Supermodular Games

Jonathan Levin

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These notes develop the theory of supermodular games. Supermodular games are those characterized by "strategic complementarities" — roughly, this means that when one player takes a higher action, the others want to do the same. Supermodular games are interesting for several reasons. First, they encompass many applied models. Second, they have the remarkable property that many solution concepts yield the same predictions. Finally, they tend to be analytically appealing — they have nice comparative statics properties and behave well under various learning rules. Much of the theory is due to Topkis (1979), Vives (1990) and Milgrom and Roberts (1990).

1 Monotone Comparative Statics

We take a brief detour to review monotone comparative statics, starting with the property of increasing differences (or supermodularity). For this, suppose $X \subset \mathbb{R}$ and T some partially ordered set.

Definition 1 *A function $f : X \times T \to \mathbb{R}$ has **increasing differences** in (x,t) if for all $x' \geq x$ and $t' \geq t$,*

$$f(x',t') - f(x,t') \geq f(x',t) - f(x,t).$$

What does this mean? If f has increasing differences in (x,t), then the incremental gain to choosing a higher x (i.e. x' rather than x) is greater when t is higher. That is, $f(x',t) - f(x,t)$ is nondecreasing in t. You can check that increasing differences is symmetric — an equivalent statement is that if $t' > t$, then $f(x,t') - f(x,t)$ is nondecreasing in x.

Note that f need not be nicely behaved, nor do X and T need to be intervals. For instance, we could have $X = \{0,1\}$ and just a few parameter values $T = \{0,1,2\}$. If, however, f *is* nicely behaved, we can re-write increasing differences in terms of derivatives.

Lemma 1 *If f is twice continuously differentiable, then then f has increasing differences in (x,t) if and only if $t' \geq t$ implies that $f_x(x,t') \geq f_x(x,t)$ for all x, or alternatively that $f_{xt}(x,t) \geq 0$ for all x, t.*

A central question in monotone comparative statics is to identify when:

$$x(t) = \arg\max_{x \in X} f(x,t)$$

will be non-decreasing (or increasing) in t. The main result we will use is due to Topkis (1968).

Theorem 1 *Let $X \subset \mathbb{R}$ be compact and T a partially ordered set. Suppose $f : X \times T \to \mathbb{R}$ has increasing differences in (x,t), and is upper semi-continuous in x.*[1] *Then (i) for all t, $x(t)$ exists and has a greatest and least element $\overline{x}(t)$ and $\underline{x}(t)$; and (ii) if $t' \geq t$, then $x(t') \geq x(t)$ in the sense that $\overline{x}(t') \geq \overline{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.*

Proof. *(i)* Fix t, and pick $x^1 \leq x^2 \leq ...$, with each $x^k \in x(t)$, and let $\overline{x} = \lim_{k\to\infty} x^k$. Then for all $x \in X$,

$$f(x^k,t) \geq f(x,t) \qquad \Rightarrow \qquad f(\overline{x},t) \geq f(x,t)$$

by continuity. Thus, $\overline{x} \in x(t)$. It follows that $x(t)$ must have a largest (and by the same argument, smallest) element.

(ii) Fix t and t', and let $x \in x(t)$ and $x' \in x(t')$ to be two maximizers. By the fact that x maximizes $f(x,t)$,

$$f(x,t) - f(\min(x,x'),t) \geq 0.$$

This implies (check the two cases that $x \geq x'$ and $x \leq x'$) that:

$$f(\max(x,x'),t) - f(x',t) \geq 0,$$

so by supermodularity

$$f(\max(x,x'),t') - f(x',t') \geq 0.$$

Thus, $\max(x,x')$ maximizes $f(\cdot,t')$. Now if we pick $x = \overline{x}(t)$ and $x' = \overline{x}(t')$, an immediate implication is that $x' \geq x$. A similar argument applies to the lowest maximizers. *Q.E.D.*

Topkis' Theorem says that if f has increasing differences, then the set of maximizers $x(t)$ is increasing in t in the sense that both the highest and lowest maximizers will not decrease if t increases.

[1] Recall that a function $f : X \to \mathbb{R}$ is upper semi-continuous at x_0 if for any ε, there exists a neighborhood $U(x_0)$ such that $x \in U(x_0)$ implies that $f(x) < f(x_0) + \varepsilon$. The function f is upper semi-continuous if it is upper semi-continuous at each $x_0 \in X$.

2 Supermodular Games

We now introduce the notion of a supermodular game, or game with strategic complementarities.

Definition 2 *The game* $(S_1, ..., S_I; u_1, .., u_I)$ *is a* ***supermodular game*** *if for all* i*:*

- S_i *is a compact subset of* $\mathbb{R}$*;*
- u_i *is upper semi-continuous in* s_i, s_{-i}*.*
- u_i *has increasing differences in* (s_i, s_{-i})*.*

Applying Topkis' Theorem in this context shows immediately that each player's best response function is increasing in the actions of other players.

Corollary 1 *Suppose* (S, u) *is a supermodular game, and let*

$$BR_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

Then

(i) $BR_i(s_{-i})$ *has a greatest and least element* $\overline{BR}_i(s_{-i})$, $\underline{BR}_i(s_{-i})$.

(ii) If $s'_{-i} \geq s_{-i}$*, then* $\overline{BR}_i(s'_{-i}) \geq \overline{BR}_i(s_{-i})$ *and* $\underline{BR}_i(s'_{-i}) \geq \underline{BR}_i(s_{-i})$.

2.1 Examples

1. (Investment Game) Suppose firms $1, ..., I$ simultaneously make investments $s_i \in \{0, 1\}$ and payoffs are:

$$u_i(s_i, s_{-i}) = \begin{cases} \pi\left(\sum_{j=1}^{I} s_j\right) - k & \text{if } s_i = 1 \\ 0 & \text{if } s_i = 0 \end{cases}$$

   where π is increasing in *aggregate* investment.

2. (Bertrand Competition) Suppose firms $1, ..., I$ simultaneously choose prices, and that

$$D_i(p_i, p_{-i}) = a_i - b_i p_i + \sum_{j \neq i} d_{ij} p_j$$

   where $b_i, d_{ij} \geq 0$. Then $S_i = \mathbb{R}^+$ and $\pi(p_i, p_{-i}) = (p_i - c_i) D_i(p_i, p_{-i})$ has $\left(\partial^2 \pi_i\right) / (\partial p_i \partial p_j) = d_{ij} \geq 0$. So the game is supermodular.

3. (Cournot Competition) Cournot duopoly is supermodular if we take

$$\begin{aligned} s_1 &= \text{Firm 1's quantity} \\ s_2 &= \text{Negative of Firm 2's quantity} \end{aligned}$$

4. (Diamond Search Model) Consider a simplified version of Diamond's (1982) search model (suggested by Milgrom and Roberts, 1990). There are I agents who exert effort looking for trading partners. Let e_i denote the effort of agent i, and $c(e_i)$ the cost of this effort, where c is increasing and continuous. The probability of finding a partner is $e_i \cdot \sum_{j \neq i} e_j$ and the cost is $c(e_i)$. Then:

$$u_i(e_i, e_{-i}) = e_i \cdot \sum_{j \neq i} e_j - c(e_i)$$

has increasing differences in e_i, e_{-i} so the game is supermodular.

2.2 Solving the Bertrand Game

Consider the Bertrand game from above, where firms 1 and 2 choose prices p_1, p_2. Suppose they have zero marginal costs, and that $D_i(p_i, p_j) = 1 - 2p_i + p_j$. Then

$$\Pi_i(p_i, p_j) = p_i \left[1 - 2p_i + p_j\right].$$

Note that

$$\frac{\partial \Pi_i}{\partial p_i}(p_i, p_j) = 1 - 4p_i + p_j$$

Let's apply iterated strict dominance to this game.

Set $S_i^0 = [0, 1]$.

- If $p_i < \frac{1}{4}$, then $\frac{\partial \Pi_i}{\partial p_i} > 1 - 4\frac{1}{4} + p_j \geq 0 \Rightarrow$ any $p_i < \frac{1}{4}$ is strictly dominated.
- If $p_i > \frac{1}{2}$, then $\frac{\partial \Pi_i}{\partial p_i} < 1 - 4\frac{1}{2} + p_j \leq 0 \Rightarrow$ any $p_i > \frac{1}{2}$ is strictly dominated.

So $S_i^1 = [\frac{1}{4}, \frac{1}{2}]$. Note that this is the interval of best-responses: $BR_i(p_j) \in [\frac{1}{4}, \frac{1}{2}]$.

Let $S_i^k = [\underline{s}^k, \overline{s}^k]$, where

$$\underline{s}^k = \frac{1}{4} + \frac{\underline{s}^{k-1}}{4} = \frac{1}{4} + \frac{1}{16} + \frac{\underline{s}^{k-2}}{16} = ... = \frac{1}{4} + \frac{1}{16} + ... + \frac{1}{4^k} + \frac{\underline{s}^0}{4^k}$$

$$\overline{s}^k = \frac{1}{4} + \frac{\overline{s}^{k-1}}{4} = \frac{1}{4} + \frac{1}{16} + \frac{\overline{s}^{k-2}}{16} = ... = \frac{1}{4} + \frac{1}{16} + ... + \frac{1}{4^k} + \frac{\overline{s}^k}{4^k}$$

So

$$\lim_{k\to\infty} \underline{s}^k = \lim_{k\to\infty} \overline{s}^k = \frac{1}{3}.$$

So $(\frac{1}{3}, \frac{1}{3})$ is the only Nash equilibrium, and the unique rationalizable profile.

3 Main Result

We now use the properties of supermodular games to show that the correspondence between rationalizable and Nash strategies in the Bertrand example is significantly more general than might appear at first glance.

Theorem 2 *Let (S, u) be a supermodular game. Then the set of strategies surviving iterated strict dominance has greatest and least elements $\underline{s}, \overline{s}$ and $\underline{s}, \overline{s}$ are both Nash equilibria.*

Corollary 2 *This implies the following.*

1. *Pure strategy NE exist in supermodular games*
2. *The largest and smallest strategies compatible with ISD, rationalizability, correlated equilibrium and Nash equilibrium are the same.*
3. *If a supermodular game has a unique NE, then it is dominance solvable (& lots of learning or adjustment rules will converge to it (e.g. best-response dynamics)).*

Proof. As in the example, we iterate the best response mapping. Let $S^0 = S$, and let $s^0 = (s_1^0, ..., s_I^0)$ be the largest element of S. Let $s_i^1 = \overline{BR}_i(s_{-i}^0)$, and $S_i^1 = \{s_i \in S_i^0 : s_i \leq s_i^1\}$. If $s_i \notin S_i^1$, i.e. $s_i > s_i^1$, then it is *dominated* by s_i^1 when $s_{-i} \in S_{-i}^0$ because (by increasing differences and the fact that s_i' is the biggest maximizer)

$$u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) \leq u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) < 0$$

Note that $s_i^1 = \overline{BR}_i(s_{-i}^0)$ and $s_i^1 \leq s_i^0$.

Iterating this argument, define

$$s_i^{k+1} = \overline{BR}_i(s_{-i}^k) \qquad \text{and} \qquad S_i^{k+1} = \left\{ s_i \in S_i : s_i \leq s_i^{k+1} \right\}$$

Now, if $s^k \leq s^{k-1}$, this implies that $s_i^{k+1} = \overline{BR}_i(s_{-i}^k) \geq \overline{BR}_i(s_{-i}^{k-1}) = s_i^k$. So by induction, s_i^k is a decreasing sequence for each i. Define:

$$\overline{s}_i = \lim_{k \to \infty} s_i^k$$

This limit exists and only strategies $s_i \leq \overline{s}_i$ are undominated.

Similarly, we can start with $s^0 = (s_1^0, ..., s_I^0)$ the smallest elements in S and identify $\underline{s}$.

To complete the proof, we need to show that $\overline{s} = (\overline{s}_1, ..., \overline{s}_I)$ is a Nash equilibrium. Then for all i, s_i,

$$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k)$$

Taking limits as $k \to \infty$,

$$u_i(\overline{s}_i, \overline{s}_{-i}) \geq u_i(s_i, \overline{s}_{-i}).$$

Q.E.D.

4 Properties of Supermodular Games

A useful propery of supermodular games is that we can use monotonicity to prove comparative statics results. Our first result shows how changes in parameters that affect the marginal returns to action shift the equilibria of a supermodular game.

- A supermodular game (S, u) is indexed by t if each players payoff function is indexed by $t \in T$, some ordered set, and for all i, $u_i(s_i, s_{-i}, t)$ has increasing differences in (s_i, t).

Proposition 1 *Suppose (S, u) is a supermodular game indexed by t. The largest and smallest Nash equilibria are increasing in t.*

Proof. Let $\overline{BR}(s, t) : S \times T \to S$ be the largest best response function as defined above for the game with parameter t. Then $\overline{BR}_i(s, t)$ is i's best response to s_{-i} given parameter value t and is nondecreasing in s and t by Topkis' Theorem. Thus $\overline{BR}(s, t)$ is nondecreasing. Every Nash equilibrium satisfies $\overline{BR}(s, t) \geq s$, and moreover $\overline{s}(t) = \sup\{s : \overline{BR}(s, t) \geq s\}$ is the largest first point of $\overline{BR}(s, t)$ and hence the largest Nash equilibrium (formally this follows from Tarski's Fixed Point Theorem). Since $\overline{BR}(s, \cdot)$ is nondecreasing, $\overline{s}$ is nondecreasing. A similar argument proves the result for the smallest Nash equilibrium. *Q.E.D.*

Because there is a positive feedback between the strategic choices of different players in a supermodular game, there are often multiple equilibria. The second property we consider a welfare theorem that is particularly useful when considering such games.

- A supermodular game (S, u) has *positive spillovers* if for all i, $u_i(s_i, s_{-i})$ is increasing in s_{-i}.

Proposition 2 *Suppose (S, u) is a supermodular game with positive spillovers. Then the Nash equilibrium are ordered in accordance with Pareto preference.*

This result implies that the largest Nash equilibrium is Pareto-preferred among the set of all Nash equilibria. Nevertheless, it need not be Pareto optimal among the set of all strategy profiles.

We have now seen that the greatest and least equilibria in a supermodular game are pure strategy nash equilibria and that it is possible to obtain nice comparative statics results for these equilibria. But what about mixed strategy equilibria? Echenique and Edlin (2003) show that when a supermodular game has mixed strategy equilibria, these equilibria are always "unstable" under a variety of dynamic adjustment processes, thus justifing a focus on pure strategy equilibria.

Their idea can be seen using Battle of the Sexes as an example.

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Recall that Battle of the Sexes has two pure nash equilibria (B, B) and (F, F) and a mixed equilibrium $(\frac{2}{3}B + \frac{1}{3}F, \frac{1}{3}B + \frac{2}{3}F)$. To make this a supermodular game, we need to define an order on the strategy sets. Let $F >_i B$ for both players. Then $u_i(s_i, s_{-i})$ has increasing differences in (s_i, s_{-i}).

In the mixed equilibrium it is crucial that player 1 believes that player 2 is playing exactly $\frac{1}{3}B + \frac{2}{3}F$. If player 1 believes player 2 will play F with probability $2/3 + \varepsilon$, even for $\varepsilon > 0$ small, then player 1 will strictly prefer F. Similarly, if player 2 believes 1 will play F with any probability above $1/3$, player 2 will strictly prefer F.

Now, imagine the players play repeatedly, with player 1 initially believing 2 will play F with probability $2/3 + \varepsilon$ and player 2 initially believing 1 will play F with probability $1/3 + \eta$. Both will play F. If they then adjust their beliefs so they put more weight on their opponent's playing F (I'm purposely being a little loose about the dynamic adjustment process here), they will play F again in the next period, and so on until they always play F and have moved away from mixed strategy beliefs.

This situation is not contrived. The more general point is that so long as player i adjusts his beliefs toward j playing F when j does play F, and so long as i's response to this change is to herself play F more often, then any move toward (F, F) (or toward (B, B)) and away from the mixed equilibrium is self-reinforcing, and many reasonable dynamic processes will move away from the mixed equilibrium toward a pure equilibrium.[2]

5 Comments

1. (Extensions) These results extend to games where players have multi-dimensional strategy spaces. If $S_i \subset \mathbb{R}^n$, we need two further assumptions. First, for all i, S_i must be a *complete sublattice*; second, for all i, u_i must be supermodular in s_i as well as having increasing differences in (s_i, s_{-i}). For precise definitions, see that Monotone Comparative Statics handout. The results also extend to the case where u_i satisfies the single crossing property in (s_i, s_{-i}) as opposed to the stronger assumption of increasing differences (see Milgrom and Shannon, 1994).

2. (Comparing Fixed Points) Milgrom and Roberts (1994) use similar arguments to derive comparative statics for models where equilibria are the solutions to some equation $f(x, t) = 0$. Roughly, they show that if f is increasing in t and continuous (in a weak sense) in x, then the largest fixed point of $f(x, t) = 0$ is increasing in t. Thus their results provide analogues of Proposition 1 for another class of models.

References

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[2] Because our comparative statics results refer to the highest and lowest equilibria, you might also ask what we should make of interior equilibria. Echenique (2002) uses a related stability idea to argue that under reasonable dynamic adjustment processes, our comparative statics predictions should carry through even if players don't always end up at the lowest (or highest) equilibrium.

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