Using KenKen to Build Reasoning Skills [1]

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1 Introduction

KenKen is a puzzle whose solution requires a combination of logic and simple arithmetic and combinatorial skills. The puzzles range in difficulty from very simple to incredibly difficult. Students who get hooked on the puzzle will find themselves practicing addition, subtraction, multiplication and division facts. Specifically, for this paper, a KenKen puzzle is an $n \times n$ grid divided into *cages* using heavy lines. Each cage has a mathematical clue that consists of a number and one of the four arithmetic operations, $+, \times, -, \div$. A solution is an $n \times n$ array of the numbers 1 through n such that no two of the same numbers appear in any row or column and the clues are satisfied by the numbers in the cells of each cage. Any arrangement of the numbers from 1 to n satisfying the sudoku-like requirement of non-duplication is called a Latin Square.

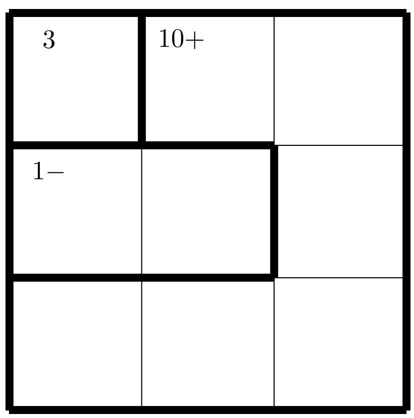
The purpose of this paper is two-fold, to point out some advantages of using numerical puzzles, especially those like KenKen that encourage students to practice arithmetic as they build reasoning skills, and to develop some natural extensions of the regular puzzle that require the puzzler to develop some higher level mathematical skills. Many teachers have found that KenKen has the potential to engage their weakest students, and those student learn two great lessons: first, they practice arithmetic without realizing it, and second, they develop the habit of persevering when they are unable to solve the puzzle immediately

The paper is divided into .... parts. In the first part, we show how reasoning about products and sums enables the solution to easy KenKens. In the second part,

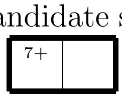
[1]This is a modified version of a workshop presented at the 2011 NCTM Annual Conference.

we study ??? specific ideas we can use to attack moderate to hard problems. These ideas, which we call tactics, are (1) parity and fault lines, (2) counting, (3) stacked cages, (4) X-wing (the name is borrowed from Sudoku) (5) parallel and orthogonal cages, (6) the unique candidate rule and (7) pair analysis. To be removed: Finally, in the last part we discuss extensions and generalizations of KenKen. In particular, TurboKenKen is a KenKen-like puzzle with an initially unknown alphabet set, Backwards KenKen, in which the distribution of characters is given but the cages are not, Primal KenKen, where the characters are all prime numbers, and Abstract KenKen, where the alphabet of characters need not be numbers, or in case they are number, they are combined using operation other than the usual arithmetic operations.

Here's a sample problem to be sure you're on the same page with me. Its a 3×3 with one clueless cage. You're supposed to distribute the number $1, 2$ and 3 in each row and each column in such a way that the sum of the numbers in the six cells of the 10+ cage is 10. Interestingly there is only one way to do this. This uniqueness of solution is one of the characteristics of KenKen. Try is now before we go further.

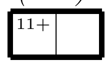


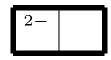
2 Tactics

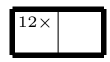
Throughout this discussion, with only a few exceptions, we'll consider 6×6 puzzles. In these puzzles, the digits 1 through 6 must be distributed along each row and column so that no digit appears more than once in each row and column. In addition, the mathematical clues must be satisfied. A cage may have just one or may have several candidate sets. This depends on the clue and also the size of the puzzle. For example, 

has candidate sets $\{1, 6\}, \{2, 5\}, \{3, 4\}$ when it is part of a 6×6 puzzle. But if it appears in a 4×4 puzzle, it has a unique set of candidates.

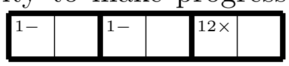
3 Parity and fault lines

A *fault line* is a heavy line that cuts entirely through the puzzle. Fault lines often provide the opportunity to use parity or other ideas because they cut the puzzle into a smaller puzzle of manageable size. *Parity* refers to evenness and oddness of a cage. Specifically, the parity of a cage C is *even (odd)* if the sum of the entries of the cage is an even (odd) number. For example, 

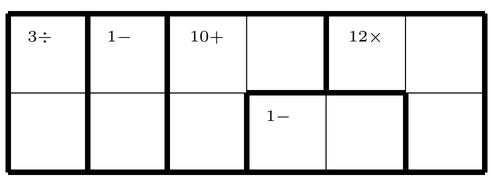
is an odd cage the sum of the entries is 11, which is an odd number. Some two-cell cages have determined parity even though the candidates are not determined. For example, 

is an even cage because the entries are either both even or both odd. On the other hand there are two-cell cages that can be either even or odd. For example, 

has two pairs of candidates, $\{2, 6\}$ and $\{3, 4\}$.

So how can we use parity to make progress towards a solution? Consider the row from a 6×6 KenKen: 

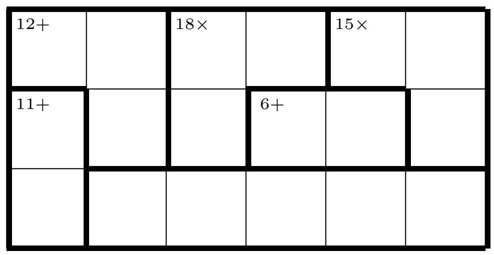
Because the sum $1 + 2 + 3 + 4 + 5 + 6 = 21$, the row must have exactly one or exactly three odd cages. Since the two [1−] cages are odd, so must be the [12×] cage. There is another way to look at the problem of determining the candidates for the [12×] cage. If we put 2 and 6 is the [12×] cage, where would the 1 go. Since 1 can go only with 2 in a [1−] cage, the $\{2, 6\}$ cannot be the set for the [12×] cage. But consider the two-row KenKen fragment below.



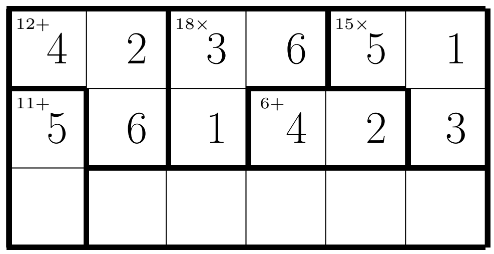
The set of the [12×] cage is one of $\{1, 3, 4\}$, $\{1, 2, 6\}$, or $\{2, 2, 3\}$, the last two of which are odd. But the two cages [3÷] and [10+] are even and the two [1−] cages are both odd. The sum of the entries in the two rows is 42, so the number of odd cages must be even. Therefore the [12×] cage can have only the digits $1, 3$, and 4.

A fault line is a heavy line that cuts entirely through the puzzle. Fault lines often provide the opportunity to use parity or other ideas because they cut the puzzle into a smaller puzzle of manageable size.

Of course we can sometimes use parity when there are no fault lines. Consider the puzzle part below:

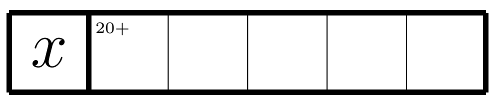

12+
18×
15×
11+
6+

Notice that all three of the cages [18×], [6+] and [12+] are even cages while [15×] is an odd cage. Therefore the entry in the top cell of the [11+] cage must be odd. One (non-unique) solution is

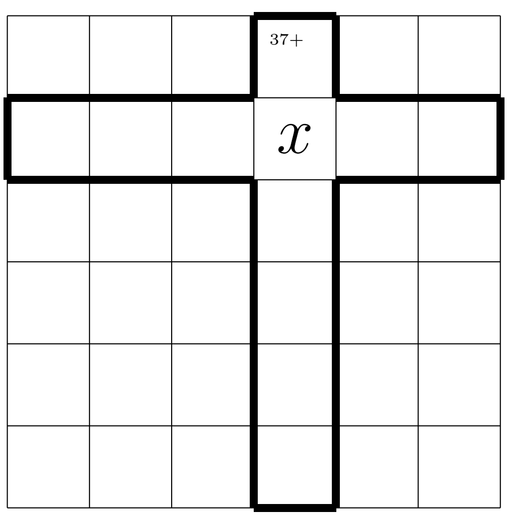


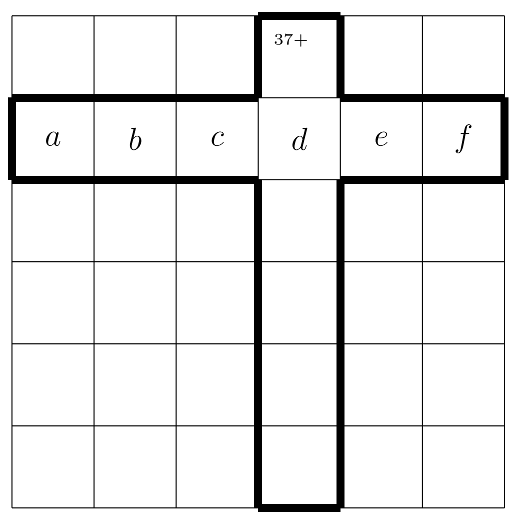
4 Counting

Consider the 6×6 KenKen fragment below. Find the digit that goes in the cell with the x.



Of course, the sum of the entries in each row is $1 + 2 + \cdots + 6 = 21$. So the cell with the x must be exactly $21 - k$. You'll see more examples of this idea below.

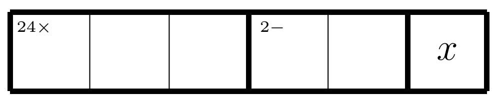




The sum of the row entries, $a+b+c+d+e+f$ is 21, so the sum of the 5 non-d entries in the column must be $37-21=16$. Hence $d=21-16=5$.

5 Stacked Cages

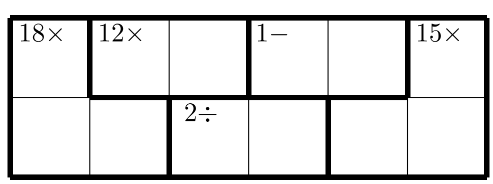
Some puzzles have two or more cages confined to a single line (a row or a column). In this case, we call the cages *stacked*, and we can often take advantage of this situation. Consider the fragment below.



Parity does not help much. All we know from parity is that x is even. This follows from the fact that [24×] is odd (it's either $\{1,4,6\}$ or $\{2,3,4\}$) and [2−] is even as we saw above. Since the sum of each line in a 6×6 puzzle is 21, the entry x must be even. But we can learn more as follows. The cage [24×] contains the 4 of its row. Therefore, the [2−] cage does not contain 4, from which it follows that $3 \in \{[2-]\}$. But in this case, it now follows that $\{[24\times]\} = \{1,4,6\}$. Now we can see that $\{[2-]\} = \{3,5\}$, and from this it follows that $x=2$.

6 The X-wing strategy

The X-wing strategy refers to the fact that no k parallel lines can have more than k copies of a given symbol. In the sample case below, we use the fact that there are at most two 2's in the two rows, and then use parity and counting to finish the problem. Find the candidate sets for each cage.



The candidate multisets for [15×] and [18×] all contain 3, so the cage [12×] cannot contain a 3. Therefore $\{[12\times]\} = \{2,6\}$. Now the 4 in the top row must be in the [1−] cage, and it cannot go with a 3 so $\{[1-]\} = \{4,5\}$. The rest is straightforward.

18× 1, 3	12× 2, 6	2, 6	1− 4, 5	4, 5	15× 1, 3, 5
1, 3, 6	1, 3, 6	2÷ 2, 4	2, 4	1, 3, 5	1, 3, 5

7 Parallel and Orthogonal Cages

Suppose a two-cell cage [$n*$] appears in two parallel lines in the same position within the line. For example,

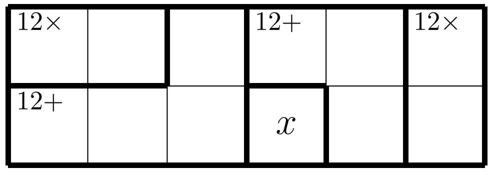
$n*$					
$n*$					

Then the required uniqueness of the solution implies that the two cages cannot be filled with the same two-element set. Consider the example below. Find the value of x.

4−		15+			
4−		10+			48×
			x		

Solution: First note, as above that the two 4− cages cannot be filled with the same two-digit set. The two candidate sets are $\{1, 5\}$ and $\{2, 6\}$. The upper [4−] gets $\{1, 5\}$ so parallel cage idea puts $\{2, 6\}$ in lower [4−] which rules all possible entries in $b6$ except $b6 = 4$, so in row bb we have $(2 + 6) + (10 - x) + 4 = 21$. It follows that $\{10+]\} = \{1, 1, 3, 5\}$, and $x = 1$.

A simple example of orthogonality is shown below.

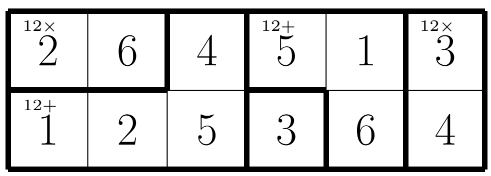


Find the value of x.

Solution: The value of x is 3. Note that the cage [12×] has two candidate sets, $\{3, 4\}$ and $\{2, 6\}$. The two [12×] cages are *orthogonal*. That is, they are oppositely oriented and together have three cells on the same line. These two cages cannot be filled the same way (why), so there must be one of each candidate set. In this case, we can count all the cages to find x.

$$x = 42 - \sum[12x] - \sum[12\times] - 12 - 12 = 42 - 7 - 8 - 12 - 12 = 3.$$

Of course, we don't know which [12×] cage contributes 7 and which one contributes 8. One possible complete solution is



Of course fragments need not have unique solutions.

8 Unique Candidate Rule

This name was suggested by Tom Davis. It refers to the rule that once $n - 1$ copies of a digit are in place, the location of the last one is determined. There are several variations of this. One example is given. Use the Unique Candidate Rule to find the (unique) 3×3 Latin Square with the given values.

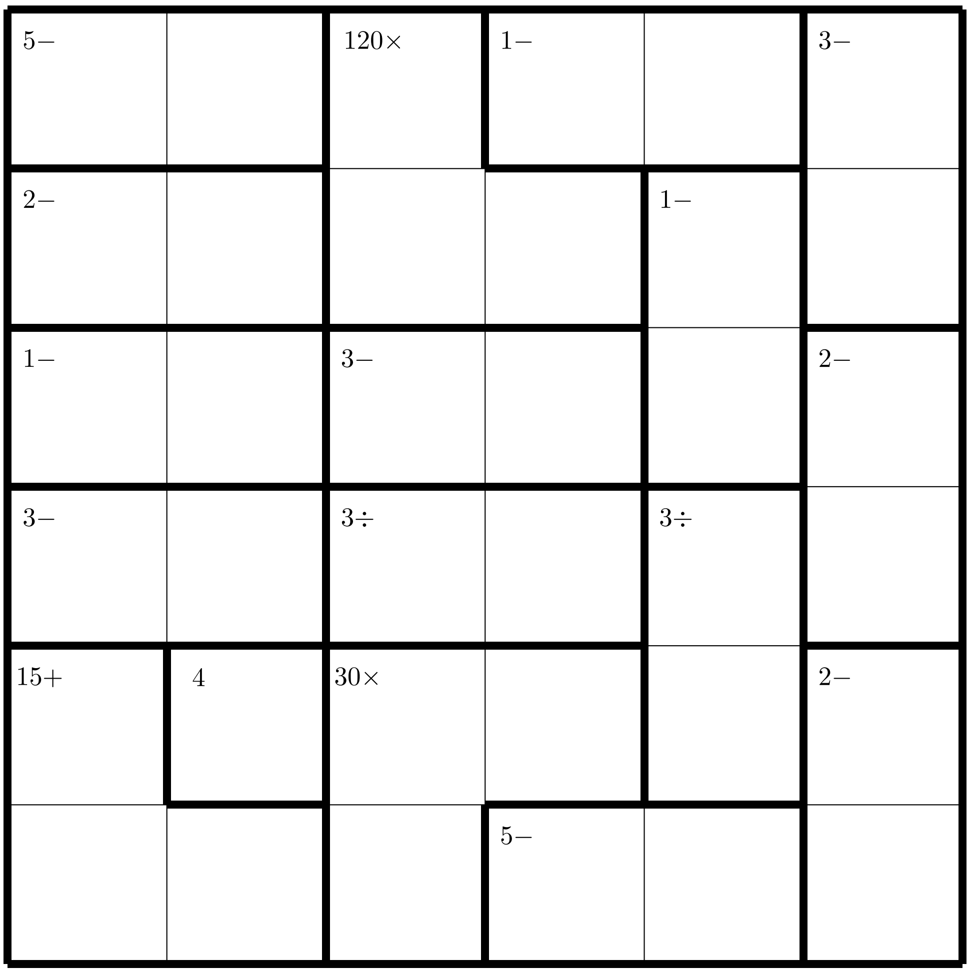
		2
1		

Solution:

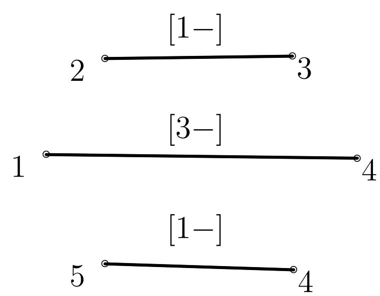
2	3	1
3	1	2
1	2	3

9 Pair Analysis

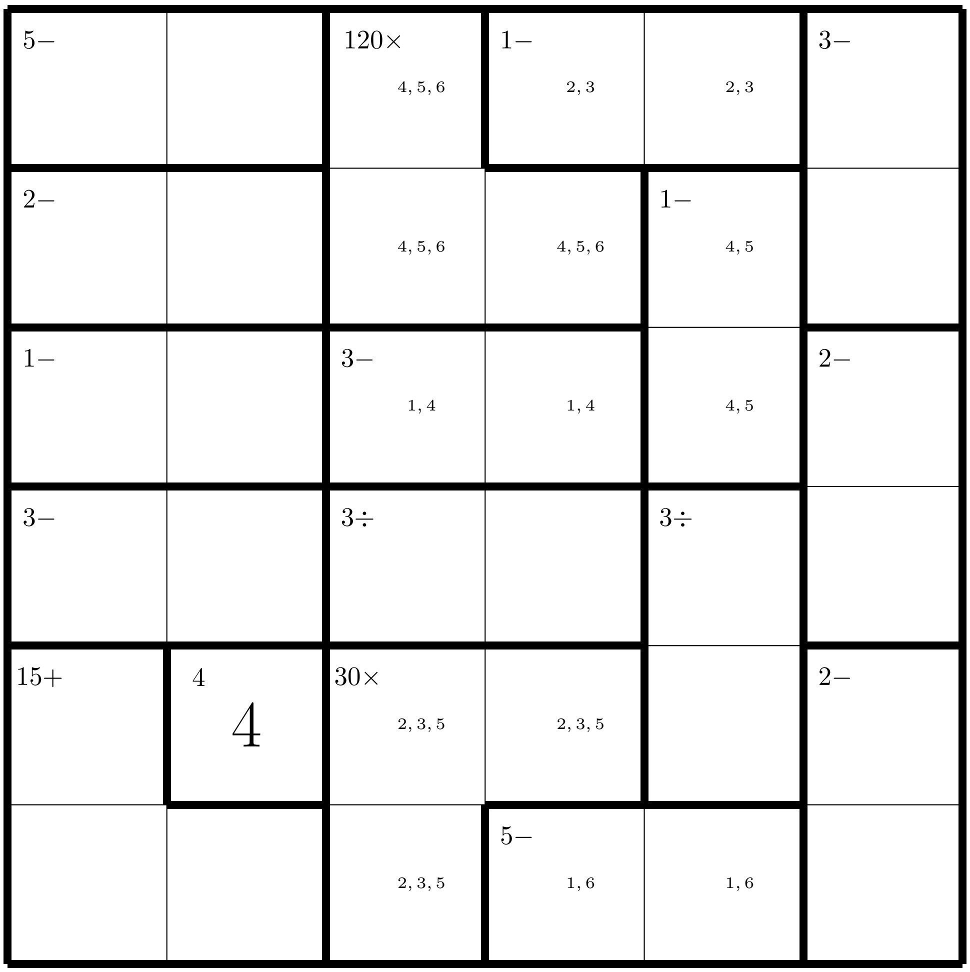
Here's a 6×6 challenge that appeared in the print edition of the New York Times on September 2, 2011. Notice that there are two vertical fault lines. See if you can make use of them and other ideas we've discussed to solve this puzzle.



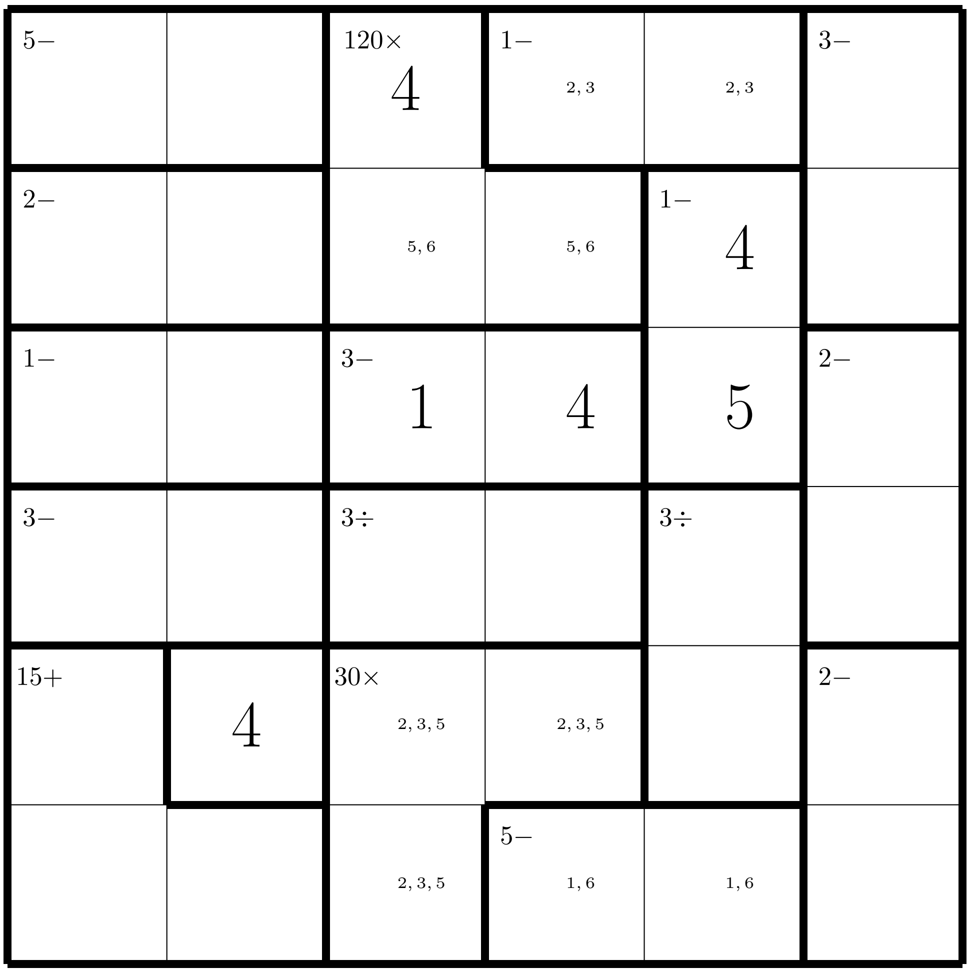
Solution: Consider the 18 cells in columns 3, 4 and 5. Two of the cages in these columns have just one candidate set. They are [120×], which has only $\{4, 5, 6\}$, and [5−], which has only $\{1, 6\}$. But notice that the two [3÷] cages are orthogonal, which means they must be different, so one is $\{1, 3\}$ and the other is $\{2, 6\}$. This means that all three of the 6's in these three columns are accounted for, which implies that $\{[30\times]\} = \{2, 3, 5\}$. Now together these five cages account for the following multiset: $1, 1, 2, 2, 3, 3, 4, 5, 5, 6, 6, 6$. That leaves exactly the multiset $1, 2, 3, 4, 4, 5$ left to fill the other three 2-cell cages. The other cages are [1−], [1−], and [3−]. We build a labeled graph to help us assign these candidates to one another. To do this, simply list the six digits in circular fashion and then start connecting them in pairs so that one pair differs by 3 and the other two differ by 1.



Notice that this is the only pairing that works. If we paired the 2 and the 5 in the [3−] cage, there would not be a way to match the 1 with another digit that differs from it by 1. Also, notice that the $\{2, 3\}$ candidate set for one of the [1−] cages in incompatible with both [3−] candidate sets (stacked cages), and therefore this set $\{2, 3\}$ must occupy the [1−] cage at the top. So now we can make great progress.


5−
120×
4, 5, 6
1−
2, 3
2, 3
3−
2−
4, 5, 6
4, 5, 6
1−
4, 5
1−
3−
1, 4
1, 4
4, 5
2−
3−
3÷
3÷
15+
4
4
30×
2, 3, 5
2, 3, 5
2−
2, 3, 5
5−
1, 6
1, 6

Solution: At this stage several cells have been resolved.

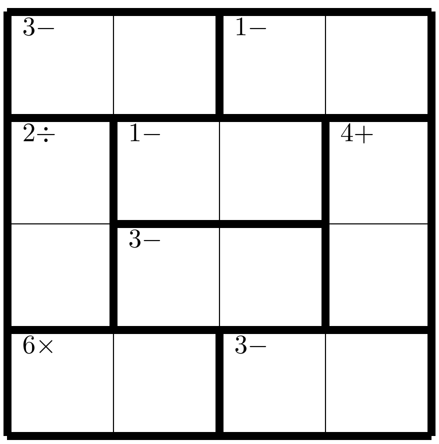


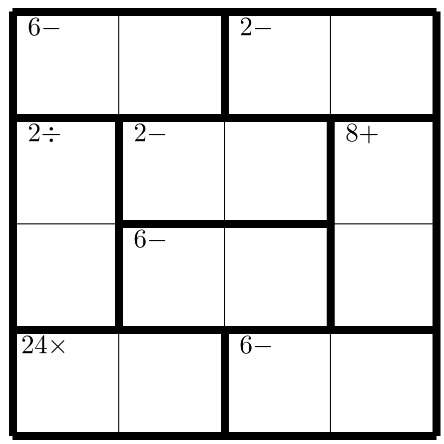
Solution: The complete solution is given below.

5− 1	6	120× 4	1− 2	3	3− 5
2− 3	1	6	5	1− 4	2
1− 2	3	3− 1	4	5	2− 6
3− 5	2	3÷ 3	1	3÷ 6	4
15+ 6	4 4	30× 5	3	2	2− 1
4	5	2	5− 6	1	3

10 Isomorphic Puzzles

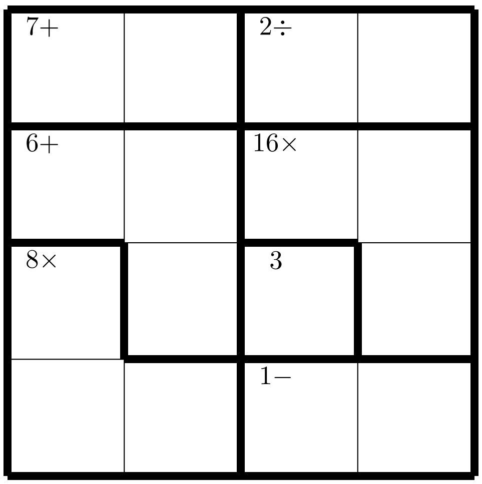
Consider the two 4×4 puzzles below. The first requires distributing the digits 1 through 4 and the second, the digits $2, 4, 6$, and 8.

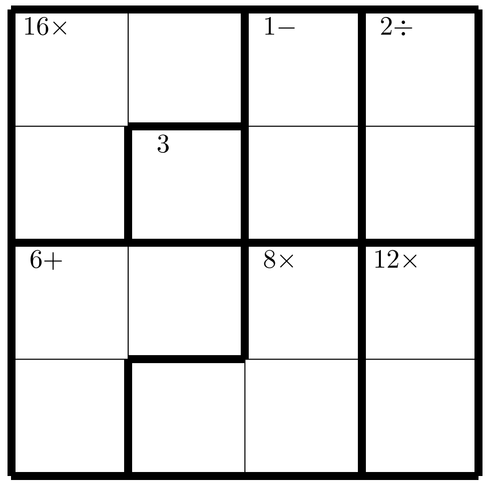




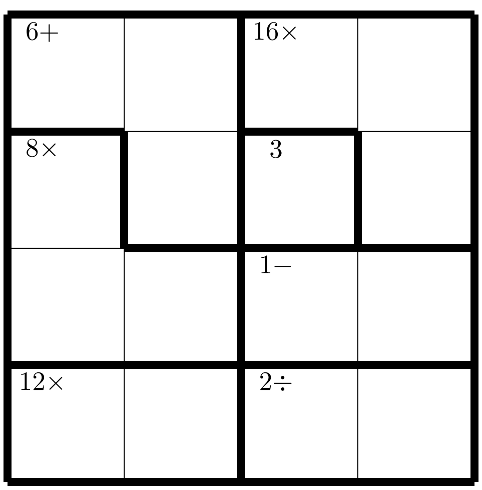
Look at the relation between the additive and subtractive cages. In the second, the clues are all twice as big. But the division cage has the same clue and the multiplicative cages are [6×] in the first and [24×] in the second. This is an example of an isomorphic pair of puzzles. The doubling function $f(n) = 2n$ maps the solutions to the former to that of the later.

Look at the two 4×4 puzzles below. Prove that they are isomorphic or tell why they are not.



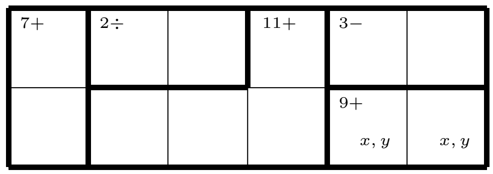


Solution: They are isomorphic. The following puzzle, obtained by moving the top row to the bottom and replacing the 7+ clue with the clue 12×, might help you see why.

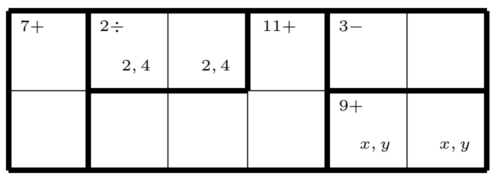


11 Exercises

1. Consider the 6×6 KenKen fragment. Find the candidates for the [9+] cage.

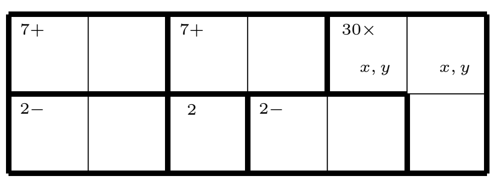


Solution: The answer is $\{x, y\} = \{4, 5\}$. The reasoning goes like this. The four cages [7+], [11+], [3−], and [9+] are odd. Since the sum of the two rows is 42, the number of odd cages is even. Hence the [2÷] cage is even. So we can fill in the two cells as follows.



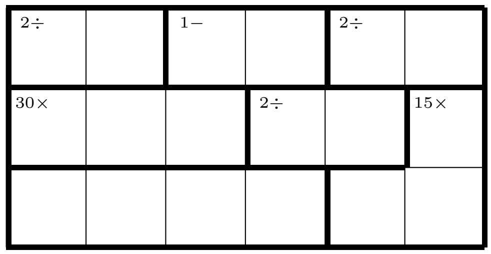
Now it follows that the [3−] cage must be filled with $\{3, 6\}$. Next note that the top left cell can be either 1 or 5. Either way we fill this cell, we are left with $\{x, y\} = \{4, 5\}$.

2. Consider the 6×6 KenKen fragment. Find the candidates for the value of x and y.

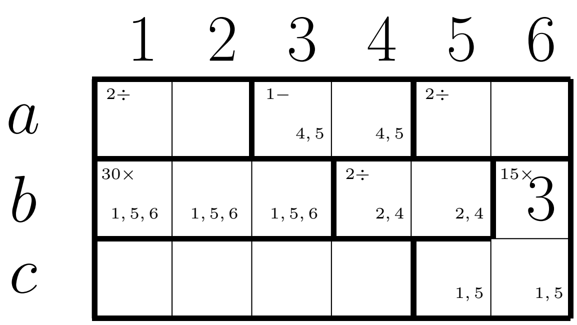


Solution: One of the [2−] cages has even entries and the other odd entries. Since 2 is not available, the cage with even entries has 4 and 6. The cage with odd entries has a 3, and either 1 or 5, leaving 5 or 1 for the bottom right corner. If that bottom right corner is 1, then the other two cells of the [30×] cage are 5 and 6, but that would give the top row a sum of $7 + 7 + 11 = 26 > 21$, so we have a contradiction. The right bottom must be the number 5. Now that means $\{x, y\} = \{1, 6\}$, since the sum of x and y must be 7.

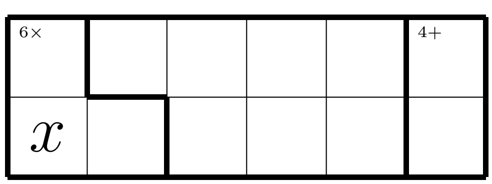
3. Consider the 6×6 KenKen fragment. Find the candidates for four cages in the fragment.



Solution: First consider the top row. The 5 must be part of the [1−] cage, and since both the [2÷] cages have entries whose sums are multiples of 3, so does the [1−] cage. Hence we have the candidates $\{4, 5\}$ for the [1−] cage. Since the number 15 is odd, it can have only odd divisors, so position $c6$ is odd. Since $\{[30\times]\}$ is either $\{2, 3, 5\}$ or $\{1, 5, 6\}$, both of which have even sums, the [2÷] cage in row b must be even, which leaves only $\{2, 4\}$. This forces $\{[30\times]\} = \{1, 5, 6\}$ and $b6 = 3$.



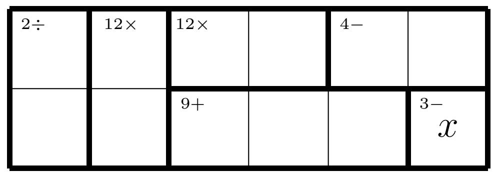
4. This idea came to me from John Watkins (who credits Barry Cipra). Find the value of x.



Solution: The [4+] cells must be 1 and 3 and the only candidate set for the [6×] cage is $\{1, 2, 3\}$. Now the unique solution requirement means that the parallel pairs $a1, b1$ cannot be the same as $a6, b6$, so either $a1 = 2$ or $b1 = 2$. But if $a1 = 2$ then the orthogonal pairs $b1, b2$ would be the same as $a6, b6$ which we have seen provides a contradiction. Therefore, $x = 2$.

5. In this final example, we show how using a combination of the ideas above can solve a very demanding problem. Consider the 6×6 KenKen fragment. Find

the value of x.



Solution: The answer is $x = 6$. The reasoning goes like this. The set of the [4−] cage is $\{1, 5\}$ because one of the numbers in $a2, a3, a4$ is either a 2 or a 6. By orthogonality, one of the [12×] cages is $\{3, 4\}$ and the other is $\{2, 6\}$. Now the cage [2÷] has three candidate sets, $\{1, 2\}$, $\{2, 4\}$, and $\{3, 6\}$. What do all these have in common? Their sums are all multiples of 3. And so is 42. So the number x that goes in position $b6$ makes $\sum[2\div]+(3+4)+(2+6)+(1+5)+9+x$ a multiple of 3. But the 3 must be used in the [9+] cage along with the 1 and the 5.