RAPIDITY AND INVARIANT CROSS SECTIONS

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The article describes in detail two relativistic concepts which have important applications in high energy physics: rapidity and invariant cross sections. Simple derivations and applications are shown, which could be used in introducing these topics to students in courses on special relativity or high energy physics.

INTRODUCTION

Rapidity and invariant cross sections are frequently used by high energy physicists in describing the production of particles in nuclear reactions. Both of these concepts are quite simple and easy to understand, but they are usually ignored in common textbooks dealing with special relativity[1,2,3] or high energy physics.[4] Lack of familiarity with these concepts may become an obstacle in understanding theoretical and experimental results in high energy research. Perhaps it is time to add rapidity and invariant cross sections to the list of traditional topics which are taught in courses on special relativity or high energy physics.

This paper is an attempt to provide a simple and systematic explanation of both rapidity and invariant cross sections. Some of the formulations were based on references (5) and (6), while others were influenced by lectures at the Physics Department of Brookhaven National Laboratory, (7,8). It is assumed that the reader is familiar with basic equations of special relativity, which are reviewed in Table 1; simple derivations of these can be found elsewhere, for example, in reference (9).

For convenience, ordinary units are often disregarded in the high energy community and the so called "natural" system of units is used instead. In this system c is dimensionless and equal to one. Quantities such as P and M_0 then have units of energy, but are often still referred to as linear momentum and rest mass, respectively. This may lead to confusion for those unaccustomed to the "natural" system of units. In order to help clear up this ambiguity, P and M_0 (in natural units) are distinguished in Table 1 and throughout this paper from p and m_0 (in standard units). Practical energy units, MeV or Gev, are used in the literature instead of the SI units.

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ADDING ONE-DIMENSIONAL VELOCITIES

Rapidity, as its name implies, is related to velocity. It is a dimensionless variable, y, describing the rate at which a particle is moving with respect to a chosen reference point situated on the line of motion. Mathematically, it is defined as

$$y = \tanh^{-1}\beta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta} \tag{1}$$

where $\beta = v/c$ and v is the velocity. The equivalence of hyperbolic and logarithmic forms is demonstrated in Appendix 1. The dependence of rapidity on the velocity ratio is graphically illustrated in Figure 1.

It is often convenient to express the rapidity of a particle in terms of its total energy and momentum. To do so one may replace β in the above equation by P/E (see Table 1) and obtain

$$y = \tanh^{-1}\frac{P}{E} = \frac{1}{2}\ln\frac{P+E}{P-E} \tag{2}$$

As a simple illustration, consider the following question. What is the rapidity of a proton whose total energy, E, is 2000 MeV? The rest mass energy of a proton is 938 MeV and, according to the last relation of Table 1,

$$P = \sqrt{(2000^2 - 938^2)} = 1766.4\,\text{MeV}$$

Thus, according to equation (2) the rapidity is equal to 1.39. Alternatively, the velocity of the proton could be calculated ($\beta = P/E = .8832$), and equation (1) could be used to obtain the same result. Note that positive and negative rapidities correspond to positive and negative velocities with respect to a chosen axis. Also note that due to the logarithmic dependence of y on E the magnitudes of rapidity are rather small, even for the most energetic particles. In principle, however, the range of rapidities is unlimited.

What is the advantage of using rapidity instead of other kinematic quantities, such as v or β ? To answer this question consider two frames of reference: S, which is at rest in a laboratory, and S', which moves with respect to S at a constant speed v_0, as illustrated in Figure 2. Suppose that a particle has a velocity v' with respect to S' and that we want to find its velocity, v, with respect to S. The well known relativistic solution of this problem is given by the Lorentz transformation:

$$v = \frac{v' + v_0}{1 + (v_0 v'/c^2)} \tag{3}$$

In the nonrelativistic limit (that is, when $v'v_0 << c^2$) this becomes

$$v = v' + v_0 \tag{4}$$

Thus velocities are additive only when they are very small in comparison with the speed of light. In general they are not additive. This can be contrasted with rapidities, which are *always* additive, even when v and v_0 are approaching the speed of light. That is, the relation

$$y = y' + y_0 \tag{5}$$

is correct at all velocities. To demonstrate this we will show that equation (5) leads directly to equation (3). Let us first write equation (5) in terms of β by using equation (1):

$$\frac{1}{2}\ln\frac{1+\beta}{1-\beta} = \frac{1}{2}\ln\frac{1+\beta'}{1-\beta'} + \frac{1}{2}\ln\frac{1+\beta_0}{1-\beta_0} \tag{6}$$

or as

$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta'}{1-\beta'}\right)\left(\frac{1+\beta_0}{1-\beta_0}\right) \tag{7}$$

This equation, in turn, leads to

$$\beta = \frac{\beta' + \beta_0}{1 + \beta'\beta_0}$$

which becomes equation (3) after multiplying both sides by c. The fact that equation (5) leads directly to equation (3) shows the additivity of rapidity under Lorentz transformations, and that the rapidity formulation simply and naturally incorporates the properties of the velocity transformation. This can be further illustrated with the following numerical example: find v when $v_0 = 0.9c$ and $v' = 0.5c$. According to equation (3) $v = 0.965c$. The same result can be obtained by using rapidities:

$$y_0 = \frac{1}{2}\ln\left(\frac{1+0.9}{1-0.9}\right) = 1.472$$

$$y = \frac{1}{2}\ln\left(\frac{1+0.5}{1-0.5}\right) = 0.549$$

$$y = y' + y_0 = 0.549 + 1.472 = 2.021$$

so that

$$\beta = \tanh y = \tanh(2.021) = 0.965$$

An immediate consequence of equation (5) is that *differences in rapidities are invariant.* This can be shown, for example, by applying equation (5) to two different particles so that:

$$y_2 = y_2' + y_0$$

$$y_1 = y_1' + y_0$$

Side by side subtraction shows

$$y_2 - y_1 = y_2' - y_1' \tag{8}$$

This equation says that, for example, if the difference in the rapidities of two particles is 1.5 in one frame of refrence, it will be 1.5 in all other frames. Equation (8) also applies to the differential element of rapidity dy; the invariance of dy will be important later in understanding invariant cross sections.

The following analogy may be useful to emphasize the difference between additive and non-additive quantities. Consider two coordinate systems, x_1,y_1, and x_2,y_2, shown in Figure 3. There are two ways of describing their mutual inclination: by the angle, α, and by the slope of the y_2 axis with respect to y_1 (tangent of α). Each description is adequate, but angles are more convenient for many problems because they are additive while their tangents are not. Two rotations by 10 degrees are equivalent to one rotation by 20 degrees, but two rotations by tan(10) are not equivalent to one rotation by tan(20).

GENERALIZATION TO THREE DIMENSIONS

The one-dimensional definition of rapidity can now be generalized to describe motions of particles in the real multi-dimensional world. This is important in high energy physics where many particles may be produced from a collision of two particles. For example, more than 200 particles are emitted when a stationary lead nucleus is struck by an oxygen projectile of 3200 GeV [10] at a small impact parameter. The velocity vectors of these emitted particles are of course not all parallel to each other and equation (3) applies only to components of velocity parallel to the z axis, which is normally chosen along the direction of the relative motion of the colliding particles. The components perpendicular to that axis remain the same in all longitudinal frames of reference, i.e., those parallel to the z axis.

The rapidity of a particle in three-dimensional space is defined as:

$$y = \tanh^{-1} \beta_z = \frac{1}{2} \ln \frac{1+\beta_z}{1-\beta_z} = \frac{1}{2} \ln \frac{E+P_z}{E-P_z} \tag{9}$$

where β_z and P_z are the components of the velocity ratio and momentum variable parallel to the z axis. This is consistent with the preliminary, one-dimensional definition of rapidity of equation (2) except that β_z and P_z are now used instead of β and P. In fact, equation (9) is a formal definition of rapidity while equation (2), which is a particular case of that general definition, is a convenient pedagogical device to introduce the new concept. It is important to emphasize that unlike velocity, rapidity is not a vector; it is a scalar quantity associated with the z axis.

It turns out that E and P_z can separately be expressed as functions of rapidity. To demonstrate this we can write the last equation from Table 1 as:

$$E^2 = P^2 + M_0^2 = M_t^2 + P_z^2 \tag{10}$$

where

$$P^2 = P_x^2 + P_y^2 + P_z^2$$

and

$$M_t^2 = M_0^2 + P_x^2 + P_y^2 \tag{11}$$

Note that M_t, known as the "transverse mass", is a Lorentz invariant quantity because both P_x and P_y are perpendicular to the z axis, while M_0 is a constant. The quantity $(P_x^2+P_y^2)^{1/2}$ is the invariant transverse momentum variable, usually denoted as P_t. Equation (10) can now be written as

$$\left(\frac{E}{M_t}\right)^2 - \left(\frac{P_z}{M_t}\right)^2 = 1 \qquad (12)$$

By comparing with a familiar relation between hyperbolic functions,

$$\cosh^2 y - \sinh^2 y = 1$$

we can tentatively assign

$$E = M_t \cosh y \qquad (13)$$

and

$$P_z = M_t \sinh y \qquad (14)$$

The validity of these important relations can be confirmed by showing that they are consistent with the definition of rapidity. Dividing equation (14) with equation (13), we have

$$\frac{P_z}{E} = \tanh y$$

which is the same as the defining equation (9) because $P_z/E = \beta_z$ (see Table 1).

Note that we now have a simple relation between the derivative of P_z with respect to rapidity y for fixed M_t (or fixed P_t) and the energy:

$$\left(\frac{dP_z}{dy}\right)_{M_t} = M_t \cosh y = E \qquad (15)$$

Also note that the invariant variable, M_t, can be explicitly introduced into the definition of rapidity by modifying equation (9) as follows:

$$y = \frac{1}{2}\ln\frac{E+P_z}{E-P_z} = \frac{1}{2}\ln\frac{(E+P_z)(E+P_z)}{(E-P_z)(E+P_z)} = \ln\frac{E+P_z}{M_t} \qquad (16)$$

It is important to keep in mind that rapidity of a particle depends not only on the magnitude of its velocity but also on the polar angle, θ, with respect to the beam axis. Specifically, equation (9) can be written as

$$y = \frac{1}{2}\ln\frac{1+\beta\cos\theta}{1-\beta\cos\theta} \qquad (17)$$

PSEUDORAPIDITIES

Observe that for ultrarelativistic particles the value of rapidity (according to equation (17)) is determined only by the angles of emission, θ,because for these particles β is one or very nearly so. For example, any ultrarelativistic particle emitted at an angle, θ , equal to

60 degrees will have rapidity, $y = 1.5$. This leads to the formal definition of pseudorapidity, denoted by η:

$$\eta = \frac{1}{2} \ln \frac{1 + \cos\theta}{1 - \cos\theta} = \ln \cot \theta/2 \tag{18}$$

Note that in the limit as θ goes to zero, the pseudorapidity of a particle approaches infinity. Except at angles near 0^o and 180^o, rapidities and pseudorapidities are numerically indistinguishable when velocities are very close to c. Naturally the pseudorapidity defined by the above equation can also be calculated for slower particles but in that case y and η are no longer nearly identical. Note that zero-mass particles, such as photons and neutrinos, are ultrarelativistic ($v = c$) at all energies and consequently their pseudorapidities and rapidities are always *exactly* equal.

Experimentally, pseudorapidity is an easy quantity to measure; all that is required is knowledge of the polar angles of emission of the produced particles. Rapidity, however, requires knowledge of both the polar angles of emission *and* the velocities of the particles. Many experiments are not designed to extract the quantities necessary for complete rapidity measurements; those that are can usually only make such measurements in limited angular regions. Therefore, particle density distributions are often given in terms of $dN/d\eta$ instead of dN/dy; however, for high energy particles the two distributions will be almost identical, except at angles near 0 and 180 degrees.

The dependence of η on θ is plotted in Figure 4; it shows that pseudorapidities, like rapidities, are always positive for particles emitted into the forward hemisphere and negative for those emitted into the backward hemisphere. It also shows how the slope of the η versus θ curve depends on the angle:

$$\frac{d\eta}{d\theta} = -\frac{1}{\sin\theta/2} \tag{19}$$

Thus a narrow range of angles close to 0 degrees or 180 degrees is always associated with a broad range of pseudorapidities. This is important when spatial distributions of emitted particles are described in terms of pseudorapidities rather than in terms of angles, as is often done in high energy publications, such as reference (11).

REACTION CROSS SECTIONS

Probabilities of nuclear reactions can be expressed by "effective areas", called cross sections. An elementary tutorial on that subject is found, for example, in reference (12). Cross sections (σ, measured in barns or millibarns) are used to describe total yields of reactions regardless of energies of emitted particles or of their spatial distributions. On the other hand, differential cross sections, such as $d\sigma/dE$ (in mb/MeV) and $d\sigma/d\theta$ (in mb/radian), are used in studies of energy and spatial distributions of the emitted particles.

Angular (spatial) distributions can also be described in terms of $d\sigma/d\Omega$, where Ω stands for the solid angle expressed in steradians (str). For example, $d\sigma/d\Omega$ might be equal to

10 mb/str at 30 degrees and 3 mb/str at 45 degrees. Double differential cross sections, $d^2\sigma/dEd\theta$ (mb/MeV-rad) and $d^2\sigma/dEd\Omega$ (mb/MeV-str) are also used frequently in order to display energy spectra at specific angles or angular distributions for specific energies. In practice the determination of the double differential cross section, $d^2\sigma/d\theta dE$, at a given E and θ, is performed by measuring the number of particles, d^2N, emitted into the angular interval between θ and $\theta + d\theta$, and whose energies are between E and $E + dE$. Once the number d^2N is known a differential cross section can be calculated from it with knowledge of the incident beam intensity and the number of atoms per unit area in the target.

A more detailed description of reaction products may involve triple differential cross sections, such as $d^3\sigma/dP_xdP_ydP_z$. The way of subdividing a reaction cross section into smaller parts (differential cross sections) depends on what is to be learned or emphasized in a given study. The same can be said about the choice of independent variables, such as v, E, P_t, θ, y, etc. against which the cross sections may be plotted in order to uncover important information about the reactions. No matter which kinematic variables are chosen, however, the sum of all the parts (integration of the differential cross sections) must be equal to the total number of millibarns associated with the probability of a given reaction.

A total reaction cross section, σ, is an invariant quantity; its value does not change when expressed in different frames of reference. Differential cross sections, on the other hand, may or may not be invariant. For example, neither $d\sigma/dE$ nor $d\sigma/d\Omega$ are invariant. Transformations of common differential cross sections from one frame of reference to another are described in many textbooks, such as references (13), (9) and (14). These transformations are often difficult and cumbersome. Furthermore, they are impossible to perform when information about the velocity of a particular frame is not known, which is a common situation in practical research. It is thus desirable to use specific types of differential cross sections which remain the same in all frames of reference.

INVARIANT CROSS SECTIONS

In general, the existence of invariant quantities is very important; it demonstrates that the relatavistically subjective reality can be described in objective terms. This aspect of special relativity has not been sufficiently emphasized in most textbooks. In fact, many misconceptions about the theory [15] would perhaps be avoided if it were named the "invariance theory", as preferred by Einstein. [16,17] The important idea is that it is possible to construct invariant quantities by using appropriate combinations of quatities which by themselves are non-invariant. For example, it is well known that the total energy, E, and linear momentum, p, of a particle, taken separately, are not invariant but a combination of them can be used to construct an invariant quantity called the energy-momentum four-vector. The length of this four-vector in any frame is equal to the rest mass of the particle. Likewise, the half-life Δt of a muon and the distance Δz traveled in that time depend on the frame of reference, while the square of the time-space interval, $\Delta z^2 - (c\Delta t)^2$, is the same for all inertial observers.

When P is used as a kinematic variable, the triple differential cross section can be written as $d^3\sigma/dP_xdP_ydP_z$. The last quantity is often denoted as $d^3\sigma/dP^3$, with an understanding that dP^3 is the elementary "volume" in P space, as illustrated in Figure 5. It turns out that the product of $d^3\sigma/dP^3$ and E is also an invariant quantity. This happens (see Appendix 2) because changes in E associated with transformations from one Lorentz frame to another are inversely proportional to changes in the $d^3\sigma/dP^3$. The product

$$\sigma_{inv} = E\frac{d^3\sigma}{dP^3} \tag{20}$$

is called the invariant differential cross section. Since both E and P have units of energy, σ_{inv} is usually expressed in (mb/GeV^2). Sometimes dp (rather than dP) is used in the definition of the invariant cross section; in that case the units are (mb-c^3/GeV^2), as in Figure 6.

Actually, there are several other kinds of differential cross sections which also happen to be invariant. For example, consider $d\sigma/dy$, where y is rapidity defined by equation (9). This quantity is invariant because, as was shown by equation (8), the differential element of rapidity dy is invariant. The same is true for the pseudorapidity distributions of ultrarelativistic particles, formulated in terms of $d\sigma/d\eta$. Azimuthal distributions, $d\sigma/d\phi$ or $d^2\sigma/d\phi d\eta$ are also invariant because azimuthal angles are confined to planes perpendicular to the z axis. The term "invariant differential cross section", however, usually refers to the combination of non-invariant quantities defined in equation(20).

Note that the invariant cross section can also be used at non-relativistic energies, where it can be reduced to the partially integrated quantity (see Appendix 2)

$$\sigma'_{inv} = \frac{1}{P}\frac{d^2\sigma}{dT d\Omega} \tag{21}$$

In this equation T stands for the kinetic energy and Ω for the solid angle, thus the units of σ'_{inv} are (mb/GeV^2-str). A good illustration for the use of the above equation at low energies can be found, for example, in a study of alpha particles and protons emitted from the Ar + Al reaction at 190 MeV [18]. This study revealed that the velocity of the source of these particles coincided with the velocity of an equilibrated composite system, and not, for example, with velocities of fission-like fragments. This in turn lead to a discovery that highly excited composite systems may be unusually large in comparison with stable nuclei.

Returning to equation (20), which can be used at all energies, and combining it with equation (15), gives

$$\sigma_{inv} = \frac{d^3\sigma}{dP_xdP_ydP_z/E} = \frac{d^3\sigma}{dP_xdP_ydy}$$

In polar coordinates of the P_x, P_y plane this can be written as

$$\sigma_{inv} = \frac{d^3\sigma}{P_t dP_t dy d\phi}$$

For an azimuthally isotropic system, integration over ϕ angles leads to another partially integrated quantity

$$\sigma''_{inv} = \frac{d^2\sigma}{2\pi P_t dP_t dy}$$

which has units of (mb/GeV^2).

Rapidity first came to be used by particle phyicists approximately twenty years ago, and has since become increasingly important, as illustrated in Figure 6. This figure displays the distribution of rapidities of positive kaons produced in proton-proton collisions in the Intersecting Storage Rings at CERN. Note that by plotting the invariant cross sections as a function of the difference between the maximum rapidity (essentially, the beam rapidity) and the rapidity of the detected kaons, y, the data are automatically transformed into the rest frame of the incident protons. The remarkable overlapping of data points (diamonds, circles, etc), collected at total beam energies differing by more than a factor of two, would not be seen if the data were plotted as a function of y in the laboratory frame of reference. The fact that all points lie on one smooth curve indicates that, at least in the range covered, the rapidity distribution of the kaons is independent of the total available energy. Rapidity distributions of other particles, such as pions and anti-protons, also summarized in Reference (19), follow the same trend.

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APPENDIX 1: Equivalence of Hyperbolic and Logarithmic Forms of Rapidity

The hyperbolic functions of any variable y are defined as

$$\sinh y = \frac{e^y - e^{-y}}{2} \tag{1a}$$

$$\cosh y = \frac{e^y + e^{-y}}{2} \tag{1b}$$

$$\tanh y = \frac{\sinh y}{\cosh y} \tag{1c}$$

The definition of rapidity given in equation (9) of the text is equivalent to

$$\sinh y = \beta_z \gamma_z \tag{1d}$$

because the hyperbolic sine can be expressed by the hyperbolic tangent as

$$\sinh y = \frac{\tanh y}{(1 - \tanh^2 y)^{\frac{1}{2}}} = \frac{\beta_z}{(1 - \beta_z^2)^{\frac{1}{2}}} = \beta_z \gamma_z$$

Likewise,

$$\cosh y = \gamma_z \tag{1e}$$

because the hyperbolic cosine can be expressed by the hyperbolic tangent as

$$\cosh y = \frac{1}{(1 - \tanh^2 y)^{\frac{1}{2}}} = \frac{1}{(1 - \beta_z^2)^{\frac{1}{2}}} = \gamma_z$$

In order to prove the equivalence of the hyperbolic and logarithmic forms of the rapidity definition, we can add ($1d$) to ($1e$) and obtain

$$\cosh y + \sinh y = \gamma_z(1 + \beta_z)$$

which leads to

$$e^y = \gamma_z(1 + \beta_z)$$

and consequently to

$$y = \ln\Big(\gamma_z(1 + \beta_z)\Big) = \ln\left(\frac{1 + \beta_z}{(1 - \beta_z^2)^{\frac{1}{2}}}\right) = \frac{1}{2}\ln\left(\frac{1 + \beta_z}{1 - \beta_z}\right) \tag{1f}$$

Note that only positive sign solution was retained in $(1f)$ so that the sign of y is the same as that of β_z. Also note that in the nonrelativistic approximation y is equal to β_z because the limit of $\ln(1+\beta_z)-\ln(1-\beta_z)$ is $2\beta_z$ when β_z approaches zero. This is illustrated in Figure 1.

APPENDIX 2: Invariance of $E\left(\frac{d^3\sigma}{dP^3}\right)$ and Partially Integrated Cross Sections

The invariance of equation (20) of the text can be shown most easily by writing the expression completely in terms of invariant quantities. In cylindrical coordinates, (see Figure 5), the "volume element", dP^3 can be expressed as

$$dP^3 = P_t dP_t dP_z d\phi \tag{2a}$$

where

$$P_t = (P_x^2 + P_y^2)^{1/2} \tag{2b}$$

Clearly, P_t and $d\phi$ are Lorentz invariant with respect to longitudinal transformations because they are defined in a plane perpendicular to z axis. Therefore, in cylindrical coordinates we can write

$$E\left(\frac{d^3\sigma}{dP^3}\right) = \frac{d^3\sigma}{P_t dP_t d\phi dP_z/E}$$

Note that dP_z in the volume element is taken with P_t fixed (again see Figure 5). By using equation (15) of the text, we have $dP_z/E = dy$, so that

$$E\left(\frac{d^3\sigma}{dP^3}\right) = \frac{d^3\sigma}{P_t dP_t d\phi dy} \tag{2c}$$

As mentioned previously, P_t,dP_t and $d\phi$ are invariant; dy is invariant by equation (8) of the text. The invariance of the right side of the above equation demonstrates that the left side is also invariant under a Lorentz transformation.

Naturally, dP^3 can also be expressed in spherical coordinates, P, θ, and ϕ as shown in the lower part of Figure 5. In that case

$$dP^3 = PdP\sin\theta d\theta d\phi$$

For azimuthally symmetric distributions the above can be integrated over all angles ϕ yielding

$$dP^2 = 2\pi PdP\sin\theta d\theta$$

so that in terms of the remaining variables, P and θ, one has

$$\sigma'_{inv} = \frac{Ed^2\sigma}{2\pi PdP\sin\theta d\theta}$$

This can be further simplified for nonrelativistic particles for which $E = M_0 + T$, where T is the kinetic energy (equal to $P^2/2M_0$) and where $T << M_0$:

$$\sigma'_{inv} = \frac{M_0 d^2\sigma}{2\pi P dP \sin\theta d\theta} = \frac{M_0 d^2\sigma}{P dP d\Omega}$$

Note that $d\Omega = 2\pi \sin\theta d\theta$ is the elementary solid angle covering polar angles from θ to $\theta + d\theta$. Replacing PdP with $M_0 dT$ leads to Equation (21) of the text,

$$\sigma'_{inv} = \frac{d^2\sigma}{P dT d\Omega}$$

which is a useful form of the invariant cross section for axially isotropic reactions at low energies.

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Figure Captions

Figure 1

Dependence of rapidity y and $(\gamma - 1)$ on v/c.

Figure 2

Frame S' moving with respect to frame S at velocity v_0

Figure 3

Angles are additive but their tangents are not.

Figure 4

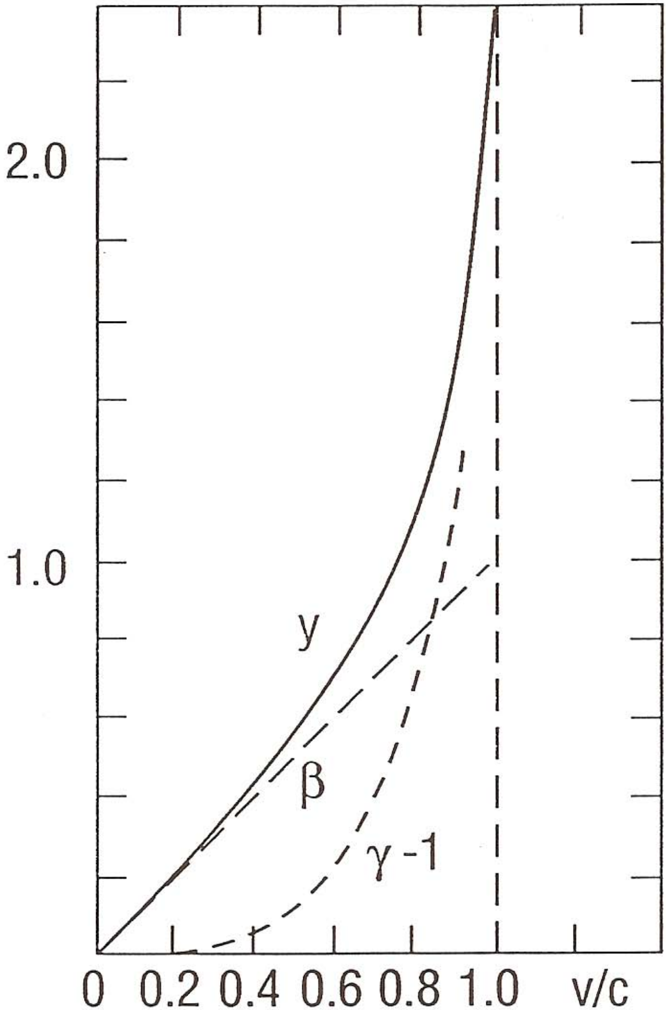
Pseudorapidity versus polar angle.

Figure 5

P - space volume elements in rectangular (left), cylindrical (right), and spherical (below) coordinates.

Figure 6

Invariant production cross sections for positive kaons measured at the ISR for a fixed p_t of 0.4 GeV/c, plotted as a function of the rapidity shift in the frame of the incident protons. The data were summarized from different experiments in Reference 19. The diamonds correspond to a total energy of the two colliding beams of 23 GeV; squares, 31 GeV; circles, 45 GeV; and triangles, 53 GeV.


2.0
1.0
y
β
γ -1
0
0.2
0.4
0.6
0.8
1.0
v/c

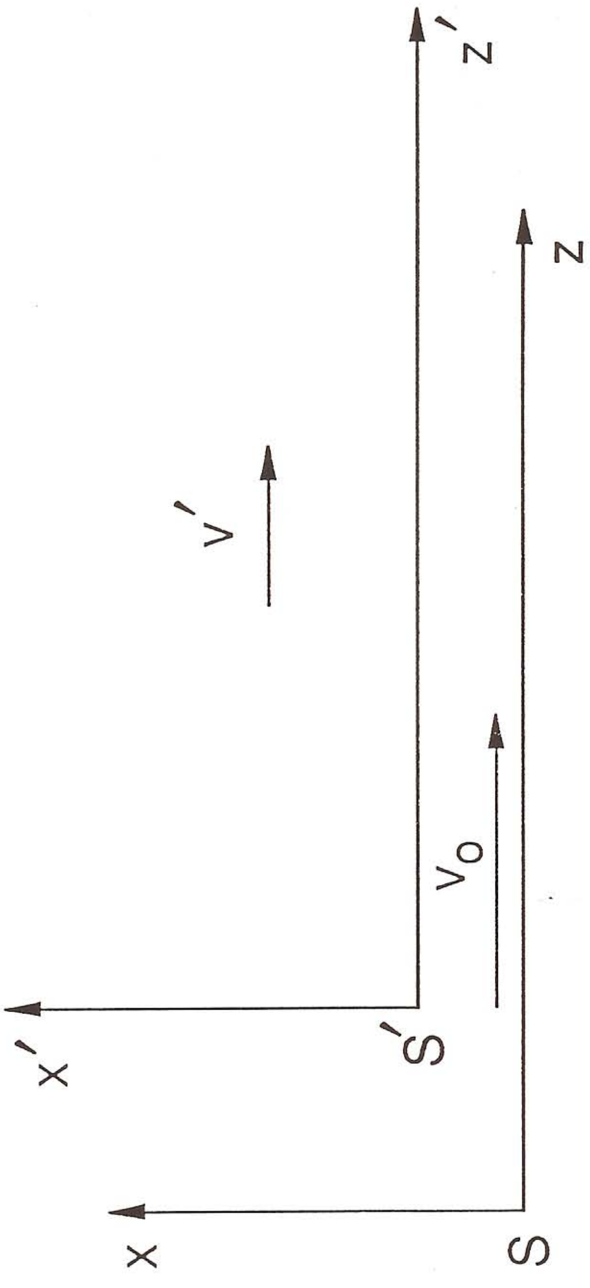
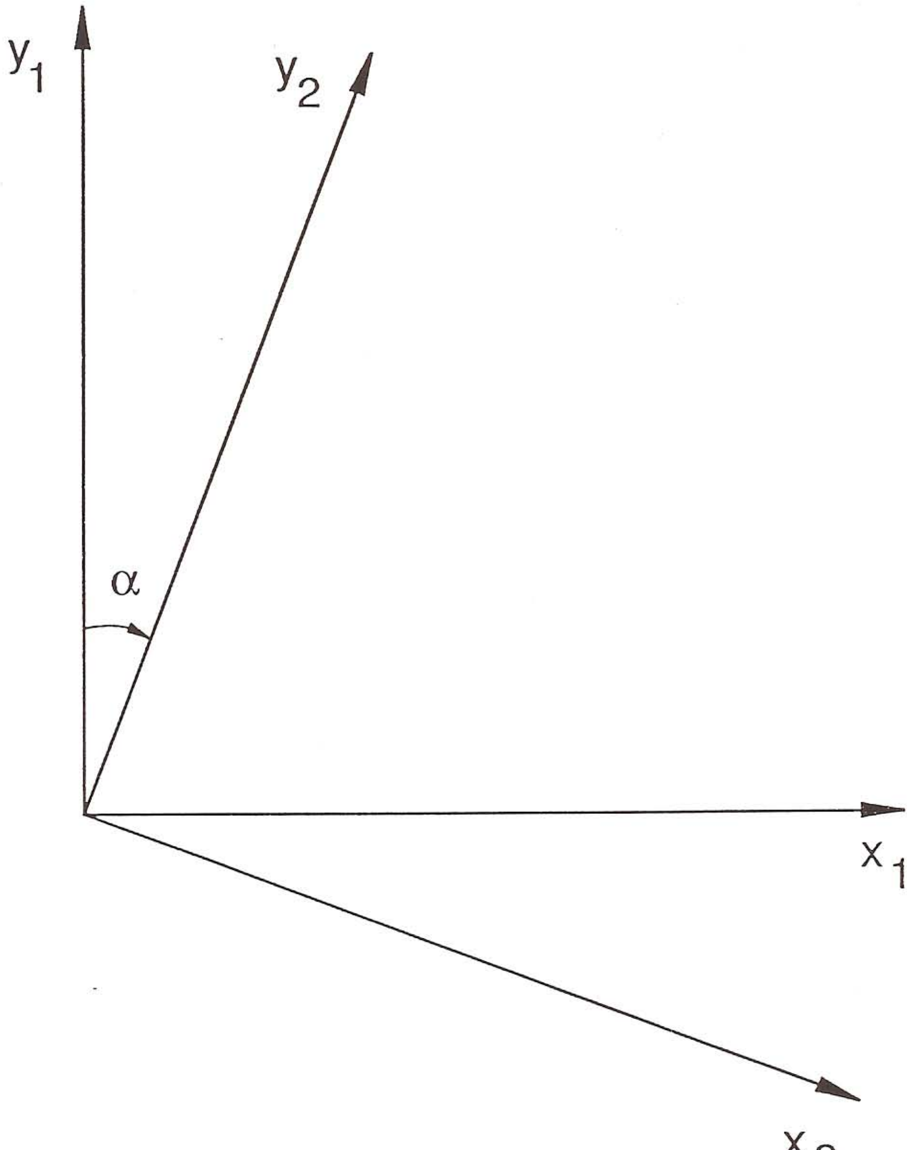


Fig-2



y_1
y_2
α
x_1
x_2

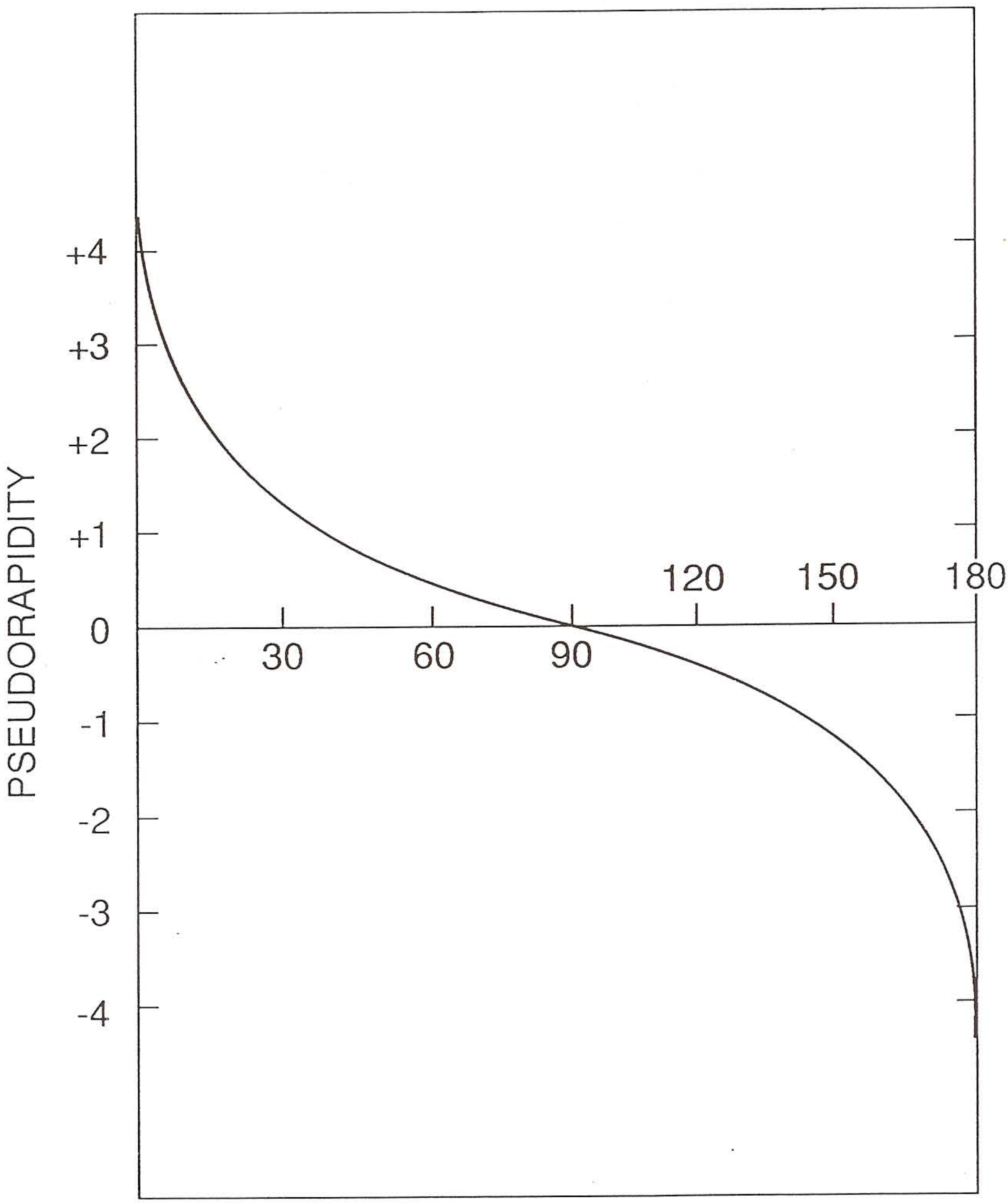


Fig 4

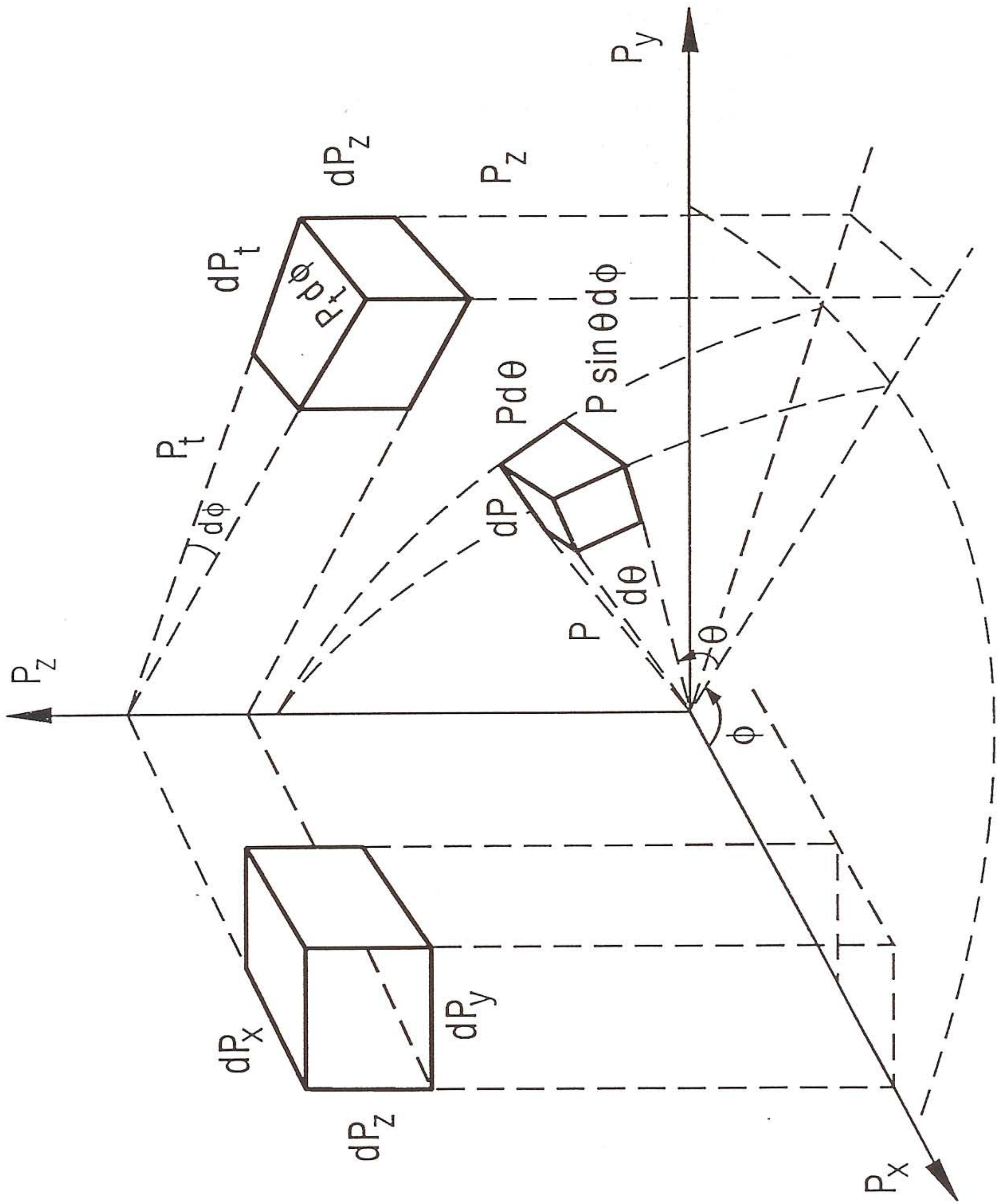


Fig 5

$\beta = v/c$	velocity ratio (dimensionless)
$\gamma = (1-\beta^2)^{-1/2}$	gamma factor (dimensionless)
m_0	rest mass ($\mathrm{GeV}/\mathrm{c}^2$)
$M_0 = m_0c^2$	rest mass energy (GeV)
$m = \gamma m_0$	relatavistic mass ($\mathrm{GeV}/\mathrm{c}^2$)
$E = mc^2 = \gamma M_0$	total energy (GeV)
$T = E - M_0 = (\gamma - 1)M_0$	kinetic energy (GeV)
$p = \gamma m_0 v$	linear momentum (GeV/c)
$P = cp = \beta E = \beta\gamma M_0$	linear momentum variable (GeV)
$P^2 = E^2 - M_0^2$	

Table 1

Basic Relatavistic Quantities and Relations