

# Brief review of complex numbers

These notes collect some basic facts about complex numbers.

## 1 Representations

### Imaginary and complex numbers

We start by introducing a symbol  $i$  that represents the squareroot of  $-1$ , *i.e.*,  $i^2 = -1$ . For some strange reason, electrical engineers use the symbol  $j$  instead of  $i$ . (Maybe because  $i$  could be confused with current?) It's a dumb tradition, but I'll respect it here.

We refer to a number of the form  $bj$ , where  $b$  is real, as *imaginary*. A *complex number* has the form  $a + bj$ , where  $a$  and  $b$  are real. Some examples are:  $j$ ,  $0$ ,  $-0.0023 + 0.553j$ ,  $-3.44$ , and  $-10^6j$ .

### Real and imaginary parts

If  $c = a + bj$ , where  $a$  and  $b$  are real, we refer to  $a$  as the *real part* of  $c$ , and  $b$  as the *imaginary part* of  $c$ . The real and imaginary parts are denoted

$$\Re(c) = a, \quad \Im(c) = b.$$

Two complex numbers are equal if and only if their real parts are equal and also their imaginary parts are equal. Another way to say this is that a complex number has only one representation in the form  $a + bj$ .

We can think of complex numbers as vectors with two (real) components, *i.e.*, real and imaginary. Thus we can plot complex numbers in the *complex plane*, with the horizontal ( $x$ -) axis denoting real part and the vertical ( $y$ -) axis showing imaginary part.

### Conjugate

The *conjugate* of a complex number  $c = a + bj$  is denoted  $\bar{c}$  or  $c^*$ , and is defined by  $\bar{c} = a - bj$ . Geometrically,  $\bar{c}$  is constructed by reflecting  $c$  through the real (*i.e.*, horizontal) axis.

### Absolute value

The *magnitude*, *absolute value*, or *length* of a complex number is defined as

$$|a + bj| = \sqrt{a^2 + b^2}.$$

Note that the magnitude is always nonnegative, and equals zero only if the complex number is zero (*i.e.*,  $0 + 0j$ ). The magnitude of a complex number is the Euclidean distance from the origin in the complex plane to the point.

## Angle

The *angle* or *phase* or *argument* of the complex number  $a + bj$  is the angle, measured in radians, from the point  $1 + 0j$  to  $a + bj$ , with counterclockwise denoting positive angle. The angle of a complex number  $c = a + bj$  is denoted  $\angle c$ :

$$\angle c = \arctan b/a.$$

Several comments are in order. First, angles that differ by a multiple of  $2\pi$  are considered equal. Second, the formula above uses the *four quadrant arctan*, and is much better written as  $\angle c = \arctan(b, a)$  (often expressed as `atan2(b, a)` in computer languages). The angle of the complex number 0 is undefined. As examples,  $\angle 1.23 = 0$  (but we could just as well write  $\angle 1.23 = -4\pi$ ),  $\angle j = \pi/2$ ,  $\angle(1 + j) = \pi/4$ ,  $\angle(1 - j) = -\pi/4$ .

## Addition

We define addition of complex numbers as

$$(a + bj) + (c + dj) = (a + c) + (b + d)j,$$

*i.e.*, the real and imaginary parts add separately. This corresponds to vector addition in the complex plane. Subtraction is defined similarly.

We can express the real and imaginary parts in terms of addition and conjugates as

$$\Re(c) = \frac{c + \bar{c}}{2}, \quad \Im(c) = \frac{c - \bar{c}}{2j}.$$

The *triangle inequality* states that the absolute value of a complex sum is no more than the sum of the absolute values, *i.e.*,

$$|f + g| \leq |f| + |g|$$

where  $f$  and  $g$  are complex numbers. The *distance* between two complex numbers  $f$  and  $g$  is given by  $|f - g|$  (which coincides with the Euclidean distance between  $f$  and  $g$  in the complex plane).

## Multiplication

Complex multiplication is based on the identity  $j^2 = -1$ :

$$(a + bj)(c + dj) = ac + adj + bcj + bdj^2 = (ac - bd) + (ad + bc)j.$$

Division is defined similarly: if  $f$ ,  $g$ , and  $h$  are complex numbers, then  $f = g/h$  means  $fh = g$ .

The inverse of a complex number  $f$  means  $1/f$ . If  $f = a + bj$ , where  $a$  and  $b$  are real, the inverse can be expressed as

$$1/f = \frac{a}{a^2 + b^2} - j \frac{b}{a^2 + b^2}$$

(we'll see later where this formula comes from).

The absolute value of a product is the product of the absolute values, *i.e.*,  $|fg| = |f||g|$ , where  $f$  and  $g$  are complex numbers. The angle of a product is the sum of the angles, *i.e.*,  $\angle(fg) = \angle f + \angle g$ .

Complex addition, subtraction, multiplication, and division obey the same standard rules as real numbers: commutative, distributive, associative, etc.

We can express the magnitude using the conjugate as  $|c| = \sqrt{\bar{c}c}$ . (In particular, the product of a complex number with its conjugate is always real and nonnegative.) Since  $\bar{c}c = |c|^2$ , we can express the inverse explicitly as

$$1/c = \frac{\bar{c}}{|c|^2}.$$

## Complex exponential

With addition and multiplication, we can define polynomials of complex numbers. Thus for example  $p(x) = 1 + x - x^2$  evaluated at  $x = 1 + j$  yields

$$p(1 + j) = 1 + (1 + j) + (1 + j)^2 = 2 + 3j.$$

We can define more general functions of complex numbers via power series expansions. Perhaps the most important case is the *complex exponential*:

$$e^{a+bj} = e^a \cos b + (e^a \sin b)j.$$

One important special case is

$$e^{j\phi} = \cos \phi + j \sin \phi,$$

where  $\phi$  is real. For example, we have  $e^{j\pi} = -1$ .

The formula for the exponential shows that

$$|e^c| = e^{\Re c}, \quad \angle e^c = \Im c.$$

Thus, we define the logarithm of a (nonzero) complex number  $c$  as  $\log c = \log |c| + j\angle c$ , so that  $e^{\log c} = c$ . Note that the imaginary component of  $\log c$  is ambiguous; we can freely add any multiple of  $2\pi$ . Thus we can say that

$$\log(1 - j) = \log \sqrt{2} - j(\pi/4 + 2\pi k)$$

where  $k$  is any integer. The ambiguity in the (imaginary part of the) logarithm requires care.

## Exponential representation

From the exponential formula we see that we can express the complex number  $c = a + bj$  in terms of the complex exponential as  $c = re^{j\theta}$ , where

$$r = |c|, \quad \theta = \angle c.$$

The form  $c = re^{j\theta}$  is called the *exponential* or *polar* of the complex number  $c$  (since  $(r, \theta)$  are the polar coordinates of the point with rectangular coordinates  $(a, b)$ ).

Note that  $r$  and  $\theta$  can be thought of as related to the real and imaginary parts of the complex logarithm of  $c$ . It's not too surprising, therefore, that the exponential representation of a product is easily expressed:

$$(re^{j\theta})(qe^{j\phi}) = (rq)e^{j(\theta+\phi)}.$$

In words: the magnitudes of a product multiply; the angles add. The conjugate and inverse can be expressed as

$$\bar{c} = re^{-j\theta}, \quad 1/c = (1/r)e^{-j\theta},$$

where  $c = re^{j\theta}$ .

This gives us a geometric picture of complex multiplication. Multiplication by a real number *scales* or stretches the number (reversing orientation if the real number is negative). Multiplication by  $j$  *rotates* a number by  $\pi/2 = 90^\circ$ ; more generally multiplication by

$$e^{j\theta} = \cos \theta + j \sin \theta$$

rotates complex numbers (counterclockwise) by the angle  $\theta$ .