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x_i} \quad \text{for } i = 1 \text{ to } 3, \tag{2.53}$$

where x_i and p_i are the components of $\mathbf{x}$ and $\mathbf{p}$.)
Substituting for $\mathrm{d}\mathbf{x}/\mathrm{d}t$ and $\mathrm{d}\mathbf{p}/\mathrm{d}t$ into the continuity equation,

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\boldsymbol{\cdot}\left(f\,\frac{\partial H}{\partial \mathbf{p}}\right) + \frac{\partial}{\partial \mathbf{p}}\boldsymbol{\cdot}\left(-f\,\frac{\partial H}{\partial \mathbf{x}}\right) = 0 \ .$$

For a star moving in a gravitational potential Φ, the Hamiltonian is

$$H = \frac{p^2}{2m} + m\,\Phi(\mathbf{x}) = \frac{\mathbf{p}\boldsymbol{\cdot}\mathbf{p}}{2m} + m\,\Phi(\mathbf{x}) \ . \tag{2.54}$$

where $\mathbf{p}$ is its momentum and m is its mass. Differentiating,

$$\begin{aligned}\frac{\partial H}{\partial \mathbf{p}} &= \frac{\mathrm{d}}{\mathrm{d}\mathbf{p}}\left(\frac{\mathbf{p}\cdot\mathbf{p}}{2m}\right) + \frac{\mathrm{d}}{\mathrm{d}\mathbf{p}}(m\Phi) \\ &= \frac{\mathbf{p}}{m} + 0 \qquad \text{because } \Phi(\mathbf{x},t) \text{ is independent of } \mathbf{p} \\ &= \frac{\mathbf{p}}{m}\end{aligned}$$

$$\text{and} \quad \begin{aligned}\frac{\partial H}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}}\left(\frac{p^2}{2m}\right) + m\frac{\partial \Phi}{\partial \mathbf{x}} \\ &= 0 + m\frac{\partial \Phi}{\partial \mathbf{x}} \qquad \text{because } p^2 = \mathbf{p}\cdot\mathbf{p} \text{ is independent of } \mathbf{x} \\ &= m\frac{\partial \Phi}{\partial \mathbf{x}}\ .\end{aligned}$$

Substituting for $\partial H/\partial \mathbf{p}$ and $\partial H/\partial \mathbf{x}$,

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}}\cdot\left(f\,\frac{\mathbf{p}}{m}\right) - \frac{\partial}{\partial \mathbf{p}}\cdot\left(fm\,\frac{\partial \Phi}{\partial \mathbf{x}}\right) &= 0 \\ \therefore\quad \frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial f}{\partial \mathbf{x}} - m\,\frac{\partial \Phi}{\partial \mathbf{x}}\cdot\frac{\partial f}{\partial \mathbf{p}} &= 0\end{aligned}$$

because $\mathbf{p}$ is independent of $\mathbf{x}$, and because $\partial\Phi/\partial\mathbf{x}$ is independent of $\mathbf{p}$ since $\Phi \equiv \Phi(\mathbf{x},t)$.

But the momentum $\mathbf{p} = m\,\mathrm{d}\mathbf{x}/\mathrm{d}t$ and the acceleration is $\frac{1}{m}\mathrm{d}\mathbf{p}/\mathrm{d}t = -\,\partial\Phi/\partial\mathbf{x}$ (the gradient of the potential).

$$\therefore\quad \frac{\partial \Phi}{\partial \mathbf{x}} = -\,\frac{1}{m}\,\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\ .$$

$$\text{So,} \quad \begin{aligned}\frac{\partial f}{\partial t} + \frac{m}{m}\,\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{x}} - m\left(-\frac{1}{m}\,\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right)\cdot\frac{\partial f}{\partial \mathbf{p}} &= 0 \\ \therefore\quad \frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{p}} &= 0\ .\end{aligned}$$

The left-hand side is the differential $\mathrm{d}f/\mathrm{d}t$. So,

$$\boxed{\frac{\partial f}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}\cdot\frac{\partial f}{\partial \mathbf{p}} \equiv \frac{\mathrm{d}f}{\mathrm{d}t} = 0} \qquad (2.55)$$

— the collisionless Boltzmann equation.

While this equation is called the collisionless Boltzmann equation (or CBE) in stellar dynamics, in Hamiltonian dynamics it is known as Liouville's theorem.

2.14 The implications of the Collisionless Boltzmann Equation

The collisionless Boltzmann equation tells us that $\mathrm{d}f/\mathrm{d}t = 0$. This means that the density in phase space, f, does not change with time for a test particle. Therefore if we follow a star in orbit, the density f in 6-dimensional phase space around the star is constant.

This simple result has important implications. If a star moves inwards in a galaxy as it follows its orbit, the density of stars in space increases (because the density of stars in the galaxy is greater closer to the centre). $\mathrm{d}f/\mathrm{d}t = 0$ then tells us that the spread of stellar velocities around the star will increase to keep f constant. Therefore the velocity dispersion around the star increases as the star moves inwards. The velocity dispersion is therefore larger in regions of the galaxy where the density of stars is greater. Conversely, if a star moves out from the centre, the density of stars around it will decrease and the velocity dispersion will decrease to keep f constant.

The collisionless Boltzmann equation, and the Poisson equation (which is the gravitational analogue of Gauss's law in electrostatics) together constitute the basic equations of stellar dynamics:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \ , \qquad \nabla^2\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}) \ , \tag{2.56}$$

where f is the distribution function, t is time, $\Phi(\mathbf{x}, t)$ is the gravitational potential at point $\mathbf{x}$, $\rho(\mathbf{x}, t)$ is the mass density at point $\mathbf{x}$, and G is the constant of gravitation.

The collisionless Boltzmann equation applies because star-star encounters do not change the motions of stars significantly over the lifetime of a galaxy, as was shown in Section 2.5. Were this not the case and the system were collisional, the CBE would have to be modified by adding a "collisional term" on the right-hand side.

Though f is a density in phase space, the full form of the collisionless Boltzmann equation does not necessarily have to be written in terms of $\mathbf{x}$ and $\mathbf{p}$. We can express $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$ in any set of six variables in phase space. You should remember that f is always taken to be a density in six-dimensional phase space, even in situations where it is a function of fewer variables. For example, if f happens to be a function of energy alone, it is not the same as the density in energy space.

2.15 The Collisionless Boltzmann Equation in Cylindrical Coordinates

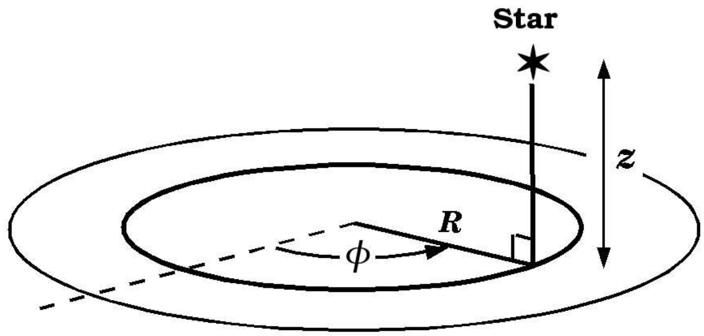
[This section is not examinable.]

So far we have considered Cartesian coordinates (x, y, z, v_x, v_y, v_z). However, the form

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t}\frac{\partial f}{\partial v_i}\right) = 0 \ ,$$

for the collisionless Boltzmann equation of Equation 2.46 applies to any coordinate system.

For a galaxy, it is often more convenient to use cylindrical coordinates with the centre of the galaxy as the origin.



The coordinates of a star are (R, ϕ, z). A cylindrical system is particularly useful for spiral galaxies like our own where the $z = 0$ plane is set to be the Galactic plane. (Note the use of a lower-case ϕ as a coordinate angle, whereas elsewhere we have used a capital Φ to denote the gravitational potential.)

The collisionless Boltzmann equation in this system is

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\mathrm{d}R}{\mathrm{d}t}\frac{\partial f}{\partial R} + \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\partial f}{\partial \phi} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial f}{\partial z} + \frac{\mathrm{d}v_R}{\mathrm{d}t}\frac{\partial f}{\partial v_R} + \frac{\mathrm{d}v_\phi}{\mathrm{d}t}\frac{\partial f}{\partial v_\phi} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\frac{\partial f}{\partial v_z}$$

$$= 0 \ , \tag{2.57}$$

where v_R, v_ϕ, and v_z are the components of the velocity in the R, ϕ, z directions.

We need to replace the differentials of the velocity components with more convenient terms. $\mathrm{d}v_R/\mathrm{d}t$, $\mathrm{d}v_\phi/\mathrm{d}t$ and $\mathrm{d}v_z/\mathrm{d}t$ are related to the acceleration $\mathbf{a}$ (but are not actually the components of the acceleration for the R and ϕ directions). The velocity and acceleration in terms of these differentials in a cylindrical coordinate system are

$$\begin{aligned}
\mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}t}\hat{\mathbf{e}}_R + R\frac{\mathrm{d}\phi}{\mathrm{d}t}\hat{\mathbf{e}}_\phi + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{e}}_z \\
\mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right)\hat{\mathbf{e}}_R + \left(2\frac{\mathrm{d}R}{\mathrm{d}t}\frac{\mathrm{d}\phi}{\mathrm{d}t} + R\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2}\right)\hat{\mathbf{e}}_\phi \\
&\quad + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\hat{\mathbf{e}}_z
\end{aligned} \tag{2.58}$$

where $\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$ are unit vectors in the R, ϕ and z directions (a standard result for any cylindrical coordinate system, and for *any* velocity, acceleration or force, gravitational or any other kind: see Appendix C5). Representing the velocity as $\mathbf{v} = v_R\hat{\mathbf{e}}_R + v_\phi\hat{\mathbf{e}}_\phi + v_z\hat{\mathbf{e}}_z$ and equating coefficients of the unit vectors,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = v_R \ , \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{v_\phi}{R} \ , \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = v_z \ . \tag{2.59}$$

The acceleration can be related to the gravitational potential Φ with $\mathbf{a} = -\boldsymbol{\nabla}\Phi$ (because the only forces acting on the star are those of gravity). In a cylindrical coordinate system,

$$\boldsymbol{\nabla} \equiv \hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi\frac{1}{R}\frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z} \ . \tag{2.60}$$

Using this result and equating coefficients, we obtain,

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2} - R\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = -\frac{\partial\Phi}{\partial R}\,, \qquad 2\,\frac{\mathrm{d}R}{\mathrm{d}t}\,\frac{\mathrm{d}\phi}{\mathrm{d}t} + R\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = -\frac{1}{R}\,\frac{\partial\Phi}{\partial\phi}\,, \qquad \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -\frac{\mathrm{d}\Phi}{\mathrm{d}z}$$

Rearranging these and substituting for $\mathrm{d}R/\mathrm{d}t$, $\mathrm{d}\phi/\mathrm{d}t$ and $\mathrm{d}z/\mathrm{d}t$ from 2.59, we obtain,

$$\frac{\mathrm{d}v_R}{\mathrm{d}t} = -\frac{\partial\Phi}{\partial R} + \frac{v_\phi^2}{R}\,, \qquad \text{and} \qquad \frac{\mathrm{d}v_z}{\mathrm{d}t} = -\frac{\partial\Phi}{\partial z}\,,$$

and with some more manipulation,

$$\begin{aligned}\frac{\mathrm{d}v_\phi}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(R\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big) = \frac{\mathrm{d}R}{\mathrm{d}t}\,\frac{\mathrm{d}\phi}{\mathrm{d}t} + R\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = v_R\,\frac{v_\phi}{R} + \Big(-\frac{1}{R}\,\frac{\partial\Phi}{\partial\phi} - 2\,\frac{\mathrm{d}R}{\mathrm{d}t}\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\Big)\\ &= \frac{v_R\,v_\phi}{R} - \frac{1}{R}\,\frac{\partial\Phi}{\partial\phi} - 2\,v_R\,\frac{v_\phi}{R} = -\frac{1}{R}\,\frac{\partial\Phi}{\partial\phi} - \frac{v_R\,v_\phi}{R}\,. \qquad (2.61)\end{aligned}$$

Substituting these into Equation 2.57, we obtain,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + v_R\,\frac{\partial f}{\partial R} + \frac{v_\phi}{R}\,\frac{\partial f}{\partial\phi} + v_z\,\frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial\Phi}{\partial R}\right)\frac{\partial f}{\partial v_R}&\\ - \frac{1}{R}\left(v_R v_\phi + \frac{\partial\Phi}{\partial\phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial\Phi}{\partial z}\,\frac{\partial f}{\partial v_z} = 0\,, \qquad (2.62)\end{aligned}$$

This is the collisionless Boltzmann equation in cylindrical coordinates. This form relates f to observable parameters $(R, \phi, z, v_R, v_\phi, v_z)$ and the potential Φ.

In many practical cases, particularly spiral galaxies, Φ will be independent of ϕ, so $\partial\Phi/\partial\phi = 0$ (but not if we include spiral arms where the potential will be slightly deeper).

2.16 Orbits of Stars in Galaxies

2.16.1 The character of orbits

The term orbit is used to describe the trajectories of stars within galaxies, even though they are very different to Keplerian orbits such as those of planets in the Solar System. The orbits of stars in a galaxy are usually not closed paths and in general they are three dimensional (they do not lie in a plane). They are often complex. In general they are highly chaotic, even if the galaxy is in equilibrium.

The orbit of a star in a spherical potential, to consider the simplest example, is confined to a plane perpendicular to the angular momentum vector of the star. It is, however, not a closed path and has an appearance that is usually described as a rosette. In axisymmetric potentials (e.g. an oblate elliptical galaxy) the orbit is confined to a plane that precesses. This plane is inclined to the axis of symmetry and rotates about the axis. The orbit within the plane is similar to that in a spherical potential.

Triaxial potentials can have orbits that are much more complex. Triaxial potentials often have the tendency to tumble about one axis, which leads to chaotic star orbits.

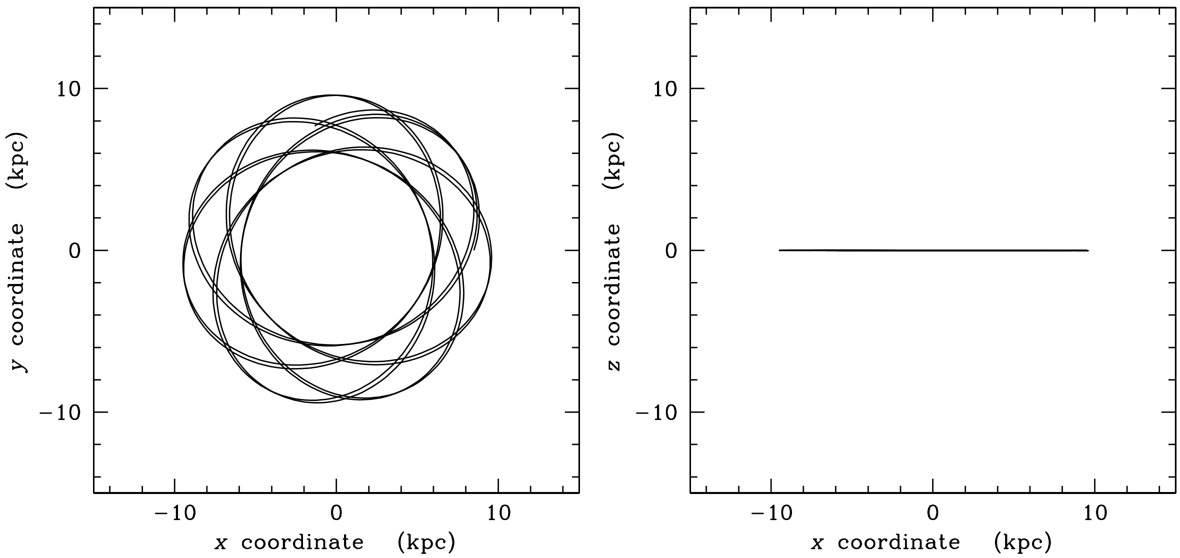


Figure 2.2: An example of the orbit of a star in a spherical potential. An example star has been put into an orbit in the $x - y$ plane. Its orbit follows a "rosette" pattern, but it remains in the $x - y$ plane. [These diagrams were plotted using data generated assuming a Plummer potential: the potential lacks a deep central cusp.]

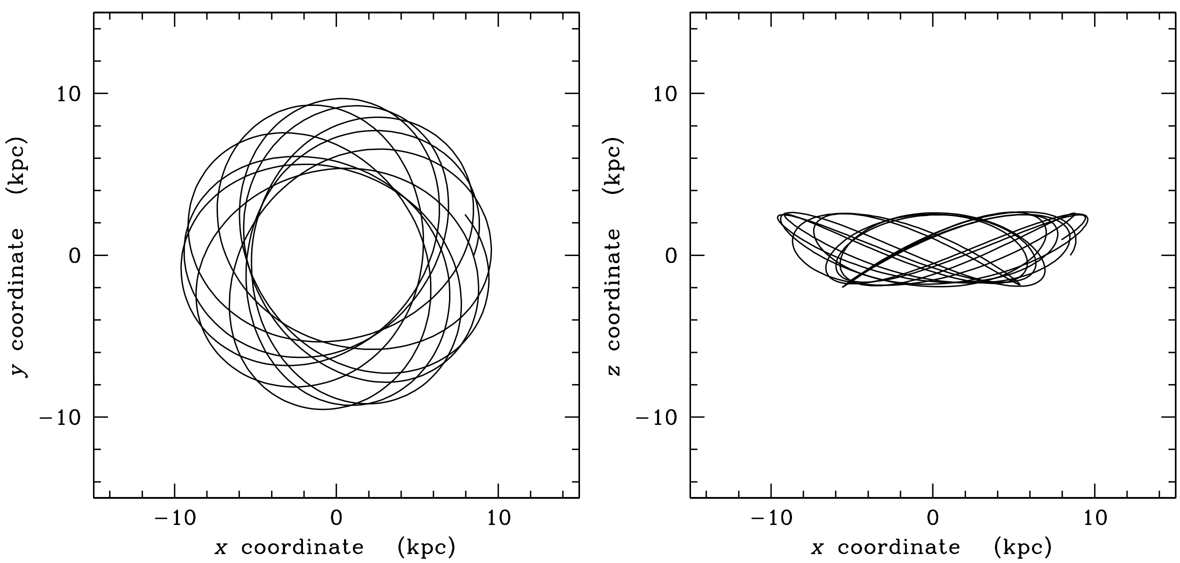


Figure 2.3: The orbit of a star in a flattened (oblate) potential. An example star has been put into an orbit inclined to the $x - y$ plane. The galaxy is flattened in the z direction with an axis ratio of 0.7. The orbit follows a "rosette" pattern, but the plane of the orbit precesses. This illustrates the trajectory of a star in an oblate elliptical galaxy, for example.

2.16.2 The chaotic nature of many orbits

In chaotic systems, stars that initially move along similar paths will diverge, eventually moving along very different orbits. The divergence in their paths is exponential in time, which is the technical definition of chaos in dynamical systems. Their motion shows

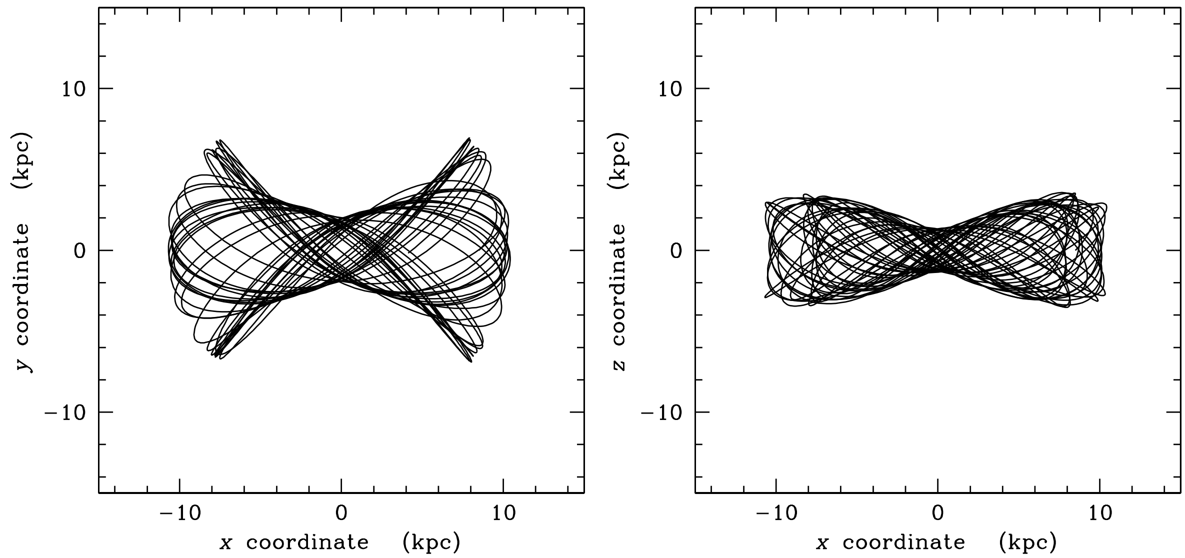


Figure 2.4: The orbit of a star in a triaxial potential. An example star has been put into an orbit inclined to the $x-y$ plane. The galaxy has different dimensions (different scale sizes) in each of the x, y and z directions. The orbit is complex and it maps out a region of space. This illustrates the trajectory of a star in a triaxial elliptical galaxy, for example. (This simulation extends over a longer time period than those of Figures 2.2 and 2.3.)

a stretching and folding in phase space. This can be so even if there is no collective motion of stars at all (f in equilibrium).

This stretching and folding in phase space can be appreciated using an analogy. When making bread, a baker's dough behaves essentially as a fluid. Dough is incompressible, but that does not prevent the baker stretching it in one direction and shrinking it in others, and then folding it back. So while the dough keeps much the same overall shape, particles initially nearby within it can be dispersed to widely different parts of it, through the repeated stretching and folding. The same stretching and folding operation can take place for stars in phase space. In fact it appears that phase space is typically riddled with regions where f gets stretched in one direction while being shrunk in others. Thus nearby orbits tend to diverge, and the divergence is exponential in time.

Simulations show that the timescale for divergence (the e-folding time) is $T_{diverge} \sim T_{cross}$, the crossing time, and gets shorter for higher star densities.

However, in some special cases, there is no chaos. These systems are said to be *integrable.*

If the dynamics is confined to one real-space dimension (hence two phase-space dimensions) then no stretching-and-folding can happen, and orbits are regular. So in a spherical system all orbits are regular. In addition, there are certain potentials (usually referred to as Stäckel potentials) where the dynamics decouples into three effectively one-dimensional systems; so if some equilibrium f generates a Stäckel potential, the orbits will stay chaos-free. Also, small perturbations of non-chaotic systems tend to produce only small regions of chaos,[1] and orbits may be well described through

[1]If you ever come across the 'KAM theorem', that's basically it.

perturbation theory.

2.16.3 Integrals of the motion

To solve the collisionless Boltzmann equation for stars in a galaxy, we need further constraints on the position and velocity. This can be done using *integrals of the motion*. These are simply functions of the star's position $\mathbf{x}$ and velocity $\mathbf{v}$ that are constant along its orbit. They are useful in potentials $\Phi(\mathbf{x})$ that are constant over time. The distribution function f is also constant along the orbit and can be written as a function of integrals of the motion.

Examples of integrals of the motion are:

- The total energy. The mechanical energy E of a particular star in a potential is constant over time, so $E(\mathbf{x}, \mathbf{v}) = \frac{1}{2}mv^2 + m\Phi(\mathbf{x})$. Because this is dependent on the mass of the star, it is more normal to work with the energy per unit mass, which will be written as E_m here. So $E_m = \frac{1}{2}v^2 + \Phi$ is a constant.

- In an axisymmetric potential (e.g. our Galaxy), the z-component of the angular momentum, L_z, is conserved. Therefore L_z is an integral of the motion in such a potential.

- In a spherical potential, the total angular momentum $\mathbf{L}$ is constant. Therefore $\mathbf{L}$ is an integral of the motion in this potential, and the x, y and z components of $\mathbf{L}$ are each integrals of the motion.

An orbit is said to be *regular* if it has as many isolating integrals that can define the orbit unambiguously as there are spatial dimensions.

2.16.4 Isolating integrals and integrable systems

The collisionless Boltzmann equation tells us that $\mathrm{d}f/\mathrm{d}t = 0$ (Section 2.14). As was discussed earlier, if we move with a star in its orbit, f is constant locally as the star passes through phase space at that instant in time. But if the system is in a steady state (the potential is constant over time), f is constant along the star's path *at all times*. This means that the orbits of stars map out constant values of f.

An integral of the motion for a star (e.g. energy per unit mass, E_m) is constant (by definition). They therefore define a 5-dimensional hypersurface in 6-dimensional phase space. The motion of a star is confined to that 5-dimensional surface in phase space. Therefore f is constant over that hypersurface.

A different value of the isolating integral (e.g. a different value of E_m) will define a different hypersurface. In turn, f will be different on this surface. So f is a function of the isolating integral, i.e. $f(x, y, z, v_x, v_y, v_z) = \mathrm{fn}(I_1)$ where I_1 is an integral of the motion. I_1 here "isolates" a hypersurface. Therefore the integral of the motion is known as an *isolating integral*.

Integrals that fail to confine orbits are called "non-isolating" integrals. A system is integrable if we can define isolating integrals that enable the orbit to be determined.

In integrable systems there are significant simplifications. Each orbit is (i) confined to a three-dimensional toroidal subspace of six-dimensional phase space, and (ii) fills its torus evenly.[2] Phase space itself is filled by nested orbit-carrying tori—they have to be nested, since orbits can't cross in phase space. Therefore the time-average of each orbit is completely specified once we have specified which torus it is on; this takes three numbers for each orbit, and these are called 'isolating integrals' – they are constants for each orbit of course. Think of the isolating integrals as a coordinate system that parameterises orbital tori; transformations to a different set of isolating integrals is like a coordinate transformation.

If isolating integrals exist, then any f that depends only on them will automatically satisfy the collisionless Boltzmann equation. Conversely, since orbits fill their tori evenly, any equilibrium f cannot depend on location *on* the tori, it can only depend on the tori themselves, i.e., on the isolating integrals. This result is known as the Jeans Theorem.

2.16.5 The Jeans Theorem

The Jeans Theorem is an important result in stellar dynamics that states the importance of integrals of the motion in solving the collisionless Boltzmann equation for gravitational potentials that do not change with time. It was named after its discoverer, the English astronomer, physicist and mathematician Sir James Hopwood Jeans (1877–1946). [3]

It states that any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of the motion in the galaxy's potential, and any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation.

This means that in a potential that does not change with time, we can express the collisionless Boltzmann equation in terms of integrals of motion, and then solve for the distribution function f in terms of those integrals of motion. We can then convert the solution of f in terms of the integrals to a solution for f in terms of the space and velocity coordinates. For example, if the energy per unit mass E_m and total angular momentum components L_x and L_y are constant for each star in some potential, then we can solve for f uniquely as a function of E_m, L_x and L_y. Then we can convert from E_m, L_x and L_y to give f as a function of (x, y, z, v_x, v_y, v_z).

You should be wary of Jeans' theorem, especially when people tacitly assume it, because as we saw, it assumes that the system is integrable, which is in general not the case.

2.17 Spherical Systems

2.17.1 Solving for *f* in spherical galaxies

The Jeans Theorem does apply in spherical systems of stars, such as spherical elliptical galaxies. As a consequence, f can depend on (at most) three integrals of motion in a spherical system. The simplest case is for f to be a function of the energy of the stars only. (Since we are considering bound systems, $f = 0$ for $E > 0$ always: any

[2]These two statements are important results from Hamiltonian dynamical systems which we won't try to prove here. But the statements that follow in this section are straightforward consequences of (i) and (ii).

[3]Much of this work was published by Jeans in the Monthly Notices R.A.S., 76, 70, 1915.

stars that did have $E > 0$ will have escaped from the galaxy.) To find an equilibrium solution, we only have to satisfy Poisson's equation $\nabla^2\Phi = 4\pi G\rho$.

The total energy of a star of mass m moving with a velocity $\mathbf{v}$ is $E = \frac{1}{2}mv^2 + m\Phi$, where Φ is the gravitational potential at the point where the star is situated. Here it is more convenient to use the *energy per unit mass* $E_m = \frac{1}{2}v^2 + \Phi$.

A spherical galaxy can be described very simply by a spherical polar coordinate system (r, θ, ϕ) with the origin at the centre. Poisson's equation relates the Laplacian of the gravitational potential Φ at a point to the local mass density ρ as $\nabla^2\Phi = 4\pi G\rho$. In a spherical polar coordinate system the Laplacian of any scalar function $A(r, \theta, \phi)$ is

$$\nabla^2 A \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A}{\partial\phi^2} \quad (2.63)$$

(a standard result from vector calculus: see Appendix C).

In a spherically symmetric galaxy that does not change with time, the potential is a function of the radial distance r from the centre only. So $\partial\Phi/\partial\theta = 0$ and $\partial\Phi/\partial\phi = 0$. Therefore,

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) . \quad (2.64)$$

Substituting this into the Poisson equation,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho . \quad (2.65)$$

The distribution function f is related to the number density n of stars by

$$n = \int f\,\mathrm{d}^3\mathbf{v}$$

(from Equation 2.39), and in this case f is a function of energy per unit mass: $f = f(E_m)$. We can relate this to the density ρ using $\rho = \overline{m}\,n$ where $\overline{m}$ is the mean mass of a star, giving,

$$\rho = \overline{m}\int f\,\mathrm{d}^3\mathbf{v} . \quad (2.66)$$

Note that here we are assuming that mass is in the form of stars only: there is no dark matter here. This integral is over all velocities. We can convert from $\mathrm{d}^3\mathbf{v}$ to $\mathrm{d}v$, where $v \equiv |\mathbf{v}|$ by considering a thin spherical shell in a space defined by the three velocity components, which gives $\mathrm{d}^3\mathbf{v} = 4\pi v^2\mathrm{d}v$. So

$$\rho = 4\pi\,\overline{m}\int fv^2\,\mathrm{d}v . \quad (2.67)$$

Note that this integration is over all velocities *at a particular point* in the galaxy. It can be performed over velocity at each and every point in the galaxy, so this ρ is $\rho(r)$. (Here v is the magnitude of the velocity vector $\mathbf{v}$, so $v \geq 0$ always.)

We must determine the limits on this integral. For any particular point in the galaxy (i.e. any value of r), the minimum possible velocity is $v = 0$, which occurs when a star moving on a radial orbit reaches its maximum distance from the centre at that point. The maximum velocity at this position occurs when a star has the greatest possible energy ($E_m = 0$, which would allow a star to move out from the point to arbitrary distance: any star with energy per unit mass $E_m > 0$ will be moving faster than the escape velocity for that location and will escape from the galaxy). Therefore, using $E_m = \frac{1}{2}v^2 + \Phi(r)$, the maximum velocity (for $E_m = 0$) is $v = \sqrt{-2\Phi(r)}$ (remember that $\Phi(r)$ is negative, so that $-2\Phi(r)$ is positive). So the integration at this point in space is from velocity $v = 0$ to $\sqrt{-2\Phi(r)}$. So,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = (4\pi)^2\, G\,\overline{m}\int_0^{\sqrt{-2\Phi(r)}} f\, v^2\,\mathrm{d}v \ . \tag{2.68}$$

We can convert this integral over velocity to an integral over energy per unit mass. $E_m = \frac{1}{2}v^2 + \Phi$ gives $\mathrm{d}E_m = v\,\mathrm{d}v$ at a fixed position (and hence for a constant $\Phi(r)$). The maximum possible energy per unit mass is 0 (because any stars with $E_m > 0$ will have escaped long ago), while the minimum possible value at a radius r would be given by a star that is stationary at that point: $E_m = \Phi(r)$ (which is of course negative). So, at any radius r,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = (4\pi)^2\sqrt{2}\, G\,\overline{m}\int_{\Phi(r)}^{0}\sqrt{E_m - \Phi(r)}\ f(E_m)\,\mathrm{d}E_m \ , \tag{2.69}$$

on substituting $v = \sqrt{2(E_m - \Phi)}$.

It is usual in Equation 2.68 to take $f(v)$ as given and to try to solve for $\Phi(r)$ and hence $\rho(r)$; this is a nonlinear differential equation. In Equation 2.69 we would normally take Φ as given, and try to solve for $f(E_m)$; this is a linear integral equation.

There are $f(E_m)$ models in the literature, and you can always concoct a new one by picking some $\rho(r)$, computing $\Phi(r)$ and then solving Equation 2.69 numerically. Note that the velocity distribution is isotropic for any $f(E_m)$. If f depends on other integrals of motion, say angular momentum L or its z component, or both – thus $f(E_m, L^2, L_z)$ – then the velocity distribution will be anisotropic, and there are many examples of these around too.

2.17.2 Example of a spherical, isotropic distribution function: the Plummer potential

As discussed earlier, the Plummer potential has a gravitational potential Φ and a mass density ρ at a radial distance r from the centre that are given by

$$\Phi(r) = -\frac{GM_{tot}}{\sqrt{r^2+a^2}} \ , \qquad \rho(r) = \frac{3M_{tot}}{4\pi}\frac{a^2}{(r^2+a^2)^{5/2}} \ , \qquad \text{(2.32) and (2.33)}$$

where M_{tot} is the total mass of the galaxy and a is a constant. The distribution function f for the Plummer model is related to the density ρ by Equation 2.67. It can be shown that these $\Phi(r)$ and $\rho(r)$ forms give a solution,

$$f(E_m) = \frac{24\sqrt{2}}{7\pi^3}\frac{a^2}{G^5 M_{tot}^4\,\overline{m}}\,(-E_m)^{\frac{7}{2}} \ . \tag{2.70}$$

This can be verified by inserting in Equation 2.68, which can be done with some mathematical work. This result gives the distribution function f as a function only of the energy per unit mass E_m. To calculate f for any point (x, y, z, v_x, v_y, v_z) in phase space, we need only to calculate E_m from these coordinates and then calculate the value of f associated with that E_m.

2.17.3 Example of a spherical, isotropic distribution function: the isothermal sphere

The isothermal sphere was introduced in Section 2.10.3. The density profile was given in Equation 2.38. The isothermal sphere is defined by analogy with a Maxwell-Boltzmann gas, and therefore the distribution function as a function of the energy per unit mass E_m is given by,

$$f(E_m) = \frac{n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{E_m}{\sigma^2}\right) = \frac{n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\frac{1}{2}v^2 + \Phi}{\sigma^2}\right) , \qquad (2.71)$$

where σ is a velocity dispersion and acts in this distribution like a temperature does in a gas. n_0 is a constant. Integrating over velocities gives

$$\begin{aligned} n(r) &= \int f \, \mathrm{d}^3\mathbf{v} = \int_0^\infty f \, . \, 4\pi \, v^2 \, \mathrm{d}v = \frac{4\pi \, n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\Phi}{\sigma^2}\right) \int_0^\infty v^2 \exp\left(-\frac{v^2}{2\sigma^2}\right) \mathrm{d}v \\ &= \frac{4\pi \, n_0}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\Phi}{\sigma^2}\right) . \left(\frac{\sigma^3}{4}\sqrt{8\pi}\right) = n_0 \exp\left(-\frac{\Phi(r)}{\sigma^2}\right) , \qquad (2.72) \end{aligned}$$

using the standard integral $\int_0^\infty x^2 \mathrm{e}^{-ax^2} \mathrm{d}x = \sqrt{\pi/a^3}/4$. (Note that the isothermal distribution includes stars with speeds from $v = 0$ to ∞, so our integration is from zero to infinity in this case, instead of the 0 to $\sqrt{-2\Phi}$ used in the more realistic general case in Equation 2.68. In practice, no stable galaxy will have stars with speeds larger than $\sqrt{-2\Phi(r)}$ at a point a distance r from the centre because these stars would be travelling faster than escape velocity.)
Converting this to density $\rho(r)$ using $\rho = \overline{m}\, n$, where $\overline{m}$ is the mean mass of the stars, we get,

$$\rho(r) = \rho_0 \exp\left(-\frac{\Phi(r)}{\sigma^2}\right) , \quad \text{and equivalently,} \quad \Phi(r) = -\,\sigma^2 \ln\left(\frac{\rho(r)}{\rho_0}\right) , \qquad (2.73)$$

where ρ_0 is a constant (with $\rho_0 \equiv n_0\, \overline{m}$). Using this, Poisson's equation ($\nabla^2\Phi = 4\pi G\rho$) in a spherically symmetric potential on substituting for $\mathrm{d}\Phi/\mathrm{d}r$ becomes,

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}}{\mathrm{d}r}\left(-\,\sigma^2 \ln\left(\frac{\rho}{\rho_0}\right)\right)\right) = 4\pi G\, \rho \, ,$$

which simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) = -\,\frac{4\pi G}{\sigma^2}\, r^2\, \rho \, . \qquad (2.74)$$

This is a second-order differential equation in ρ and r. One solution to this is

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2} , \qquad (2.38)$$

which is known as the singular isothermal sphere. As already commented in Section 2.10.3, the isothermal sphere has infinite mass! (A side effect of this is that the boundary condition $\Phi(\infty) = 0$ cannot be used, which is why we needed the redundant-looking constant ρ_0 in Equations 2.71 and 2.72.) Nevertheless, this isothermal sphere density profile is often used as a model, with some large-r truncation assumed, for the dark matter haloes of disc galaxies.

The same $\rho(r)$ can be produced by many different f, all having different velocity distributions.

2.18 Observable and Measurable Quantities

The phase space distribution f is usually very difficult to measure observationally, because of the challenges of measuring the distribution of stars over space and particularly over velocity. Velocity components along the line of sight can be measured spectroscopically from a Doppler shift. However, transverse velocity components cannot be measured directly for galaxies beyond our own (or at least beyond the Local Group). As a function of seven variables (six of the phase space, plus time), the function f can be awkward to compute theoretically. It is therefore more convenient to use quantities related to f.

The number density n of stars in space can be measured observationally by counting more luminous stars for nearby galaxies, or from the observed intensity of light for more distant galaxies. Star counts combined with estimates of the distances of individual stars can provide n as a function of position within our Galaxy. For a distant galaxy, converting the intensity along the line of sight of the *integrated light* from large numbers of stars in to number densities – a process known as deprojection – requires assumptions about the stellar populations and their three-dimensional distribution. Nevertheless, reasonable attempts can be made in many cases.

Spectroscopy provides mean velocities $\langle v_l \rangle$ along the line of sight through a galaxy, and the widths of absorption lines provide velocity dispersions σ_l along the line of sight. These mean velocities will be weighted according to the numbers of stars. In our own Galaxy, it is usually possible to calculate the mean velocity and the dispersions of the velocity about the mean value.

The velocity dispersion is an important concept. Observational data can provide the velocity dispersion in perpendicular directions, at least within our own Galaxy. So in our Galaxy, at each point in space, we might have the mean values of the velocity components, $\langle v_R \rangle$, $\langle v_\phi \rangle$ and $\langle v_z \rangle$, in the R, ϕ and z directions, plus the dispersions σ_R, σ_ϕ and σ_z, of the velocity components about their mean values, expressed as standard deviations. Velocity dispersions are often represented by the *velocity dispersion ellipsoid*. This is an idealised representation of the dispersions as a three-dimensional ellipsoid, where the distance from the origin in any particular direction is the size of the velocity dispersion in that direction.

When working with galaxies other than our own, we sometimes consider *isotropic velocity distributions* (particularly for elliptical galaxies). In this case, the velocity dispersions in each direction are the same: $\sigma_r = \sigma_\theta = \sigma_\phi$, using a spherical coordinate system (r, θ, ϕ) here. We may abbreviate these equal components simply as σ, and these will be the same as the velocity dispersion σ_l along our line of sight. If the mean velocities are zero, i.e. $\langle v_r \rangle = \langle v_\theta \rangle = \langle v_\phi \rangle = 0$ (as will be the case if the galaxy is in a steady state and has no net rotation), then $\sigma^2 = \langle v_r^2 \rangle = \langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle$. The mean of the square of the space velocity will be $\langle v^2 \rangle = \langle v_r^2 \rangle + \langle v_\theta^2 \rangle + \langle v_\phi^2 \rangle$. Therefore, $\langle v^2 \rangle = 3\sigma^2$ in a steady-state with no net rotation.

Although researchers often use the velocity dispersions in three perpendicular directions (such as σ_R, σ_ϕ and σ_z), a full description of the dynamics of stars requires a *velocity dispersion tensor* σ_{ij}. This will discussed in detail later.

It is therefore much more convenient to calculate quantities involving number densities n, mean velocities and velocity dispersions from f. These quantities can then be compared with observations more directly. A series of equations called the Jeans Equations allow this to be done.

2.19 The Jeans Equations

The Jeans Equations relate number densities, mean velocities, velocity dispersions and the gravitational potential. They were first used in stellar dynamics by Sir James Jeans in 1919.

It is useful to derive equations for the quantities

$$\begin{aligned} n &= \int f \, \mathrm{d}^3\mathbf{v} \ , \\ n \, \langle v_i \rangle &= \int v_i \, f \, \mathrm{d}^3\mathbf{v} \ , \\ n \, \sigma_{ij}^2 &= \int (v_i - \langle v_i \rangle) \, (v_j - \langle v_j \rangle) \, f \, \mathrm{d}^3\mathbf{v} \ , \end{aligned} \tag{2.75}$$

by taking moments of the collisionless Boltzmann equation (expressed in the Cartesian variables x_i and v_i). σ_{ij} is a velocity dispersion tensor: it is discussed in more detail below.

The collisionless Boltzmann equation gives (Equation 2.46)

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \right) = 0 \, ,$$

or equivalently,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0 \, ,$$

on substituting for the components of acceleration from $\mathrm{d}\mathbf{v}/\mathrm{d}t = -\boldsymbol{\nabla}\Phi$.

To derive the first of the Jeans Equations, we shall consider the zeroth moment by integrating this equation over all velocities.

$$\int \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) \mathrm{d}^3\mathbf{v} = \int 0 \, . \, \mathrm{d}^3\mathbf{v} \ . \tag{2.76}$$

$$\therefore \quad \int \frac{\partial f}{\partial t} \, \mathrm{d}^3\mathbf{v} + \sum_{i=1}^{3} \int v_i \frac{\partial f}{\partial x_i} \, \mathrm{d}^3\mathbf{v} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \, \mathrm{d}^3\mathbf{v} = 0$$

(with the right hand being zero because it is a definite integral). Some of these terms can be simplified, particularly by noting the integration is performed over all velocities at each position and time.

$$\text{But} \qquad \int \frac{\partial f}{\partial t}\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial}{\partial t} \int f\, \mathrm{d}^3\mathbf{v} \qquad \text{because } t \text{ and } v_i\text{'s are independent}$$

$$= \frac{\partial n}{\partial t} \qquad \text{because } n = \textstyle\int f\, \mathrm{d}^3\mathbf{v},$$

$$\text{and} \qquad \int v_i \frac{\partial f}{\partial x_i}\, \mathrm{d}^3\mathbf{v} \;=\; \int \frac{\partial (v_i f)}{\partial x_i}\, \mathrm{d}^3\mathbf{v} \qquad \text{because } v_i\text{'s and } x_i\text{'s are independent}$$

$$= \frac{\partial}{\partial x_i} \int v_i\, f\, \mathrm{d}^3\mathbf{v} \qquad \text{because } x_i\text{'s and } v_i\text{'s are independent}$$

$$= \frac{\partial\, (n\, \langle v_i \rangle\,)}{\partial x_i} \qquad \text{on substituting } n\langle v_i \rangle = \textstyle\int v_i f \mathrm{d}^3\mathbf{v}$$

$$\text{and} \qquad \int \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i}\, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i}\, \mathrm{d}^3\mathbf{v} \qquad \text{because } x_i\text{'s and } \Phi \text{ are independent of } v_i\text{'s}$$

$$= \frac{\partial \Phi}{\partial x_i}\, (0) \qquad \text{because } f \longrightarrow 0 \text{ as } |v_i| \longrightarrow \infty \quad \text{(by analogy with the divergence theorem)}$$

$$= 0 \; .$$

Substituting for these terms,

$$\boxed{\frac{\partial n}{\partial t} + \sum_{i=1}^{3} \frac{\partial n \langle v_i \rangle}{\partial x_i} = 0 \; ,} \tag{2.77}$$

which is a continuity equation. This is the first of the Jeans Equations.

To derive the second of the Jeans Equations, we consider the first moment of the collisionless Boltzmann equation by multiplying by v_i and integrating over all velocities. Multiplying the C.B.E. throughout by v_i, we obtain,

$$v_i \frac{\partial f}{\partial t} + v_i \sum_{j=1}^{3} v_j \frac{\partial f}{\partial x_j} - v_i \sum_{j=1}^{3} \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} = 0 \; , \tag{2.78}$$

where the summation is performed over an integer j because we have introduced a velocity component v_i. Note that the use of v_i means that we are considering one particular velocity component only at this stage, i.e. one value of i from $i = 1, 3$. Integrating this over all velocities,

$$\int \left(v_i \frac{\partial f}{\partial t} + \sum_{j=1}^{3} v_i\, v_j \frac{\partial f}{\partial x_j} - \sum_{j=1}^{3} v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} \right) \mathrm{d}^3\mathbf{v} = \int 0 \,.\, \mathrm{d}^3\mathbf{v} \; . \tag{2.79}$$

$$\therefore \quad \int v_i \frac{\partial f}{\partial t}\, \mathrm{d}^3\mathbf{v} + \sum_{j=1}^{3} \int v_i\, v_j \frac{\partial f}{\partial x_j}\, \mathrm{d}^3\mathbf{v} - \sum_{j=1}^{3} \int v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\mathbf{v} = 0 \; .$$

$$\text{But} \quad \int v_i \, \frac{\partial f}{\partial t} \, \mathrm{d}^3\mathbf{v} \;=\; \int \frac{\partial (v_i \, f)}{\partial t} \, \mathrm{d}^3\mathbf{v} \quad \text{because } v_i \text{ and } t \text{ are independent}$$

$$= \frac{\partial}{\partial t} \int v_i \, f \, \mathrm{d}^3\mathbf{v}$$

$$= \frac{\partial}{\partial t}(n \, \langle v_i \rangle) \qquad \text{because } n \, \langle v_i \rangle = \textstyle\int v_i \, f \, \mathrm{d}^3\mathbf{v},$$

$$\text{and} \quad \int v_i \, v_j \, \frac{\partial f}{\partial x_j} \, \mathrm{d}^3\mathbf{v} \;=\; \int \frac{\partial}{\partial x_j}(v_i \, v_j \, f) \, \mathrm{d}^3\mathbf{v} \quad \text{because } v_i \text{ and } v_j \text{ are independent of } x_i$$

$$= \frac{\partial}{\partial x_j} \int v_i v_j f \, \mathrm{d}^3\mathbf{v} \quad \text{because } x_i \text{ and } v_i\text{'s are independent}$$

$$= \frac{\partial \, (n \, \langle v_i v_j \rangle \,)}{\partial x_j} \quad \text{on substituting } n\langle v_i v_j \rangle = \textstyle\int v_i v_j f \, \mathrm{d}^3\mathbf{v}$$

$$\text{and} \quad \int v_i \, \frac{\partial \Phi}{\partial x_j} \, \frac{\partial f}{\partial v_j} \, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_j} \int v_i \, \frac{\partial f}{\partial v_j} \, \mathrm{d}^3\mathbf{v} \quad \text{because } x_j\text{'s and } \Phi \text{ are independent of } v_i\text{'s}$$

$$\text{But} \quad \frac{\partial (v_i f)}{\partial v_j} \;=\; v_i \, \frac{\partial f}{\partial v_j} \;+\; f \, \frac{\partial v_i}{\partial v_j} \qquad \therefore \quad v_i \, \frac{\partial f}{\partial v_j} \;=\; \frac{\partial (v_i f)}{\partial v_j} \;-\; f \, \frac{\partial v_i}{\partial v_j}$$

$$\text{and} \quad \frac{\partial v_i}{\partial v_j} \;=\; 1 \quad \text{if } i = j$$

$$= 0 \quad \text{if } i \neq j \quad \text{because } v_i \text{ and } v_j \text{ are independent if } i \neq j$$

$$\therefore \quad \frac{\partial v_i}{\partial v_j} \;=\; \delta_{ij}$$

$$\therefore \quad v_i \, \frac{\partial f}{\partial v_j} \;=\; \frac{\partial (v_i f)}{\partial v_j} \;-\; \delta_{ij} \, f \quad .$$

$$\text{So} \quad \int v_i \, \frac{\partial \Phi}{\partial x_j} \, \frac{\partial f}{\partial v_j} \, \mathrm{d}^3\mathbf{v} \;=\; \frac{\partial \Phi}{\partial x_j} \int \left(\frac{\partial (v_i f)}{\partial v_j} \;-\; \delta_{ij} \, f \right) \, \mathrm{d}^3\mathbf{v}$$

$$= \frac{\partial \Phi}{\partial x_j} \left(\int \frac{\partial (v_i f)}{\partial v_j} \, \mathrm{d}^3\mathbf{v} \;-\; \delta_{ij} \int f \, \mathrm{d}^3\mathbf{v} \right)$$

$$= \frac{\partial \Phi}{\partial x_j} \left(0 \;-\; \delta_{ij} \, n \right) \; \text{because } v_i f \longrightarrow 0 \text{ as } |v_i| \longrightarrow \infty$$

$$= -\frac{\partial \Phi}{\partial x_j} \, \delta_{ij} \, n \quad .$$

Substituting for these terms,

$$\frac{\partial (n \langle v_i \rangle)}{\partial t} \;+\; \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big(n \langle v_i v_j \rangle \big) \;-\; \sum_{j=1}^{3} \left(-\frac{\partial \Phi}{\partial x_i} \, \delta_{ij} \, n \right) \;=\; 0 \quad .$$

So,

$$\boxed{\frac{\partial (n \langle v_i \rangle)}{\partial t} \;+\; \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big(n \langle v_i v_j \rangle \big) \;=\; -\frac{\partial \Phi}{\partial x_i} \, n \;,} \tag{2.80}$$

for each of $i = 1, 2, 3$. This is the second of the Jeans Equations.

We need to introduce a tensor velocity dispersion σ_{ij} defined so that

$$n\,\sigma_{ij}^2 \;\equiv\; \int \big(v_i - \langle v_i\rangle\big)\,\big(v_j - \langle v_j\rangle\big)\; f\; \mathrm{d}^3\mathbf{v}\ , \tag{2.81}$$

for $i, j = 1, 3$ (see Equation 2.75 above). This is used to represent the spread of velocities in each direction. It is a symmetric tensor and we can choose some coordinate system in which it is diagonal (i.e. $\sigma_{11} \neq 0,\ \sigma_{22} \neq 0,\ \sigma_{33} \neq 0$, but all the other elements are zero). This is known as the *velocity ellipsoid*. For example, in a cylindrical coordinate system, we might use elements such as $\sigma_{RR}, \sigma_{\phi\phi}$ and σ_{zz}. If the velocity dispersion is isotropic, $\sigma_{11} = \sigma_{22} = \sigma_{33}$, which we might simplify by writing as σ only.

Rearranging Equation 2.81 and multiplying out,

$$\begin{aligned}
\sigma_{ij}^2 &= \frac{1}{n}\int \big(v_i - \langle v_i\rangle\big)\,\big(v_j - \langle v_j\rangle\big)\; f\; \mathrm{d}^3\mathbf{v} \\
&= \frac{1}{n}\int \Big(\ v_iv_j - v_i\langle v_j\rangle - \langle v_i\rangle v_j + \langle v_i\rangle\,\langle v_j\rangle\ \Big)\; f\; \mathrm{d}^3\mathbf{v} \\
&= \frac{1}{n}\int v_iv_j\; f\; \mathrm{d}^3\mathbf{v} \;-\; \frac{1}{n}\int v_i\,\langle v_j\rangle\; f\; \mathrm{d}^3\mathbf{v} \;-\; \frac{1}{n}\int \langle v_i\rangle\, v_j\; f\; \mathrm{d}^3\mathbf{v} \\
&\qquad + \frac{1}{n}\int \langle v_i\rangle\,\langle v_j\rangle\; f\; \mathrm{d}^3\mathbf{v} \\
&= \frac{1}{n}\int v_iv_j\; f\; \mathrm{d}^3\mathbf{v} \;-\; \langle v_j\rangle\,\frac{1}{n}\int v_i\; f\; \mathrm{d}^3\mathbf{v} - \;\langle v_i\rangle\,\frac{1}{n}\int v_j\; f\; \mathrm{d}^3\mathbf{v} \\
&\qquad + \langle v_i\rangle\,\langle v_j\rangle\frac{1}{n}\int f\; \mathrm{d}^3\mathbf{v} \qquad \text{because } \langle v_i\rangle \text{ and } \langle v_j\rangle \text{ are constants} \\
&= \langle v_iv_j\rangle \;-\; \langle v_j\rangle\,\langle v_i\rangle \;-\; \langle v_i\rangle\,\langle v_j\rangle \;+\; \langle v_i\rangle\,\langle v_j\rangle \qquad \text{from Equation 2.75.}
\end{aligned}$$

So,

$$\boxed{\ \sigma_{ij}^2 \;=\; \langle v_iv_j\rangle \;-\; \langle v_i\rangle\,\langle v_j\rangle\ }\ . \tag{2.82}$$

This can be used to find $\langle v_iv_j\rangle$ using

$$\langle v_iv_j\rangle \;=\; \sigma_{ij}^2 \;+\; \langle v_i\rangle\,\langle v_j\rangle\ .$$

Substituting for $\langle v_iv_j\rangle$ into the second of the Jeans Equations (Equation 2.80),

$$\frac{\partial(n\langle v_i\rangle)}{\partial t} \;+\; \sum_{j=1}^{3}\left[\frac{\partial}{\partial x_j}\big(n\sigma_{ij}^2\big) \;+\; \frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big)\right] \;=\; -\,\frac{\partial\Phi}{\partial x_i}\,n\ ,$$

for each of $i = 1, 2$ and 3. Therefore,

$$\langle v_i\rangle\,\frac{\partial n}{\partial t} \;+\; n\,\frac{\partial\langle v_i\rangle}{\partial t} \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\sigma_{ij}^2\big) \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big) \;=\; -\,\frac{\partial\Phi}{\partial x_i}\,n\ . \tag{2.83}$$

We can eliminate the 1st and 4th terms using the first of the Jeans Equations (Equation 2.77). Multiplying that equation throughout by $\langle v_i\rangle$,

$$\begin{aligned}
\langle v_i\rangle\,\frac{\partial n}{\partial t} \;+\; \langle v_i\rangle\,\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\langle v_j\rangle\big) &\;=\; 0 \\
\therefore\quad \langle v_i\rangle\,\frac{\partial n}{\partial t} \;+\; \sum_{j=1}^{3}\langle v_i\rangle\,\frac{\partial}{\partial x_j}\big(n\langle v_j\rangle\big) &\;=\; 0 \qquad (2.84)
\end{aligned}$$

$$\text{But} \quad \frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big) \;=\; \langle v_i\rangle\,\frac{\partial}{\partial x_j}\big(n\langle v_j\rangle\big) \;+\; n\,\langle v_j\rangle\,\frac{\partial\langle v_i\rangle}{\partial x_j}$$

Substituting for $\langle v_i\rangle\,\frac{\partial}{\partial x_j}\big(n\langle v_j\rangle\big)$,

$$\langle v_i\rangle\,\frac{\partial n}{\partial t} \;+\; \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big) \;-\; n\,\langle v_j\rangle\,\frac{\partial\langle v_i\rangle}{\partial x_j}\right) \;=\; 0$$

$$\therefore \quad \langle v_i\rangle\,\frac{\partial n}{\partial t} \;+\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\langle v_i\rangle\langle v_j\rangle\big) \;=\; n\sum_{j=1}^{3}\langle v_j\rangle\,\frac{\partial\langle v_i\rangle}{\partial x_j}\;.$$

Substituting this into Equation 2.83, we obtain,

$$\boxed{n\,\frac{\partial\langle v_i\rangle}{\partial t} \;+\; n\sum_{j=1}^{3}\langle v_j\rangle\,\frac{\partial\langle v_i\rangle}{\partial x_j} \;=\; -\,n\,\frac{\partial\Phi}{\partial x_i} \;-\; \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\big(n\sigma_{ij}^2\big)\;,} \tag{2.85}$$

where i can be any of 1, 2 or 3. This is a third Jeans Equation.
This can also be expressed as,

$$\frac{\mathrm{d}\langle\mathbf{v}\rangle}{\mathrm{d}t} \;=\; -\,\boldsymbol{\nabla}\Phi \;-\; \frac{1}{n}\,\boldsymbol{\nabla}\cdot(n\boldsymbol{\sigma}^2)\;, \tag{2.86}$$

where $\langle\mathbf{v}\rangle$ is the mean velocity vector, t is the time, Φ is the potential, n is the number density of stars and $\boldsymbol{\sigma}^2$ represents the tensor σ_{ij}^2. Note that here $\mathrm{d}/\mathrm{d}t$ is not $\partial/\partial t$, but

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\;\left(\equiv\frac{D\mathbf{v}}{Dt}\right) \;=\; \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\;, \tag{2.87}$$

which is sometimes called the convective derivative; it is also sometimes written as D/Dt to emphasise that it is not simply $\frac{\partial}{\partial t}$.

This is similar to the Euler equation in fluid dynamics. An ordinary fluid has

$$\frac{\mathrm{d}\langle\mathbf{v}\rangle}{\mathrm{d}t} \;=\; -\,\boldsymbol{\nabla}\Phi \;-\; \frac{\boldsymbol{\nabla}P}{\rho} \;+\; \text{viscous terms}\;, \tag{2.88}$$

where the pressure P arises because of the high rate of molecular encounters, which also leads to the equation of state, and P is isotropic. In stellar dynamics, the stars behave like a fluid in which $\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\sigma}^2)$ behaves like a pressure, but it is anisotropic. Indeed, this anisotropy is the reason that it is represented by a tensor, whereas in an ordinary fluid the pressure is represented by a scalar. A related fact is that in the flow of an ordinary fluid the particle paths and streamlines coincide, whereas stellar orbits and the streamlines $\langle\mathbf{v}\rangle$ do not generally coincide.

The Jeans Equations have been represented here in terms of the number density n of stars. However, it is possible to work instead with the mean mass density in space ρ instead of n. The Jeans Equations can be used for all stars in a galaxy, but sometimes they are used for subpopulations in our Galaxy (e.g. G dwarfs, K giants). If they are used for subpopulations, Φ remains the total gravitational potential of all matter (including dark matter), but the velocities and number densities refer to the subpopulations.

2.20 The Jeans Equations in an Axisymmetric System, e.g. the Galaxy

Using cylindrical coordinates (R, ϕ, z) and assuming axisymmetry (so $\partial/\partial\phi = 0$), the second Jeans Equation is

$$\frac{\partial}{\partial t}\big(n\langle v_R\rangle\big) + \frac{\partial}{\partial R}\big(n\langle v_R^2\rangle\big) + \frac{\partial}{\partial z}\big(n\langle v_R v_z\rangle\big) + \frac{n}{R}\Big(\langle v_R^2\rangle - \langle v_\phi^2\rangle\Big) = -\,n\,\frac{\partial\Phi}{\partial R}$$

for the R direction,

$$\frac{\partial}{\partial t}\big(n\langle v_\phi\rangle\big) + \frac{\partial}{\partial R}\big(n\langle v_R v_\phi\rangle\big) + \frac{\partial}{\partial z}\big(n\langle v_\phi v_z\rangle\big) + \frac{2\,n}{R}\langle v_R v_\phi\rangle = 0$$

for the ϕ direction,

$$\frac{\partial}{\partial t}\big(\,n\langle v_z\rangle\,\big) + \frac{\partial}{\partial R}\big(\,n\langle v_R v_z\rangle\,\big) + \frac{\partial}{\partial z}\big(\,n\langle v_z^2\rangle\,\big) + \frac{n\,\langle v_R v_z\rangle}{R} = -\,n\,\frac{\partial\Phi}{\partial z}$$

for the z direction. (2.89)

In a steady state, where the potential does not change with time, we can use $\partial/\partial t = 0$. This axisymmetric form of the second Jeans Equation is useful in spiral galaxies, such as our own Galaxy, provided that we neglect any change in the potential in the ϕ direction (although there might be a ϕ dependence if the potential is deeper in the spiral arms).

Meanwhile, the first of the Jeans Equations in a cylindrical coordinate system with axisymmetric symmetry ($\partial/\partial\phi = 0$) is,

$$\frac{\partial n}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\big(\,Rn\langle v_R\rangle\,\big) + \frac{\partial}{\partial z}\big(\,n\langle v_z\rangle\,\big) = 0\ . \qquad (2.90)$$

2.21 The Jeans Equations in a Spherically Symmetric System

The second Jeans Equation in a steady-state ($\partial/\partial t = 0$) spherically-symmetric ($\partial/\partial\phi = 0$, $\partial/\partial\theta = 0$) galaxy in a spherical polar coordinate system (r, θ, ϕ) is

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\,n\,\langle v_r^2\rangle\,\Big) + \frac{n}{r}\Big[\,2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\,\Big] = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\ . \qquad (2.91)$$

This might be used, for example, for a spherical elliptical galaxy.

(This equation will just be stated here: it will not be derived. It can be derived from the collisionless Boltzmann equation expressed in spherical coordinates using similar methods to the Cartesian Jeans equations discussed above.)

We can calculate the gradient in the potential in this spherical case very simply. Using the general result that the acceleration due to gravity is $\mathbf{g} = -\boldsymbol{\nabla}\Phi$, that $g = GM(r)/r^2$ in a spherically symmetric system where $M(r)$ is the mass interior to the radius r, and that $\boldsymbol{\nabla}\Phi = \mathrm{d}\Phi/\mathrm{d}r$ in a spherical system, we get $\mathrm{d}\Phi/\mathrm{d}r = GM(r)/r^2$.

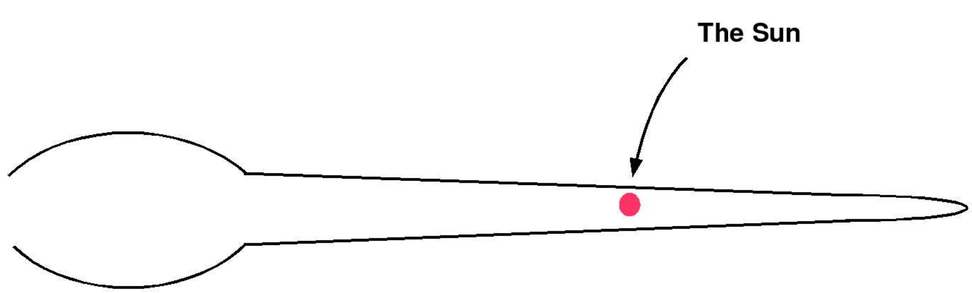
As a simple test to see whether this really does work, let us make a crude model of our Galaxy's stellar halo. We shall assume that the halo is spherical, assume a logarithmic potential of the form $\Phi(r) = v_0^2 \ln r$ where v_0 is a constant, assume that the velocity components are isotropic (i.e. $\langle v_r^2\rangle = \langle v_\theta^2\rangle = \langle v_\phi^2\rangle = \sigma^2$, where σ is a constant), and assume that the star number density of the halo can be approximated by $n(r) \propto r^{-l}$ where l is a constant. Equation 2.91 now becomes

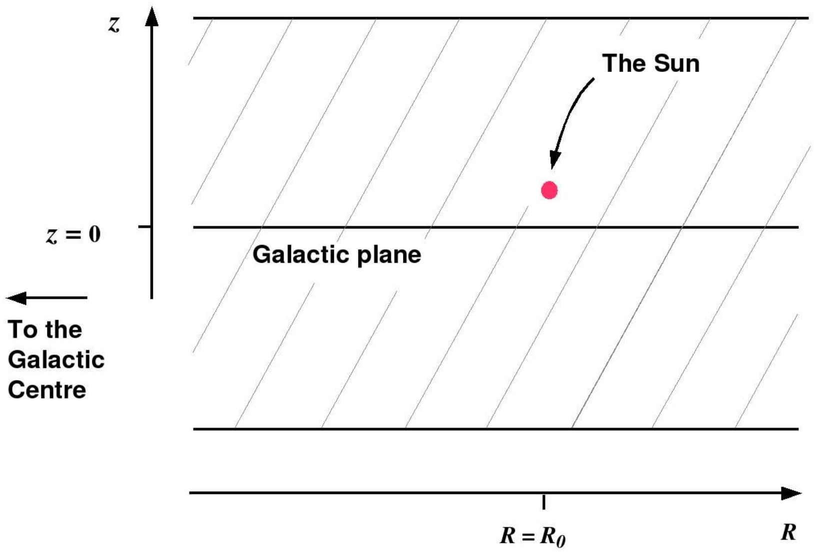
$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\sigma^2\right) + \frac{n}{r}(0) = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\,, \qquad \therefore \quad \sigma^2\,\frac{\mathrm{d}n}{\mathrm{d}r} = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\,,$$

on substituting for the velocity terms. Using $\Phi = v_0^2 \ln r$ and $n(r) = kr^{-l}$ (where k is a constant) we get $\mathrm{d}\Phi/\mathrm{d}r = v_0^2/r$ and $\mathrm{d}n/\mathrm{d}r = -lkr^{-l-1}$. Substituting for these and cancelling r, we obtain $\sigma = v_0/\sqrt{l}$. For the Milky Way's halo, observations show that $n \propto r^{-3.5}$ (i.e. $l = 3.5$), while v_0 as measured from gas on circular orbits is 220 km s^{-1}, and rotation is negligible (which is a requirement for $v_\theta^2 = \sigma^2$ etc.). So we expect $\sigma \simeq 220\ \mathrm{km\,s^{-1}}/\sqrt{3.5} \simeq 120\ \mathrm{km\,s^{-1}}$. And it is.

2.22 Example of the Use of the Jeans Equations: the Surface Mass Density of the Galactic Disc

The Jeans Equations can be applied to our Galaxy to measure the surface mass density of the Galactic disc at the solar distance from the centre using observations of the velocities of stars along the line of sight lying some distance above or below the Galactic plane. The surface mass density is the mass per unit area of the disc when viewed from from a great distance. It is expressed in units of kg m^{-2}, or more commonly solar masses per square parsec ($M_\odot \mathrm{pc}^{-2}$). This analysis is important because it allows the quantity of dark matter in the disc to be estimated. Determining whether there is dark matter in the Galactic disc is a very important constraint on the nature of dark matter.





The second Jeans Equation in a cylindrical coordinate system (R, ϕ, z) centred on the Galaxy, with $z = 0$ in the plane and $R = 0$ at the Galactic Centre states for the z direction that

$$\frac{\partial(\, n\langle v_z\rangle\,)}{\partial t} + \frac{\partial(\, n\langle v_R v_z\rangle\,)}{\partial R} + \frac{\partial(\, n\langle v_z^2\rangle\,)}{\partial z} + \frac{n\langle v_R v_z\rangle}{R} = -\,n\,\frac{\partial \Phi}{\partial z}$$

(Equation 2.89), where n is the star number density, v_R and v_z are the velocity components in the R and z directions, $\Phi(R, z, t)$ is the Galactic gravitational potential and t is time.
The Galaxy is in a steady state, so n does not change with time. Therefore the first term $\partial(n\langle v_z\rangle)/\partial t = 0$.
Observations show that

$$\frac{\partial(\, n\langle v_R v_z\rangle\,)}{\partial R} \simeq 0 \quad \text{and} \quad \frac{n\langle v_R v_z\rangle}{R} \simeq 0 \ ,$$

as is to be expected because of the cancelling of positive and negative terms of the z-components of the velocity. Therefore,

$$\frac{\partial(\, n\langle v_z^2\rangle\,)}{\partial z} = -\,n\,\frac{\partial \Phi}{\partial z} \ . \qquad (2.92)$$

$\langle v_z^2\rangle$ is the mean square velocity in the direction perpendicular to the Galactic plane.
Poisson's equation gives $\nabla^2\Phi = 4\pi G\rho$, where ρ is the mass density at a point. In cylindrical coordinates the Laplacian is

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial \phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

(from Appendix C).
If we observe stars directly above and below the Galactic plane, all at the same Galactocentric radius R, we can neglect the $\partial\Phi/\partial R$ and $\partial^2\Phi/\partial\phi^2$ terms.

$$\therefore \quad \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho \ ,$$

and substituting for $\partial\Phi/\partial z$ from Equation 2.92 into this,

$$\frac{\partial}{\partial z}\left(-\frac{1}{n}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\right) = 4\pi G\rho \ .$$

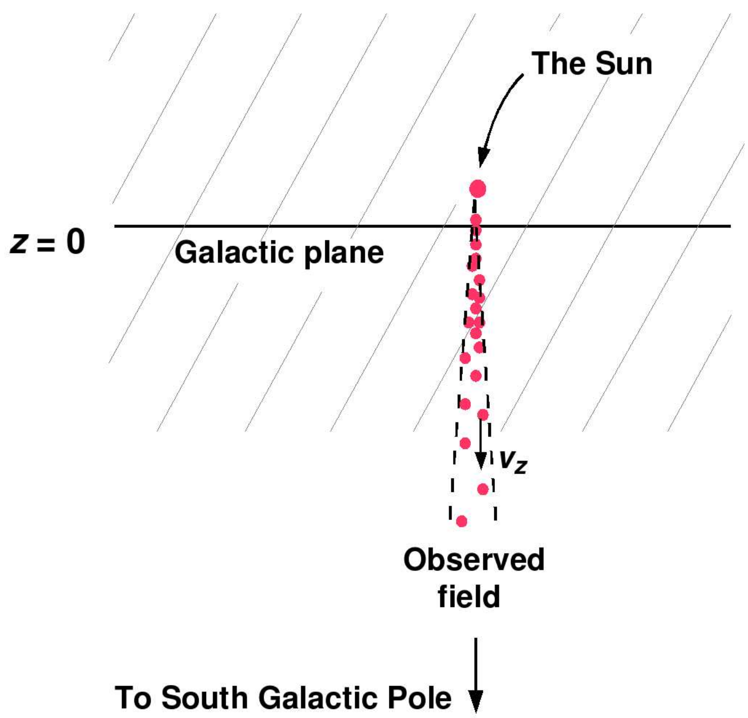
Integrating perpendicular to the Galactic plane from $-z$ to z, the surface mass density within a distance z of the plane at a Galactocentric radius R is

$$\begin{aligned}\Sigma(R,z) &= \int_{-z}^{z}\rho\,\mathrm{d}z' = \int_{-z}^{z}\frac{1}{4\pi G}\frac{\partial}{\partial z}\left(-\frac{1}{n}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\right)\mathrm{d}z' \\ &= -\frac{1}{4\pi G}\left[\frac{1}{n}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\right]_{z'=-z}^{z} = -\frac{1}{2\pi Gn}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\Bigg|_z \ ,\end{aligned}$$

assuming symmetry about $z = 0$. Therefore the surface mass density within a distance z of the plane at the solar Galactocentric radius R_0 is

$$\Sigma(R_0,z) = -\frac{1}{2\pi Gn}\frac{\partial}{\partial z}\left(n\langle v_z^2\rangle\right)\Bigg|_z \ . \qquad (2.93)$$

If the star densities n can be measured as a function of height z from the plane and if the z-component of the velocities v_z can be measured as spectroscopic radial velocities, we can solve for $\Sigma(R_0, z)$ as a function of z. This gives, after modelling the contribution from the dark matter halo, the mass density of the Galactic disc.



The analysis can be performed on some subclass of stars, such as G giants or K giants. In this case the number density n of stars in space is that of the subclass. Number counts of stars towards the Galactic poles, combined with estimates of the distances to individual stars, give n. Spectroscopic observations give radial velocities (the velocity components along the line of sight) through the Doppler effect. By observing towards the Galactic poles, the radial velocities are the same as the v_z components.

This analysis gives $\Sigma(R_0, z)$ as a function of z. The value increases with z as a greater proportion of the stars of the disc are included, until all the disc matter is included. $\Sigma(R_0, z)$ will still increase slowly with z beyond this as an increasing amount mass from the dark matter halo is included. Indeed, it is necessary to determine the contribution $\Sigma_d(R_0)$ from the disc alone to the observed data. An additional complication is that in measuring $\partial(n\langle v_z^2\rangle)/\partial z$ as a function of z, we are dealing with the differential of observed quantities. This means that the effects of observational errors can be considerable.

The first measurement of the surface density of the Galactic disc was carried out by Oort in 1932. More modern attempts were carried out in the 1980s by Bahcall and by Kuijken and Gilmore. There has been considerable debate about the interpretation of results. Early studies claimed evidence of dark matter in the Galactic disc, but more recently some consensus has developed that there is little dark matter in the disc itself, apart from the contribution from the dark matter halo that extends into the disc. A modern value is $\Sigma_d(R_0) = 50 \pm 10\ M_\odot\ \mathrm{pc}^{-2}$. The absence of significant dark matter in the disc indicates that dark matter does not follow baryonic matter closely on a small scale, a very important result.

2.23 N-body Simulations

An alternative approach that can be adopted to study the dynamics of stars in galaxies is to use N-body simulations. In these analyses, the system of stars is represented by a large number of particles and computer modelling is used to trace the dynamics of these particles under their mutual gravitational attractions. These simulations usually determine the positions of the test particles at each of a series of time steps, calculating the changes in their positions between each step. It is possible to add further particles to trace gas and dark matter, although the gas must be made collisional.

The individual particles in a galaxy simulation, however, do not correspond to stars. It is impossible to represent every star in a galaxy in N-body simulations. In practice, the limits on computational power allow only $\sim 10^5$ to 10^8 particles, whereas there may be as many as 10^{12} stars in the galaxies being modelled. The appropriate interpretation of simulation particles is as Monte-Carlo samplers of the distribution function f.

In Section 2.6 we found that the ratio of the relaxation time to the crossing time was $T_{relax}/T_{cross} \sim N/12 \ln N$ for a system of N particles. It follows that a system that is modelled computationally by too few particles will have a relaxation time that is too short, and may experience the effects of two-body encounters. As a consequence, the particles in N-body simulations have to be made collisionless artificially. The standard way of doing this is to replace the $1/r$ gravitational potential of each particle by $(r^2+a^2)^{-\frac{1}{2}}$, which amounts to smearing out the mass on the 'softening length' scale a.

Early N-body computer codes performed calculations for each time step that took a time that depended on the number N of particles as N^2. Modern codes perform faster computations by treating distant particles differently to nearby particles. "Tree codes" combine the effects of number of distant particles together. This increases their efficiency and the computation times scale as only $N \ln N$.

N-body simulations are widely used now to study the evolution of galaxies, and an active research area at present is to incorporate gas dynamics in them. In contrast to standard N-body methods, smoothed particle hydrodynamics (SPH) are often used to study the gas in galaxies. Modern simulations include the effects of dark matter alongside stars and gas. They can follow the collapse of clumps of dark matter in the early Universe that led to the formation of galaxies. Simulations can also follow the growth of structure in the Universe as gravitational attraction produced the clustering of galaxies observed today. N-body simulations can model the effects of large changes in gravitational potentials, whereas analytical methods can find these more challenging.