Soft variable-structure controls: a survey

J. Adamy, A. Flemming

Darmstadt University of Technology, Institute for Automatic Control
Landgraf-Georg-Straße 4, 64283 Darmstadt, Germany

Abstract

Variable-structure controls are normally understood to be controls that have sliding modes and robustness as their main objective. In addition to sliding-mode controls, there are also variable-structure controls, which were developed for the purpose of intentionally precluding sliding modes and achieving high regulation rates and short settling times. Two types of such controls may be distinguished, variable-structure controls that switch between different parameters and a systematic further development of them called "soft variable-structure controls" that continuously vary controllers' parameters or structures and achieve nearly time-optimal control performance. This paper surveys soft variable-structure controls, compares them to other controls, taking a submarine dive-control as an example, and presents an outlook on their auspicious further development.

Keywords: Nonlinear control, soft variable-structure control, nearly time-optimal control, piecewise-linear control

1 Introduction and history

In this section, we briefly describe the history and objectives of discontinuous controls, beginning with early heuristic approaches, and continuing with a description of systematic control methods and the unified framework of hybrid systems. We conclude by depicting soft variable-structure controls as a systematic further development of discontinuous controls.

1.1 Early discontinuous controls

Beyond the initial examples of discontinuous controls, Philon's oil lamp in the ancient Greece in the third century BC and two-step level controls for fluids, which were developed by the brothers Banu Musa for the Caliph of Baghdad in the ninth century (cf. Mayr, 1969, 1970), the earliest precursors of variable-structure controls were controls incorporating relays. Such controllers for steam engines and ships were described in (Poncelet, 1826; Fink, 1865; Farcot, 1873). Vishnegradsky (1878) presented a first theoretical examination and, in further work, Léauté (1885, 1891) described early attempts at employing the phase plane. An important application is the Tirrill-controller that was developed by A. A. Tirrill at General Electric in 1902 in order to control electric generators. In the following years, the theory of controllers incorporating relays and discontinuities was further elaborated (Házen, 1934; Andronov & Chaikin, 1937, 1949; Oldenbourg & Sartorius, 1948; Truxal, 1955; Smith, 1958; Zypkin, 1958).

Based on the knowledge of these relay controls, controls switching between several subcontrollers were developed over the period 1940 through 1960 and those controls might well be the first to be rightly called "variable-structure controls" (VSC). In this first development phase of VSC, the scientists of that time considered second-order plants and commonly employed geometric methods in the phase plane. Their objective was improving control-system performance, i.e., achieving shorter settling times and reducing overshooting, than achievable using a single linear controller only.

Flügge-Lotz was one of the first to investigate such systems, and she and Taylor expressed their intention in the following sentence that remains valid to this day: "The decision to introduce nonlinear components in a control system may stem from the desire to have a system whose performance is less tied to history than that of linear control systems of comparable power handling" (Flügge-Lotz & Taylor, 1956). Beyond some early work (Bilharz, 1941; Flügge-Lotz & Klotter, 1943, 1948, 1949), typical examples of the basic ideas developed over that period may be found in, e.g., (McDonald, 1950; Flügge-Lotz, 1953; Flügge-Lotz & Wunsch, 1955; Flügge-Lotz & Taylor, 1956; Ostrovsky 1960).

The ideas of this first development phase were seminal

⋆ This paper was not presented at any IFAC meeting. Corresponding author J. Adamy. Tel. +49-(0)6151-16-3442. Fax +49-(0)6151-16-2507.

Email address: JAdamy@iat.tu-darmstadt.de (J. Adamy).

for succeeding methods in control theory: (1) an early contribution towards adaptive control known as "dual-mode control" (McDonald, 1952), (2) time-optimal control, accompanied by initial important results, in (Bushaw, 1952, 1953; Fel'dbaum, 1953), and (3) sub-optimal VSC lacking sliding modes aimed at achieving short settling times (Kiendl, 1972; Becker, 1979).

1.2 Systematic design methods: sliding-mode control, VSC precluding sliding modes, and switching supervisory control

A second development phase started with the discovery that high-frequency switching between the various controllers involved, the so-called "sliding mode," may occur in the case of discontinuous VSC. If the control system involved is operating in this mode, it drifts down a switching curve towards the origin. In this sliding mode, the control is robust with respect to variations of plant parameters. This robustness is often the main objective of using sliding-mode control. However, the disadvantage of using sliding-mode control systems is the high-frequency switching occurring in their actuator control signals, which will normally reduce actuator service life and severely limit their applicability in practical use (Corless, 1994; Denker & Kaynak, 1994; Young, Utkin, Özgüner, 1999). In addition to robustness, tracking, adaptive model following, and observers having sliding modes are further objectives (Hung, Gao & Hung, 1993; Hsu, Costa & Lizarralde, 1993; Zinober, 1994; Edwards & Spurgeon, 1998; Choi, Misawa & Young, 1999).
Nikol'skii (1934) was the first to use the term "sliding mode". A comprehensive theory of systems and controls having sliding modes was devised by Emelyanov (1959, 1967, 1969), Emelyanov and Fedotova (1962), and Emelyanov and Kostyleva (1964) in the Soviet Union. Filippov (1960, 1964, 1988) developed a mathematical theory for such differential equations having discontinuous right-hand sides. Sliding-mode theory has since been continually elaborated upon by Utkin, Itkis, and others (Utkin, 1977, 1983, 1992; Itkis, 1976; Hung, Gao & Hung, 1993; Yu & Xu, 2000; Ha et al, 2001). One of those extensions smoothes the operation of sliding-mode controllers to the point where no undesirable high-frequency switching occurs, an improvement that is termed "quasi-sliding-mode" or "pseudo-sliding-mode" control (Hung, Gao & Hung, 1993; Zinober, 1994; Edwards & Spurgeon, 1998).
VSC for linear systems that were based on a principle that intentionally precludes sliding modes were devised under a third development phase. The motivation for these controls was achieving regulation rates that would be much better, and settling times that would be much shorter, than those of linear controls and closer to those of optimal controls, along with less design and implementation effort than would be required in the case of optimal controls. Furthermore, avoiding violating control and state constraints was considered, and an important feature of the controls developed. The intention in devising these controls was rather similar to that of the researchers involved in the initial development phase, and Flügge-Lotz (1968) described this intention in an early vision: "It soon became apparent that, although optimal controls are not easy to realize, discontinuous suboptimal controls can be still profitable."
The starting points of that third phase, which commenced in 1972, were the works by Kiendl (1972) on "suboptimal controllers having piecewise-linear structures" and Kiendl and Schneider (1972). The controllers they devised switch over to using faster-acting linear subcontrollers during regulation cycles, which allows much more rapid regulation than when a single linear controller only is employed. Two different types of such controllers were developed.
The first type switches from subcontrollers to faster ones at certain times. A version of this type expanded to allow use of MIMO-systems appears in (Kreitner, 1980, 1982). An extension to asymmetric control constraints appears in (Opitz, 1982). The second type consists of a set of nested, positively invariant sets and a set of linear controllers in order that each controller will be assigned to one of the nested sets. During a regulation cycle, the trajectory runs from a positively invariant set into the next simultaneously activating the assigned controller and then into the next nested set, and so on. Stelzner (1987) described a design method for such controllers employing piecewise-linear Lyapunov functions. These controllers having nested, positively invariant sets were also developed in later works (Wredenhagen & Bélanger, 1994; De Doná et al, 1999; Benzaouia & Baddou, 1999), and their authors were obviously uninformed about the earlier works of Kiendl (1972), Kiendl and Schneider (1972), and Stelzner (1987), which were published in German only.
The investigations of Becker (1977, 1978, 1979), Xing-Huo and Chun-Bo (1987), and Zimmer (1987) are to be regarded as falling under this third development phase, cf. also (Föllinger, 1998), and determined optimal selection strategies. The controls devised by Itschner (1975, 1977) also fall within areas covered by those works. Maximizing system performance, while simultaneously avoiding violating state constraints during regulation periods were also the objectives of controls described in (Kapasouris, Athans & Stein, 1988; Tan, 1991). All VSC of this third development phase were developed in order to arrive at higher regulation rates and lower settling times than would be achievable using linear controls.
Kiendl (1972) and Wredenhagen and Bélanger (1994) called their controllers "piecewise-linear controllers" and, indeed, they may be regarded as both VSC and a subclass of piecewise-linear systems. The latter includes a wide range of different systems, structures, and objectives, including locally linearized systems, relay controls, and certain types of electrical circuits (Sontag, 1981; Banks & Khathur, 1989; Koutsoukos, 2000; Koutsoukos & Antsaklis, 2002; Johansson, 2003).

A fourth development phase established switching between several controllers as a method of adaptive control theory. These adaptive control techniques are known as "multi-model adaptive control," "switching supervisory control (SSC)," and "universal controller" (Miller & Davison, 1989; Ryan, 1994), and are not normally regarded as part of VSC theory. They differ from the variable-structure approach both in their objective, which is largely achieving adaptability of the controllers employed, and in the structures of their control systems. In the case of multi-model adaptive control, switching is controlled by variations of the plant. However, switching of VSC, both those with and without sliding modes, is controlled by the plant's state vector.

Switching in adaptive control was first introduced by Mårtensson (1986), and further developments will be found in, e.g., (Fu & Barmish, 1986; Middleton, Goodwin, Hill & Mayne, 1988; Morse, Mayne & Goodwin, 1992; Pait & Morse, 1994; Mareels & Polderman, 1996; Morse, 1996; Angeli & Mosca, 2002). Morse (1995) presented an overview of some switching controls called "prerouted," "hysteresis," "dwell-time," and "cyclic" switching controls. Narendra and Balakrishnan (1994, 1997) described such "multi-model adaptive controls" that had the additional objectives of improving transient response and retaining stability. In (Skafidas et al, 1999), switching controls were also used for controlling stochastic plants.

1.3 Hybrid control: a unified framework

Discontinuous controls were developed for quite different purposes, e.g., rapid regulation and short settling times, robustness, plant stabilization, or adaptability. Consequentially, their design methods are quite different, and the notation and classifications employed have thus far been inconsistent. Although terms, such as "variable-structure control," "sliding-mode control," "discontinuous control," "switching control," "piecewise-linear control," "variable-parameter control," "switching supervisory control," and "switching adaptive control" are commonly encountered, they do not always have the same meanings.

All of these switching controls and variable-structure controls have become part of the theory of hybrid systems over the past decade. Hybrid systems are a mixture of real-time continuous and discrete-event systems (Blondel & Tsitsiklis, 1999, 2000; van der Schaft & Schumacher, 2000; Engell, Frehse & Schnieder 2002; Savkin & Evans, 2002; Morari, Baotic & Borrelli, 2003) and provide a unified framework (Branicky, Borkar & Mitter, 1994, 1998; Mignone, Bemporad & Morari, 1999; Heemels, De Schutter & Bemporad, 2001) encompassing differing areas of system and control theory, such as relay control systems, bang-bang control, piecewise-linear systems, timed and hybrid automata, simulation and modeling languages, rule-based control systems, multi-model adaptive control, and variable-structure systems, including those control systems mentioned above. Hybrid systems are also used in the field of fuzzy control (Stanciulescu, 1999, 2002; Qin & Jia, 2002) and in the field of recurrent fuzzy systems (Kempf & Adamy, 2004), which are closely related to finite automata (Adamy & Kempf, 2003; Kempf & Adamy 2003).

Beyond some other early research work on hybrid systems (Ören, 1977), one of the first references (perhaps the first) covering these systems in which they are called "hybrid" is that of Witsenhausen (1966), who defined a special class thereof. However, the research mainstream and the evolution of work in the field as an independent field of research was initiated at a later date at a workshop held at the University of Santa Clara, CA, USA, in 1986. The results of that workshop were published in 1987 (IEEE Report, 1987). Table 1 summarizes the history that have been described thus far.

1.4 Soft VSC: a systematic improvement on discontinuous controls

Switching between different controllers allows, among other things, improving the performance, i.e., the regulation rates and settling times, of control systems, compared to the case where only a single controller is employed, provided that suitable controllers and a suitable switching strategy are chosen. Note that this was the objective of, and the result obtained by, the pioneers in the first and third development phase.

Since the control performance potentially attainable by suitably designed VSC lacking sliding modes increases with the total number of controllers employed, as many controllers as possible are employed in order to arrive at high levels of control performance. One thus ends up employing infinitely many controllers and switching among them at infinitesimally short time intervals, which yields either continuously varying controller parameters or continuously varying controller structures. Controls of this type are also termed "continuous variable-structure controls" or, as they were originally termed by Franke (1982b) "soft variable-structure controls." These controls are descended from improved discontinuous VSC of the third development phase, and their primary purposes is achieving rapid regulation and settling times that are nearly as short as those of time-optimal controls. No sliding modes can arise in the case of such controls, since they are precluded by the latter's principle of operation. Furthermore, their actuator control signals are smooth and lack discontinuities, in contrast to the case of switching controls.

In a phase that commenced in 1972, Kiendl, Franke, and a number of other authors developed controls of this type (Kiendl, 1972; Kiendl & Schneider, 1972; Albers, 1983; Franke, 1982a, 1982b, 1983, 1986; Opitz, 1984, 1986a, 1986b, 1987; Adamy, 1991; Kiendl & Scheel, 1991; Niewels & Kiendl, 2000; Niewels, 2001, 2002). Apart from the main objective of achieving rapid regulation, the robustness of soft VSC was also examined

Table 1
Genesis and milestones of the various development phases

220 BC	Philon's oil lamp
840	Banu Musa's two-step controls for fluid-levels
1826	relay controls for steam engines
1885	development of the phase-plane method
1902	Tirrill-regulator for electric generators
1934	first sliding-mode control
1950	switching controls achieving short settling times (start of the first VSC-development phase)
1959	theory of sliding-mode controls targeting robustness (start of the second development phase)
1960	Fillipov's theory covering discontinuous differential equations
1972	VSC precluding sliding modes yielding short settling times (start of the third development phase, which was based on the intention of the first development phase)
1972	first soft VSC-concept yielding much shorter settling times than earlier controls
1982	establishing soft VSC as a systematic control method (descended from the controls of the third development phase)
1986	switching supervisory control having adaptability as main objective (start of the fourth development phase)
1986	development of hybrid system theory commences at a workshop held in Santa Clara, CA, USA
1994	hybrid systems established as a unified framework for a wide class of continuous-discontinuous structures

and proven (Franke, 1983, 1986; Niewels, 2002; Niewels & Kiendl, 2003). In view of their purely continuous behavior, soft VSC are not hybrid systems. Applications of soft VSC to, e.g., a DC-motor, a hydraulic drive, a crane, or a submarine will be found in (Franke, 1983; Borojevic, Garces & Lee, 1984; Lee, 1985; Borojevic, Naitoh & Lee, 1986; Adamy, 1991; Niewels, 2002). Furthermore, soft VSC is in a sense related to continuous gain scheduling approaches (Rugh & Shamma, 2000).
This paper surveys soft VSC. These controls are often further developments of switching controls and, in this case, we start off by describing these precursors before proceeding to describe the soft VSC derived from them in order to illustrate the relationships between discontinuous and soft VSC and provide a readily comprehensible introduction to the subject matter. The scientists who had developed soft VSC published their results in German only and, in spite of our thorough searches for works on soft VSC published by other authors, we have been unable to find any other such works. Since the primary literature is available in German only, we will describe soft VSC in a manner that will be readily comprehensible, and will design the controls without making use of any additional information.
In the next section, we shall describe the objectives, capabilities, and advantages of soft VSC. Soft VSC incorporating nested, positively invariant sets and their further development employing implicit Lyapunov functions shall be described in Section 3. In Section 4, we shall explain the operation of a soft VSC employing a dynamic selection strategy. In Section 5, we shall take a look at a soft VSC having a saturation range whose bounds are varied by the plant's state vector. None of these soft VSC violate control constraints. In order to illustrate their rapid regulation process and the control performance obtained, we apply these controls to the case of a submarine, where they are employed for controlling dive depth.

2 What are soft variable-structure controls and why are they useful?

In order to explain the structure and advantages of soft VSC, let us start off by considering the case of discontinuous variable-structure controls or switching controls for n-dimensional linear plants,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \tag{1}$$

having the control constraints

$$|u| \leq u_0. \tag{2}$$

The controller

$$u = \mathcal{F}(\mathbf{x}, p), \tag{3}$$

where $\mathcal{F}$ is a general operator, depends on the state vector, $\mathbf{x}$, of the plant and a selection parameter, p, that is computed by a selection strategy or supervisor, i.e.,

$$p = S(\mathbf{x}), \tag{4}$$

defined by a discontinuous function, S. The selection strategy switches between k different selection parameters, p, or a finite number, k, of subcontrollers, $\mathcal{F}(\mathbf{x}, p)$, respectively. Fig. 1 illustrates such a control.
The objectives of such discontinuous VSC are often improved settling times, in the case of VSC lacking sliding modes, robustness, in the case of sliding-mode controls, or adaptability, in the case of switching supervisory controls. However, their disadvantage is the discontinuities occurring in their control signals, u, which occur both in the case of discontinuous VSC lacking sliding modes and, particularly, in the case of sliding-mode controls, and their high-frequency switching that, in most cases, reduces the service lives of actuators.
This disadvantage disappears if the parameter p is chosen such that it depends on a continuous function, S,

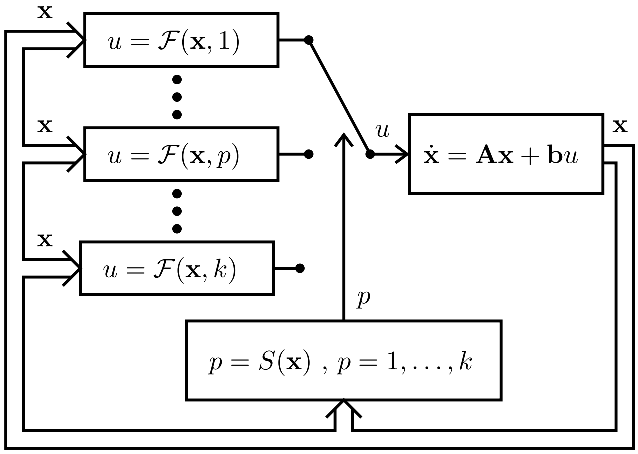


Fig. 1. Block schematic of a discontinuous variable-structure control.

in which case, we will have a continuous set of subcontrollers, $\mathcal{F}(\mathbf{x}, p)$, i.e., an infinite number, instead of a finite number, of subcontrollers and a smooth control signal. In such cases, we call the VSC a "soft VSC." In case of soft VSC, we can generalize the selection strategy of Eq. (4) to obtain

$$S(\mathbf{x}, p^{(n)}, \ldots, p) = 0, \tag{5}$$

which also includes dynamic behavior and implicit equations. Fig. 2 illustrates its structure.

Note that the soft VSC that have been developed to date are based on discontinuous VSC-methods lacking sliding modes, as mentioned in the introduction, and not on the theory of sliding-mode control or switching supervisory control. Their objectives are thus short settling times and high regulation rates, although other objectives may be taken into account in future soft VSC. This fact is important if the intention of current soft VSC, which is totally different from that of sliding-mode controls or switching supervisory controls, is to be understood.

In view thereof, it may be seen that both their continuous control signals and larger numbers of controllers represent benefits, which is why the infinite number of controllers allows reducing settling times to levels that would be unachievable if there were a finite number of subcontrollers.

To make this more readily apparent, let us start off by considering the case of employing a linear controller,

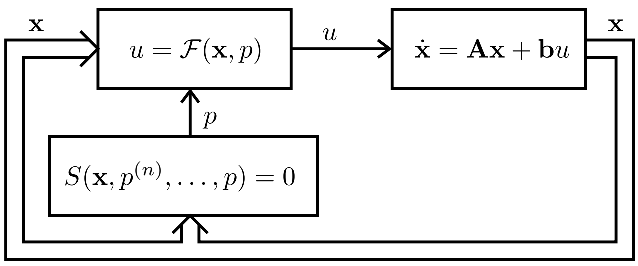


Fig. 2. Block schematic of soft variable structure controls.

$u = -\mathbf{k}^T\mathbf{x}$, on the plant of Eq. (1), whose control signal is constrained by Eq. (2). This linear controller has the disadvantage that its control signal, u, monotonically decreases during regulation cycles where the state variables, x_i, involved monotonically decrease. This linear controller thus cannot make efficient use of the available range, $|u| \leq u_0$, of the control signal, and, consequentially, regulation will not be as fast as it would be if the control signal's constraints were fully exploited. However, switching between subcontrollers will allow improving this situation, provided that suitable subcontrollers and an appropriate selection strategy have been designed.

In order to illustrate this fact, let us now consider an example having a finite number, k, of linear controllers, $u = -\mathbf{k}_p^T\mathbf{x}$, $p = 1, ..., k$. Each subcontroller, together with Eq.(1), leads to a control system,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}_p^T)\mathbf{x} = \hat{\mathbf{A}}_p\mathbf{x}, \tag{6}$$

where the subcontrollers, $\mathbf{k}_p$, have been chosen such that the n eigenvalues, $\lambda_{p,j}$, of $\hat{\mathbf{A}}_p$, where $\mathrm{Re}\{\lambda_p\} < 0$, are given by

$$\lambda_{p+1,j} = h\lambda_{p,j} \quad , h > 1. \tag{7}$$

These controllers thus accelerate the control system's behavior when p increases, while simultaneously causing a similar behavior, since the eigenvalue configuration remains the same (Adamy, 1991; Adamy & Könemund, 2001, 2002; Könemund et al, 1998, Niewels, 2001, 2002). Now we choose a suitable selection strategy precluding sliding modes that switches from a subcontroller, $\mathbf{k}_p$, to another subcontroller, $\mathbf{k}_{p+1}$, such that the control constraint, $|u| \leq u_0$, will be fully exploited, but without being violated, and such that more rapid regulation will be obtained after each switching.

It should be obvious that employing an infinite number of subcontrollers, $\mathbf{k}(p)$, by using a continuous selection variable, $p = S(\mathbf{x})$, incorporating a continuous function, S, causes the absolute values of the eigenvalues to also continuously increase with p, and that the regulation rate will thus be greater, and the settling time shorter, than would be the case if a switching controller were employed, and much greater and much shorter, respectively, than would be the case if a single linear controller were employed.

Soft VSC even allow achieving settling times close to those of time-optimal controls, which will be illustrated by means of an example to be considered in the sections that follow. In contrast to time-optimal control, much less effort is required for designing and implementing soft VSC. Their second objective is allowing a continuous control signal, which neither switching control systems nor time-optimal control systems provide. Apart from these two, original objectives, robustness has also been demonstrated for these controls.

3 VSC employing nested and implicit Lyapunov functions

The initial concept for a soft VSC proposed by Kiendl (1972) and Kiendl and Schneider (1972) is a systematic extension of their discontinuous VSC lacking sliding modes. The subcontrollers of this discontinuous VSC are chosen in a manner rather similar to those described in the example of the preceding section. The control's selection strategy is based on nested, positively invariant sets. In order to explain this control, we start off by describing the discontinuous case in the section that follows. Stelzner (1987) improved the method for designing such controls. As mentioned in the introduction, Wredenhagen and Bélanger (1994) developed the same VSC in a later work.

In a second section, we describe the concept of a soft VSC employing nested Lyapunov functions proposed by Kiendl and Schneider. This concept was taken up and further developed into soft VSC employing implicit Lyapunov functions in (Adamy, 1991). In the process of extending this kind of control, the final step has thus far been that of (Niewels & Kiendl, 2000, 2003; Niewels, 2001, 2002), where the implicit Lyapunov approach was generalized to the case of VSC employing multivalued Lyapunov functions and robustness was proven.

Since the direct method of Lyapunov's stability theory (Hahn, 1967; Rouche, Habets & Laloy, 1977; Sastry, 1999) is essential to all controls described in this paper, we shall briefly review it here for the convenience of readers. We start with the following well-known theorem:

Theorem 1 *The differential equation,* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *with a continuous function* $\mathbf{f}$*, having an equilibrium state,* $\mathbf{x} = \mathbf{0}$*, has a unique solution. If there exists a function* $v(\mathbf{x})$ *having continuous partial derivatives and if*

(A1) $v(\mathbf{0}) = 0$,

(A2) $v(\mathbf{x}) > 0 \quad , \mathbf{x} \neq \mathbf{0}$,

(A3) $\dot{v}(\mathbf{x}) < 0 \quad , \mathbf{x} \neq \mathbf{0}$,

then the equilibrium state, $\mathbf{x} = \mathbf{0}$*, will be asymptotically stable and* $v(\mathbf{x})$ *will be called a "Lyapunov function."*

For stable linear systems, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, it will always be possible to compute a Lyapunov function, $v(\mathbf{x}) = \mathbf{x}^T\mathbf{R}\mathbf{x}$, having a positive-definite matrix, $\mathbf{R}$, by solving the so-called "Lyapunov equation,"

$$\mathbf{A}^T\mathbf{R} + \mathbf{R}\mathbf{A} = -\mathbf{Q}, \tag{8}$$

for an arbitrary positive-definite matrix, $\mathbf{Q}$. If there exists a Lyapunov function, $v(\mathbf{x})$, for a system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and

$$G = \{\mathbf{x} \mid v(\mathbf{x}) < c\} \tag{9}$$

is bounded, then G is a positively invariant set that is also called a "Lyapunov region," i.e., a region for which every trajectory that starts therein never leaves it.

3.1 Discontinuous VSC employing nested Lyapunov functions

Let us consider linear plants (1) and linear subcontrollers, $u = -\mathbf{k}_p^T\mathbf{x}$, along with a control-parameter restriction, $|u| \leq u_0$, and a selection strategy, $p = S(\mathbf{x})$. Furthermore, we only consider bounded sets, $\mathbf{X}_0$, of possible initial vectors $\mathbf{x}(t = 0)$, since $\mathbf{X}_0 = \mathbb{R}^n$ is normally not of practical interest. The control mode consists of three major elements (Kiendl, 1972; Kiendl & Schneider, 1972):

(B1) A family of k linear state controllers, $u = -\mathbf{k}_p^T\mathbf{x}$, each of which leads to stable control loops,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}_p^T)\mathbf{x}, \quad p = 1, \ldots, k \tag{10}$$

whose response times decrease with increasing index, p.

(B2) For each control loop (10), there should exist a Lyapunov region,

$$G_p = \{\mathbf{x} \mid v_p(\mathbf{x}) < c_p\}, \tag{11}$$

where c_p determines the size of G_p. Furthermore, G_p should be such that all $\mathbf{x} \in G_p$ satisfy the relation, $|u| = \left|\mathbf{k}_p^T\mathbf{x}\right| \leq u_0$, restricting the values of the control parameters.

(B3) The Lyapunov regions, G_p, should be nested in accordance with

$$G_{p+1} \subset G_p \qquad , p = 1, \ldots, k-1, \tag{12}$$

with an increasing index, p.

Fig. 3 depicts an example of a family of nested Lyapunov regions, where the dimensions of the respective individual regions, G_p, have been chosen such that each of them is tangent to the straight lines, $\mathbf{k}_p^T\mathbf{x} = u_0$ and $\mathbf{k}_p^T\mathbf{x} = -u_0$, arising from the relation, $\left|\mathbf{k}_p^T\mathbf{x}\right| \leq u_0$, restricting the values of the control parameter. This tangential construction guarantees a good exploitation of the control signal's constraints during regulation cycles. The largest region G_p must include $\mathbf{X}_0$.

The selection strategy, $p = S(\mathbf{x})$, then determines each of the zones, $Z_p = G_p \backslash G_{p+1}$, within which the current state vector, $\mathbf{x}(t)$, will be found. The control vectors, $\mathbf{k}_p$, associated with each zone, Z_p, are then activated.

The trajectory is thus confined to increasingly smaller regions, G_p, during regulation cycles, where the associated controller, $\mathbf{k}_p$, will be activated upon entry into each region, G_p. Since the controllers involved lead to control loops (10) with response times that decrease with increasing index, p, regulation proceeds much faster than if a single controller only were employed. The running from a Lyapunov region G_p into the next smaller one that occurs is also the reason why sliding modes cannot occur. Note that this control can be carried out for all $\mathbf{x} \in G_1$, and, thus, $\mathbf{X}_0 \subseteq G_1$ should be satisfied. For a $\mathbf{x} \notin G_1$ it might be that the control constraints $|u| \leq u_0$ will be violated.

Methods for designing such controls will be found in (Kiendl, 1972; Kiendl & Schneider, 1972; Stelzner, 1987; Wredenhagen & Bélanger, 1994). The initial step (B1) in their design is determining the control vectors, $\mathbf{k}_p$, which may be accomplished by choosing suitable sets of eigenvalues, $\lambda_{p,j}$, for each control loop (10). Faster-acting controllers may be obtained by, e.g., choosing $\lambda_{p+1,j} = h \cdot \lambda_{p,j}$ with $h > 1$ as in Eq. (7).

The second step (B2) in the design is choosing a Lyapunov region (11) for each control loop (10), which may be accomplished by, e.g., employing quadratic Lyapunov functions, $v_p(\mathbf{x}) = \mathbf{x}^T\mathbf{R}_p\mathbf{x}$. The matrix, $\mathbf{R}_p$, comes from the Lyapunov equation,

$$\hat{\mathbf{A}}_p^T\mathbf{R}_p + \mathbf{R}_p\hat{\mathbf{A}}_p = -\mathbf{Q}_p \qquad , \hat{\mathbf{A}}_p = \mathbf{A} - \mathbf{b}\mathbf{k}_p^T. \tag{13}$$

The $\mathbf{Q}_p$ appearing in Eq. (13) should be positive-definite matrices. $\mathbf{Q}_{p+1} = \mathbf{Q}_p$ is frequently a reasonable choice. The Lyapunov region G_p from Eq. (11) will then be an ellipse determined by the matrix $\mathbf{R}_p$.

Since the condition $\left|\mathbf{k}_p^T\mathbf{x}\right| \leq u_0$ is to be satisfied for all $\mathbf{x} \in G_p$ and to be fully exploited, c_p should be chosen such that the hyperplanes $\pm\mathbf{k}_p^T\mathbf{x} = u_0$ are tangent to the ellipse,

$$G_p = \left\{\mathbf{x} \mid \mathbf{x}^T\mathbf{R}_p\mathbf{x} < c_p\right\}. \tag{14}$$

In order to determine this c_p, we solve the optimization problem $\mathbf{x}^T\mathbf{R}_p\mathbf{x} \rightarrow \max$, subject to the restrictions

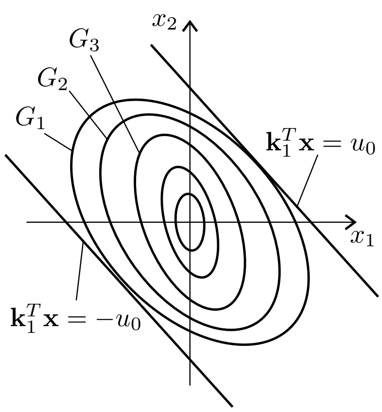


Fig. 3. A family of nested Lyapunov regions.

$\pm\mathbf{k}_p^T\mathbf{x} = u_0$, whose solution yields the value sought,

$$c_p = \frac{u_0^2}{\mathbf{k}_p^T\mathbf{R}_p^{-1}\mathbf{k}_p}. \tag{15}$$

Under the third step (B3) in the design procedure, it should be verified that all k regions, G_p, are nested, where a nesting condition may be formulated for this purpose. Two regions, G_p and G_{p+1}, will be specified by a point, $\mathbf{x}$, that satisfies the inequalities

$$\frac{\mathbf{x}^T\mathbf{R}_p\mathbf{x}}{c_p} < \frac{\mathbf{x}^T\mathbf{R}_{p+1}\mathbf{x}}{c_{p+1}} < 1. \tag{16}$$

If all points that satisfy Eq. (16), then $G_{p+1} \subset G_p$, which will be precisely the case if the matrix

$$\frac{\mathbf{R}_{p+1}}{c_{p+1}} - \frac{\mathbf{R}_p}{c_p} \tag{17}$$

is positive definite. This condition should be verified for all $p = 1, \ldots, k-1$.

Kiendl (1972) and Stelzner (1987) also employed parallelepipeds as an alternative to ellipsoidal Lyapunov regions. These latter result if one employs $v(\mathbf{x}) = ||\mathbf{W}\mathbf{x}||_\infty$ as the Lyapunov function, where $||\mathbf{z}||_\infty$ is the maximum norm of a vector $\mathbf{z}$ (Kiendl, Adamy & Stelzner, 1992; Łoskot, Polánski & Rudnicki, 1998).

3.2 *Soft VSC employing implicit Lyapunov functions: basics and stability*

Infinitely densely nesting the Lyapunov regions for the final discontinuous VSC described in the section immediately above yields a soft VSC (Kiendl, 1972; Kiendl & Schneider, 1972). As for the discontinuous case, the control concept involved here consists of three elements:

(C1) A continuous family of linear state controllers, $u = -\mathbf{k}^T(p)\mathbf{x}$, all of which lead to stable control loops,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T(p))\mathbf{x}. \tag{18}$$

(C2) For each control loop in Eq. (18), a Lyapunov region, $G(p)$, must be determined for which the restriction, $|u| = \left|\mathbf{k}^T(p)\mathbf{x}\right| \leq u_0$, on the control parameter, u, should be satisfied for all $\mathbf{x} \in G(p)$.

(C3) The Lyapunov regions, $G(p)$, involved should be infinitely densely nested, i.e., $G(p+\varepsilon) \subset G(p)$ should hold true for every small $\varepsilon > 0$, which implies that the size of $G(p)$ will decrease with increasing p.

Unfortunately, the soft VSC concept above yields no concrete method for designing such controls. A suitable design method was developed in (Adamy, 1991) and will

be described below.
The Lyapunov regions, $G(p)$, involved may be defined by

$$G(p) = \{\mathbf{x} \mid g(p, \mathbf{x}) < 0\}\,, \tag{19}$$

together with a suitable function $g(p, \mathbf{x})$. The control vector, $\mathbf{k}(p)$, associated with a Lyapunov region, $G(p)$, will be activated whenever the trajectory, $\mathbf{x}(t)$, enters $G(p)$, which will occur whenever $\mathbf{x}(t)$ lies on the border,

$$\partial G(p) = \{\mathbf{x} \mid g(p, \mathbf{x}) = 0\}\,, \tag{20}$$

of $G(p)$. The parameter, p, determining $\mathbf{k}(p)$ will thus be determined for each $\mathbf{x}(t)$ during regulation cycles by the implicit equation

$$g(p, \mathbf{x}) = 0. \tag{21}$$

In regard to the control, g must satisfy two conditions:

(C4) For each $\mathbf{x}$, Eq. (21) must be uniquely solvable for p.

(C5) The function g should be chosen such that the control loop (18), (21) will be stable.

Satisfying Condition (C4) is necessary in order to be able to assign one, and only one, value, p, to each state vector $\mathbf{x}$. If this is not the case, either Eq. (21) has no solution and no p will exist for a state vector $\mathbf{x}$, or Eq. (21) has several solutions p and several controllers $\mathbf{k}(p)$ will thus be assigned to a single state vector, $\mathbf{x}$. In these irregular cases, the control described above obviously cannot be implemented. However, Condition (C4) will be satisfied if a function, $p = p(\mathbf{x})$, is defined by Eq. (21).
Without loss of generality, Conditions (C1), (C2), and (C3) may be reformulated such that the size of $G(p)$ decreases as the parameter p decreases and $p = 0$ for $\mathbf{x} = \mathbf{0}$. We will then be able to satisfy Condition (C5), since p will also decrease along every trajectory of the control loop (18), (21) and

$$v = p \tag{22}$$

will be a Lyapunov function of the control loop,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T(v))\mathbf{x} = \hat{\mathbf{A}}(v)\mathbf{x}, \tag{23}$$

$$g(v, \mathbf{x}) = 0, \tag{24}$$

resulting from Eqs. (18), (21), and (22). Fig. 4 illustrates the structure of this control loop.
If Eq. (24) is to actually define a Lyapunov function, this implicit equation must satisfy two conditions that guarantee that (C4) and (C5) will be satisfied: (1) It must implicitly define a function, $v(\mathbf{x})$. (2) This function, $v(\mathbf{x})$, must be an implicit Lyapunov function of the system of Eq. (23). These conditions will be met if the

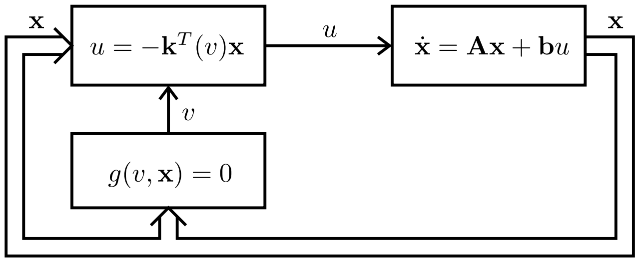


Fig. 4. Structure of the control loop employing an implicitly defined Lyapunov function v as selection strategy.

following theorem on implicit Lyapunov functions given and proven in (Adamy, 1991) holds true for Eqs. (23) and (24):

Theorem 2 *The differential equation,* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *with a continuous function* $\mathbf{f}$*, having an equilibrium state,* $\mathbf{x} = \mathbf{0}$*, has a unique solution for every initial state taken from a neighborhood,* U_1*, of the origin. Within a region,*

$$H = \{(v, \mathbf{x}) \mid 0 < v < \bar{v},\ \mathbf{x} \in U_0 \backslash \{\mathbf{0}\}\}, \tag{25}$$

where $U_0 \subseteq U_1$ *is a neighborhood of the origin, there exists a continuous function,* $g(v, \mathbf{x})$*, that is differentiable with respect to* v *and* $\mathbf{x}$ *such that*

(D1) for $\mathbf{x} \to \mathbf{0}$ *the limit* $v \to 0+$ *results from* $g(v, \mathbf{x}) = 0$ *and*

(D2) $\lim_{v \to 0+} g(v, \mathbf{x}) > 0$ *and* $\lim_{v \to \bar{v}-} g(v, \mathbf{x}) < 0 \quad \forall \mathbf{x} \in U_0 \backslash \{\mathbf{0}\}$.

If the pair of conditions

(D3) $-\infty < \dfrac{\partial g(v, \mathbf{x})}{\partial v} < 0,$

(D4) $\dfrac{\partial g(v, \mathbf{x}(t))}{\partial t} < 0 \qquad \forall\, (v, \mathbf{x}),$ *where* $g(v, \mathbf{x}) = 0,$

are then both satisfied within H*, then the equilibrium state,* $\mathbf{x} = \mathbf{0}$*, will be asymptotically stable.*
Furthermore, the equation $g(v, \mathbf{x}) = 0$ *implicitly defines a function,* v*, where* $0 < v(\mathbf{x}) < \bar{v}$*, within* $U_0 \backslash \{\mathbf{0}\}$ *that may be made continuous at* $\mathbf{x} = \mathbf{0}$ *by defining* $v(\mathbf{0}) = 0$*. This continued function is a Lyapunov function for the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *within* U_0*.*

This theorem is mainly a combination of the well known implicit-function theorem (Courant & John, 2000) and Theorem 1. We can always assume $\bar{v} = 1$ without loss of generality.
The above conditions may be elucidated as follows: Condition (D3) provides that g will monotonically decrease for constant $\mathbf{x}$, governed by v. Due to (D2), g thus has one, and only one, zero for $v \in\ ]0, \bar{v}\,[$. Regarding the implicit-function theorem, the equation $g(v, \mathbf{x}) = 0$ thus implicitly defines a function, $v(\mathbf{x}) > 0$, within H. Due to

(D1), at $\mathbf{x} = \mathbf{0}$, $v(\mathbf{x})$ is either defined by $v = 0$ or may be made continuous. Condition (C4) will thus be satisfied. Since the implicit equation (24) has only a single solution on the interval $]0, \bar{v}[$, the values $g(\varepsilon, \mathbf{x})$ and $g(\bar{v} - \varepsilon, \mathbf{x})$ have opposite signs for every sufficiently small ε and every $\mathbf{x} \in G(\bar{v} - \varepsilon)$. The bracketed solution may thus be numerically computed employing the bisection method or the more rapidly converging Pegasus method (Dowell & Jarratt, 1972), either of which will always yield the correct result. A typical example of how g varies with v is shown in Fig. 5.

Applying the implicit-function theorem and introducing the abbreviations $g_t(v, \mathbf{x}(t)) = \partial g(v, \mathbf{x}(t))/\partial t$ and $g_v(v, \mathbf{x}) = \partial g(v, \mathbf{x})/\partial v$, we then have for the time derivative of $v(\mathbf{x})$:

$$\dot{v}(\mathbf{x}) = -\frac{g_t(v, \mathbf{x}(t))}{g_v(v, \mathbf{x})}. \tag{26}$$

From Eq. (26) and Conditions (D3) and (D4), we have $\dot{v}(\mathbf{x}) < 0$. v is thus a Lyapunov function for $\dot{\mathbf{x}} = f(\mathbf{x})$, and Condition (C5) will also be satisfied.
Furthermore, Condition (D3) implies that

$$g(v_1, \mathbf{x}) \leq g(v_2, \mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{27}$$

for $v_1 > v_2$, and thus that all regions, $G(v) = \{\mathbf{x} \mid g(v, \mathbf{x}) < 0\}$, will be nested and Condition (C3) will be satisfied.
Condition (D4), i.e., $g_t(v, \mathbf{x}) = \dot{\mathbf{x}}^T \cdot \text{grad}_{\mathbf{x}} g(v, \mathbf{x}) < 0$, obviously implies that all trajectories will enter into the respective region, $G(v)$, involved. Thus, the regions $G(v)$ are Lyapunov regions and Condition (C2) will be satisfied, provided that we adjust the size of $G(v)$ such that all $\mathbf{x} \in G(v)$ satisfy $|\mathbf{k}^T \mathbf{x}| \leq u_0$. The latter will always be possible.
The theorem on implicit Lyapunov functions is of major importance to control design (23), (24), since if its conditions are satisfied, then the design conditions, (C2), (C3) and (C4), (C5), will also be satisfied. However, this theorem yields only general conditions that must be met in order to design a stable control. That means that suitable functions, $\mathbf{k}(v)$ and $g(v, \mathbf{x})$, will have to be chosen, a procedure that will be covered in the next section.

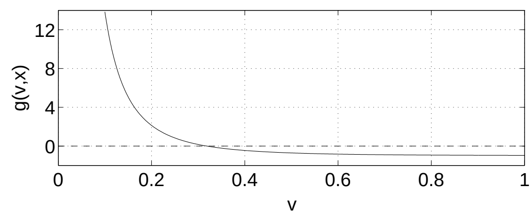


Fig. 5. A plot of the function, $g(v, \mathbf{x})$, of the submarine example appearing in Section 3.4 versus the variable v for a fixed vector, $\mathbf{x} = [0 \quad 0 \quad -0.001]^T$.

3.3 Soft VSC employing implicit Lyapunov functions: subcontrollers and selection strategy

The design of a specific control proceeds as follows: One assumes, without loss of generality, that the linear system (1) is in controllable standard form, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{28}$$

or have been transformed into the above.
We now choose the control vector, $\mathbf{k}(v)$, of Eq. (23) such that the eigenvalues, λ_i, of the plant (23) will be shifted onto rays, $\lambda_i(v) = \lambda_i(1)v^{-1}$, that start at $\lambda_i(1)$ and proceed toward negative infinity with decreasing v, as illustrated in Fig. 6. Note that (C1) will now be satisfied. The above choice of $\mathbf{k}(v)$ leads to faster linear control subsystems (23), since v decreases during regulation cycles. As in the discontinuous case of Section 3.1, the aim here is achieving increasingly higher regulation rates during regulation cycles.
In order to achieve these ray-like eigenvalue paths, $\lambda_i(v)$, we need to formulate the control vector, $\mathbf{k}(v)$, as follows:

$$\mathbf{k}(v) = \begin{bmatrix} \hat{a}_0 v^{-n} - a_0 \\ \hat{a}_1 v^{-(n-1)} - a_1 \\ \vdots \\ \hat{a}_{n-1} v^{-1} - a_{n-1} \end{bmatrix}, \tag{29}$$

where the $\hat{a}_i$ are the coefficients of the characteristic polynomial of $\hat{\mathbf{A}}(v = 1)$, as given by Eq. (23). Arranging the coefficients, a_i, of the plant in a vector,

$$\mathbf{a}^T = (a_0, a_i, ..., a_{n-1}), \tag{30}$$

and the coefficients, $\hat{a}_i$, of the system controlled by $\mathbf{k}(1)$ in a vector,

$$\hat{\mathbf{a}}^T = (\hat{a}_0, \hat{a}_i, ..., \hat{a}_{n-1}), \tag{31}$$

allows writing the control vector, $\mathbf{k}(v)$, in the form,

$$\mathbf{k}(v) = \mathbf{D}^{-1}(v)\hat{\mathbf{a}} - \mathbf{a}, \tag{32}$$

containing the diagonal matrix,

$$\mathbf{D}(v) = \text{diag}\,(v^n, \ldots, v^2, v). \tag{33}$$

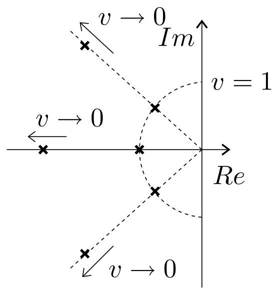


Fig. 6. Eigenvalue configurations of the system $\hat{\mathbf{A}}(v)$.

Inserting Eqs. (28) and (32) into Eq. (23), the system matrix of the control system may be written in the form

$$\hat{\mathbf{A}}(v) = \frac{1}{v}\mathbf{D}(v)\hat{\mathbf{A}}_1\mathbf{D}^{-1}(v), \quad \hat{\mathbf{A}}_1 = \hat{\mathbf{A}}(1). \tag{34}$$

Under a further design step, we need to choose suitable Lyapunov regions, $G(v) = \{\mathbf{x} \mid g(v, \mathbf{x}) < 0\}$, as described under design Condition (C2). Ellipses, $G(v) = \{\mathbf{x} \mid \mathbf{x}^T\mathbf{R}(v)\mathbf{x} - 1 < 0\}$, are the most widespread Lyapunov regions and are also suitable here. We multiply the quadratic form by an additional function, $e(v)$, where $e(v) > 0$, in order to affect the size of every ellipse, $G(v)$, such that the pair of hyperplanes given by $|\mathbf{k}^T(v)\mathbf{x}| = u_0$ will be tangent to the ellipse, and obtain:

$$G(v) = \{\mathbf{x} \mid g(v, \mathbf{x}) = e(v) \cdot \mathbf{x}^T\mathbf{R}(v)\mathbf{x} - 1 < 0\}. \tag{35}$$

Similarly to the discontinuous case of Eq. (15), we obtain ellipses tangent to the two hyperplanes given by $|\mathbf{k}^T(v)\mathbf{x}| = u_0$ using

$$e(v) = \frac{\mathbf{k}^T(v)\mathbf{R}^{-1}(v)\mathbf{k}(v)}{u_0^2}, \tag{36}$$

where the only difference compared to the discontinuous case is that $\mathbf{k}$ and $\mathbf{R}$ are continuous functions depending on v.

3.4 Satisfying the theorem on implicit Lyapunov functions

Recall that the implicit Lyapunov function,

$$g(v, \mathbf{x}) = e(v) \cdot \mathbf{x}^T\mathbf{R}(v)\mathbf{x} - 1 = 0, \tag{37}$$

must satisfy the conditions of Theorem 2. Considering Condition (D4), which guarantees that $G(v)$ will be a Lyapunov region, we have from Eq. (37) that

$$e(v)\mathbf{x}^T\left[\hat{\mathbf{A}}^T(v)\mathbf{R}(v) + \mathbf{R}(v)\hat{\mathbf{A}}(v)\right]\mathbf{x} < 0. \tag{38}$$

Since we provided that $e(v) > 0$, this equation leads to the Lyapunov equation,

$$\hat{\mathbf{A}}^T(v)\mathbf{R}(v) + \mathbf{R}(v)\hat{\mathbf{A}}(v) = -\mathbf{Q}(v), \tag{39}$$

where $\mathbf{Q}(v)$ is a positive-definite matrix. Using Eq. (34), we obtain:

$$\begin{aligned}&\mathbf{D}^{-1}(v)\hat{\mathbf{A}}_1^T\mathbf{D}(v)\mathbf{R}(v) + \mathbf{R}(v)\mathbf{D}(v)\hat{\mathbf{A}}_1\mathbf{D}^{-1}(v)\\ &= -v \cdot \mathbf{Q}(v).\end{aligned} \tag{40}$$

This Lyapunov equation depends on v, and we are now interested in eliminating this dependency in order to obtain an equation that is mathematically easier to handle. For this purpose, we choose the positive-definite matrices, $\mathbf{R}(v)$ and $\mathbf{Q}(v)$, in the following form:

$$\mathbf{R}(v) = \mathbf{D}^{-1}(v)\mathbf{R}_1\mathbf{D}^{-1}(v), \tag{41}$$

$$\mathbf{Q}(v) = \frac{1}{v}\mathbf{D}^{-1}(v)\mathbf{Q}_1\mathbf{D}^{-1}(v), \tag{42}$$

where $\mathbf{R}_1$ and $\mathbf{Q}_1$ must be positive-definite matrices. Inserting Eq. (41) and (42) into Eq. (40), Condition (D4) of Theorem 2 takes on the form:

$$\hat{\mathbf{A}}_1^T\mathbf{R}_1 + \mathbf{R}_1\hat{\mathbf{A}}_1 = -\mathbf{Q}_1, \tag{43}$$

which has then advantage that all matrices have constant coefficients.
The choice of $\mathbf{R}(v)$ in Eq. (41) has the consequence that the function $e(v)$ of Eq. (36) becomes a polynomial of order $2n$ and may be written in the form:

$$\begin{aligned}e(v) &= \frac{1}{u_0^2}\left[\mathbf{k}^T(v)\mathbf{D}(v)\mathbf{R}_1^{-1}\mathbf{D}(v)\mathbf{k}(v)\right]\\ &= \frac{1}{u_0^2}\,[\,\mathbf{a}^T\mathbf{D}(v)\mathbf{R}_1^{-1}\mathbf{D}(v)\mathbf{a}\\ &\quad - 2\hat{\mathbf{a}}^T\mathbf{R}_1^{-1}\mathbf{D}(v)\mathbf{a} + \hat{\mathbf{a}}^T\mathbf{R}_1^{-1}\hat{\mathbf{a}}\,],\end{aligned} \tag{44}$$

using Eqs. (32) and (41) in Eq. (36).
In the next step, we consider Condition (D3) of Theorem 2, which guarantees that all $G(v)$ will be nested, and apply Eq. (37), obtaining

$$-\infty < e'(v) \cdot \mathbf{x}^T\mathbf{R}(v)\mathbf{x} + e(v) \cdot \mathbf{x}^T\frac{\partial\mathbf{R}(v)}{\partial v}\mathbf{x} < 0 \tag{45}$$

for Condition (D3). Since $e(v) > 0$ and $\mathbf{R}(v)$ is positive definite, this condition will be satisfied if

$$\max_{v\in[0,\bar{v}]} e'(v) \le 0, \tag{46}$$

and if

$$\frac{\partial\mathbf{R}(v)}{\partial v} \tag{47}$$

is a negative-definite matrix. Inequality (46) may be easily checked by computing the maxima of e' on $[0, \bar{v}]$, since $e(v)$ and its derivative, $e'(v)$, are polynomials.

The matrix $\mathbf{R}(v)$ may be differentiated using Eq. (41), and we then obtain for Eq. (47):

$$\frac{\partial \mathbf{R}(v)}{\partial v} = \frac{\partial \mathbf{D}^{-1}(v)\mathbf{R}_1\mathbf{D}^{-1}(v)}{\partial v} = \frac{1}{v}\mathbf{D}^{-1}(v)\cdot(\mathbf{N}\mathbf{R}_1 + \mathbf{R}_1\mathbf{N})\cdot\mathbf{D}^{-1}(v), \quad (48)$$

where $\mathbf{N} = \mathrm{diag}\ (-n, \ldots, -1)$. This matrix (48) will be negative definite if the matrix

$$\mathbf{N}\mathbf{R}_1 + \mathbf{R}_1\mathbf{N} = -\mathbf{S}_1, \quad (49)$$

is negative definite. As in the case of Eq. (43), Eq. (49) is independent of v, which is again a consequence of the choice of $\mathbf{R}(v)$ used in Eq. (41).
Based on several simple considerations involving limiting values, it may be shown that (D1) and (D2) of Theorem 2 are satisfied.
Summarizing the results presented above, Theorem 2 is satisfied and designing [1] a soft VSC employing implicit Lyapunov functions necessitates satisfying Eqs. (43), (46), and (49). How this may be accomplished will be covered in the next section.

3.5 Computing controller parameters

The considerations above imply that a soft VSC employing implicit Lyapunov functions consists of the control loop

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T(v))\mathbf{x} \quad , \mathbf{k}(v) = \mathbf{D}^{-1}(v)\hat{\mathbf{a}} - \mathbf{a} \quad (50)$$

and the selection strategy

$$g(v, \mathbf{x}) = e(v)\cdot\mathbf{x}^T\mathbf{R}(v)\mathbf{x} - 1 = 0, \quad (51)$$

where

$$e(v) = \frac{1}{u_0^2}\left[\ \mathbf{a}^T\mathbf{D}(v)\mathbf{R}_1^{-1}\mathbf{D}(v)\mathbf{a} - 2\hat{\mathbf{a}}^T\mathbf{R}_1^{-1}\mathbf{D}(v)\mathbf{a} + \hat{\mathbf{a}}^T\mathbf{R}_1^{-1}\hat{\mathbf{a}}\ \right], \quad (52)$$

$$\mathbf{R}(v) = \mathbf{D}^{-1}(v)\mathbf{R}_1\mathbf{D}^{-1}(v), \quad (53)$$

and $e(v)$ is a polynomial of order $2n$ or less. Note that a soft VSC employing implicit Lyapunov functions will be operative for $0 < v < \bar{v}$, i.e., will operate as a soft VSC for all $\mathbf{x} \in G(\bar{v})$, where $G(\bar{v})$ is the largest of all nested Lyapunov regions, $G(v)$. We may choose $\bar{v} = 1$ without loss of generality, as mentioned above.

[1] In addition to ellipsoidal Lyapunov regions $G(v)$, regions with other types of shapes, e.g., polyhedra, may also be employed in Eq. (37) by utilizing vector norms (Adamy, 1991). The robustness of this type of soft VSC has been shown in (Niewels, 2002; Niewels & Kiendl, 2003).

Computing the parameters of this control involves choosing a suitable vector, $\hat{\mathbf{a}}$, in a first step and a matrix, $\mathbf{R}_1$, in a second step, and verifying $\mathbf{X}_0 \subseteq G(1)$ in a third step. Let us now consider these steps in detail.
Step 1: Recall that the vector, $\hat{\mathbf{a}}$, consists of the coefficients of the characteristic polynomial of the system matrix, $\hat{\mathbf{A}}_1 = \hat{\mathbf{A}}(v = 1)$. The best way to choose $\hat{\mathbf{a}}$ is selecting eigenvalues, $\lambda_i(1)$, of $\hat{\mathbf{A}}_1$ such that the linear system, $\dot{\mathbf{x}} = \hat{\mathbf{A}}_1\mathbf{x}$, will exhibit good control performance.
Furthermore, we need to allow for the fact that the resultant control vector, $\mathbf{k}(1) = \hat{\mathbf{a}} - \mathbf{a}$, will be chosen such that any $\mathbf{x} \in \mathbf{X}_0$, where $\mathbf{X}_0$ is the set of all possible initial state vectors, $\mathbf{x}(t = 0)$, satisfies the control constraints, $\left|\mathbf{k}^T\mathbf{x}\right| \le u_0$.
Step 2: We need to choose the matrix $\mathbf{R}_1$ such that Eqs. (43), (46), and (49) will be satisfied. Fortunately, there will normally be a set of matrices, $\mathbf{R}_1$, that satisfy these conditions. We thus choose from those matrices that matrix, $\mathbf{R}_1$, that yields the region, $G(1) = \left\{\mathbf{x} \mid e(1)\cdot\mathbf{x}^T\mathbf{R}_1\mathbf{x} - 1 < 0\right\}$, having the maximum volume. That region will also be the largest region within which we may apply soft VSC. Since the volume of the ellipsoid, $G(1)$, is proportional to $1/\sqrt{e^n(1)\det\mathbf{R}_1}$, we need to solve the constrained optimization problem

$$\max_{\mathbf{R}_1} \frac{1}{\sqrt{e^n(1)\det\mathbf{R}_1}} \quad (54)$$

subject to the three restrictions,

$$\begin{aligned} &\hat{\mathbf{A}}_1^T\mathbf{R}_1 + \mathbf{R}_1\hat{\mathbf{A}}_1 = -\mathbf{Q}_1, \\ &\mathbf{N}\mathbf{R}_1 + \mathbf{R}_1\mathbf{N} = -\mathbf{S}_1, \\ &\max_{v\in[0,\bar{v}]} e'(v) \le 0, \end{aligned} \quad (55)$$

where $\mathbf{Q}_1$ and $\mathbf{S}_1$ are arbitrary positive-definite matrices. Note that this will be the case if their respective smallest eigenvalues, $\lambda_{\min}(\mathbf{Q}_1)$ and $\lambda_{\min}(\mathbf{S}_1)$, are both positive. Also note that $e'(v)$ depends on $\mathbf{R}_1$. The optimization problem may be analytically solved in simple cases only. Solution of the optimization problem (54), (55) normally employs numerical methods involving, e.g., a log-barrier function (McCormick, 1983),

$$B(\mathbf{R}_1) = \frac{1}{\sqrt{e^n(1)\det\mathbf{R}_1}} + r\left[\ \ln(\lambda_{min}(\mathbf{Q}_1)) + \ln(\lambda_{min}(\mathbf{S}_1)) + \ln(-\max_{v\in[0,\bar{v}]} e'(v))\ \right], \quad (56)$$

that transforms the restricted problem (54), (55) into an unrestricted one,

$$\max_{\mathbf{T}} B(\mathbf{R}_1 = \mathbf{T}^T\mathbf{T}), \quad (57)$$

where the matrix $\mathbf{R}_1$ is composed of a triangular matrix, $\mathbf{T}$, and its transpose. In contrast to the case of the elements of the positive-definite matrix $\mathbf{R}_1$, the nonzero

elements of $\mathbf{T}$ may be arbitrary real numbers. The parameter r determines the steepness of the barrier. The problem (57) may now be solved using a hill-climbing method or an evolutionary algorithm (Schwefel, 1995).
Step 3: We have to check whether $\mathbf{X}_0 \subseteq G(1)$ is satisfied. If this is not the case we have to restart with Step 1.
In many cases, the restrictions (55) may be met without solving the optimization problem itself by choosing a positive-definite matrix, $\mathbf{Q}_1$, and computing the matrix $\mathbf{R}_1$ from $\hat{\mathbf{A}}_1^T \mathbf{R}_1 + \mathbf{R}_1 \hat{\mathbf{A}}_1 = -\mathbf{Q}_1$. The Lyapunov equation, $\mathbf{N}\mathbf{R}_1 + \mathbf{R}_1 \mathbf{N} = -\mathbf{S}_1$, will frequently be satisfied for this $\mathbf{R}_1$, since $\mathbf{N}$ is a diagonal matrix. Furthermore, the third restriction of (55) will also be met in many cases. However, this simplified method will not normally yield the optimal solution of (54), (55).

3.6 A typical example: a submarine dive control

We consider a submarine built by Kockumation AB, Sweden, which should be equipped with a dive control in order to allow controlling dive depth. The model for this submarine, along with a linear and a saturated dive control, have been described by Gutman and Hagander (1985). The linear, time-invariant, state-space model in controllable standard form,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -0.005 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad (58)$$

describes its vertical motion. Its dive depth is represented by x_1, its vertical velocity by x_2, and its vertical acceleration by x_3.
The control parameter, u, is confined to the range

$$|u| \leq u_0 = 2.5 \cdot 10^{-5}. \qquad (59)$$

Consider the set of possible initial state vectors, $\mathbf{x}(0)$, given by

$$\mathbf{X}_0 = \{\mathbf{x} \mid |x_1| \leq 10, |x_2| \leq 0.05, |x_3| \leq 0.0046\}. \qquad (60)$$

Design of a soft VSC employing implicit Lyapunov functions may then proceed as in the preceding section.
Step 1: We choose the eigenvalue configuration

$$\begin{aligned} \lambda_1(1) &= -0.00369 \\ \lambda_{2/3}(1) &= -0.00246 \pm j0.00492 \end{aligned} \qquad (61)$$

which leads to

$$\hat{\mathbf{a}}^T = [1.1165 \cdot 10^{-7} \quad 4.8413 \cdot 10^{-5} \quad 8.6100 \cdot 10^{-3}]. \qquad (62)$$

The vectors $\mathbf{a}^T = [0 \quad 0 \quad 0.005]$ and $\hat{\mathbf{a}}$ jointly define the control vector, $\mathbf{k}(v) = \mathbf{D}^{-1}(v)\hat{\mathbf{a}} - \mathbf{a}$.

Step 2: Solving either of the optimization problems (54), (55) or (57) yields a finally optimized matrix,

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 2.5921 \cdot 10^2 & 7.9935 \cdot 10^3 \\ 2.5921 \cdot 10^2 & 9.9489 \cdot 10^4 & 6.5779 \cdot 10^6 \\ 7.9935 \cdot 10^3 & 6.5779 \cdot 10^6 & 9.7977 \cdot 10^8 \end{bmatrix}. \qquad (63)$$

Step 3: Note that $G(1)$ includes $\mathbf{X}_0$ and that control may thus be carried out for any initial state $\mathbf{x} \in \mathbf{X}_0$.
According to (50), (51), (52), and (53) and the parameters determined above, the closed loop system is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1.1165 \cdot 10^{-7}}{v^3} & -\frac{4.8413 \cdot 10^{-5}}{v^2} & -\frac{8.6100 \cdot 10^{-3}}{v} \end{bmatrix} \mathbf{x} \qquad (64)$$

and the implicit equation $g(v, \mathbf{x}) = 0$ leads to the polynomial

$$\begin{aligned} v^{2n} g(v, \mathbf{x}) = {} & c_6(\mathbf{x})v^6 + c_5(\mathbf{x})v^5 + c_4(\mathbf{x})v^4 + c_3(\mathbf{x})v^3 \\ & + c_2(\mathbf{x})v^2 + c_1(\mathbf{x})v + c_0 = 0, \end{aligned} \qquad (65)$$

which is the selection strategy, where

$$\begin{aligned} c_0(\mathbf{x}) &= 1.7803 \cdot 10^{-4} x_1^2 \\ c_1(\mathbf{x}) &= 0.0923 x_1 x_2 - 2.7854 \cdot 10^{-4} x_1^2 \\ c_2(\mathbf{x}) &= 2.8461 x_1 x_3 + 17.7119 x_2^2 + 1.3924 \cdot 10^{-4} x_1^2 \\ &\quad - 0.1444 x_1 x_2 \\ c_3(\mathbf{x}) &= 0.0722 x_1 x_2 + 2.3421 \cdot 10^3 x_2 x_3 - 4.4531 x_1 x_3 \\ &\quad - 27.7120 x_2^2 \\ c_4(\mathbf{x}) &= 1.7443 \cdot 10^5 x_3^2 + 13.8531 x_2^2 + 2.2261 x_1 x_3 \\ &\quad - 3.6645 \cdot 10^3 x_2 x_3 \\ c_5(\mathbf{x}) &= 1.8319 \cdot 10^3 x_2 x_3 - 2.7291 \cdot 10^5 x_3^2 \\ c_6(\mathbf{x}) &= 1.3643 \cdot 10^5 x_3^2 - 1. \end{aligned} \qquad (66)$$

The polynomial (65), whose coefficients vary with $\mathbf{x}$, has only one root in $]0, \bar{v} = 1[$. The value of the implicit Lyapunov function $v \in]0, \bar{v} = 1[$ can thus alway reliably determined from Eq. (65) as mentioned in Section 3.2.
In practical applications, we would discontinue using the VSC at some low value of $\underline{v}$, e.g., $\underline{v} = 0.05$, and carry on from there using the linear control vector, $\mathbf{k}(\underline{v})$, rather continuing to use the VSC until $v \approx 0$, since the latter would lead to computational problems occurring in conjunction with Eq. (64).
A detailed comparison of the soft VSC above to time-optimal control, linear control, saturated linear control and the soft VSC described in the subsequent sections, is given in Fig. 13(a), 13(b), and 14, presenting plots of dive depth, x_1, and of the actuator control signals, u, involved, and in Section 6.

4 Bilinear and dynamic soft VSC

The starting point of soft VSC that employ a differential equation as their selection strategy was the work of Becker (1977, 1978, 1979) regarding how a time-optimal switching strategy might be computed for two given subcontrollers and a given plant. The major result of his work was that computing such an optimal switching strategy is a very time-consuming task and of little use for practical applications. He also proposed a simple Lyapunov-based switching control and interpreted it as a bilinear control system. Longchamp (1980) as well as Xing-Huo and Chun-Bo (1987) cover similar controls. The aims of these switching controls are achieving high regulation rates and precluding sliding modes. Unfortunately, the latter cannot be guaranteed.
Franke (1982a, 1982b, 1983, 1986) was inspired by these bilinear VSC, and his idea was reliably precluding sliding modes by making the switching variable, p, continuous using a differential equation, $\dot{p} = f(p, \mathbf{x})$. He presented examples of their high regulation rates and proved their robustness. We call these soft VSC "dynamic soft VSC," due to their dynamic selection strategy, $\dot{p} = f(p, \mathbf{x})$. An application of these soft VSC is described in (Borojevic, Garces & Lee, 1984; Lee, 1985; Borojevic, Naitoh & Lee, 1986). In order to guarantee their stability, Franke used Lyapunov's direct method. Opitz (1984, 1986a, 1986b, 1987) extended the control described above to the point where only a single output-parameter vector or value, and no state vector, is required for regulation. Proof of stability is based on hyperstability theory in this case.
In the next section, we shall start off by describing Becker's discontinuous bilinear VSC in order to both illustrate the historical background and provide a readily comprehensible introduction to the motivation that led Franke to his soft VSC.

4.1 Bilinear VSC

Once again, we consider linear plants, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$. Becker (1979) employed a pair of linear subcontrollers, $u = -\mathbf{k}_p^T\mathbf{x}$, and switched between them such that a Lyapunov function of the control loop invariably takes on its least value. Cf. also the depiction appearing in (Föllinger, 1998). He thus employed a selection strategy involving a quadratic switching surface and a selection variable, p, that could take on either of the pair of values 1 and -1. The control equation for this approach is

$$u = -\frac{1}{2}\left[(\mathbf{k}_2 + \mathbf{k}_1)^T + p\,(\mathbf{k}_2 - \mathbf{k}_1)^T\right]\mathbf{x}. \tag{67}$$

For $p = 1$, $\mathbf{k}_2$ will be active; for $p = -1$, $\mathbf{k}_1$ will be active. Inserting Eq. (67) into $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$, we obtain the control loop,

$$\dot{\mathbf{x}} = \hat{\mathbf{A}}\mathbf{x} + p\mathbf{M}\mathbf{x}, \tag{68}$$

which is bilinear in p and $\mathbf{x}$, where

$$\hat{\mathbf{A}} = \mathbf{A} - \frac{1}{2}\mathbf{b}\,(\mathbf{k}_2 + \mathbf{k}_1)^T, \quad \mathbf{M} = -\frac{1}{2}\mathbf{b}\,(\mathbf{k}_2 - \mathbf{k}_1)^T. \tag{69}$$

The control vectors, $\mathbf{k}_2$ and $\mathbf{k}_1$, are chosen such that $\hat{\mathbf{A}}$ will be stable. In the following, we shall assume that no sliding mode exists for this bilinear control system.
Becker then employs a Lyapunov function, $v(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$, for the system of Eq. (68). In order that the equilibrium state, $\mathbf{x} = \mathbf{0}$, will be globally asymptotically stable, we demand that

$$\dot{v}(\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\mathbf{P}\dot{\mathbf{x}} < 0. \tag{70}$$

Note, that we must exclude all $\mathbf{x}$ at the discontinuity of the switching surface in Eq. (70). In order to complete the stability considerations, we have to consider differential inclusions (Filippov, 1988). Since these considerations do not restrict the results in the following and to simplify matters we do not describe these considerations here.
Substituting $\dot{\mathbf{x}}$ in Eq. (70) using Eq. (68) we obtain

$$\begin{aligned}\dot{v}(\mathbf{x}) = {} & \mathbf{x}^T(\hat{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\hat{\mathbf{A}})\mathbf{x} \\ & + p\mathbf{x}^T\left(\mathbf{M}^T\mathbf{P} + \mathbf{P}\mathbf{M}\right)\mathbf{x} < 0\end{aligned} \tag{71}$$

that must be satisfied for all $\mathbf{x} \in \mathbb{R}^n \backslash \{0\}$ not lying on the switching surface. We thus choose a positive-definite matrix, $\mathbf{Q}$, in

$$\hat{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\hat{\mathbf{A}} = -\,\mathbf{Q} \tag{72}$$

in order to provide that the first term in Eq. (71) will be negative, which will allow using Eq. (72) to compute the positive-definite matrix, $\mathbf{P}$.
We then choose a selection strategy, $p = S(\mathbf{x})$, such that $\dot{v}(\mathbf{x})$ will be minimized, which will be the case if

$$p = -\mathrm{sgn}\left(\mathbf{x}^T\left(\mathbf{M}^T\mathbf{P} + \mathbf{P}\mathbf{M}\right)\mathbf{x}\right), \tag{73}$$

in which case, the Lyapunov function will decrease as rapidly as possible during regulation cycles. The motivation for this choice is the implication that a rapidly decreasing Lyapunov function normally leads to $\mathbf{x}$ rapidly reaching the equilibrium point, $\mathbf{x} = \mathbf{0}$.
Becker's VSC-system consists of Eq. (68), combined with the selection strategy of Eq. (73). He recommended selecting the pair of subcontrollers such that they will differ from one another as much as possible (Becker, 1979; Föllinger, 1998) in order to allow achieving more rapid regulation and shorter settling times than in case where a single linear controller only is employed.
As Becker noted, the design of the above control is not aimed at allowing sliding modes, although such may well occur. He thus sought criteria that would allow determining whether a given control system (68), (73) will

exhibit a sliding mode. Unfortunately, the condition for the occurrence of sliding modes that he obtained is a sufficient condition only, such that sliding-modes can never be reliably precluded.
Franke (1981) proved that the above control is robust against disturbances and parameter variations in the absence of sliding modes. As mentioned above, we described Becker's bilinear VSC in view of their historical relevance to the development of dynamic soft VSC, rather than their theoretical or practical significance.

4.2 Dynamic soft VSC: basics and stability

The problems that arise in the case of the bilinear VSC above are that undesirable sliding modes might occur and that their occurrence cannot be precluded. In order to remedy that situation, Franke (1982a, 1982b, 1983, 1986) made Becker's discontinuous selection variable, p, continuous by means of a differential equation

$$\dot{p}=f(p,\mathbf{x}). \tag{74}$$

Switching and sliding modes can thus no longer occur, and the bilinear VSC becomes a soft VSC. In addition to smoothly varying actuator control signals, another benefit of this approach is the infinite number of subcontrollers available for reducing settling times and accelerating regulation.
Franke also considered linear plants [2] ,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \tag{75}$$

duly allowing for the control constraints,

$$|u| \leq u_{0.}. \tag{76}$$

Once again, a bounded set $\mathbf{X}_0$ of possible initial values $\mathbf{x}(t = 0)$ is considered. Similarly to the case of the controller of Eq. (67), the variable-structure controller is

$$u = -(\mathbf{k}+p{\cdot}\mathbf{l})^T\mathbf{x}. \tag{77}$$

Combining the equations for the plant (75) and the controller (77), we obtain

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T - p\mathbf{b}\mathbf{l}^T)\mathbf{x} = (\mathbf{A}_0 - p\mathbf{b}\mathbf{l}^T)\mathbf{x}, \tag{78}$$

where the control vector, $\mathbf{k}$, is chosen such that $\mathbf{A}_0 = \mathbf{A} - \mathbf{b}\mathbf{k}^T$ will be stable. Using Eqs. (74) and (78), we obtain the differential equation,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} (\mathbf{A}_0 - p\mathbf{b}\mathbf{l}^T)\mathbf{x} \\ f(p,\mathbf{x}) \end{bmatrix}, \tag{79}$$

for the entire control system shown in Fig. 7.
The stability, compliance with control constraints, and performance of this control system are largely dependent upon the selection strategy employed, i.e., upon the function f. Most of the following considerations will thus be involved in constructing such a function, f.
We demand that the control system (79) has a single asymptotically stable equilibrium point,

$$\begin{bmatrix} \mathbf{x} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}. \tag{80}$$

In order to comply with this demand, we consider a positive-definite quadratic form,

$$v(p,\mathbf{x}) = \mathbf{x}^T\mathbf{R}\mathbf{x}+qp^2, \tag{81}$$

with the intention of choosing the function f such that $v(p,\mathbf{x})$ will be a Lyapunov function for the control system (79) that will guarantee asymptotic stability. Using Theorem 1, the latter will be the case if

$$\dot{v}(p,\mathbf{x}) = \dot{\mathbf{x}}^T\mathbf{R}\mathbf{x} + \mathbf{x}^T\mathbf{R}\dot{\mathbf{x}}+2qp\dot{p} < 0. \tag{82}$$

Using Eq. (79), we obtain

$$\begin{aligned}\dot{v}(p,\mathbf{x}) = {} & \mathbf{x}^T(\mathbf{A}_0^T\mathbf{R} + \mathbf{R}\mathbf{A}_0)\mathbf{x} - p\mathbf{x}^T(\mathbf{l}\mathbf{b}^T\mathbf{R} + \mathbf{R}\mathbf{b}\mathbf{l}^T)\mathbf{x} \\ & + 2pqf(p,\mathbf{x}) < 0,\end{aligned} \tag{83}$$

which yields

$$\begin{aligned}\dot{v}(p,\mathbf{x}) = {} & \mathbf{x}^T(\mathbf{A}_0^T\mathbf{R} + \mathbf{R}\mathbf{A}_0)\mathbf{x} \\ & + 2p\left[-\mathbf{x}^T\mathbf{R}\mathbf{b}\mathbf{l}^T\mathbf{x}+qf(p,\mathbf{x})\right] < 0.\end{aligned} \tag{84}$$

In order to simplify this expression, we define a function, r, such that

$$-\mathbf{x}^T\mathbf{R}\mathbf{b}\mathbf{l}^T\mathbf{x}+qf(p,\mathbf{x}) = -p\cdot r(p,\mathbf{x}). \tag{85}$$

Using this Eq. (85) in Eq. (84), we obtain

$$\dot{v}(p,\mathbf{x}) = \mathbf{x}^T(\mathbf{A}_0^T\mathbf{R} + \mathbf{R}\mathbf{A}_0)\mathbf{x} - 2p^2r(p,\mathbf{x}) < 0. \tag{86}$$

Eq.(86) will obviously be satisfied, and the control system (79) will be stable if

$$\mathbf{A}_0^T\mathbf{R} + \mathbf{R}\mathbf{A}_0 = -\mathbf{Q} \tag{87}$$

[2] In this paper, we describe a slightly simplified version of Franke's soft VSC in order to allow explaining its main features in the simplest-possible terms. The original version included methods for designing controllers for linear MIMO systems, systems with distributed parameters, and considerations related to command signals.

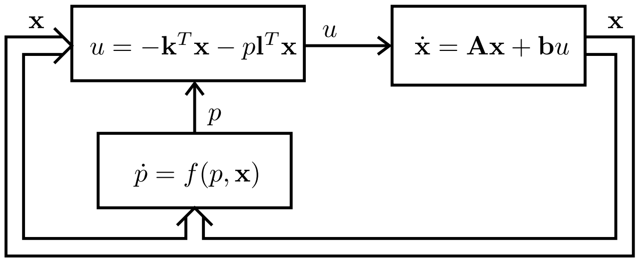


Fig. 7. Structure of the control system conforming to dynamic soft VSC.

is a negative-definite matrix and

$$r(p,\mathbf{x})>0. \tag{88}$$

Since $\mathbf{A}_0$ has exclusively eigenvalues with negative real parts, there always exists a positive-definite solution, $\mathbf{R}$, for an arbitrarily chosen positive-definite matrix, $\mathbf{Q}$. The first stability condition (87) will thus always be satisfiable. The second stability condition (88) is both easy to satisfy, since we can arbitrarily choose a positive function, r, in Eq. (86), and allows choosing r such that the control constraints (76) involved will be met, which we shall do in the next section.

4.3 Dynamic soft VSC: selection strategy

From Eq. (85), we obtain the selection strategy

$$\dot{p}=f(p,\mathbf{x})=\frac{\mathbf{x^T R b l^T x}-p\cdot r(p,\mathbf{x})}{q}, \tag{89}$$

where we intend to choose the function r such that, as mentioned above, compliance with the control constraints, $|u|\le u_0$, will be maintained, which is equivalent to maintaining compliance with the inequalities

$$-u_0\le -\mathbf{k}^T\mathbf{x}-p\cdot\mathbf{l}^T\mathbf{x}\le u_0 \tag{90}$$

obtained from Eqs. (76) and (77). We convert Eq. (90) into

$$\frac{-u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}}\le p\le\frac{u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}} \qquad \text{for } \mathbf{l}^T\mathbf{x}>0, \tag{91}$$

$$\frac{u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}}\le p\le\frac{-u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}} \qquad \text{for } \mathbf{l}^T\mathbf{x}<0 \tag{92}$$

and obtain constraints for the selection variable, p, instead of constraints imposed on actuator control signals, u.

The bounds defined by Eqs. (91) and (92) become very large when $\mathbf{x}$ approaches the equilibrium point $\mathbf{x}=\mathbf{0}$, in which case, the selection variable, p, is allowed to take on very large values that can cause implementation problems and strongly amplify noise and disturbances in the actuator control signal in Eq. (77). Since these are undesirable effects, we impose additional restrictions on p by demanding that

$$-P\le p\le P, \tag{93}$$

where P is a large positive number.

Combining the inequalities (91), (92) and (93), we obtain

$$\alpha(\mathbf{x})\le p\le\beta(\mathbf{x}), \tag{94}$$

where the bounds involved depend upon the state vector, $\mathbf{x}$, as follows:

$$\alpha(\mathbf{x})=\begin{cases}\dfrac{u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}} & ,\mathbf{l}^T\mathbf{x}\le\dfrac{-u_0+\mathbf{k}^T\mathbf{x}}{P}\\ -P & ,\dfrac{-u_0+\mathbf{k}^T\mathbf{x}}{P}<\mathbf{l}^T\mathbf{x}<\dfrac{u_0+\mathbf{k}^T\mathbf{x}}{P},\\ \dfrac{-u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}} & ,\mathbf{l}^T\mathbf{x}\ge\dfrac{u_0+\mathbf{k}^T\mathbf{x}}{P}\end{cases} \tag{95}$$

$$\beta(\mathbf{x})=\begin{cases}\dfrac{-u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}} & ,\mathbf{l}^T\mathbf{x}\le\dfrac{-u_0-\mathbf{k}^T\mathbf{x}}{P}\\ P & ,\dfrac{-u_0-\mathbf{k}^T\mathbf{x}}{P}<\mathbf{l^T}\mathbf{x}<\dfrac{u_0-\mathbf{k}^T\mathbf{x}}{P}.\\ \dfrac{u_0-\mathbf{k}^T\mathbf{x}}{\mathbf{l}^T\mathbf{x}} & ,\mathbf{l}^T\mathbf{x}\ge\dfrac{u_0-\mathbf{k}^T\mathbf{x}}{P}\end{cases} \tag{96}$$

Fig. 8 presents plots of the functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ for the case where they are functions of a single variable, x.

We next choose the function, $r(p,\mathbf{x})>0$, used in the selection strategy (89) such that inequality (94) will be satisfied. Recall that compliance with the control constraints (76) will also be maintained in this case.

A suitable choice for r is the function

$$r(p,\mathbf{x})=\begin{cases}\mu(1-\dfrac{\alpha(\mathbf{x})}{p})+\mu_0\dfrac{\alpha(\mathbf{x})}{p} & ,p\le\alpha(\mathbf{x})\\ \mu_0 & ,\alpha(\mathbf{x})<p<\beta(\mathbf{x})\\ \mu(1-\dfrac{\beta(\mathbf{x})}{p})+\mu_0\dfrac{\beta(\mathbf{x})}{p} & ,p\ge\beta(\mathbf{x})\end{cases} \tag{97}$$

such that Eq. (89) represents an anti-windup system, as shown in Fig. 9. The constant $\mu\gg 1$ determines the

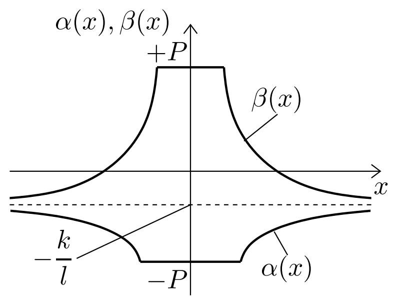


Fig. 8. The functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ for the case where they are functions of a single variable, x.

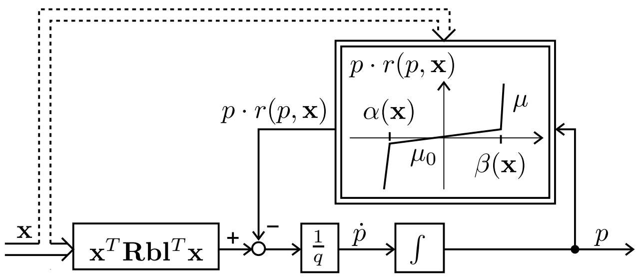


Fig. 9. Block schematic of the differential equation, $\dot{p} = f(p, \mathbf{x})$, representing an anti-windup system.

slope within the anti-windup region that prevents the value of the integrator's output parameter from falling below $\alpha(\mathbf{x})$ or exceeding $\beta(\mathbf{x})$. The constant $0 < \mu_0 \ll 1$ determines the slope in the region where the anti-windup is inactive. Note that with this choice both the constraints (90), (94) and the stability condition (88) will be satisfied.

Remaining to be resolved is the matter of why the dynamic soft VSC given by Eqs. (75), (77), and (89) has a higher regulation rate and a shorter settling time than a single linear controller. To illustrate these features, assume that a disturbance, $\mathbf{x}_0 = \mathbf{x}(t = 0)$, is to be attenuated to $\mathbf{x} = \mathbf{0}$ and that the selection variable is $p(t = 0) \approx 0$ at the beginning of the regulation process. In this particular case, we have $p \cdot r(p(t = 0), \mathbf{x}_0) \approx 0$ and, consequentially, from Eq. (89), we have that

$$\dot{p} \approx \frac{\mathbf{x^T R b l^T x}}{q}. \tag{98}$$

$|p|$ will then increase during the regulation process and, consequentially, the derivative, $\dot{v}$, appearing in Eq. (86) will decrease to the point where $\dot{v} \ll 0$. The Lyapunov function, v, will thus rapidly decrease and tend toward zero, yielding a very rapid regulation process.

The linear subsystems, $\mathbf{A} - \mathbf{b}\mathbf{k}^T - p\mathbf{b}\mathbf{l}^T$, might become unstable during the regulation process, which is not a restriction, since stability will still be guaranteed.

At the conclusion of the regulation cycle, the state vector, $\mathbf{x}$, will continually decrease to smaller values. The term $\mathbf{x^T R b l^T x}$ appearing in Eq. (89) will thus approximately equal zero, in which case, we obtain $\dot{p} \approx -p \cdot r(p, \mathbf{x})/q$ for Eq. (89). Since $p \cdot r(p, \mathbf{x})/q$ is positive for $p > 0$ and negative for $p < 0$, the parameter p tends toward zero and the active linear controller will largely consist of the $\mathbf{k}$ appearing in Eq. (78).

4.4 Computing controller parameters

Summarizing the results of the foregoing sections, a dynamic soft VSC consists of the plant (75) controlled by the soft variable-structure controller (77), i.e.,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}^T - p\mathbf{b}\mathbf{l}^T)\mathbf{x}, \tag{99}$$

where p is determined by the selection strategy (89),

$$\dot{p} = \frac{\mathbf{x^T R b l^T x} - p \cdot r(p, \mathbf{x})}{q}, \tag{100}$$

where the function r is given by Eq. (97). In order to design this control, we need to determine the linear control vectors $\mathbf{k}$ and $\mathbf{l}$, the matrix $\mathbf{R}$, the parameter q, and the parameters P, μ, μ_0 of the function r. Unfortunately, Franke did not describe a method for choosing these parameters. We propose the following five steps:

Step 1: Choose a control vector, $\mathbf{k}$, such that $\mathbf{A}_0 = \mathbf{A} - \mathbf{b}\mathbf{k}^T$ will be stable and will exhibit good control performance, i.e., there will be only slight overshooting and settling times will be short. Furthermore, $\mathbf{k}$ will have to be chosen such that any initial state vector, $\mathbf{x} \in \mathbf{X}_0$, will satisfy the control constraints, $\left|\mathbf{k}^T\mathbf{x}\right| \leq u_0$. Stable control can then take place for all $\mathbf{x} \in \mathbf{X}_0$.

Step 2: Choose an control vector, $\mathbf{l}$: Plotting a root locus depending on p for the system might be a helpful heuristic approach to choosing a control vector, $\mathbf{l}$, since it would yield information on the eigenvalues of the linear subsystems (99). However, no relationships between $\mathbf{l}$ and control performance have become known to date.

Step 3: The positive-definite matrix $\mathbf{R}$ may be computed using Eq. (87). We thus need to choose an arbitrary positive-definite matrix $\mathbf{Q}$, e.g., the identity matrix, and find a solution to the Lyapunov equation (87).

Step 4: The parameters, μ, μ_0, and P, of the function $r(p, \mathbf{x})$ need to be chosen. A small value of μ_0 satisfying $0 < \mu_0 \ll 1$ should be chosen. The slope, μ, of the anti-windup should be large, since it will then guarantee the efficiency of the anti-windup and compliance with the actuator constraints, $\pm u_0$. The bounds, $\pm P$, of the selection variable should also be chosen large.

Step 5: The parameter q allows affecting the rate of variation of p in Eq. (100). A suitable value for q may be determined by simulating the dynamic soft VSC system for different values, q, and choosing that value that yields the best-possible results.

In some cases, performance may be optimized by repeating the five steps above.

4.5 Continuation of the example of the dive control

According to the procedure described above, the following steps are carried out for designing a dive control for the submarine of Section 3.6:

Step 1: We choose the eigenvalues $\lambda_{1,2,3} = -3.0 \cdot 10^{-3}$ for $\mathbf{A}_0$ and compute

$$\mathbf{k}^T = [2.70 \cdot 10^{-8} \quad 2.70 \cdot 10^{-5} \quad 4.00 \cdot 10^{-3}]. \tag{101}$$

If this control vector is employed, the constraints, $|u| \leq u_0$, will be satisfied for all initial states, $\mathbf{x} \in \mathbf{X}_0$.

Step 2: The chosen vector

$$\mathbf{l}^T = [7.9918 \cdot 10^{-7} \quad 2.5836 \cdot 10^{-4} \quad 2.7840 \cdot 10^{-2}] \tag{102}$$

yields good control behavior of the closed-loop system.
Step 3: We choose $\mathbf{Q} = \mathrm{diag}(1, 1, 1)$ and obtain the matrix sought

$$\mathbf{R} = \begin{bmatrix} 6.8750 \cdot 10^{6} & 1.8750 \cdot 10^{9} & 1.8519 \cdot 10^{11} \\ 1.8750 \cdot 10^{9} & 6.4815 \cdot 10^{11} & 6.9445 \cdot 10^{13} \\ 1.8519 \cdot 10^{11} & 6.9445 \cdot 10^{13} & 7.7162 \cdot 10^{15} \end{bmatrix} \quad (103)$$

from Eq. (87).
Step 4: The parameter $\mu_0 = 1 \cdot 10^{-2}$ was chosen very small and $\mu = 1 \cdot 10^{6}$ very large. The value of parameter, P, must be a large number: $P = 1 \cdot 10^{2}$.
Step 5: This step consists of scaling using q in order to adjust the rate of variation of the variable p. After several simulation runs, we found that $q = 1 \cdot 10^{4}$ was a reasonable choice. The determination of the parameters of the dynamic soft VSC has thus been completed.
For comparing the dynamic soft VSC with the other controls, see again Fig. 13(a), 13(b), and 14 and the detailed comparison in Section 6.

5 Soft VSC with variable saturation

The superior control performance of the VSC described above is largely attributable to the fact that the control parameter, u, available within the constraints $|u| \leq u_0$ may be efficiently utilized, since even for small excursions, $\mathbf{x}$, $|u|$ will remain close to u_0, which is not the case for linear controllers. The VSC described by Albers (1983) is based on this same principle.

5.1 Soft VSC with variable saturation: fundamentals of the control concept

Here, once again, linear plants

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \quad (104)$$

subject to a restriction, $|u| \leq u_0$, on their control parameter, u, and a bounded set $\mathbf{X}_0$ of possible initial vectors $\mathbf{x}(t = 0)$ are considered. The starting point is a pair of state controllers, $\mathbf{k}_1$ and $\mathbf{k}_2$, along with a saturation term,

$$u = u_1 + u_2, \quad (105)$$

$$u_1 = -\mathbf{k}_1^T \mathbf{x}, \quad (106)$$

$$u_2 = -\mathrm{sat}(u_s(\mathbf{x}), \tilde{u}) \quad , \tilde{u} = \mathbf{k}_2^T \mathbf{x}, \quad (107)$$

combined with variable limits, $\pm u_s(\mathbf{x})$, imposed on the saturation term

$$\mathrm{sat}\;(u_s(\mathbf{x}), \tilde{u}) = \begin{cases} u_s(\mathbf{x}) & , \tilde{u} \geq u_s(\mathbf{x}) \\ \tilde{u} & , |\tilde{u}| < u_s(\mathbf{x}) \\ -u_s(\mathbf{x}) & , \tilde{u} \leq -u_s(\mathbf{x}) \end{cases} . \quad (108)$$

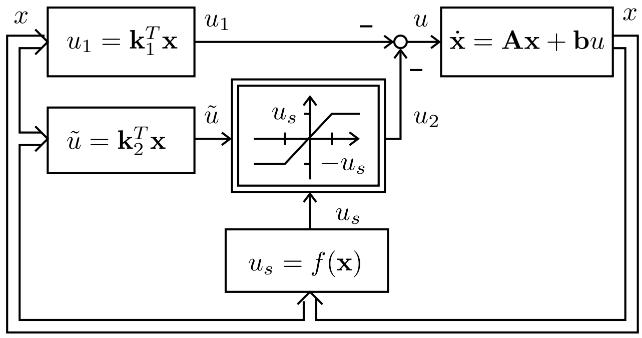


Fig. 10. Structure of the VSC with variable saturation.

Fig. 10 depicts the structure of this control, which operates in the following manner: The variable limit, $u_s(\mathbf{x})$, will be chosen such that $u_s(\mathbf{x}) = 0$ for large vectors, $\mathbf{x}$. Regulation for large vectors $\mathbf{x}$ will thus be linear, and employs u_1. In this particular case, the available range of the actuator control signal, u, falling within the constraints, $|u| \leq u_0$, will be efficiently utilized by taking $u = -\mathbf{k}_1^T \mathbf{x}$.
In the case of moderate vectors, $\mathbf{x}$, the actuator control signal, $u_1 = -\mathbf{k}_1^T \mathbf{x}$, will no longer efficiently utilize the available range, $[-u_0, u_0]$. The regulation rate will thus not be as high as it might be. In order to improve this situation, the nonlinear component, u_2, is blended into the actuator control signal, u, appearing in Eq. (105). The variable-saturation limit, $u_s(\mathbf{x})$, will shift from $u_s(\mathbf{x}) = 0$ for large $\mathbf{x}$ to a positive value $u_s(\mathbf{x}) > 0$ for moderate $\mathbf{x}$ in order to allow that. Regulation will then take place faster than when u_1 alone is employed, where the nonlinear component, u_2, should increase as excursions of the state vector, $\mathbf{x}$, decrease.
The control will become linear once again for very small excursions of the state vectors, $\mathbf{x}$, since $\mathbf{x}$ will be so small that $|\tilde{u}| = \left|\mathbf{k}_2^T \mathbf{x}\right| < u_s(\mathbf{x})$. Regulation will then be accomplished employing $u = -\left(\mathbf{k}_1^T + \mathbf{k}_2^T\right) \mathbf{x}$, where the control vector, $\mathbf{k}_1 + \mathbf{k}_2$, is chosen such that regulation will be stable and more rapid than when $\mathbf{k}_1$ alone is employed.
We can reformulate the nonlinear control law (105), (106), and (107) such that its soft VSC-character will become more apparent and the structure of the control loop will correspond to that shown in Fig. 2. We thus rewrite Eq. (105) to yield

$$u = -\mathbf{k}_1^T \mathbf{x} - p \mathbf{k}_2^T \mathbf{x}, \quad (109)$$

where

$$p = \frac{u_s(\mathbf{x})}{\mathbf{k}_2^T \mathbf{x}} \mathrm{sat}\left(1, \frac{\mathbf{k}_2^T \mathbf{x}}{u_s(\mathbf{x})}\right), \quad (110)$$

and

$$\text{sat}\left(1, \frac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})}\right) = \begin{cases} 1 & , \dfrac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})} \geq 1 \\ \dfrac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})} & , \left|\dfrac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})}\right| < 1 \\ -1 & , \dfrac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})} \leq -1 \end{cases} . \qquad (111)$$

The parameter, p, is the selection variable determining the linear subcontrollers appearing in Eq. (109). Fig. 11 depicts the structure of the reformulated system.

Note that the parameter p can only vary over the range $0 \leq p \leq 1$, since Eq. (110) implies that $p = 1$ for $\left|\mathbf{k}_2^T\mathbf{x}/u_s(\mathbf{x})\right| < 1$ and $0 \leq p \leq 1$ for $\left|\mathbf{k}_2^T\mathbf{x}/u_s(\mathbf{x})\right| \geq 1$. Fig. 12 illustrates how the parameter p and control component u_2 are interrelated for a fixed $u_s(\mathbf{x})$.

In the case of the control loop, we have from $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u$ and Eq. (109) that

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k}_1^T - p\mathbf{b}\mathbf{k}_2^T)\mathbf{x} = \hat{\mathbf{A}}(p)\mathbf{x}. \qquad (112)$$

The next major task is choosing, or devising, the selection strategy appearing in Eq. (110), i.e., choosing the variable limit $u_s(\mathbf{x})$, such that the soft VSC (110), (112) will satisfy the following two conditions:

(E1) The restrictions $|u| \leq u_0$ must apply.

(E2) We presume that (E1) holds. The stability of the equilibrium state, $\mathbf{x} = \mathbf{0}$ of the control system (110), (112), must then be guaranteed, where merely an asymptotic stability of all possible trajectories that start within a region

$$G = \left\{\mathbf{x} \mid \mathbf{x}^T\mathbf{R}\mathbf{x} \leq v_G\right\}, \qquad (113)$$

rather than globally asymptotic stability, is required. The matrix $\mathbf{R}$ is positive definite.

Consequently, $\mathbf{X}_0$ should be included in G. How the two Conditions (E1) and (E2) may be satisfied will be taken up in the next section.

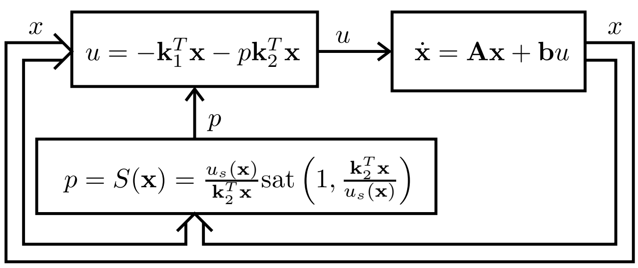


Fig. 11. Structure of the reformulated VSC.

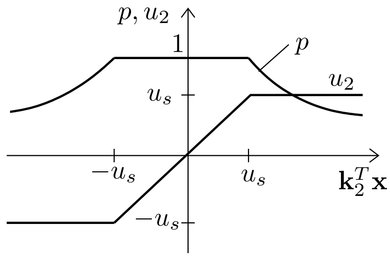


Fig. 12. Plots of the variable parameters p and u_2.

5.2 *Soft VSC with variable saturation: stability and selection strategy*

One starts off by assuming that Condition (E1) will be satisfied and then derives stability conditions for the control loop (112). The equilibrium state, $\mathbf{x} = \mathbf{0}$, of this control loop will be asymptotically stable if

$$v(\mathbf{x}) = \mathbf{x}^T\mathbf{R}\mathbf{x}, \qquad (114)$$

is a Lyapunov function within G, i.e., if $\dot{v}(\mathbf{x}) < 0$ within G, for a suitable positive-definite matrix, $\mathbf{R}$. G will then be a Lyapunov region of the control system. The condition $\dot{v}(\mathbf{x}) < 0$ yields the Lyapunov equation

$$\hat{\mathbf{A}}^T(p)\mathbf{R} + \mathbf{R}\hat{\mathbf{A}}(p) = -\mathbf{Q}(p), \qquad (115)$$

which will always lead to positive-definite matrices, $\mathbf{Q}(p)$, over a particular interval, $[p_{\min}, 1]$, where $p_{min} \geq 0$, since the control system (112) has been designed above such that it will be stable for $p = 1$. Stable control may then take place for all $p \in [p_{\min}, 1]$, and we will thus obtain the broadest-possible stability range if $p_{min} = 0$.

Since the matrix $\hat{\mathbf{A}}(p)$ linearly depends upon p, all we need to do is verifying that the two matrices, $\mathbf{Q}(p_{\min})$ and $\mathbf{Q}(1)$, are positive definite in order to guarantee that $\mathbf{Q}(p)$ will be positive definite for all $p \in [p_{\min}, 1]$ (Garofalo, Celentano & Glielmo, 1993).

The sufficient condition for stable control,

$$p_{\min} \leq p \leq 1, \qquad (116)$$

limits the acceptable choices of the variable-saturation function, $u_s(\mathbf{x})$, employed in the selection strategy (110), as we shall see below. We shall consider the case where $\left|\mathbf{k}_2^T\mathbf{x}/u_s(\mathbf{x})\right| \geq 1$ in Eq. (111), i.e., the saturated case in Eq. (111). We then have that

$$\text{sat}\left(1, \frac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})}\right) = \pm 1 \quad , \text{ for } \left|\frac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})}\right| \geq 1. \qquad (117)$$

We thus obtain for Eq. (110)

$$p = \frac{u_s(\mathbf{x})}{\left|\mathbf{k}_2^T\mathbf{x}\right|} \quad , \text{ for } \left|\frac{\mathbf{k}_2^T\mathbf{x}}{u_s(\mathbf{x})}\right| \geq 1. \qquad (118)$$

Employing (118) in the condition of (116) and allowing for the fact that $p \leq 1$ will be satisfied for any u_s, we obtain the condition

$$u_s(\mathbf{x}) \geq p_{\min} \left|\mathbf{k}_2^T \mathbf{x}\right|. \tag{119}$$

Eq. (119) provides an initial criterion for choosing $u_s(\mathbf{x})$. In order to make this condition more specific and eliminate the dependence of the right-hand side of Eq. (119) upon $\mathbf{x}$, we need to estimate the maximum value that may be attained by the $\left|\mathbf{k}_2^T \mathbf{x}\right|$ appearing in Eq. (119). Since exclusively values of $\mathbf{x} \in G$ are of interest and G is a Lyapunov region, the maximum value of $|\mathbf{k}_2^T \mathbf{x}|$ lies on the boundary of G. We may thus determine this maximum of $|\mathbf{k}_2^T \mathbf{x}|$ by solving the optimization problem

$$\left|\mathbf{k}_2^T \mathbf{x}\right| \to \max \quad , \text{with} \quad \mathbf{x}^T \mathbf{R} \mathbf{x} = v_G. \tag{120}$$

Solution of the problem (120) yields

$$\max_{\mathbf{x}^T \mathbf{R} \mathbf{x} = v_G} \left|\mathbf{k}_2^T \mathbf{x}\right| = \sqrt{v_G \mathbf{k}_2^T \mathbf{R}^{-1} \mathbf{k}_2}. \tag{121}$$

Eq. (119) will thus be satisfied for $\mathbf{x} \in G$ if, along with Eq. (121),

$$u_s(\mathbf{x}) \geq p_{\min} \sqrt{v_G \mathbf{k}_2^T \mathbf{R}^{-1} \mathbf{k}_2}, \tag{122}$$

which is the stability condition that must be taken into account in choosing $u_s(\mathbf{x})$, holds.
In addition to satisfying this condition (122), $u_s(\mathbf{x})$ must also be chosen such that Condition (E1), which has thus far been presumed to be satisfied, will also be satisfied, i.e., that

$$|u| = |u_1 + u_2| \leq u_0 \tag{123}$$

will also be satisfied. Eq. (123) will be satisfied if

$$|u_1| + |u_2| \leq u_0. \tag{124}$$

The magnitude of the term, $|u_2|$, appearing in Eq. (124) may be estimated as follows:

$$|u_2| \leq u_s(\mathbf{x}). \tag{125}$$

The maximum value of $|u_1|$ that might occur over a trajectory that starts within G may also be estimated employing the Lyapunov function (114). In order to estimate it, we consider a trajectory $\tilde{\mathbf{x}}(t)$ of the control loop that starts at the boundary of the Lyapunov region

$$E(\mathbf{x}) = \{\tilde{\mathbf{x}} \mid \tilde{\mathbf{x}}^T \mathbf{R} \tilde{\mathbf{x}} \leq v(\mathbf{x})\}, \tag{126}$$

within which $\tilde{\mathbf{x}}$ will never leave that ellipsoid, and $|u_1| = \left|\mathbf{k}_1^T \tilde{\mathbf{x}}\right|$ will thus assume its maximal value at the boundary of the region $E(\mathbf{x})$. Proceeding similarly to the case of Eq. (120), this value may be determined from the optimization problem

$$\left|\mathbf{k}_1^T \tilde{\mathbf{x}}\right| \to \max \quad , \text{with} \quad \tilde{\mathbf{x}}^T \mathbf{R} \tilde{\mathbf{x}} = v(\mathbf{x}). \tag{127}$$

The solution is

$$\max_{\tilde{\mathbf{x}}^T \mathbf{R} \tilde{\mathbf{x}} = v(\mathbf{x})} \left|\mathbf{k}_1^T \tilde{\mathbf{x}}\right| = \sqrt{v(\mathbf{x}) \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1}. \tag{128}$$

It thus follows that

$$|u_1| = \left|\mathbf{k}_1^T \tilde{\mathbf{x}}\right| \leq \sqrt{v(\mathbf{x}) \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1} \quad \text{for } \tilde{\mathbf{x}} \in \mathbf{E}(\mathbf{x}). \tag{129}$$

Inserting Eqs. (125) and (129) into Eq. (124), we obtain

$$\sqrt{v(\mathbf{x}) \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1} + u_s(\mathbf{x}) \leq u_0. \tag{130}$$

Eq. (130) will surely be satisfied if one chooses a $u_s(\mathbf{x})$ given by

$$u_s(\mathbf{x}) = u_0 - \sqrt{v(\mathbf{x}) \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1}, \tag{131}$$

which is the saturation function, $u_s(\mathbf{x})$, sought. The entire soft VSC will then be given by Eqs. (110), (112), and (131).
The stability condition (122) may now be more accurately formulated employing Eq. (131):

$$u_0 \geq p_{\min} \sqrt{v_G \mathbf{k}_2^T \mathbf{R}^{-1} \mathbf{k}_2} + \sqrt{v(\mathbf{x}) \mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1}, \tag{132}$$

which may be further simplified such that it is independent of $\mathbf{x}$. The maximum value of $v(\mathbf{x})$ for $\mathbf{x} \in G$ is v_G. Eq. (132) will thus surely be satisfied if

$$u_0 \geq \sqrt{v_G} \cdot \left(p_{\min} \sqrt{\mathbf{k}_2^T \mathbf{R}^{-1} \mathbf{k}_2} + \sqrt{\mathbf{k}_1^T \mathbf{R}^{-1} \mathbf{k}_1}\right). \tag{133}$$

In the following section, we shall summarize the results and describe the procedure for computing a soft VSC with variable saturation.

5.3 Computing controller parameters

In view of the results obtained in the preceding sections above, soft VSC exploiting variable saturation consist of the plant (104) and the controller (109), i.e.,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{k}_1^T \mathbf{b} - p \mathbf{k}_2^T \mathbf{b}) \mathbf{x}, \tag{134}$$

where it follows from equations (110) and (131) that the variable parameter p will be determined by

$$p = \frac{u_s(\mathbf{x})}{\mathbf{k}_2^T \mathbf{x}} \text{sat}\left(1, \frac{\mathbf{k}_2^T \mathbf{x}}{u_s(\mathbf{x})}\right) \tag{135}$$

and

$$u_s(\mathbf{x}) = u_0 - \sqrt{v(\mathbf{x})\mathbf{k}_1^T\mathbf{R}^{-1}\mathbf{k}_1} \quad , v(\mathbf{x}) = \mathbf{x}^T\mathbf{R}\mathbf{x}. \tag{136}$$

Designing a soft VSC with variable saturation thus involves choosing the two control vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ and the matrix $\mathbf{R}$. Simple means for designing the controller will be described in the four steps appearing below.
Step 1: We need to pick a linear controller, $\mathbf{k}_1$, that yields already good control performance and is such that $\mathbf{A} - \mathbf{k}_1^T\mathbf{b}$ will become a stable system.
Step 2: Assuming the best case, $p_{min} = 0$, and allowing for the considerations applying to Eq. (115), we solve the Lyapunov equation

$$\hat{\mathbf{A}}^T(0)\mathbf{R} + \mathbf{R}\hat{\mathbf{A}}(0) = -\mathbf{Q}(0), \tag{137}$$

where $\mathbf{Q}(0)$ is an arbitrarily chosen positive-definite matrix, and obtain $\mathbf{R}$.
Step 3: We need to determine the maximum value of v_G occurring in the stability condition (133), which for $p_{min} = 0$ may be simplified to

$$u_0 \geq \sqrt{v_G} \cdot \sqrt{\mathbf{k}_1^T\mathbf{R}^{-1}\mathbf{k}_1}, \tag{138}$$

such that the Lyapunov region, $G = \{\mathbf{x} \mid \mathbf{x}^T\mathbf{R}\mathbf{x} \leq v_G\}$, will be large enough that $\mathbf{X}_0 \subseteq G$. Stable control may then take place for any $\mathbf{x} \in \mathbf{X}_0$. If this condition cannot be met, the design procedure must be restarted, starting with Step 1, but, this time, employing a smaller control vector $\mathbf{k}_1$.
Step 4: We need to find the control vector, $\mathbf{k}_2$ in $\hat{\mathbf{A}}(1) = \mathbf{A} - \mathbf{b}\mathbf{k}_1^T - \mathbf{b}\mathbf{k}_2^T$, such that the Lyapunov equation

$$\hat{\mathbf{A}}^T(1)\mathbf{R} + \mathbf{R}\hat{\mathbf{A}}(1) = -\mathbf{Q}(1), \tag{139}$$

will yield a positive-definite matrix, $\mathbf{Q}(1)$. A condition is that the control exploiting $\mathbf{k}_1 + \mathbf{k}_2$ should operate much faster than if $\mathbf{k}_1$ alone is employed.
In the worst case, i.e., if no solutions satisfying the stability conditions, (137), (138) and (139), can be found, varying the value of p_{min} in Eq. (133) and restarting the design procedure should lead to a feasible parameter combination.

5.4 The conclusion of the example of the submarine

Reconsidering the submarine, design of its dive control will proceed as described in the preceding section.
Step 1: We choose the same eigenvalues as in Eq. (61) and the same control vector $\mathbf{k}_1 = \mathbf{k}(1)$ with

$$\mathbf{k}_1^T = [1.1165 \cdot 10^{-7} \quad 4.8413 \cdot 10^{-5} \quad 3.6100 \cdot 10^{-3}] \tag{140}$$

employed for the implicit soft VSC in Section 3.6.
Step 2: Since we are using the same controller, we also obtain the same matrix $\mathbf{R}$ of Eq. (63) as in Section 3.6.
Step 3: This step shows that $\mathbf{X}_0 \subseteq G$. Condition (138) will be satisfied for the $\mathbf{R}$ appearing above.
Step 4: We choose the control vector

$$\mathbf{k}_2^T = [4.4605 \cdot 10^{-7} \quad 1.4542 \cdot 10^{-4} \quad 1.5000 \cdot 10^{-2}] \tag{141}$$

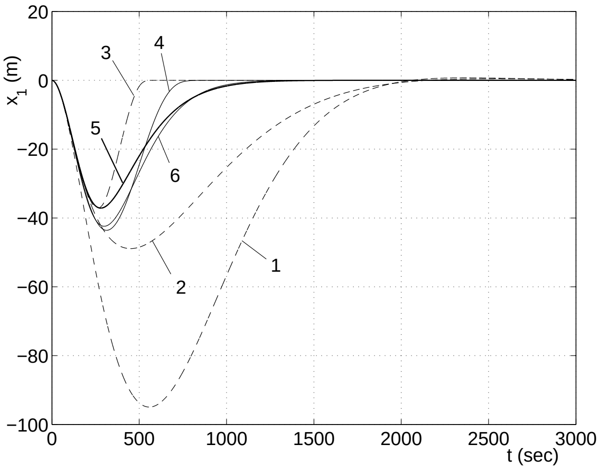
which satisfies the demands of Step 4.
The subsequent section compares this control with the others.

6 Comparing the controls

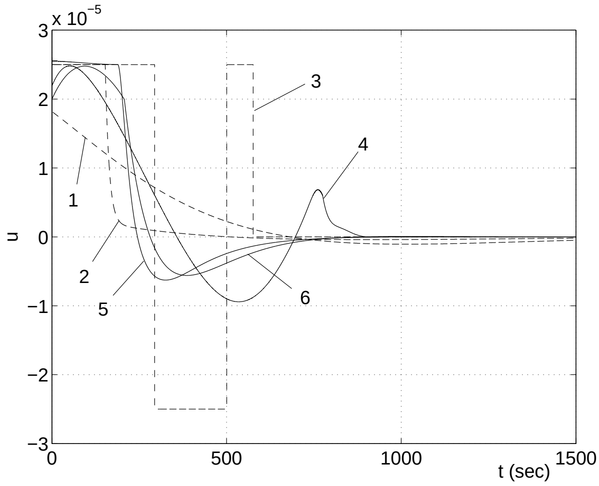
Fig. 13(a) presents plots of dive depth, x_1, of the submarine for the case of regulation from the initial state, $\mathbf{x}^T = [0 \quad 0 \quad -0.004]$, which is a downward acceleration that was also considered in (Gutman & Hagander, 1985), and continuing until the equilibrium state, $\mathbf{x}^T = \mathbf{0}$, is reached for the cases where the different soft VSC, a linear, a time-optimal (Athans & Falb, 1966), and a saturated linear control (Gutman & Hagander, 1985), respectively, are employed. Comparing their respective control performance in terms of settling time and regulation rate, we see that the VSC yields control performance similar to that of time-optimal controls.
The saturated linear control's behavior is better than that of the linear control, but both yield performance that is vastly inferior to those of the time-optimal control and the VSC in respect to settling time, regulation rate, and control over dive depth.
Fig. 13(b) depicts the actuator control signals, u, of the respective controls. It may be seen that the soft VSC exploit the feasible range of values falling within the interval $\pm u_0$ much better than the linear control or saturated linear control, although not as well as the time-optimal control. However, the latter lacks a smoothly varying actuator control signal.

For small initial vectors, the differences between the settling times and regulation rates of soft VSC and those of linear controls become larger due to the former's much better exploitation of the feasible range, $\pm u_0$, of actuator control signals. The dive depth, x_1, for an initial state, $\mathbf{x}^T = [0 \quad 0 \quad -0.0008]$, which is one-fifth of the one previously considered, is shown in Fig. 14.

Summarized, soft VSC are capable of achieving extremely short settling times, T_S, as the typical example above we have chosen from a set of other examples yielding similar results demonstrates. In many cases, their regulation curves, $\mathbf{x}(t)$, are close to those of time-optimal controls. In particular, for soft VSC with implicit Lyapunov functions this is the normal case. However, this cannot be guaranteed, and, unfortunately, no estimation of the settling times of all soft VSC is known yet. Saturated linear controllers also achieve short settling times. This advantage over linear controls becomes more obvious as disturbances or initial vectors, $\mathbf{x}(0)$, become smaller.
A further benefit of soft VSC is the smoothness of



(a) Dive depth, x_1, as a function of time, t.



(b) Actuator-control parameter, u, as a function of time, t.

Fig. 13. Results of regulation from an initial state, $\mathbf{x}^T = [0 \quad 0 \quad -0.004]$, to the equilibrium state, $\mathbf{x}^T = \mathbf{0}$, employing linear control (1), saturated linear control (2), time-optimal control (3), soft VSC employing implicit Lyapunov functions (4), dynamic soft VSC (5), soft VSC with variable saturation (6).

their actuator control signals, u, where "smoothness" includes lack of discontinuities and chattering, and low signal variation rates. The bang-bang curves, u, of time-optimal controls have some discontinuities that actuators are either incapable of handling, or have hard times handling, in practical cases.

Compared to linear state controllers, the effort involved in designing soft VSC is greater than when the pole-placement method and only slightly greater than when LQC-methods are employed for designing the linear controls. However, much less design effort is required than in case of time-optimal feedback controls.

The effort involved in implementing soft VSC is also much less than in case of time-optimal feedback controls. In the case of dynamic soft VSC and soft VSC with variable saturation, some simple nonlinearities will have to be implemented. In the case of soft VSC with implicit Lyapunov functions, implementation is slightly more complex, since we need to solve an implicit equation online. However, since this equation can always be solved and the solver can be implemented on a digital unit that is currently normally used for control tasks, this implicit equation is not a drawback. Of course, the linear controller and the saturated linear controller are simplest to implement.

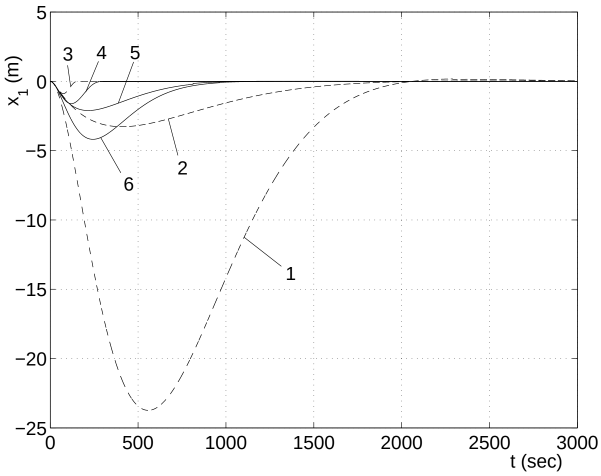


Fig. 14. Dive depth, x_1, as a function of time, t. Regulation from the initial state $\mathbf{x}^T = [0 \quad 0 \quad -0.0008]$ to the equilibrium state $\mathbf{x}^T = \mathbf{0}$ using linear control (1), saturated linear control (2), time-optimal control (3), soft VSC employing implicit Lyapunov functions (4), dynamic soft VSC (5), soft VSC with variable saturation (6).

7 Open problems and challenges

Soft VSC is one of the newer branches of VSC, and its evolution has not yet come to an end. We list below the questions remaining open and the enhancements that would be desirable:

(1) It might prove useful if the methods for computing the parameters of the soft VSC described here were to be improved and their design could be simplified.

(2) In the case of those soft VSC described here, linear state-space controllers and three different types of selection strategies, an implicit function, a differential equation, and an explicit function, have been employed. Development of new soft VSC incorporating other controllers and selection strategies is another challenge.

(3) Soft VSC have thus far been developed for linear plants only, although there is an auspicious potential for extending them to the control of nonlinear plants. This

represents yet another challenge to further research.
(4) Short settling times and high regulation rates are the main objectives of soft VSC. The robustness of soft VSC has also been partially examined, but has not yet become part of design procedures.
(5) Soft VSC with linear subcontrollers may be interpreted as bilinear systems that are linear in the state space variables, x_i, and linear in the selection variable, p, but "bilinear" in the sense that both are considered simultaneously. It will be of interest for the theory of soft VSC to examine them in this regard.

8 Summary and conclusions

This review covers soft variable-structure controls and their design. Soft variable-structure controls represent a systematic advance on the variable-structure controls lacking sliding modes that were formerly developed in order to achieve superior regulation rates and preclude sliding modes. Consequentially, the regulation rates and settling times of soft variable-structure controls are even better than those of their discontinuous precursors and can be nearly equal those of time-optimal controls. However, soft variable-structure controls are much easier to compute and implement and have smoothly varying actuator control signals. Soft variable-structure controls have been developed solely for linear plants, although the concepts involved may also be applied to nonlinear plants. This might well be the major challenge to future work on advancing soft variable-structure controls.

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