BASIC APPLICATIONS OF WEAK KÖNIG'S LEMMA IN FEASIBLE ANALYSIS

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Abstract. In the context of a feasible theory for analysis, we investigate three fundamental theorems of analysis: the Heine/Borel covering theorem for the closed unit interval, and the uniform continuity and the maximum principles for real valued continuous functions defined on the closed unit interval.

§1. The three results. The business of reverse mathematics is to investigate the logico-mathematical strength of the various theorems of ordinary mathematics. This investigation is usually carried over the second-order base theory RCA_0 – a theory whose proof-theoretic strength is that of primitive recursive arithmetic. In this article, we investigate three basic theorems of analysis over a *feasible* base theory, i.e., a theory whose provably total functions (with appropriate graphs) are the polynomial time computable functions. Our feasible base theory is BTFA, a theory introduced by Ferreira in a paper entitled "A feasible theory for analysis" [8]: we presuppose familiarity with the notation and results of that paper and an acquaintance with the basic features of research in reverse mathematics (as exposed in the relevant sections of chapters II, III and IV of [10]). Notice that the first-order part of the intended model of BTFA is $2^{<\omega}$, the set of finite sequences of zeros and ones (also called binary words or strings), as opposed to the more traditional setting of the natural numbers. As it happens, we find the binary setting more perspicuous for dealing with theories concerned with sub-exponential classes of computational complexity. The first-order part of a model of BTFA is denoted by $\mathbb{W}$ (for *words*).

Given a formula A of the language of BTFA and x a distinguished (first-order) variable, we say that A defines an infinite subtree of $\mathbb{W}$, and write $Tree_\infty(A_x)$, if

$$\forall x \forall y (A(x) \wedge y \subseteq x \to A(y)) \wedge \forall n \in \mathbb{T}\, \exists x (\ell(x) = n \wedge A(x)),\ ^1$$

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where $\mathbb{T}$ denotes the tally part of the domain $\mathbb{W}$. If, besides $Tree_\infty(A_x)$, one also has

$$\forall x \forall y (A(x) \wedge A(y) \rightarrow x \subseteq y \vee y \subseteq x),$$

then we say that A defines an infinite path, and write $Path(A_x)$. Weak König's lemma for boundedly defined trees, denoted by $\Sigma^b_\infty - \mathsf{WKL}$, is the principle

$$Tree_\infty(A_x) \rightarrow \exists X(Path(X) \wedge \forall x(x \in X \rightarrow A(x))), \tag{1}$$

where A is a bounded formula and $Path(X)$ abbreviates the more cumbersome $Path((x \in X)_x)$. The following theorem was proven in [8]:

Theorem. *The theory* $\mathsf{BTFA} + \Sigma^b_\infty - \mathsf{WKL}$ *is conservative over* BTFA *with respect to* Π^1_1*-sentences.*

The reader should keep in mind two noteworthy features of the above theorem. Firstly, the theorem concerns infinite binary trees defined by *bounded* formulas (i.e., Σ^b_∞-formulas). Therefore, these trees need not exist as sets in BTFA (bounded formulas of the language of BTFA define in the standard model precisely the sets of the Meyer-Stockmeyer polynomial hierarchy – see [6] and [5]; *pace* the resolution of outstanding problems in computational complexity, the theory BTFA does not have enough comprehension to define these sets). Secondly, the path whose existence weak König's lemma guarantees is, of course, a *set*. Insofar as the infinite binary trees considered in the ordinary setting of reverse mathematics are sets, we have in the above theorem a new phenomenon. As a matter of fact, the restriction of (1) to sets (i.e., to formulas $A(x)$ of the form $x \in X$) is not sufficient for the ordinary studies of analysis within the framework of feasibility, as Theorem 1 below indicates. Two restrictions of (1) play an important role in the sequel: the above referred restriction to sets (the principle simply denoted by WKL), and the restriction to Π^b_1-formulas (the principle $\Pi^b_1 - \mathsf{WKL}$).

The last part of Ferreira's unpublished thesis [5] investigates three basic theorems of analysis in the Cantor space setting: the Heine/Borel covering theorem, the uniform continuity theorem, and the maximum principle. The discussion of these theorems in the Cantor space setting is specially natural within BTFA because its elements (viz. the infinite paths through the binary tree) and topology mesh very well with feasible constructions. In the real number setting, on the other hand, it appears that some technical rabble is unavoidable. We now list the three main results of this paper. The pertinent formalizations of the concepts of analysis used in the statements of the theorems will be provided in the next sections.

Theorem 1. *Over* BTFA*, the Heine/Borel theorem for* $[0, 1]$ *is equivalent to* $\Pi^b_1 - \mathsf{WKL}$*.*

Theorem 2. *Over* BTFA*, the principle that every (total) real valued continuous function defined on* $[0, 1]$ *is uniformly continuous implies* WKL *and is implied by* $\Pi^b_1 - \mathsf{WKL}$*.*

Observe that in Theorem 2 above, we do not have a perfect match – there is a gap that we were unable to fill.

The amount of induction present in BTFA is induction on notation for NP-predicates. Formally,

$$A(\epsilon) \wedge \forall x(A(x) \rightarrow A(x0) \wedge A(x1)) \rightarrow \forall x A(x),$$

where $A(x)$ is a Σ_1^b-formula. We obtain a (seemingly) stronger theory if we also admit the "slow" induction scheme:

$$A(\epsilon) \wedge \forall x(A(x) \rightarrow A(S(x))) \rightarrow \forall x A(x), \tag{2}$$

where $A(x)$ is also a Σ_1^b-formula. Let us explain the "successor" function S: the elements of $\mathbb{W}$ can be ordered according to length and, within the same length, lexicographically; this yields a discrete linear order, provably so in BTFA, with least element ϵ; by definition, $S(x)$ is the *next* element after x in this order. In the framework of Buss' bounded arithmetic [1], the scheme (2) corresponds to the $\Sigma_1^{\mathsf{b}} - \mathsf{IND}$ scheme, the mark of Buss' theory T_2^1 (more of this in the last section). Finally:

THEOREM 3. *Over* $\mathsf{BTFA} + \Sigma_\infty^b - \mathsf{WKL}$*, the following are equivalent:*

(a) *Every continuous real valued function defined on* $[0,1]$ *has a maximum.*
(b) *Every continuous real valued function defined on* $[0,1]$ *has a supremum.*
(c) *The induction scheme* (2) *for* Σ_1^b*-formulas.*

It should be remarked that the above three theorems are generalizations of similar results over the base theory RCA_0 (well, Theorem 3 is void in this setting). More precisely, they coincide with those results provided induction for Σ_1^0-predicates is available (i.e., provided we have essentially RCA_0).[2]

§2. Preliminaries. As we told in the opening section, we assume familiarity with the theory BTFA and, in particular, with its formal language of binary words. In the following paragraphs, we will briefly sketch how to formalize the basic notions of analysis in BTFA. This was first outlined by Yamazaki in [11], and it is done in some detail in [4].

In feasible theories (more generally, in theories in which exponentiation is not a total function), we must distinguish between *tally* numbers and *dyadic* numbers. The distinction and interplay between these two sorts of numbers is a very important feature of what follows. The reader not used to making this distinction should proceed in a cautious pace, making perfectly clear for herself whether a particular passage concerns dyadic or tally arithmetic. We now briefly sketch this distinction.

Dyadic natural numbers $y \in \mathbb{N}_2$ are represented by binary strings of zeros and ones of the form $1x$ (with $x \in \mathbb{W}$) or by the empty string ϵ. If x is $x_1 x_2 \cdots x_{n-1}$, where each x_i is 0 or 1, then we should view y as the number $\sum_{i=0}^{n-1} x_i 2^{n-i-1}$, where $x_0 = 1$. The empty string represents the number zero: as usual, this number is denoted by 0 (no confusion should arise between the number 0, which is the empty string ϵ, and the string 0). The basic processes of arithmetic (i.e., sum, multiplication, modified subtraction and long division) have polynomial time computable algorithms that can be formalized in BTFA. Actually, the arithmetic of $\mathbb{N}_2$ is exactly encapsulated by Buss' theory S^1_2 (for this latter theory, see [1]). The tally natural numbers are, on the other hand, just the tally strings, i.e., the elements of $\mathbb{T}$. This set is also denoted by $\mathbb{N}_1$, and we usually reserve the letters k, m, n for the members of this set. The arithmetic of $\mathbb{N}_1$ is straightforward: zero is given by ϵ, successor by concatenation with 1, addition $+$ by concatenation, multiplication by $\times$, and the less than or equal relation by $\subseteq$. With these definitions, the system $\mathbb{N}_1$ becomes a model of the well-known bounded arithmetic theory $\mathsf{I}\Delta_0$.

A *dyadic rational number* is a triple of the form $(\pm, x, y)$, where x (resp., y) is the empty string or a string starting with 1 (resp., ending with 1). We assume that the triples are coded as strings in a smooth way. If $x = x_0x_1 \cdots x_{n-1}$ and $y = y_0y_1 \cdots y_{m-1}$, where each x_i and y_j is 0 or 1, then we should view the triple $(\pm, x, y)$ as representing the rational number $\pm(\sum_{i=0}^{n-1} x_i 2^{n-i-1} + \sum_{j=0}^{m-1} \frac{y_j}{2^{j+1}})$. We usually write this number in radix notation: $\pm x_0x_1 \cdots x_{n-1}.y_0 \cdots y_{m-1}$. Given $x \in \mathbb{W}$, it is useful to denote by x^* the word x with its rightmost zeros chopped off. Thus, $.x^*$ is a dyadic rational number: it is actually the number $\sum_{i<\ell(x)} \frac{x_i}{2^{i+1}}$, where x_i is the $(i+1)$-th bit of the word x (for a tally i less than $\ell(x)$). It poses no problem to naturally define a structure of ordered ring in the set $\mathbb{D}$ of dyadic rational numbers. In this ring, the numbers of the form

$$(+, 1\underbrace{00\ldots0}_{n \text{ zeros}}, \epsilon) \quad \text{and} \quad (+, \epsilon, \underbrace{00\ldots0}_{n-1 \text{ zeros}}1),$$

where $n \in \mathbb{N}_1$, are (respectively) cofinal in the set of dyadic rational numbers, and co-initial in the set of positive dyadic rational numbers $\mathbb{D}^+$. As usual, these numbers are denoted (respectively) by 2^n, and $\frac{1}{2^n}$ or 2^{-n}. Observe that although $\mathbb{D}$ is not a field, we can always divide by tally powers of 2 there.

Definition. (BTFA) We say that a function $\alpha : \mathbb{N}_1 \mapsto \mathbb{D}$ is a *real number* if $|\alpha(n) - \alpha(m)| \leq 2^{-n}$ for all $n \leq m$. Two real numbers α and β are said to be *equal*, and we write $\alpha = \beta$, if $\forall n \in \mathbb{N}_1 |\alpha(n) - \beta(n)| \leq 2^{-n+1}$.

This definition is taken from Yamazaki [11]. It follows closely the definition of real numbers given in [10], with the noteworthy feature that it requires that the domain of a real number be the set of tally numbers (in theories which prove the totality of the exponential function, $\mathbb{N}_1$ and $\mathbb{N}_2$ are essentially the same thing; thus, the above definition coincides with the usual definition over RCA_0). Within BTFA, it is easy to embed $\mathbb{D}$ into the real numbers (into $\mathbb{R}$). The basic arithmetic operations can be defined on $\mathbb{R}$ so that BTFA proves that $\mathbb{R}$ is a real closed ordered field (see [4]). An alternative definition for real numbers would be to consider the so-called *dyadic real numbers*. A dyadic real number is a triple $(\pm, x, X)$, where $x \in \mathbb{N}_2$ and X is an infinite path. The usual (radix point) notation for these numbers is $\pm x.X$. Informally, such dyadic real numbers give rise to the real numbers $\pm(\sum_{i=0}^{n-1} x_i 2^{n-i-1} + \sum_{i=0}^{\infty} \frac{X(i)}{2^{i+1}})$, where $X(i)$ is the $(i+1)$-th bit of X (for $i \in \mathbb{N}_1$). One can associate to each dyadic real number a real number (as in the definition above) in a natural way.

The reader should compare the next definition with Simpson's definition of continuous real function in [10] (and also with Yamazaki's different – and inequivalent – definition in [11]).

Definition. Within BTFA, a (code for a) *continuous partial function* from $\mathbb{R}$ into $\mathbb{R}$ is a set os quintuples $\Phi \subseteq \mathbb{W} \times \mathbb{D} \times \mathbb{N}_1 \times \mathbb{D} \times \mathbb{N}_1$ such that:

1. if $(x,n)\Phi(y,k)$ and $(x,n)\Phi(y',k')$, then $|y - y'| \leq 2^{-k} + 2^{-k'}$;
2. if $(x,n)\Phi(y,k)$ and $(x',n') < (x,n)$, then $(x',n')\Phi(y,k)$;
3. if $(x,n)\Phi(y,k)$ and $(y,k) < (y',k')$, then $(x,n)\Phi(y',k')$;

where $(x,n)\Phi(y,k)$ abbreviates the $\exists\Sigma_1^b$-relation $\exists w(w,x,n,y,k) \in \Phi$, and where the notation $(x',n') < (x,n)$ means that $|x - x'| + 2^{-n'} < 2^{-n}$.

The next two definitions should be standard. For the record, we write them down.

Definition. (BTFA) Let Φ be a continuous partial real function of a real variable. We say that a real number α is in the domain of Φ and, with abuse of language, write $\alpha \in dom(\Phi)$, if

$$\forall k \in \mathbb{N}_1 \exists n \in \mathbb{N}_1 \exists x, y \in \mathbb{D}(|\alpha - x| < 2^{-n} \wedge (x,n)\Phi(y,k)).$$

Definition. (BTFA) Let Φ be a continuous partial real function of a real variable, and let α be a real number in the domain of Φ. We say that a real number β is the *value of* α *under the function* Φ, and write $\Phi(\alpha) = \beta$, if

$$\forall x, y \in \mathbb{D}\, \forall n, k \in \mathbb{N}_1((x,n)\Phi(y,k) \wedge |\alpha - x| < \frac{1}{2^n} \to |\beta - y| \leq \frac{1}{2^k}).$$

The following fact is basic, although it does not come easy (see [4]) because minimization along the binary words is not available in BTFA:

Theorem. (BTFA) *Let Φ be a continuous partial real function of a real variable and let α be a real number in the domain of Φ. Then there is a unique real number β such that $\Phi(\alpha) = \beta$.*

The following proposition is handy. The reader should note that the fact that T below is a (set) tree plays a crucial role in the proof given below. For instance, were T merely a set of words of equal length (considered as end notes of the obvious Σ_1^b-tree), then the construction given in the argument below wouldn't go through.

Proposition 1. (BTFA) *Let T be a subtree of $\mathbb{W}$ with no infinite paths. Given a function $f : \mathbb{W} \mapsto \mathbb{D}_0^+$, there is a continuous (total) function defined on $[0,1]$ such that, for all end nodes x of T, $\Phi(.x^*) = f(x)$. Moreover, we can take Φ with the following property: For all $\alpha \in [0,1]$, there is an end node x of T such that $\Phi(\alpha) \leq f(x)$.*

PROOF. Let us call two end nodes x and y of T *consecutive* if $.x^* < .y^*$ and for no end node z of T, $.x^* < .z^* < .y^*$. For the sake of uniformity, introduce an imaginary node μ such that, for all $w \in \mathbb{W}$, $.w^* < .\mu^* = 1$. If need be (i.e., if no string of zeros is an end node of T), let us also introduce another imaginary node ν of T such that, for all $w \in \mathbb{W}$, $0 = .\nu^* < .w^*$. The idea is to define $\Phi(.x^*) = f(x)$ for all end nodes x of T (putting $f(1) = 0$ and, if need be, $f(0) = 0$) and, otherwise, define $\Phi(\alpha)$ by piecewise linearity, i.e., for α in the closed interval $[.x^*, .y^*]$, where x and y are consecutive end nodes of T, define

$$\Phi(\alpha) = f(x) + \frac{\alpha - .x^*}{.y^* - .x^*}(f(y) - f(x)).$$

Such a continuous function has the desired properties, and it is standard to construct a continuous function code for Φ *provided* that the following two conditions hold:

1. For all $w \in \mathbb{W}$, there are consecutive end nodes x and y of T such that $.x^* \leq .w^* \leq .y^*$.
2. One can check in polynomial time (within BTFA) whether, on inputs $x, y, w \in \mathbb{W}$, x and y are consecutive end nodes of T such that $.x^* \leq .w^* \leq .y^*$.

The fact that the above two conditions hold in BTFA should be clear for a reader of like mind: The consecutive nodes x and y of (1) can be found by a suitable (bounded) recursions along the tally part; the ternary relation of (2) can be described *via* a subword quantification formula. $\dashv$

The idea of the following proposition is well-known:

PROPOSITION 2. $\mathsf{BTFA} + \Sigma^{\mathsf{b}}_{\infty} - \mathsf{WKL}$ *proves weak König's lemma* (*i.e., the scheme* (1)) *for trees defined by* Π^0_1*-formulas. Similarly,* $\mathsf{BTFA} + \Pi^{\mathsf{b}}_1 - \mathsf{WKL}$ *proves weak König's lemma for trees defined by* $\forall\Pi^b_1$*-formulas.*

PROOF. We can treat both cases together, since it is the very same reason that accounts for the truth of the two statements above, viz. that the classes of bounded formulas and Π^b_1-formulas are both closed under universal bounded quantifications. Let $A(x)$ be a Π^0_1-formula (resp., a $\forall\Pi^b_1$-formula) which defines an infinite subtree of $\mathbb{W}$. $A(x)$ is of the form $\forall z B(z, x)$, where B is a bounded formula (resp., a Π^b_1-formula). Define

$$T(x) := \forall z \preccurlyeq x \forall y \subseteq x B(z, y).^3$$

$T(x)$ is still a bounded formula (resp., a Π^b_1-formula), and it is easy to check that it defines an infinite subtree of $\mathbb{W}$. Hence, by hypothesis, there is a path X through this tree. It is straightforward to check that this path is also a path through the original tree defined by $A(x)$. $\dashv$

Finally, we find that it might me helpful to finish this section with four remarks on the problem of working within BTFA:

1. We cannot define functions by primitive recursion in BTFA. However, we can define functions by bounded recursion on notation and, in particular, by bounded recursion along the tally part.
2. We cannot define sets by bounded quantification, i.e., by quantification ranging over all words of length less than a given length, or (equivalently) ranging over all dyadic natural numbers less than a certain given one. We can, however, define sets by quantification ranging over all subwords of a given word, or over all tally numbers less than a given tally one.
3. Not every bounded set of words can be coded by a word. Such a possibility is, in fact, a re-statement of the totality of exponentiation (see the appendix of [7]).
4. Given a non empty set of words of equal length, BTFA does not seem to be strong enough to pick the lexicographically least (greatest) word of this set. However, BTFA is able to pick the least tally number satisfying a given Σ_1^b-formula (if the formula is satisfiable by a tally number at all).

§3. The Heine-Borel and the uniform continuity theorems. The following definition is a suitable adaptation of the definition of an open set as given in [10, p. 81]:

Definition. (BTFA) A (code for an) open set U of $\mathbb{R}$ is a set $U \subseteq \mathbb{W} \times \mathbb{D} \times \mathbb{N}_1$. We say that a real number α is an *element* of U, and write $\alpha \in U$, if

$$\exists z \in \mathbb{D} \exists n \in \mathbb{N}_1 (|\alpha - z| < \frac{1}{2^n} \wedge \exists w (w, z, n) \in U).$$

Suppose that U is an open set and that $[0,1] \subseteq U$ (i.e., every real number in the closed unit interval is an element of U). In this situation, the Heine-Borel theorem guarantees the existence of $k \in \mathbb{N}_1$ with the following property: For all $\alpha \in [0,1]$, there are $z \in \mathbb{D}$, $n \in \mathbb{N}_1$ and $w \in \mathbb{W}$ of length less than k such that $|\alpha - z| < 2^{-n}$ and $(w, z, n) \in U$.[4]

Proof of Theorem 1. The proof of both directions of this theorem are adaptations of well-known arguments. In order to prove the Heine-Borel theorem, we reason in $\mathsf{BTFA} + \Pi_1^b - \mathsf{WKL}$. Let U be an open set such that $U \subseteq [0,1]$. To each $x \in \mathbb{W}$, we associate the dyadic rational numbers $a_x = .x^*$ and $b_x = a_x + 2^{-\ell(x)}$. Consider the $\forall \Pi_1^b$-formula $T(x)$ defined by:

$$\neg \exists w \in \mathbb{W} \exists z \in \mathbb{D} \exists n \in \mathbb{N}_1 ((w, z, n) \in U \wedge z - \frac{1}{2^n} < a_x < b_x < z + \frac{1}{2^n}).$$

Clearly, $T(x)$ and $y \subseteq x$ implies $T(y)$. Suppose, in order to reach a contradiction, that there are elements of arbitrary length satisfying the above formula. Then, by the second part of Proposition 2, there is an infinite path X throught T. In this situation, it is easy to argue that the real $\alpha = \sum_{i=0}^{\infty} \frac{X(i)}{2^{i+1}}$ is an element $[0,1]$ which is not an element of U, contradicting our assumption. Therefore, the elements satisfying T have length bounded by a certain $r \in \mathbb{N}_1$.

Equivalently:

$$\forall x(T(x) \wedge \ell(x) = r \rightarrow \exists w, z, n\, (w, z, n) \in U \wedge z - \frac{1}{2^n} < a_x < b_x < z + \frac{1}{2^n}).$$

By bounded collection, we can bound the lengths of the above w, z and n by a certain $k \in \mathbb{N}_1$. Using the fact that the closed intervals $[a_x, b_x]$, for $\ell(x) = r$, cover the closed unit interval, it follows that the above k does the job.

Reciprocally, assume the Heine-Borel covering theorem. Let us consider the following adaptation of Cantor's middle-third set: let $C \subseteq [0, 1]$ consist of all real numbers of the form

$$\sum_{i=0}^{\infty} \frac{3X(i)}{4^{i+1}},$$

where X is an infinite path through $\mathbb{W}$. For each $x \in \mathbb{W}$ let

$$a_x = \sum_{i<\ell(x)} \frac{3x(i)}{4^{i+1}} \quad \text{and} \quad b_x = a_x + \frac{1}{4^{\ell(x)}}.$$

Note that these numbers are dyadic rational numbers, i.e., they are in $\mathbb{D}$ (this is the reason why we have slightly modified the definition of Cantor's middle-third set). Let,

$$a'_x = a_x - \frac{1}{4^{\ell(x)}} \quad \text{and} \quad b'_x = b_x + \frac{1}{4^{\ell(x)}}.$$

The following two properties are easy to prove (in BTFA):

i. Given X an infinite path through $\mathbb{W}$ and $x \in \mathbb{W}$, if $\alpha = \sum_{i=0}^{\infty} \frac{3X(i)}{4^{i+1}}$ is in the open interval (a'_x, b'_x), then $x \subset X$ (i.e., x is an initial segment of X).
ii. For all $\alpha \in [0, 1]$, α is not in C if, and only if, $\alpha \in (b_{x0}, a_{x1})$ for a certain $x \in \mathbb{W}$.

Take now a $\forall\Pi_1^b$-formula $T(x)$ defining a subtree of $\mathbb{W}$ with no infinite paths through it. Let $T(x)$ be of the form $\forall w\, (w, x) \in Q$, for a certain set Q. Define the open set $U = U_0 \cup U_1$, where

$$U_0 = \{(w, c'_x, 2\ell(x)) : (w, x) \notin Q\} \quad \text{and} \quad U_1 = \{(\epsilon, c_x, 2(\ell(x) + 1)) : x \in \mathbb{W}\},$$

and where $c'_x = (a'_x + b'_x)/2$ and $c_x = (b_{x0} + a_{x1})/2$. Remark that the open intervals (a'_x, b'_x) and (b_{x0}, a_{x1}) are, respectively, the open intervals with centers c'_x and c_x and radius $2^{-2\ell(x)}$ and $2^{-2(\ell(x)+1)}$. If $\alpha \in [0, 1] \setminus C$, then $\alpha \in U_1$. On the other hand, if $\alpha \in C$ then (using the fact that there are no infinite paths through T) it is easy to show that $\alpha \in U_0$. In sum, U is an open covering of $[0, 1]$. Therefore, by the Heine-Borel theorem, there is $k \in \mathbb{N}_1$ such that for all $\alpha \in [0, 1]$, there are $x \in \mathbb{D}$, $n \in \mathbb{N}_1$ and $w \in \mathbb{W}$ of length less than k with $|\alpha - x| < 2^{-n}$ and $(w, x, n) \in U$. We want to show that the elements of $T(x)$ are not of arbitrarily large length. In fact, we claim that

$$\forall x(T(x) \rightarrow \ell(x) < k).$$

Suppose not. Take x of length k satisfying T. Consider an infinite path X of $\mathbb{W}$ with $x \subset X$. Since the real $\alpha = \sum_{i=0}^{\infty} \frac{3X(i)}{4^{i+1}}$ is in C and since U covers $[0,1]$, α must be in U_0. Thus, there are w and y such that $(w, c'_y, 2\ell(y)) \in U_0$, $a'_y < \alpha < b'_y$ and $\ell(y) < k$. This implies that $y \subset X$. Thus, both x and y are initial segments of X and $\ell(y) < \ell(x)$. Therefore, $y \subseteq x$. Since we have $T(x)$, we can conclude $T(y)$. This is a contradiction. ⊣

Definition 1. (BTFA) Let $\Phi : [0,1] \mapsto \mathbb{R}$ be a (total) continuous function. We say that Φ is *uniformly continuous* if

$$\forall k \in \mathbb{N}_1 \exists m \in \mathbb{N}_1 \forall \alpha, \beta \in [0,1](|\alpha - \beta| < \frac{1}{2^m} \to |\Phi(\alpha) - \Phi(\beta)| < \frac{1}{2^k}).$$

Proposition 3. (BTFA) *Let $\Phi : [0,1] \mapsto \mathbb{R}$ be a uniformly continuous function. Then there is $n \in \mathbb{N}_1$ such that, for all $\alpha \in [0,1]$, $\Phi(\alpha) \leq 2^n$.*

Proof. By hypothesis, there is $m \in \mathbb{N}_1$ such that if $|\alpha - \beta| < 2^{-m}$ then $|\Phi(\alpha) - \Phi(\beta)| < 1$, for $\alpha, \beta \in [0,1]$. It is clear that

$$\forall x(\ell(x) = m \to \exists n \in \mathbb{N}_1\, \Phi(.x^*) < 2^n).$$

Now, the formula after the implication sign above is a Σ^0_1-formula. Thus, by bounded collection, there is $r \in \mathbb{N}_1$ such that $\forall x(\ell(x) = m \to \Phi(.x^*) < 2^r)$. Since every real in the closed unit interval is within 2^{-m} of a certain $.x^*$ for x of length m, it is clear that $n = r + 1$ does the job. ⊣

Proof of Theorem 2. We reason in BTFA. Suppose that the principle $\Pi^{\mathsf{b}}_1 - \mathsf{WKL}$ holds. Let $\Phi : [0,1] \mapsto \mathbb{R}$ be a total continuous function, and fix $k \in \mathbb{N}_1$. Consider the open set $U := \{((w,y), x, n+2) : (w, x, n, y, k+2) \in \Phi\}$. Using the fact that Φ is a total function, it is easy to argue that $[0,1] \subseteq U$. By the Heine-Borel theorem (which is available by Theorem 1), there is $m \in \mathbb{N}_1$ such that, for all $\alpha \in [0,1]$, there are $n \in \mathbb{N}_1$, $x, y \in \mathbb{D}$, all of length less than m, satisfying $|\alpha - x| < 2^{-(n+2)}$ and $(x,n)\Phi(y, k+2)$. It is now easy to argue that $|\Phi(\alpha) - \Phi(\beta)| \leq 2^{-k}$ whenever α and β are reals in the closed unit interval whose difference is less than $2^{-(m+1)}$.

Reciprocally, assume that WKL fails. Let T be a subtree of $\mathbb{W}$ with elements of arbitrarily large length but with no infinite paths. By Proposition 1, there is a continuous real function Φ defined on the closed unit interval such that $\Phi(.x^*) = 2^{\ell(x)}$, for all end nodes x of T. Therefore, Φ is unbounded. Hence, by Proposition 3, Φ is not uniformly continuous. ⊣

§4. The maximum principle. The scheme of "slow" induction (2) for Σ^b_1-formulas is equivalent (within BTFA) to a maximization principle, namely to the principle that every non empty set X of words of equal length has a lexicocraphically greatest (least) element. It is also equivalent to the following seemingly more general maximization principle: If a Σ^b_1-formula is satisfiable by a word of a given length, then there is a lexicographically greatest (least) word of that length satisfying the given formula. The latter equivalence is due to Samuel Buss [1, p. 56]. The former equivalence is explained in [5, p. 88] or in [3, lemma 5.2.7.(a)]. The same arguments given in [8] show that the theory $\mathsf{BTFA} + \Sigma^{\mathsf{b}}_{\infty} - \mathsf{WKL}$ (and, *a fortiori*, BTFA itself) augmented with the scheme of "slow" induction for Σ^b_1-formulas is Π^0_2-conservative over the theory $\Sigma^{\mathsf{b}}_1 - \mathsf{IA}$ (the theory T^1_2, in Buss' notation). The $\forall\Sigma^b_1$-consequences of this theory were studied by Buss and Krajíček [2, Section 5] and Ferreira [9]. These studies showed that the witnesses of the $\forall\Sigma^b_1$-consequences of T^1_2 are precisely the optimal solutions of *polynomial local search* problems. As it happens, these witnesses are straighforwardly computable in polynomial time using a NP-oracle, but they don't seem to be computable in polynomial time *tout court*. On the proof-theoretic side, the theory BTFA augmented with the scheme of "slow" induction for Σ^b_1-formulas is sandwiched between BTFA and BTFA together with the scheme of comprehension for Σ^b_1-formulas.

Proof of Theorem 3. We prove that (a) ⇒ (b) ⇒ (c) ⇒ (a). The first implication is trivial. It is not difficult to see that (b) implies (c). Let us assume (b). In order to prove (c), we prove instead the first maximization principle described in the paragraph above. Let X be a non-empty set of words of equal length $n \in \mathbb{N}_1$. By considering the following full (bounded) tree $T = \{w \in \mathbb{W} : \ell(w) \leq n\}$, we can apply Proposition 1 to define a continuous total function $\Phi : [0,1] \mapsto \mathbb{R}$ such that, for all words w of length n, $\Phi(w) = .w^*$ if $w \in X$, and $\Phi(w) = 0$ otherwise. Moreover, we may take Φ with the additional property that all values of Φ are majorized by a certain $.w^*$, with $w \in X$. By hypothesis, Φ has a supremum. Clearly, this is the supremum of all values of the form $.w^*$, for $w \in X$. Now, it is immediate to argue that this supremum is indeed a maximum, i.e., is of the form $.w^*$ with $w \in X$: this is the value that we were looking for.

It remains to show that (c) ⇒ (a). Assume (c) or, what is the same thing, the maximization principle for Σ^b_1-predicates described in the beginning of this section. Without loss of generality, consider a (total) continuous function $\Phi : [0,1] \mapsto [0,1]$. In order to prove that Φ has a maximum we need to make some preliminary considerations.

Given $k \in \mathbb{N}_1$, consider the open set

$$U_k := \{((w,y),x,n+1) : (w,x,n,y,k+1) \in \Phi\}.$$

By the totality of Φ, $U_k \subseteq [0,1]$. Thus, by the Heine-Borel Theorem (Theorem 1), there is $m \in \mathbb{N}_1$ such that the open set

$$U^m_k := \{((w,y),x,n+1) : w,x,y,n+1 < m \wedge (w,x,n,y,k+1) \in \Phi\}$$

already covers the interval $[0,1]$. In particular one gets $\forall k \exists m A(k,m)$, where $A(k,m)$ is the following bounded formula:

$$\forall \tilde{x}(\ell(\tilde{x}) < m \rightarrow \exists w,x,n,y < m\,(w,x,n,y,k+1) \in \Phi \wedge |.\tilde{x}^* - .x^*| < \frac{1}{2^{n+1}}).$$

Note that $A(k, m)$ implies

$$(3) \qquad \forall \alpha \in [0,1]\, \exists w, x, n, y < m\, (w, x, n, y, k+1) \in \Phi \wedge |\alpha - .x^*| < \frac{1}{2^n}.$$

Remark also, that in the situation after the bounded existential quantifier,

$$(4) \qquad |\Phi(\alpha) - \Phi(.x^*)| \le |\Phi(\alpha) - .y^*| + |.y^* - \Phi(.x^*)| \le \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}.$$

After these preliminary considerations, consider the following Π^0_1-formula:

$$\Omega(y) := \forall \tilde{x}\, [\Phi(.\tilde{x}^*) \le .y^* + \frac{1}{2^{\ell(y)-1}}]$$

$$\wedge\, \forall m (A(\ell(y), m) \to \exists x < m\, |\Phi(.x^*) - .y^*| \le \frac{1}{2^{\ell(y)-2}}).$$

We make two claims:

1. $\Omega(y) \wedge \tilde{y} \subseteq y \to \Omega(\tilde{y})$ and
2. $\forall k \in \mathbb{N}_1 \exists y (\ell(y) = k \wedge \Omega(y))$.

Assume $\Omega(y)$ and $\tilde{y} \subset y$. Since $.y^* + \frac{1}{2^{\ell(y)-1}} \le .\tilde{y}^* + \frac{1}{2^{\ell(\tilde{y})-1}}$, the first conjunct of $\Omega(\tilde{y})$ obviously holds. To argue for the second conjunct, suppose that $A(\ell(\tilde{y}), \tilde{m})$ holds. Take $m \in \mathbb{N}_1$ with $A(\ell(y), m)$, and let $x < m$ be such that $|\Phi(.x^*) - .y^*| \le \frac{1}{2^{\ell(y)-2}}$. By (4), there is $\tilde{x} < \tilde{m}$ with $|\Phi(.x^*) - \Phi(.\tilde{x}^*)| \le \frac{1}{2^{\ell(\tilde{y})}}$. Therefore:

$$|\Phi(.\tilde{x}^*) - .\tilde{y}^*| \le |\Phi(.\tilde{x}^*) - \Phi(.x^*)| + |\Phi(.x^*) - .y^*| + |.y^* - .\tilde{y}^*|$$

$$\le \frac{1}{2^{\ell(\tilde{y})}} + \frac{1}{2^{\ell(y)-2}} + \frac{1}{2^{\ell(\tilde{y})}} \le \frac{1}{2^{\ell(\tilde{y})-2}}.$$

Let us now argue for the second claim. Fix $k \in \mathbb{N}_1$ and take $m \in \mathbb{N}_1$ such that $A(k, m)$. We now use, for the only time, the hypothesis (c) in order to get the lexicographically greatest element y_k of length k satisfying the Σ^b_1-formula

$$(5) \qquad \exists w, x, n, y < m\, [\tau_k(y) = y_k \wedge (w, .x^*, n, .y^*, k+1) \in \Phi],$$

where $\tau_k(y)$ is y truncated at length k if $k \le \ell(y)$; otherwise, $\tau_k(y)$ is y concatenated with $k - \ell(y)$ zeros. Let us check that $\Omega(y_k)$ holds. Take any $\tilde{x} \in \mathbb{W}$. By (3), there are $w, x, n, y < m$ such that $(w, .x^*, n, .y^*, k+1) \in \Phi$ and $|.\tilde{x}^* - .x^*| < \frac{1}{2^n}$. Clearly, $|\Phi(.\tilde{x}^*) - .y^*| \le \frac{1}{2^{k+1}}$. Thus,

$$\Phi(.\tilde{x}^*) \le .y^* + \frac{1}{2^{k+1}} = (.y^* - .\tau_k(y)^*) + \tau_k(y)^* + \frac{1}{2^{k+1}} \le \frac{1}{2^k} + .y_k^* + \frac{1}{2^{k+1}}$$

and this is less than $.y_k^* + \frac{1}{2^{\ell(y_k)-1}}$. Now, in order to show that the second conjunct of $\Omega(y_k)$ holds, we first remark that if $\tau_k(y) = y_k$, $w, x, n, y < m$ and $(w, .x^*, n, .y^*, k+1) \in \Phi$, then

$$|\Phi(.x^*) - .y_k| \le |\Phi(.x^*) - .y^*| + |.y^* - .y_k^*| \le \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{\ell(y_k)-1}}.$$

Take $\tilde{m}$ with $A(\ell(y_k), \tilde{m})$. By (4), pick $\tilde{x} < \tilde{m}$ with $|\Phi(.x^*) - \Phi(\tilde{x}^*)| \leq \frac{1}{2^k}$. We get,

$$|\Phi(.\tilde{x}^*) - .y_k^*| \leq |\Phi(.\tilde{x}^*) - \Phi(.x^*)| + |\Phi(.x^*) - .y_k^*| \leq \frac{1}{2^k} + \frac{1}{2^{k-1}} < \frac{1}{2^{\ell(y_k)-2}}.$$

We have checked that Ω defines an infinite Π_1^0-subtree of $\mathbb{W}$. Since we are working under the assumption that $\Sigma_\infty^{\mathsf{b}} - \mathsf{WKL}$ holds, by Proposition 2 there is an infinite path Y_M throught Ω. It is straightforward to argue both that $\forall \alpha \in [0,1]\, \Phi(\alpha) \leq .Y_M$, and that $\forall n \in \mathbb{N}_1 \exists x (.Y_M \leq \Phi(.x^*) + \frac{1}{2^n})$. Let us now consider the Π_1^0-formula $T(x)$ defined as follows:

$$\forall y \in \mathbb{D} \cap [0,1]\, \forall k \in \mathbb{N}_1 [(.x^*, \ell(x) - 2)\Phi(y,k) \rightarrow |.Y_M - y| \leq \frac{1}{2^k} + \frac{1}{2^{\ell(x)}}].$$

It is easy to check that if $T(x)$ and $\tilde{x} \subseteq x$, then $T(\tilde{x})$. Now, fix $n \in \mathbb{N}_1$. Take $\tilde{x} \in \mathbb{W}$ such that $|\Phi(.\tilde{x}^*) - .Y_M| \leq \frac{1}{2^n}$. Let x be $\tau_n(\tilde{(x)})$, and suppose that for $y \in \mathbb{D} \cap [0,1]$ and $k \in \mathbb{N}_1$ we have $(.x^*, \ell(x) - 2)\Phi(y,k)$. Since $|.x^* - .\tilde{x}^*| \leq \frac{1}{2^n} < \frac{1}{2^{\ell(x)-2}}$, we get $|\Phi(.x^*) - y| \leq \frac{1}{2^k}$. Thus,

$$|.Y_M - y| \leq |.Y_M - \Phi(.\tilde{x}^*)| + |\Phi(.\tilde{x}^*) - y| \leq \frac{1}{2^{\ell(x)}} + \frac{1}{2^k}.$$

In sum, T defines an infinite Π_1^0-subtree of $\mathbb{W}$. Therefore, there in an infinite path X_M throught T. We claim that $\Phi(.X_M) = .Y_M$, which finishes the proof. Let $k \in \mathbb{N}_1$. Since $.X_M \in dom(\Phi)$, there is $n \geq k$ in $\mathbb{N}_1$ and $y \in \mathbb{D} \cap [0,1]$ such that $(.x^*, n-2)\Phi(y,k)$, where x is the initial segment of X_M of length n. Note that $T(x)$ and, hence, that $|.Y_M - y| \leq \frac{1}{2^k} + \frac{1}{2^n}$; note also that $|.X_M - x^*| < \frac{1}{2^{n-2}}$. Therefore,

$$|.Y_M - \Phi(.X_M)| \leq |.Y_M - y| + |y - \Phi(.X_M)| \leq \frac{1}{2^k} + \frac{1}{2^k} + \frac{1}{2^k}.$$

By the arbitrarity of k, we may conclude that $\Phi(.X_M) = .Y_M$. ⊣

NOTES

[1]We are slightly departing from the notation used in [8]. Here, $\ell(x)$ stands for $1 \times x$, a more friendly say of denoting the tally length of x.

[2]Induction for Σ_1^0 predicates proves the totality of exponentiation. In this latter situation, since every bounded quantification is equivalent to a subword quantification, comprehension is closed under bounded quantification and the classes Σ_∞^b and Π_1^b both collapse to sets. One more thing: the word "essentially" in the sentence to which this note refers should be understood has having the technical meaning of "same up to bi-interpretability."

[3]The formula $z \preccurlyeq x$ means that the length of z is less than or equal to the length of x, i.e., $\ell(z) \subseteq \ell(x)$. In [8] we used $z \leq x$ instead.

[4]This formulation of the Heine-Borel covering lemma with a *single* (code for an) open set U might seem odd, but it is admissible because we are also bounding the "index" variable w as well. Notice, however, that the Heine-Borel theorem only guarantees the bound k, not that the set $\{(w,z,n) : \ell(w), \ell(x), \ell(n) < k \wedge (w,z,n) \in U\}$ can be coded by an element of $\mathbb{W}$ (see the third remark at the end of section 2).

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