Homotopy types of algebraic varieties

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1 Introduction

Let k be a field. To any smooth and projective algebraic variety X over k one can associate certain *geometric* cohomology spaces $\overline{H}^*(X)$, which are finite dimensional verctor spaces over some coefficients field K. For example $H^*_{et}(\overline{X}, \mathbb{Q}_l)$ (the l-adic cohomology of $\overline{X} := X \otimes_k k^{sp}$), $H^*_{DR}(X)$ (the algebraic de Rham cohomology of X when e.g. $k = \mathbb{C}$), $H^*_{cris}(X)$ (the cristalline cohomology of X when k is of positive characteristic) .... These geometric cohomology theories $\overline{H}^*$ are the so-called *Weil cohomology theories*. They encode geometric properties of X, and are not suppose to *see* the arithmetic properties of the base field k. More precisely, the arithmetic nature of the base field k is not reflected in the spaces $\overline{H}^*(X)$ themselves but rather appears as the existence of natural additional structures on them. For example, $H^*_{et}(\overline{X}, \mathbb{Q}_l)$ comes equiped with a continuous action of the Galois group $Gal(k^{sp}/k)$, $H^*_{DR}(X)$ is endowed with a pure Hodge structure, $H^*_{cris}(X)$ has a structure of an F-isocristal over k .... The Tannakian formalism tells us furthermore that these additional structures are encoded in an action of a pro-algebraic group[1] $\mathbb{H}$ on the space $\overline{H}^*(X)$. The group $\mathbb{H}$ of course depends on the cohomology theory one chose, and in the example above is the foundamental group of the Tannakian categories of continuous finite dimension l-adic representations of $Gal(k^{sp}/k)$, of pure Hodge structures, of F-isocristals over k .... From these observations one extracts the following general principle.

Principle 1: *In algebraic geometry, (geometric) cohomology theories take their values in the category of linear representations of a certain pro-algebraic group* $\mathbb{H}$.

This principle can serve as the foundation of the theory of motives, as the category of motives is supposed to be the universal geometric cohomology theories for which $\mathbb{H}$ is the so-called *motivic Galois group.*

If we beleive that the (co)homology of an algebraic variety is only the *abelian part* of a more general homotopy type, it becomes very natural to extend the above principle to the following one.

[1]Sometimes these groups are not really groups but are rather *gerbes*, as for example in the case of cristalline cohomology. I will not make any difference in these notes and will do as if these gerbes were all trivial.

Principle 2: *For any Weil cohomology theory $\overline{H}^*$, and any smooth projective variety X, there exists a functorial* geometric homotopy type $\overline{h}(X)$, *of X with respect to the theory $\overline{H}^*$. This homotopy type $\overline{h}(X)$ (whatever it is) comes equiped with a natural action of the group $\mathbb{H}$, in such a way that the induced action on its cohomology $H^*(\overline{h}(X))$ gives back the linear representation $\overline{H}^*(X)$.*

This last principle is mathematically very imprecise, but its meaning is rather clear: the additional structures one finds on the cohomology of smooth projective varieties already exist naturally on a much richer object, their homotopy type. The group $\mathbb{H}$ being a pro-algebraic group, one sees that the object $\overline{h}(X)$, whatever it is, should not be discrete unless the action of $\mathbb{H}$ would not be interesting at all (an algebraic action of an algebraic group on a discrete set necesseraly factors throught the group of connected component, e.g. is trivial if that group is connected). In particular, one does not expect $\overline{h}(X)$ to be a topological space, a simplicial set or even a pro-simplicial set. One should rather look for a definition of $\overline{h}(X)$ in such a way that its homotopy groups $\pi_i(\overline{h}(X))$ are for example themselves pro-algebraic group over K.

Purpose of this note: *Explain the kind of structure $\overline{h}(X)$ really is, and show that principle 2 is satisfied in general.*

2 Stacks

Let C be a Grothendieck site, and $SPr(C)$ be the category of simplicial presheaves (i.e. presheaves of simplicial sets) on C. We refer to the talk of R. Jardine for the description of the local model structure on $SPr(C)$, for which the equivalences are the morphisms inducing isomorphisms on every homotopy sheaves.

Definition 2.0.1 *The* (homotopy) category of stacks *is the category $Ho(SPr(C))$.*

Recall that for a stack $F \in Ho(SPr(C))$, one can define its sheaf of connected component $\pi_0(F)$, which by definition is the sheaf on C associated to the presheaf $X \mapsto \pi_0(F(X))$. In the same way, for an object $X \in C$ and a 0-simplex $s \in F(X)$ one can define the sheaf $\pi_i(F, s)$, as the sheaf on C/X associated to the presheaf $(u : Y \to X) \mapsto \pi_i(F(Y), u^*(s))$.

A stack F is called n-truncated, if for all choice of $X \in C$ and $s \in F(X)$ one has $\pi_i(F, s) = 0$ for all $i > n$. One can easily show that the full sub-category of 0-truncated stacks in $Ho(SPr(C))$ is equivalent to the category of sheaves of sets on C. In the same way, the full sub-category of 1-truncated stacks is seen to be equivalent to the (homotopy) category of stacks in groupoids on C in the sense of Grothendieck (see R. Jardine's lecture). Keeping this in mind, an n-stack is simply an n-truncated object in $Ho(SPr(C))$, and general objects in $Ho(SPr(C))$ can be thought as an ∞-stack. This justifies the terminology *stack* to designe objects in $Ho(SPr(C))$.

The existence of the model structure on $SPr(C)$ implies several nice properties of the category $Ho(SPr(C))$, as for example existence of homotopy limits and homotopy colimits[2]. One can also show that $Ho(SPr(C))$ possesses internal Hom's, or in other words that stacks of morphisms exist. From a general point of view, $SPr(C)$ is a *model topos*, in the sense that it satisfies homotopy

[2]These are not really *properties* of $Ho(SPr(C))$, but rather additional structures. They are in any case properties of the *homotopy theory of stacks*.

analogs of Giraud's axioms characterizing categories of sheaves (see [HAGI]). This implies that the theory of stacks works in a very similar fashion than the theory of sheaves, and I will use this implicitely in the sequel of this note.

3 Schematic homotopy types

For this section I fix K a base field. The category of affine K-schemes will be denoted by Aff_K. The category Aff_K is endowed with the faithfully flat and quasi-compact topology, and gives rise to a Grothendieck site Aff_K^{ffqc}. The homotopy category of stacks over Aff_K^{ffqc} will be simply denoted by $Ho(SPr(K))$.

Definition 3.0.2 *A* schematic homotopy type (over K) *is a stack $F \in Ho(SPr(K))$ satisfying the following three conditions.*

- $\pi_0(F) = *$.
- *For any field extension $K \subset L$, and any point $s \in F(L)$, the sheaf $\pi_1(F, s)$ is represented by an affine group scheme over L.*
- *For any field extension $K \subset L$, any point $s \in F(L)$, and any $i > 1$, the sheaf $\pi_i(F, s)$ is represented by a unipotent affine group scheme over L.*

The full sub-category of $Ho(SPr(K))$ consisting of schematic homotopy types will be denoted by SHT/K.

The homotopy theory of schematic homotopy types behave well, in the sense that one can define homotopy fiber products, cohomology (with local coefficients), Postnikov decomposition, obstruction theory .... The standard properties and constructions of homotopy theory has reasonable analogs in SHT/K. We refer to [To1, Ka-Pa-To] for more details on the general theory.

Let us now assume that K is of characteristic zero. Any schematic homotopy type F possesses a *Levy decomposition*, defined in the following way. For simplicity we assume that F has a global point $* \longrightarrow F$. We can then consider the maximal pro-reductive quotient

$$\pi_1(F, *) \longrightarrow \pi_1(F, *)^{red}$$

as well as the induced morphism in SHT/K

$$\pi : F \longrightarrow K(\pi_1(F, *), 1) \longrightarrow K(\pi_1(F, *)^{red}, 1).$$

The homotopy fiber of the map π is denoted by F° and is the *universal reductive covering of F*. The natural morphism $F^\circ \longrightarrow F$ induces isomorphisms on all π_i for $i > 1$, and on the level of fundamental groups $\pi_1(F^\circ)$ is identified with the unipotent radical of $\pi_1(F, *)$. In particular, all of the homotopy sheaves $\pi_i(F^\circ)$ are unipotent.

The importance of the object F° comes from the existence of a *Curtis spectral sequence*

$$E_1^{p,q} \Rightarrow \pi_{p-q}(F^\circ),$$

for which the term $E_1^{p,q}$ only depends on the cohomology vector spaces $H^i(F^\circ, \mathbb{G}_a) := [F^\circ, K(\mathbb{G}_a, i)]$, and the differential d_1 only depends on the cup products in cohomology. This is a schematic analog of the Curtis spectral sequence in topology, relating homology and homotopy for nilpotent spaces. The existence of this spectral sequence is one of the most interesting feature of schematic homotopy types.

4 Schematic homotopy theories

We come back to our base field k, and we let $SmPr/k$ the category of smooth and projective geometrically connected algebraic varieties over k.

Definition 4.0.3 *A* schematic homotopy theory (over k and with coefficients in the field K) *is a functor*

$$h : SmPr/k \longrightarrow SHT/K$$

which satisfies the following two conditions.

1. $$\pi_i(h(Spec\ k)) = 0\ \forall\ i > 1.$$

2. *The natural morphism*
$$\pi_1(h(X)) \longrightarrow \pi_1(h(Spec\ k))$$
is surjective.

The first condition on h is equivalent to say that $h(Spec\ k)$ is a gerbe that will be denoted by $\mathbb{H}$. As I have already mentioned before I will do as if this gerbe were trivial (though in some example it is not), and therefore consider it simply as a pro-algebraic group over k. If $\mathbb{H}$ now denotes this group, then the condition on the functor h can also be written

$$h(Spec\ k) \simeq K(\mathbb{H}, 1).$$

Let us fix a schematic homotopy theory h as in Def. 4.0.3. For any $X \in SmPr/k$, one can consider the natural morphism $X \longrightarrow Spec\, k$, which induces a morphism in SHT/K

$$h(X) \longrightarrow K(\mathbb{H}, 1).$$

The homotopy fiber of this morphism will be denoted by $\overline{h}(X)$, and is called the *geometric part of the homotopy type* $h(X)$. Our condition 4.0.3 (2) insure that $\overline{h}(X)$ is a schematic homotopy type over K. Furthermore, $\overline{h}(X)$ comes equiped with a natural action of $\mathbb{H}$, and $h(X)$ can be identified with the homotopy quotient (i.e. the quotient stack)

$$h(X) \simeq [\overline{h}(X)/\mathbb{H}].$$

This gives another way of considering the schematic homotopy theory h, as being a functor

$$\overline{h} : SmPr/k \longrightarrow \mathbb{H} - SHT/K,$$

from $SmPr/k$ to the homotopy category of $\mathbb{H}$-equivariant schematic homotopy types over K. Giving h or $\overline{h}$ is equivalent.

5 Some examples

There exists a general procedure in order to construct schematic homotopy theories, based on an un-published homotopy version of Tannakian duality in which schematic homotopy types appear as Tannakian dual of certain *Tannakian model categories*. Rather than trying to explain this general process, I will rather describe some examples without mentioning how they are actually constructed.

I will need the notion of a local system on a schematic homotopy type F. By this I will mean a K-linear representation of the pro-alegbraic group $\pi_1(F)$[3]. Furthermore, for any such local system L on F, one can define $H^*(F, L)$, the cohomology of F with local coefficients (see [To1] for details).

5.1 Hodge theory

In this part we let $k = \mathbb{C}$ and $K = \mathbb{Q}$.

Recall that for any smooth and projective complex algebraic variety X, there exists a tensor category $VMHS(X)$, of variations of rational mixed Hodge structures on X. Moreover, $VMHS(X)$ is the heart of a triangulated category with t-structure $D_{MH}(X)$, of (bounded) mixed Hodge complexes on X. Recall that for an object $V \in VMHS(X)$, the absolute Hodge cohomology of X with coefficients in V is defined by

$$H^*_{Abs}(X, V) := Hom_{D_{MH}(X)}(1, V[n]).$$

Theorem 5.1.1 *For any $X \in SmPr/\mathbb{C}$, there exists a functorial schematic homotopy type $h(X) \in SHT/\mathbb{Q}$ satisfying the following two conditions.*

1. *There exists a natural equivalence of tensor categories*

$$\{Local\ systems\ on\ h(X)\} \simeq VMHS(X).$$

2. *For any local system L on $h(X)$, corresponding through the equivalence above to an object $V \in VMHS(X)$, there exists a natural isomorphism*

$$H^*(h(X), L) \simeq H^*_{Abs}(X, V),$$

compatible with extra structures such as cup products ....

The schematic homotopy type $h(X)$ of theorem 5.1.1 is called the *absolute Hodge homotopy type of X*. Both conditions of Thm. 5.1.1 can also be stated together as an equivalence of tensor traingulated categories with t-structures

$$D_{MH}(X) \simeq D_{Parf}(h(X)),$$

where $D_{Parf}(h(X))$ is a certain derived category of perfect complexes of $\mathcal{O}$-modules on the stack $h(X)$.

[3]Once again, sometimes $\pi_1(F)$ is only a gerbe, and the expression *linear representation* has then to be interpreted as *vector bundle*.

It is not hard to see using conditions (1) and (2) of Thm. 5.1.1 that one has

$$h(Spec\,\mathbb{C}) \simeq K(\mathbb{H}, 1)$$

where $\mathbb{H}$ is the Tannakian dual of the tensor category of rational mixed Hodge structures. Therefore, the functor $X \mapsto h(X)$ does define a schematic homotopy theory in the sense of Def. 4.0.3. The schematic homotopy type $\overline{h}(X)$ is now the *geometric Hodge homotopy type of* X, and comes equiped with an action of $\mathbb{H}$. Furthermore, for any local system L on $h(X)$, considered as a local system on $\overline{h}(X)$, the cohomology $H^*(\overline{h}(X), L)$ is isomorphic to the Betti cohomology $H^*(X^{top}, L)$ of the corresponding local system. The action of $\mathbb{H}$ induced on $H^*(\overline{h}(X), L)$ corresponds to a mixed Hodge structure on $H^*(X^{top}, L)$. Therefore, the object $\overline{h}(X)$, together with the action of $\mathbb{H}$ recover the Hodge theory on the cohomology of X. In the same way, one can extract from $\overline{h}(X)$ the nilpotent completion of the fundamental group of X, together with its mixed Hogde structure. In a way, the equivariant schematic homotopy type $\overline{h}(X)$ encodes all of the usual Hodge theoretic invariants of X.

5.2 Other classical theories

Let me mention that theorem 5.1.1 has analogs for other classical theories, such as l-adic cohomology or cristalline cohomology theory. The reader can reconstruct the statement by himself, simply by replacing variations of mixed Hodge structures by mixed l-adic sheaves, or F-isocristals. For details see [To1, Ol].

5.3 Motivic theory ?

Among all schematic homotopy theories, it is expected that there exists a universal one, the motivic schematic homotopy theory. Of course, it seems rather difficult to construct it without assuming certain usual conjectures (standard conjectures, vanishing conjectures ...) (thought one can always construct *something* in a formal way). The expected statement is the following motivic version of theorem 5.1.1.

Conjecture 5.3.1 *Let k be any field. There exists a schematic homotopy theory*

$$h : SmPr/k \longrightarrow SHT/\mathbb{Q}$$

satisfying the following properties.

1. *There is an equivalence of tensor categories*

$$\{Local\ systems\ on\ h(X)\} \simeq \mathcal{MM}(X),$$

where $\mathcal{MM}(X)$ is the tensor category of lisse mixed motivic sheaves on X.

2. *For any local system L on $h(X)$, corresponding through the equivalence above to an object $V \in \mathcal{MM}(X)$, there exists a natural isomorphism*

$$H^*(h(X), L) \simeq H^*_M(X, V),$$

where $H^*_M(X, V)$ is the motivic cohomology of X with coefficients in the motivic sheaf V.

From this conjecture one sees that $h(Spec\, k)$ should be of the form $K(\mathbb{H}, 1)$, for $\mathbb{H}$ a certain motivic Galois group (it is actually a gerbe) which is the Tannakian dual to the category $\mathcal{MM}(k)$ of mixed motives over k. For any $X \in SmPr/k$, one would then get a geometric part $\overline{h}(X)$, endowed with an action of $\mathbb{H}$, which encodes the motivic cohomological behaviour of X.

6 Perspectives

To finish with I will present a context of possible application of schematic homotopy theories.

Let k be any field, and let us suppose that we are given a schematic homotopy theory h and a smooth projective variety X. From the functoriality of h one gets by formal arguments a well defined map

$$\gamma : X(k) \longrightarrow \pi_0(\overline{h}(X)^{\mathbb{H}}),$$

where $\overline{h}(X)^{\mathbb{H}}$ is the simplicial set of homotopy fixed points of $\mathbb{H}$ on $\overline{h}(X)$ defined for example as the mapping space $Map(*, \overline{h}(X))$ in the category of $\mathbb{H}$-equivariant stacks. The set $\pi_0(\overline{h}(X)^{\mathbb{H}})$ can also be considered as the set of sections of the natural morphism

$$h(X) \longrightarrow h(Spec\, k)$$

up to homotopy.

Definition 6.0.2 *The* non-abelian Abel Jacobi map *(with respect to the theory h) is the map*

$$\gamma : X(k) \longrightarrow \pi_0(\overline{h}(X)^{\mathbb{H}})$$

defined above.

One interesting feature of the map γ is that it sends the set of rational points of X, which is a set without any structure, to the set $\pi_0(\overline{h}(X)^{\mathbb{H}})$ which can be expressed in purely homotopical data of X and seems therefore more structured than $X(k)$. Indeed, using a Postnikov decomposition of $\overline{h}(X)$ one constructs a natural spectral sequence

$$E_2^{p,q} = H^p(\mathbb{H}, \pi_q(\overline{h}(X))) \Rightarrow \pi_{q-p}(\overline{h}(X)^{\mathbb{H}}),$$

from the Hochschild cohomology of the group scheme $\mathbb{H}$ to the homotopy groups of $\overline{h}(X)^{\mathbb{H}}$. This spectral sequence induces a kind of *filtration* on the set $\pi_0(\overline{h}(X)^{\mathbb{H}})$, which by pull back along γ induces a kind of filtration on the set $X(k)$.

Precisely, for any integer $n \geq 0$, there exists an equivalence relation $\sim_n$ on $X(k)$, in such a way that $\sim_n$ is finer than $\sim_{n-1}$. By definition, for two rational point x and y in $X(k)$, one has $x \sim_n y$ if and only if the image of $\gamma(x)$ and $\gamma(y)$ by the projection

$$\pi_0(\overline{h}(X)^{\mathbb{H}}) \longrightarrow \pi_0((\overline{h}(X)_{\leq n})^{\mathbb{H}})$$

are the same (here $\overline{h}(X)_{\leq n}$ is the n-th Postnikov truncation of $\overline{h}(X)$). Furthemore, for any $n \geq 1$, if $x \sim_n y$, then there exists a natural obstruction

$$\delta_n(x, y) \in H^{n+1}(\mathbb{H}, \pi_{n+1}(\overline{h}(X)))$$

in order for $x \sim_{n+1} y$ to be satisfied. This can interpreted by saying that the filtration on $X(k)$ is such *its n-th graded peice injects into* $H^{n+1}(\mathbb{H}, \pi_{n+1}(\overline{h}(X)))$.

We get this way a sequence of invariants $\delta_n(x, y)$, where $\delta_n(x, y)$ is defined if $\delta_i(x, y) = 0$ for all $i < n$, and as far as I know these invariants are new. Of course, an important question is to know whether or not the filtration on $X(k)$ is exhaustive, or in other words if γ is injective. In general it is not, as γ would factor throught $\mathcal{R}$-equivalence [4]. Still the following seems to me an interesting question.

Problem 6.0.3 *Find examples of algebraic varieties X and schematic homotopy theories h such that the map*

$$\gamma : X(k) \longrightarrow \pi_0(\overline{h}(X)^{\mathbb{H}})$$

is injective (resp. surjective, resp. bijective).

Remark 6.0.4 *1.* It seems rather clear that the case where h is the motivic theory would be the most interesting one.

2. When X is a curve, Problem 6.0.3 is closely related to Grothendieck's section conjecture, that have been discussed several time during this conference. I suggest Problem 6.0.3 as a generalization of Grothendieck's section conjecture for higher dimensional varieties.

Finally, let me also mention the existence of a commutative diagram

$$\begin{array}{ccc} X(k) & \xrightarrow{\gamma} & \pi_0(\overline{h}(X)^{\mathbb{H}}) \\ \downarrow & & \downarrow \\ CH_0(X) & \xrightarrow[cycle]{} & \pi_0(\overline{C}_*(X)^{\mathbb{H}}), \end{array}$$

where $CH_0(X)$ is the Chow groups of 0-cycles on X, $\overline{C}_*(X)$ is the homology of $\overline{h}(X)$, and *cycle* is the cycle class map. The spectal sequence

$$E_2^{p,q} = H^p(\mathbb{H}, H_q(\overline{h}(X))) \Rightarrow \pi_{q-p}(\overline{C}_*(X)^{\mathbb{H}}),$$

induces a filtration on $\pi_0(\overline{C}_*(X)^{\mathbb{H}})$, and the pull back by the cycle class map of this filtration induces a filtration on $CH_0(X)$ which gives rise to the so called *higher Abel Jacobi maps* (see e.g. [Ra]). The existence of the commutative diagram above justifies the name *non-abelian Abel Jacobi map* for the map γ.

References

[Ka-Pa-To] L. Katzarkov, T. Pantev, B. Toën, *Schematic homotopy types and non-abelian Hodge theory I: The Hodge decomposition*, Preprint math.AG/0107129.

[Ol] M. Olsson, *Crystalline structure on homotopy types associated to iscocrystals*, manuscript available at http://www-math.mit.edu/ molsson/.

[4] Two points x and y in $X(k)$ are $\mathcal{R}$-equivalent if there exists a morphism $u : \mathbb{A}^1 \to X$ such that $u(0) = x$ and $u(1) = y$.

[Ra] W. Raskind, *Higher l-adic Abel-Jacobi mappings and filtrations on Chow groups*, Duke Math. J. 78 (1995), no. 1, 33–57.

[To1] B. Toën, *Champs affines*, Preprint math.AG/0012219.

[HAGI] B. Toën, G. Vezzosi, *Homotopical algebraic geometry I: Topos theory*, Preprint math.AG/0207028, to appear in Advances in Math.