INTEGRATION: THE FEYNMAN WAY

ANONYMOUS

Abstract. In this paper we will learn a common technique not often described in collegiate calculus courses. After reviewing the necessary theory, we will proceed to work through some typical examples. Throughout this process, we will see trivial integrals that can be evaluated using basic techniques of integration (such as integration by parts), however we will also encounter integrals that would otherwise require more advanced techniques such as contour integration.

1. Introduction

Many up-and-coming mathematicians, before every reaching the university level, heard about a certain method for evaluating definite integrals from the following passage in [1]:

> One thing I never did learn was contour integration. I had learned to do integrals by various methods show in a book that my high school physics teacher Mr. Bader had given me.
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> The book also showed how to differentiate parameters under the integral sign - It's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals.
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> The result was that, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me.

The method Mr. Feynman is referring to often goes by the name of *differentiating under the integral sign*, *differentiation with respect to a parameter*, or sometimes even *Feynman Integration*. However one wishes to name it, the elegance and appeal lies in how this method can be employed to evaluate seemingly complex integrals with nothing more than[1] elementary calculus.

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[1]Once one gets past the measure theory required to prove the Theorem 2.1

2. Some Key Theorems

The technique of "Feynman Integration" is a simple application of a theorem attributed to Leibniz. In this section we state the theorem in its most basic form, and end by stating a more general version that allows for even weaker hypotheses. In both cases, we address situations where the following equation (which we would love to be true) holds:

$$\frac{d}{dx}\int_Y f(x,y)dy = \int_Y \frac{\partial}{\partial x} f(x,y)dy.$$

Before stating these theorems, recall that differentiation is simply a particular example of a limit insofar as we define

$$\frac{d}{dx} f(x) =: f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h},$$

with a true definition on the far right. Thus, we see that (2) will hold whenever we may make the following statement,

$$\lim_{x\to a}\int_Y f(x,y)dy = \int_Y \lim_{x\to a} f(x,y)dy.$$

Theorem 2.1 (Elementary Calculus Version)**.** *Let $f : [a,b]\times Y \to \mathbb{R}$ be a function, with $[a,b]$ being a closed interval, and Y being a compact subset of $\mathbb{R}^n$. Suppose that both $f(x,y)$ and $\partial f(x,y)/\partial x$ are continuous in the variables x and y jointly. Then $\int_Y f(x,y)dy$ exists as a continuously differentiable function of x on $[a,b]$, with derivative*

$$\frac{d}{dx}\int_Y f(x,y)dy = \int_Y \frac{\partial}{\partial x} f(x,y)dy.$$

As mentioned above, the veracity of (2) is completely dependent upon if we can exchange the operations of limiting and integration. If we were to prove the above theorem, our argument would make full use of the compactness of Y, which of course implies *uniform* continuity. From this fact, we could show that it is justified to switch change the order of limits and integration, thus proving (2).

However, in many cases the restriction of compactness can be too severe. Often times we would like Y to be $(-\infty,a),(a,\infty),(-\infty,\infty)$,etc... In these situations, the following measure theoretic version of the above comes to our rescue:

Theorem 2.2 (Measure Theory Version)**.** *Let X be an open subset of $\mathbb{R}$, and Ω be a measure space. Suppose $f : X\times\Omega \to \mathbb{R}$ satisfies the following conditions:*

(1) *$f(x,\omega)$ is a Lebesgue-integrable function of ω for each $x \in X$.*
(2) *For almost all $\omega \in \Omega$, the derivative $\partial f(x,\omega)/\partial x$ exists for all $x \in X$.*
(3) *There is an integrable function $\Theta : \Omega \to \mathbb{R}$ such that $|\partial f(x,\omega)/\partial x| \le \Theta(\omega)$ for all $x \in X$.*

Then for all $x \in X$,

$$\frac{d}{dx}\int_\Omega f(x,\omega)d\omega = \int_\Omega \frac{\partial}{\partial x} f(x,\omega)d\omega.$$

A sketch of the proof of Theorem 2.2 would most likely make some form of a famous result from measure theory, the Dominated Convergence Theorem. This will of course provide us with the justification to switch the order of limit and integration. For the interested reader, we state the theorem whose proof may be found in [5]:

Theorem 2.3 (Dominated Convergence Theorem)**.** *Let X be a measure space, and let $\Phi, f_1, f_2, \dots$ be measurable functions such that $\int_X \Phi < \infty$ and $|f_n| \leq \Phi$ for all $n \in \mathbb{N}$. If $f_n \to f$ a.e., then f is integrable and*

$$\lim_{n\to\infty} \int_X f_n = \int_X f.$$

Before moving on to some examples, note that among the three criteria in Theorem 2.2, the first two are usually satisfied. Indeed in all of the following examples we need only check criterion 3, i.e. that f is dominated by some integral function. Once we have found the appropriate dominating function, we may safely apply Theorem 2.2 and thus "differentiate under the integral".

3. Examples

In this section we present several examples on the application of the above theorem(s). We begin with the following basic problem:

Example 3.1. *Compute the definite integral,*

$$\int_0^1 \frac{x^2-1}{\log x} dx.$$

In order to apply our theorems, we obviously need to be dealing with an integrand in two variables. In this example, we "generalize" by introducing a parameter b in the exponent of our x term. In particular, we could choose to define the following function:

$$I(b) = \int_0^1 \frac{x^b-1}{\log x} dx.$$

As long as $b > -1$, all conditions of Theorem 2.1 are satisfied and we may differentiate under the integral sign:

$$\begin{aligned} I'(b) &= \frac{d}{db}\int_0^1 \frac{x^b-1}{\log x} dx = \int_0^1 \frac{\partial}{\partial b}\left[\frac{x^b-1}{\log x}\right] dx \\ &= \int_0^1 x^b = \left.\frac{x^{b+1}}{b+1}\right|_0^1 \\ &= \frac{1}{b+1} \end{aligned}$$

whereupon integration yields

$$I(b) = \log(b+1) + C.$$

In order to find out our constant of integration, we let $b = 0$ so that our integrand is 0, implying that $C = 0$. Letting $b = 2$ will of course solve our original problem:

$$\int_0^1 \frac{x^2-1}{\log x} dx = I(2) = \log(3).$$

Example 3.2. *Compute the improper definite integral,*

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx.$$

As before, we must strategically introduce a parameter so that we can actually *use* our theorems. In this example, we "generalize" by solving the following integral

$$I(b) = \int_0^\infty \frac{\sin(x)}{x} e^{-bx} dx,$$

whereupon setting $b = 0$ and doubling will give us the desired value. But before we proceed, how do we know that we can indeed differentiate under the integral as we would hope? As mentioned in the previous section, it is clear (why?) that our integrand is Lebesgue integrable and differentiable a.e.; all that remains is to verify that it is dominated. The key here is to realize that since $|\sin(x)| \leq |x|$, this implies that

$$\left| \frac{\sin(x)}{x} e^{-bx} \right| = \left| \frac{\sin(x)}{x} \right| e^{-bx} \leq e^{-bx}.$$

Lastly since

$$\int_0^\infty e^{-bx} dx = \frac{1}{b} < \infty,$$

we have found a suitable dominating function. We are now justified in differentiating under the integral sign as follows:

$$\begin{aligned} I'(b) &= \frac{d}{db} \int_0^\infty \frac{\sin(x)}{x} e^{-bx} dx = \int_0^\infty \frac{\partial}{\partial b} \left[\frac{\sin(x)}{x} e^{-bx} \right] dx \\ &= \int_0^\infty \sin(x) e^{-bx} dx = \left. \frac{e^{-bx}(\cos(x) + b\sin(x))}{1+b^2} \right|_0^\infty \\ &= -\frac{1}{1+b^2} \end{aligned}$$

Integration of $I'(b)$ yields

$$I(b) = -\tan^{-1}(b) + C.$$

As before, we choose a strategic value $b = b_0$ in order to make our integrand vanish so that $I(b_0) = 0$. In this case, take $b = \infty$ so that $I(\infty) = 0 \Leftrightarrow C = \tan^{-1}(\infty) = \pi/2$. We thus conclude that

$$\begin{aligned} \int_{-\infty}^\infty \frac{\sin(x)}{x} dx &= 2\int_0^\infty \frac{\sin(x)}{x} dx = 2I(0) \\ &= \pi \end{aligned}$$

Example 3.3. *Compute the improper definite integral,*

$$\int_0^{\pi/2} x \cot(x) dx.$$

This particular example is tricky because it is not immediately obvious where to introduce the extra parameter. However, it turns out that the following is an appropriate choice:

$$I(b) = \int_0^{\pi/2} \frac{\tan^{-1}(b\tan(x))}{\tan(x)} dx,$$

so that we will have the answer to our original integral upon setting $b = 1$. After briefly verifying that the conditions of Theorem 2.1 are satisfied, we proceed as follows:

$$\begin{aligned} I'(b) &= \frac{d}{dx}\int_0^{\pi/2} \frac{\tan^{-1}(b\tan(x))}{\tan(x)}dx = \int_0^{\pi/2} \frac{\partial}{\partial b}\left[\frac{\tan^{-1}(b\tan(x))}{\tan(x)}\right]dx \\ &= \int_0^{\pi/2} \frac{dx}{(b\tan(x))^2+1} \\ &= \frac{\pi}{2(b+1)} \end{aligned}$$

Integrating w.r.t. b (and noting that our constant of integration will vanish) gives us

$$I(b) = \frac{\pi}{2}\log(b+1),$$

So that our original integral is obtained via

$$\int_0^{\pi/2} x\cot(x)dx = I(1) = \frac{\pi}{2}\log(2).$$

We conclude this example by performing integration by parts on our original integral. This yields the integral of another relatively famous integral often dealt with in introductory complex analysis courses:

$$\int_0^{\pi/2} \log(\sin(x))dx = -\int_0^{\pi/2} x\cot(x)dx = -\frac{\pi}{2}\log(2).$$

Example 3.4. *As our final example, we compute the following definite integral,*

$$\int_0^{\pi} e^{\cos(x)}\cos(\sin(x))dx.$$

We introduce the parameter b as follows:

$$I(b) = \int_0^{\pi} e^{b\cos(x)}\cos(b\sin(x))dx,$$

and note that all of our conditions in Theorem 2.1 are satisfied. However, before we compute as we did in the previous problems, we transform our integrand slightly so that we are working with complex exponentials:

$$\begin{aligned} I(b) &= \int_0^{\pi} e^{b\cos(x)}\cos(b\sin(x))dx = \frac{1}{2}\int_{-\pi}^{\pi} e^{b\cos(x)}\cos(b\sin(x))dx \\ &= \frac{1}{2}\int_0^{2\pi} e^{b\cos(x)}\cos(b\sin(x))dx \\ &= \Re\left[\frac{1}{2}\int_0^{2\pi} e^{be^{ix}}dx\right] \end{aligned}$$

With the problem posed in this fashion, *now* we proceed as before:

$$\begin{aligned} I'(b) &= \frac{1}{2}\frac{d}{dx}\int_0^{2\pi} e^{be^{ix}}dx = \frac{1}{2}\int_0^{2\pi} \frac{\partial}{\partial b}\left[e^{be^{ix}}\right]dx \\ &= \frac{1}{2}\int_0^{2\pi} ibe^{be^{ix}}e^{ix}dx = \frac{1}{2}e^{be^{ix}}\Big|_0^{2\pi} \\ &= 0 \end{aligned}$$

Since our derivative is 0, we know that $I(b)$ is a constant w.r.t. b. We thus conclude that

$$I(a) = I(0) = \int_0^{2\pi} dx = \pi.$$

4. Conclusion

In Section 2, we stated the key theorems related to Feynman integration with brief outlines of what would be involved in their proofs. In Section 3, we applied these theorems to evaluate some rather tough integrals that often don't lend themselves nicely to real techniques. In fact there are many, many more examples of famous integrals that most frequently solved by contour integration or perhaps even a basic series expansion coupled with the not-so-often-applied Beppo Levi Theorem. For the motivated reader, I've included a brief list of some other integrals that can be solved with creative parameterizations and a dose of differentiation under the integral:

$$\int_0^\infty e^{\left(-\frac{x^2}{y^2}-y^2\right)}dx, \quad \int_0^\infty \frac{1-\cos(xy)}{x}dx, \quad \int_0^\infty \frac{dx}{(x^2+p)^{n+1}},$$
$$\int_0^\infty e^{-x^2}dx, \quad \int_0^\infty \cos^2(x)dx, \quad \int_0^\infty \sin^2(x)dx,$$
$$\int_0^\infty \frac{\sin^2(x)}{x^2(x^2+1)}dx.$$

Notice that the middle row contains the two Fresnel Integrals, a class of integrals almost *never* solved using real methods. And of course, for the sadist with a background in differential equations, I invite you to try your luck with the last integral of the group.

References

[1] Feynman, R.P., "Surely You're Joking, Mr. Feynman!" pp. 71-72, Bantam Books, 1986.

[2] Folland, G.B., "Real Analysis: Modern Techniques and Their Applications," 2nd ed. Wiley-Interscience, 1999.

[3] Rudin, W., "Principles of Mathematical Analysis," United States: McGraw Hill, 1976.

[4] Planet Math, "Differentiation Under the Integral Sign," `http://planetmath.org/encyclopedia/DifferentiationUnderIntegralSign.html`

[5] Williams, D., "Probability with Martingales," Cambridge University Press, 1991.