5 Potential Theory

Reference: *Introduction to Partial Differential Equations* by G. Folland, 1995, Chap. 3.

5.1 Problems of Interest.

In what follows, we consider Ω an open, bounded subset of $\mathbb{R}^n$ with C^2 boundary. We let $\Omega^c = \mathbb{R}^n - \overline{\Omega}$ (the open complement of Ω). We are interested in studying the following four problems:

(a) *Interior Dirichlet Problem.*

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

(b) *Exterior Dirichlet Problem.*

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ u = g & x \in \partial\Omega^c. \end{cases}$$

(c) *Interior Neumann Problem.*

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega. \end{cases}$$

(d) *Exterior Neumann Problem.*

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega^c. \end{cases}$$

Previously, we have used Green's representation, to show that if u is a C^2 solution of the Interior Dirichlet Problem, then u is given by

$$u(x) = -\int_{\partial\Omega} g(y) \frac{\partial G}{\partial \nu_y}(x, y)\, dS(y),$$

where $G(x, y)$ is the Green's function for Ω. However, in general, it is difficult to calculate an explicit formula for the Green's function. Here, we use a different approach to look for solutions to the Interior Dirichlet Problem, as well as to the other three problems above. Again, it's difficult to calculate explicit solutions, but we will discuss existence of solutions and give representations for them.

5.2 Definitions and Preliminary Theorems.

As usual, let $\Phi(x)$ denote the fundamental solution of Laplace's equation. That is, let

$$\Phi(x) \equiv \begin{cases} -\frac{1}{2\pi} \ln|x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & n \geq 3. \end{cases}$$

Let h be a continuous function on $\partial\Omega$. The **single layer potential with moment** h is defined as

$$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)\, dS(y). \tag{5.1}$$

The **double layer potential with moment** h is defined as

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y). \tag{5.2}$$

We plan to use these layer potentials to construct solutions of the problems listed above. Notice that Green's function gives us a solution to the Interior Dirichlet Problem which is similar to a double layer potential. We will see that for an appropriate choice of h, we can write solutions of the Dirichlet problems $(a), (b)$ as *double layer potentials* and solutions of the Neumann problems $(c), (d)$ as *single layer potentials*.

First, we will prove that for a continuous function h, (5.1) and (5.2) are harmonic functions for all $x \notin \partial\Omega$.

Theorem 1. *For h a continuous function on $\partial\Omega$,*

1. *$\overline{u}$ and $\overline{\overline{u}}$ are defined for all $x \in \mathbb{R}^n$.*

2. *$\Delta\overline{u}(x) = \Delta\overline{\overline{u}}(x) = 0$ for all $x \notin \partial\Omega$.*

Proof.

1. We prove that $\overline{\overline{u}}$ is defined for all $x \in \mathbb{R}^n$. A similar proof works for $\overline{u}$.

   First, suppose $x \notin \partial\Omega$. Therefore, $\frac{\partial\Phi}{\partial\nu_y}(x-y)$ is defined for all $y \in \partial\Omega$. Consequently, for all $x \notin \partial\Omega$, we have

$$|\overline{\overline{u}}(x)| \leq |h(y)|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \left|\frac{\partial\Phi}{\partial\nu_y}(x-y)\right| dS(y) \leq C.$$

   Next, consider the case when x is in $\partial\Omega$. In this case, the term $\frac{\partial\Phi}{\partial\nu_y}(x-y)$ in the integrand is undefined at $x = y$. We prove $\overline{\overline{u}}$ is defined at this point x by showing that the integral in (5.2) still converges.

   We need to look for a bound on

$$-\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y).$$

   Recall

$$\Phi(x-y) = \begin{cases} -\frac{1}{2\pi}\ln|x-y| & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x-y|^{n-2}} & n \geq 3. \end{cases}$$

   Therefore,

$$\Phi_{y_i}(x-y) = \frac{x_i - y_i}{n\alpha(n)|y-x|^n},$$

and,

$$\frac{\partial \Phi}{\partial \nu_y}(x-y) = \nabla_y \Phi(x-y)\cdot \nu(y)$$
$$= \frac{(x-y)\cdot \nu(y)}{n\alpha(n)|y-x|^n},$$

where $\nu(y)$ is the unit normal to $\partial\Omega$ at y.

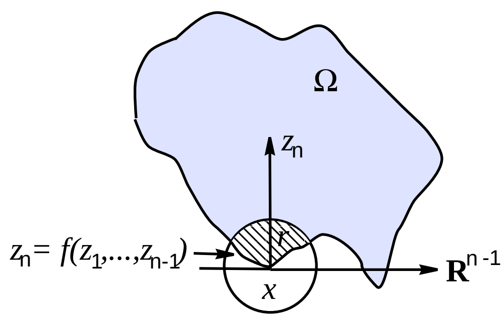
Claim: Fix $x \in \partial\Omega$. For all $y \in \partial\Omega$, there exists a constant $C > 0$ such that

$$|(x-y)\cdot \nu(y)| \le C|x-y|^2.$$

Proof of Claim. By assumption, $\partial\Omega$ is C^2. This means at each point $x \in \partial\Omega$, there exists an $r > 0$ and a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon relabelling and reorienting if necessary - we have

$$\Omega \cap B(x,r) = \{z \in B(x,r) \,|\, z_n > f(z_1,\ldots,z_{n-1})\}.$$

(See Evans - Appendix C.)



Without loss of generality (by reorienting if necessary), we may assume $x = 0$ and $\nu(x) = (0,\ldots,0,1)$. Using the fact that our boundary is C^2, we know there exists an $r > 0$ and a C^2 function $f : B(0,r) \subset \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\partial\Omega$ is given by the graph of the function f near x.

First, consider $y \in \partial\Omega$ such that $|x-y| \ge r$. In this case,

$$|(x-y)\cdot\nu(y)| \le |x-y| \le \frac{1}{r}|x-y|^2 = C(r)|x-y|^2.$$

Second, consider $y \in \partial\Omega$ such that $|x-y| \le r$. In this case, we use the fact that

$$\begin{aligned}|(x-y)\cdot\nu(y)| &= |(x-y)\cdot(\nu(x)+\nu(y)-\nu(x))| \\ &\le |(x-y)\cdot\nu(x)| + |(x-y)\cdot(\nu(y)-\nu(x))| \\ &= |y_n| + |(x-y)\cdot(\nu(y)-\nu(x))|.\end{aligned}$$

Now,

$$y_n = f(y_1,\ldots,y_{n-1})$$

where $f \in C^2$, $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, by Taylor's Theorem, we have

$$\begin{aligned}|y_n| &= |f(y_1, \ldots, y_{n-1})| \\ &\leq C|(y_1, \ldots, y_{n-1})|^2 \\ &\leq C|y|^2 \\ &= C|x-y|^2,\end{aligned}$$

where the constant C depends only on the bound on the second partial derivatives of $f(y_1, \ldots, y_{n-1})$ for $|(y_1, \ldots, y_{n-1})| \leq r$, but this is bounded because by assumption $f \in C^2(\overline{B(0,r)})$.

Next, we look at $|(x-y) \cdot (\nu(y) - \nu(x))|$. By assumption, $\partial\Omega$ is C^2 and consequently, ν is a C^1 function and therefore, there exists a constant $C > 0$ such that

$$|\nu(y) - \nu(x)| \leq C|y - x|.$$

Therefore,

$$|(x-y) \cdot (\nu(y) - \nu(x))| \leq C|y-x|^2.$$

Consequently, our claim is proven. We remark that the constant C will depend on r, but once x is chosen r is fixed. $\diamond$

Therefore, we conclude that for $x \in \partial\Omega$, all $y \in \partial\Omega$,

$$\begin{aligned}\left|\frac{\partial\Phi}{\partial\nu_y}(x-y)\right| &= \left|\frac{(x-y)\cdot\nu(y)}{n\alpha(n)|y-x|^n}\right| \\ &\leq C\frac{|x-y|^2}{|x-y|^n} \\ &= \frac{C}{|x-y|^{n-2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|-\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\,dS(y)\right| &\leq |h(y)|_{L^\infty(\partial\Omega)}\int_{\partial\Omega}\left|\frac{\partial\Phi}{\partial\nu_y}(x-y)\right|\,dS(y) \\ &\leq C\int_{\partial\Omega}\frac{1}{|x-y|^{n-2}}\,dS(y) \leq C\end{aligned}$$

using the fact that $\partial\Omega$ is of dimension $n-1$. Therefore, we conclude that $\overline{\overline{u}}$ is defined for all $x \in \partial\Omega$ and consequently for all $x \in \mathbb{R}^n$ as claimed.

2. Next, we will prove that $\Delta\overline{\overline{u}}(x) = 0$ for all $x \in \Omega$. A similar proof works to prove that $\Delta\overline{u}(x) = 0$.

   Fix $x \in \Omega$. We note that for all $y \in \partial\Omega$, $\frac{\partial\Phi}{\partial\nu_y}(x-y)$ is a smooth function. Further, using the fact that $\Phi(x-y)$ is harmonic for all $x \neq y$, we conclude that $\Delta_x\frac{\partial\Phi}{\partial\nu_y}(x-y) = 0$

for all $y \in \partial\Omega$. Therefore, using the fact that our integral is finite and $\frac{\partial\Phi}{\partial\nu_y}(x-y)$ is smooth, we conclude that

$$\begin{aligned}\Delta_x \overline{\overline{u}}(x) &= -\Delta_x \int_{\partial\Omega} h(y) \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) \\ &= -\int_{\partial\Omega} h(y) \Delta_x \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) \\ &= 0.\end{aligned}$$

□

In the above theorem, we showed that as long as h is a continuous function on $\partial\Omega$, then $\overline{u}$ and $\overline{\overline{u}}$, defined in (5.1) and (5.2), respectively, are harmonic functions on Ω. Consequently, if we can choose h appropriately so that our initial condition will be satisfied, then we can find a solution of our particular problem ((a), (b), (c), or (d)).

We claim that by choosing h appropriately, $\overline{\overline{u}}$ will give us a solution of our interior or exterior Dirichlet problem. Similarly, we will show that by choosing h appropriately, $\overline{u}$ will give us a solution of our interior or exterior Neumann problems.

For a moment, consider the interior Dirichlet problem (a). As proven above, for h a continuous function on $\partial\Omega$, $\overline{\overline{u}}$ defined in (5.2) is harmonic. Now, if we can choose h appropriately, such that for all $x_0 \in \partial\Omega$,

$$\lim_{x \in \Omega \to x_0} \overline{\overline{u}}(x) = g(x_0),$$

then we will have found a solution of the interior Dirichlet problem. Consequently, we are interested in studying the limits of $\overline{\overline{u}}$ as we approach the boundary of Ω. In order to study this, we must first prove the following lemma.

Lemma 2. (Gauss' Lemma) *Consider the double layer potential,*

$$\overline{\overline{v}}(x) = -\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y).$$

Then,

$$\overline{\overline{v}}(x) = \begin{cases} 0 & x \in \Omega^c \\ 1 & x \in \Omega \\ 1/2 & x \in \partial\Omega. \end{cases}$$

Proof. 1. First, for $x \in \Omega^c$,

$$\begin{aligned}\overline{\overline{v}}(x) &= -\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) \\ &= -\int_{\Omega} \Delta_y \Phi(x-y)\, dy \\ &= 0\end{aligned}$$

using the Divergence Theorem and the fact that $\Phi(x-y)$ is smooth for $y \in \Omega$, $x \in \Omega^c$.

2. Now, for $x \in \Omega$, $\Phi(x-y)$ is not smooth for all $y \in \Omega$. In order to overcome this problem, we fix $\epsilon > 0$ sufficiently small such that $B(x,\epsilon)$ is contained within Ω. Then on the region $\Omega - B(x,\epsilon)$, $\Phi(x-y)$ is smooth, and, consequently, we can say

$$\begin{aligned} 0 &= \int_{\Omega - B(x,\epsilon)} \Delta_y \Phi(x-y)\, dy \\ &= \int_{\partial(\Omega - B(x,\epsilon))} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) \\ &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) \end{aligned}$$

where ν is the outer unit normal to $\Omega - B(x,\epsilon)$.

As mentioned above,

$$\Phi_{y_i}(x-y) = \frac{x_i - y_i}{n\alpha(n)|y-x|^n}.$$

For $y \in \partial B(x,\epsilon)$, the outer unit normal to $\Omega - B(x,\epsilon)$ is given by

$$\nu(y) = \frac{x-y}{|x-y|}.$$

Therefore, for $y \in \partial B(x,\epsilon)$,

$$\begin{aligned} \frac{\partial \Phi}{\partial \nu_y}(x-y) &= \nabla_y \Phi(x-y) \cdot \nu(y) \\ &= \frac{x-y}{n\alpha(n)|x-y|^n} \cdot \frac{x-y}{|x-y|} \\ &= \frac{|x-y|^2}{n\alpha(n)|x-y|^{n+1}} \\ &= \frac{1}{n\alpha(n)|x-y|^{n-1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) &= \int_{\partial B(x,\epsilon)} \frac{1}{n\alpha(n)|x-y|^{n-1}}\, dS(y) \\ &= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} dS(y) \\ &= 1. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) \\ &= \int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) + 1. \end{aligned}$$

which $\implies$

$$-\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) = 1,$$

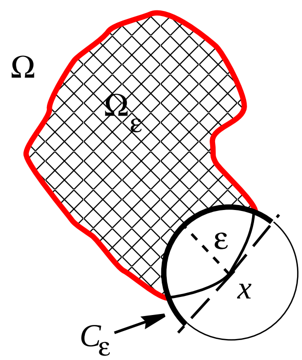
as desired.

3. Last, we consider the case $x \in \partial\Omega$. In this case, $\frac{\partial \Phi}{\partial \nu_y}(x-y)$ is not defined at $y = x$. Fix $x \in \partial\Omega$. Let $B(x,\epsilon)$ be the ball of radius ϵ about x. Let

$$\Omega_\epsilon \equiv \Omega - (\Omega \cap B(x,\epsilon)).$$

Let

$$\mathcal{C}_\epsilon \equiv \{y \in \partial B(x,\epsilon) : \nu(x) \cdot y < 0\}.$$



Let

$$\widetilde{\mathcal{C}}_\epsilon \equiv \partial\Omega_\epsilon \cap \mathcal{C}_\epsilon.$$

First, we note that

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \Delta_y \Phi(x-y)\, dy \\ &= \int_{\partial\Omega_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) \\ &= \int_{\partial\Omega_\epsilon - \widetilde{\mathcal{C}}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) + \int_{\widetilde{\mathcal{C}}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y), \end{aligned} \tag{5.3}$$

where ν_y is the outer unit normal to Ω_ϵ.

Now, first, we recall that

$$\nabla_y \Phi(x-y) = \frac{x-y}{n\alpha(n)|y-x|^n}.$$

For all $y \in \widetilde{\mathcal{C}}_\epsilon$, the outer unit normal is given by

$$\nu(y) = \frac{x-y}{|x-y|}.$$

Therefore,

$$\int_{\widetilde{\mathcal{C}}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) = \int_{\widetilde{\mathcal{C}}_\epsilon} \frac{1}{n\alpha(n)|x-y|^{n-1}}\, dS(y)$$
$$= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\widetilde{\mathcal{C}}_\epsilon} dS(y).$$

Next, we use the fact that

$$\int_{\widetilde{\mathcal{C}}_\epsilon} dS(y) \approx \int_{\mathcal{C}_\epsilon} dS(y).$$

In fact, as we will show below,

$$\int_{\widetilde{\mathcal{C}}_\epsilon} dS(y) = \int_{\mathcal{C}_\epsilon} dS(y) + O(\epsilon^n). \tag{5.4}$$

We omit the proof of (5.4) for now and will return to it below. Assuming this fact for now, we have

$$\int_{\widetilde{\mathcal{C}}_\epsilon} dS(y) = \frac{1}{2} n\alpha(n)\epsilon^{n-1} + O(\epsilon^n)$$

which implies

$$\begin{aligned} \int_{\widetilde{\mathcal{C}}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) &= \frac{1}{n\alpha(n)\epsilon^{n-1}} \left[\frac{1}{2} n\alpha(n)\epsilon^{n-1} + O(\epsilon^n)\right] \\ &= \frac{1}{2} + \frac{1}{n\alpha(n)} O(\epsilon). \end{aligned} \tag{5.5}$$

Combining (5.3) and (5.5), we have

$$0 = \int_{\partial\Omega_\epsilon - \widetilde{\mathcal{C}}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) + \frac{1}{2} + \frac{1}{n\alpha(n)} O(\epsilon),$$

which implies

$$\int_{\partial\Omega_\epsilon - \widetilde{\mathcal{C}}_\epsilon} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) = -\frac{1}{2} - \frac{1}{n\alpha(n)} O(\epsilon).$$

Taking the limit as $\epsilon \to 0^+$, we have

$$\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y) = -\frac{1}{2},$$

as claimed.

□

Now we will prove (5.4).

Claim 3. *For $\widetilde{\mathcal{C}}_\epsilon$ and $\mathcal{C}_\epsilon$ as defined above, we have*

$$\int_{\widetilde{\mathcal{C}}_\epsilon} dS(y) = \int_{\mathcal{C}_\epsilon} dS(y) + O(\epsilon^n).$$

Proof. We just need to show that the surface area of $\mathcal{C}_\epsilon - \widetilde{\mathcal{C}}'_\epsilon$ is $O(\epsilon^n)$. The surface area is approximately the surface area of the base times the height. Now the surface area of the base is $O(\epsilon^{n-2})$. Therefore, we just need to show that the height is $O(\epsilon^2)$.

Without loss of generality, we let $x = 0$. Now, by assumption, $\partial\Omega$ is C^2. Therefore, $\partial\Omega$ can be written as the graph of a C^2 function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $f(0) = 0$ and $\nabla f(0) = 0$. Therefore, if $y \in \mathcal{C}_\epsilon - \widetilde{\mathcal{C}}_\epsilon$, then

$$|y_n| \leq |f(y_1, \ldots, y_{n-1})| \leq C|(y_1, \ldots, y_{n-1})|^2 \leq C|y|^2 \leq C\epsilon^2,$$

using Taylor's Theorem. Therefore, the height is $O(\epsilon^2)$ and the claim follows. □

Now with Gauss' Lemma above, for $\overline{\overline{u}}$ defined as in (5.2) for a continuous function h, we can find the limits of $\overline{\overline{u}}(x)$ as we approach $\partial\Omega$ from the interior or the exterior. We state these results in the following theorem.

Theorem 4. *Let h be a continuous function on $\partial\Omega$. Define the double layer potential*

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y).$$

Let $x_0 \in \partial\Omega$. Then

$$\lim_{x \in \Omega \to x_0} \overline{\overline{u}}(x) = \frac{1}{2}h(x_0) + \overline{\overline{u}}(x_0)$$
$$\lim_{x \in \Omega^c \to x_0} \overline{\overline{u}}(x) = -\frac{1}{2}h(x_0) + \overline{\overline{u}}(x_0).$$

Proof. We will prove only the first case, when $x \in \Omega$. The second case works similarly.

Let $x \in \Omega$, $x_0 \in \partial\Omega$. We have

$$\begin{aligned}
\overline{\overline{u}}(x) &= -\int_{\partial\Omega} h(y) \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) \\
&= -\int_{\partial\Omega} h(y) \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) + h(x_0)\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) \\
&\quad - h(x_0)\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) \\
&= -\int_{\partial\Omega} [h(y) - h(x_0)] \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) + h(x_0) \\
&\equiv I(x) + h(x_0),
\end{aligned}$$

using the fact that

$$-\int_{\partial\Omega} \frac{\partial\Phi}{\partial\nu_y}(x-y)\, dS(y) = 1 \qquad \text{for } x \in \Omega,$$

proven in Gauss' Lemma. Similarly,

$$\begin{aligned}\overline{\overline{u}}(x_0) &= -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x_0-y)\,dS(y)\\ &= -\int_{\partial\Omega}[h(y)-h(x_0)]\frac{\partial\Phi}{\partial\nu_y}(x_0-y)\,dS(y) - h(x_0)\int_{\partial\Omega}\frac{\partial\Phi}{\partial\nu_y}(x_0-y)\,dS(y)\\ &= -\int_{\partial\Omega}[h(y)-h(x_0)]\frac{\partial\Phi}{\partial\nu_y}(x_0-y)\,dS(y) + \frac{1}{2}h(x_0)\\ &\equiv I(x_0) + \frac{1}{2}h(x_0),\end{aligned}$$

again using Gauss' Lemma. Therefore,

$$\overline{\overline{u}}(x) - \overline{\overline{u}}(x_0) = I(x) + h(x_0) - I(x_0) - \frac{1}{2}h(x_0),$$

which implies

$$\overline{\overline{u}}(x) = I(x) - I(x_0) + \frac{1}{2}h(x_0) + \overline{\overline{u}}(x_0).$$

Therefore, to prove our theorem, we need only show that

$$\lim_{x\in\Omega\to x_0}[I(x) - I(x_0)] = 0,$$

where

$$I(x) \equiv -\int_{\partial\Omega}[h(y)-h(x_0)]\frac{\partial\Phi}{\partial\nu_y}(x-y)\,dS(y).$$

Now,

$$I(x) - I(x_0) = -\int_{\partial\Omega}[h(y)-h(x_0)]\left[\frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y)\right]\,dS(y).$$

We need to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|I(x) - I(x_0)| < \epsilon$ for $|x - x_0| < \delta$.

By assumption, h is continuous, and as we know $\Phi(x-y)$ is smooth for $y \neq x$. Therefore, to get a bound on $|I(x) - I(x_0)|$, we divide $\partial\Omega$ into two pieces:

(1) $B(x_0,\gamma)\cap\partial\Omega$

(2) $\partial\Omega - \{B(x_0,\gamma)\cap\partial\Omega\}$.

We look at these two pieces below. First for (1),

$$\begin{aligned}&\left|-\int_{B(x_0,\gamma)\cap\partial\Omega}[h(y)-h(x_0)]\left[\frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y)\right]\,dS(y)\right|\\ &\qquad \leq |h(y)-h(x_0)|_{L^\infty(B(x_0,\gamma)\cap\partial\Omega)}\int_{B(x_0,\gamma)\cap\partial\Omega}\left|\frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y)\right|\,dS(y).\end{aligned}$$

By assumption, h is continuous. Therefore, for all $\widetilde{\epsilon} > 0$ there exists a $\gamma > 0$ such that $|h(y) - h(x_0)| < \widetilde{\epsilon}$ if $|y - x_0| < \gamma$. In addition,

$$\int_{B(x_0,\gamma)\cap\partial\Omega} \left| \frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y) \right| \, dS(y) \leq C$$

using the fact that $\overline{\overline{u}}$ is defined for all $x \in \mathbb{R}$. Therefore, we conclude that for any $\widetilde{\epsilon} > 0$,

$$|(1)| \leq C_1\widetilde{\epsilon}$$

for γ chosen appropriately small.

Next, for (2), we use the fact that $\frac{\partial\Phi}{\partial\nu_y}(x-y)$ is continuous in x for x away from y. Consequently, we have

$$\left| -\int_{\partial\Omega - \{B(x_0,\gamma)\cap\partial\Omega\}} [h(y) - h(x_0)] \left[\frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y) \right] \, dS(y) \right|$$
$$\leq |h(y) - h(x_0)|_{L^\infty} \left| \frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y) \right|_{L^\infty(\partial\Omega - \{B(x_0,\gamma)\cap\partial\Omega\})} \left| \int \, dS(y) \right|.$$

Now, first h is bounded on $\partial\Omega$. Therefore, $|h(y) - h(x_0)| \leq C$. Next, $\left|\int \, dS(y)\right| \leq C$. Lastly, using the fact that $\frac{\partial\Phi}{\partial\nu_y}(x-y)$ is continuous in x uniformly for y, we conclude that there exists a $\delta > 0$ such that

$$\left| \frac{\partial\Phi}{\partial\nu_y}(x-y) - \frac{\partial\Phi}{\partial\nu_y}(x_0-y) \right|_{L^\infty(\partial\Omega - \{B(x_0,\gamma)\cap\partial\Omega\})} \leq \widetilde{\epsilon},$$

for $|x - x_0| < \delta$. Therefore,

$$|(2)| \leq C_2\widetilde{\epsilon}$$

if $|x - x_0| < \delta$ where δ is chosen appropriately small.

Consequently, for $\epsilon > 0$ choose $\widetilde{\epsilon} > 0$ such that

$$C_1\widetilde{\epsilon} + C_2\widetilde{\epsilon} < \epsilon.$$

Then choosing $\gamma > 0$ sufficiently small such that

$$|(1)| \leq C_1\widetilde{\epsilon}$$

and $\delta > 0$ sufficiently small such that

$$|(2)| \leq C_2\widetilde{\epsilon}$$

when $|x - x_0| < \delta$, we conclude that

$$|I(x) - I(x_0)| \leq C_1\widetilde{\epsilon} + C_2\widetilde{\epsilon} \leq \epsilon,$$

for $|x - x_0| < \delta$, as claimed.

Therefore, we have shown that

$$\lim_{x \to x_0} [I(x) - I(x_0)] = 0.$$

Consequently,

$$\begin{aligned} \lim_{x \in \Omega \to x_0} \overline{\overline{u}}(x) &= \lim_{x \in \Omega \to x_0} [I(x) - I(x_0)] + \frac{1}{2} h(x_0) + \overline{\overline{u}}(x_0) \\ &= \frac{1}{2} h(x_0) + \overline{\overline{u}}(x_0), \end{aligned}$$

as claimed.

□

In the next section, we use this theorem to construct solutions of the interior and exterior Dirichlet problems as well as the interior and exterior Neumann problems.

5.3 Solution of Laplace's Equation as a Double or Single Layer Potential

We begin by considering the *Interior Dirichlet Problem*,

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

For a given function h, define the double-layer potential $\overline{\overline{u}}$ associated with h as

$$\overline{\overline{u}}(x) = -\int h(y) \frac{\partial \Phi}{\partial \nu_y}(x - y)\, dS(y).$$

In the previous section, we proved that $\overline{\overline{u}}$ is a harmonic function in Ω. In addition, we proved that for $x_0 \in \partial\Omega$,

$$\lim_{x \in \Omega \to x_0} \overline{\overline{u}}(x) = \frac{1}{2} h(x_0) + \overline{\overline{u}}(x_0).$$

Therefore, *if* we can find a continuous function h such that for all $x_0 \in \partial\Omega$,

$$\boxed{g(x_0) = \frac{1}{2} h(x_0) - \int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x_0 - y)\, dS(y)}$$

and we define

$$\boxed{\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(y)\, dS(y),}$$

for that choice of h, then $\overline{\overline{u}}$ will give us a solution of our interior Dirichlet problem. We will discuss the issue of existence of a solution h to this integral equation below.

Next, consider the *Exterior Dirichlet Problem*,

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ u = g & x \in \partial\Omega^c. \end{cases}$$

As proven in the previous section, for any continuous function h,

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu}(x-y)\, dS(y),$$

is harmonic in Ω^c and satisfies

$$\lim_{x \in \Omega^c \to x_0} \overline{\overline{u}}(x) = -\frac{1}{2} h(x_0) + u(x_0).$$

Therefore, *if* we can find a continuous function h such that for all $x_0 \in \partial\Omega^c$,

$$\boxed{g(x_0) = -\frac{1}{2} h(x_0) - \int_{\partial\Omega^c} h(y) \frac{\partial \Phi}{\partial \nu_y}(x_0 - y)\, dS(y),}$$

then defining

$$\boxed{\overline{\overline{u}}(x) \equiv -\int_{\partial\Omega} h(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, dS(y),}$$

for that choice of h, $\overline{\overline{u}}$ will give us a solution of our exterior Dirichlet problem.

Now, we consider the Neumann problems. We will find solutions below as single-layer potentials. Consider the *Interior Neumann Problem*,

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega. \end{cases}$$

First, we note a compatibility condition on the boundary data in order for a solution to exist. By the Divergence Theorem, we know

$$\int_{\Omega} \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS(y).$$

Therefore, in order for a solution to exist, we need

$$\int_{\partial\Omega} g(y)\, dS(y) = 0.$$

For a continuous function h, define the single-layer potential

$$\overline{u}(x) = -\int_{\partial\Omega} h(y) \Phi(y-x)\, dy.$$

From the previous section, we know that $\overline{u}$ is harmonic in Ω. In order to choose h appropriately so that our boundary condition will be satisfied, we extend the notion of *normal derivative* to points not in $\partial\Omega$ as follows. Let $x_0 \in \partial\Omega$. Let $\nu(x_0)$ be the outer unit normal to Ω at x_0. For $t < 0$, such that $x_0 + t\nu(x_0)$ is in Ω, we define

$$i^{x_0}(t) = \nabla \overline{u}(x_0 + t\nu(x_0)) \cdot \nu(x_0).$$

In a manner similar to the proof of Theorem 4 in the previous section, we can show that

$$\begin{aligned}\lim_{t\to 0^-} i^{x_0}(t) &= -\frac{1}{2}h(x_0) + \frac{\partial \overline{u}}{\partial \nu}(x_0)\\ &= -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\,dS(y).\end{aligned}$$

Therefore, *if* we can find a continuous function h such that for all $x_0 \in \partial\Omega$,

$$\boxed{g(x_0) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\,dS(y),}$$

then by defining the single-layer potential

$$\boxed{\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x - y)\,dS(y),}$$

for that choice of h, $\overline{u}$ will give us a solution of our interior Neumann problem.

Last, we consider the *Exterior Neumann Problem*,

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega. \end{cases}$$

Again, for any continuous function h, the single-layer potential

$$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x - y)\,dS(y),$$

is harmonic in Ω^c. In addition, by defining the "normal derivative" of $\overline{u}$ away from $\partial\Omega$ as described above, we can show that

$$\begin{aligned}\lim_{t\to 0^+} \partial_{\nu_x}\overline{u}(x_0 + t\nu(x_0)) &= \frac{1}{2}h(x_0) + \partial_{\nu_x}\overline{u}(x_0)\\ &= \frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\,dS(y).\end{aligned}$$

Therefore, *if* we can find a continuous function h such that for all $x_0 \in \partial\Omega$,

$$\boxed{g(x_0) = \frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0 - y)\,dS(y),}$$

then defining the single-layer potential

$$\boxed{\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x - y)\,dS(y),}$$

for that choice of h, we have found a solution of the exterior Neumann problem.

Remarks.

1. *Solvability of the Integral Equations.* We have shown for the Interior/Exterior Dirichlet Problem, the double layer potential,

$$\overline{\overline{u}}(x) = -\int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_y}(x-y)\,dS(y)$$

will solve

$$\begin{cases} \Delta u = 0 & x \in \Omega \qquad (\Omega^c \text{ for the exterior problem}) \\ u = g & x \in \partial\Omega \end{cases}$$

if h is a continuous function which solves the integral equation,

$$g(x_0) = \frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_y}(x_0-y)\,dS(y) \qquad \text{(interior Dirichlet)}$$

$$g(x_0) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_y}(x_0-y)\,dS(y) \qquad \text{(exterior Dirichlet)}$$

for all $x_0 \in \partial\Omega$.

Similarly, for the Interior/Exterior Neumann problem, the single layer potential,

$$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y),$$

will solve

$$\begin{cases} \Delta u = 0 & x \in \Omega \qquad (\Omega^c \text{ for the exterior problem}) \\ \dfrac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

if h is a continuous function which solves the integral equation

$$g(x_0) = -\frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0-y)\,dS(y) \qquad \text{(interior Neumann)}$$

$$g(x_0) = \frac{1}{2}h(x_0) - \int_{\partial\Omega} h(y)\frac{\partial \Phi}{\partial \nu_x}(x_0-y)\,dS(y) \qquad \text{(exterior Neumann).}$$

While we will not prove any results for the solvability of these integral equations, we state the following facts.

(a) For the interior/exterior Neumann problems, assuming the boundary data satisfies any necessary compatibility conditions (discussed above), then each of the integral equations has a solution.

(b) For the interior Dirichlet problem, if $\partial\Omega$ consists of one component, then there exists a solution of the integral equation for the interior Dirichlet problem.

(c) In the case when $\partial\Omega$ consists of more than one component for the interior Dirichlet problem, or in the case of the exterior Dirichlet problem, the question of existence of a solution h for the integral equations is more delicate. *If* the integral equation has a solution, then the double layer potential will give us a solution. If the integral equation does not have a solution for a particular choice of boundary data g, then we need to do a little more work to construct a solution. However, given "nice" boundary data, i.e. $g \in C(\partial\Omega)$, we can still find a solution of either of these Dirichlet problems. I refer the interested reader to *Folland, Chap. 3.*

(d) Finding a solution of the integral equations above is essentially looking for solutions h of the problems

$$(\lambda - T_i)h = g$$

where $\lambda = \pm\frac{1}{2}$

$$T_1 h(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\,dS(y)$$

$$T_2 h(x) = -\int_{\partial\Omega} h(y)\frac{\partial\Phi}{\partial\nu_x}(x-y)\,dS(y).$$

The solvability of these equations is a subject of Fredholm theory. It makes use of the fact that T_1 and T_2 are compact operators. Essentially, the solvability question is an extension of the solvability question for the finite-dimensional problem

$$(\lambda I - A)h = g$$

where A is an $n \times n$ matrix. As the solvability of the integral equations above relies on material outside the scope of this course, we will not discuss these issues here. For those interested, see Evans (Appendix D) or Folland, Chapter 3.

2. *Uniqueness.*

(a) Solutions to the interior Dirichlet problem are unique. We have shown this earlier; i.e., you can use energy methods or maximum principle, to prove this.

(b) Solutions to the interior Neumann problem are unique up to constants. We have proven this earlier as well.

(c) For the *exterior* Dirichlet and Neumann problems, we would need to impose an extra condition on the solution at infinity in order to guarantee uniqueness of solutions. For example, one can show that there exists at most one solution $u(x) \to 0$ as $|x| \to +\infty$. *Note:* If Ω^c consists of sets $\Omega_0^c, \ldots, \Omega_n^c$ where Ω_0^c is the unbounded component, solutions of the exterior Neumann problem which decay on the unbounded component Ω_0^c are unique up to constants on $\Omega_1^c, \ldots, \Omega_n^c$.

3. *Necessary conditions for Solvability of the Neumann problem.*

As discussed earlier, the *interior* Neumann problem has the following necessary compatability condition,

$$\int_{\partial\Omega} g(y)\,dS(y) = 0.$$

More generally, if Ω consists of the sets $\Omega_1, \ldots, \Omega_m$ and Ω^c consists of the sets $\Omega_0^c, \ldots, \Omega_n^c$, where Ω_0^c is the unbounded component of Ω^c, then the necessary compatability conditions for solvability of the interior/exterior Neumann problems are as follows:

$$\int_{\partial\Omega_i} \frac{\partial u}{\partial\nu}\,dS(x) = 0 \qquad i = 1, \ldots, m \qquad \text{(interior Neumann)}$$

$$\int_{\partial\Omega_i^c} \frac{\partial u}{\partial\nu}\,dS(x) = 0 \qquad i = 1, \ldots, n \qquad \text{(exterior Neumann).}$$

Note that there is no compatibility condition on the boundary data of u on the unbounded component Ω_0^c.