MASSEY PRODUCT AND ITS APPLICATIONS

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Abstract. This is the note for my talk in Graduate Student Seminar NEU. W. Massey defined Massey product as a higher order cohomology operation, which is a generalization of cup product. A first application of Massey product is in Knot theory showing that Borromean rings are linked with zero linking numbers. As another application, people can use Massey product describe the differentials in spectral sequences. There are many other important applications, such as obstruction to the formality of a space, which may be omitted in this talk.

1. Massey Products

1.1. **Cup Products review.** Given a topological space X and a commutative ring R, we can define the cohomology groups $H^*(X;R)$. There is also a product structure on $H^*(X;R)$, called cup product

$$\cup : H^p(X;R) \wedge H^q(X;R) \to H^{p+q}(X;R).$$

Example 1.1. Let $T = S^1 \times S^1$ be the torus. Then $H^0(T;R) = R$, $H^1(T;R) = R^2$ generated by $\{a, b\}$, $H^2(T;R) = R$ generated by $\{\gamma\}$. Then, we have $a \cup b = \gamma = -b \cup a$ and the other cup products are zeros. In fact, the cohomology ring $H^*(T;R)$ of torus T is the exterior algebra $\bigwedge_R[a, b]$. In general, $H^*(T^n;R) = \bigwedge_R[a_1, \cdots, a_n]$.

Cup product is not easy to compute. However, many methods can be used to compute cup product, definition combined simplicial homology, method in Hatcher's book, intersection method combined Poincare duality, product formula, even (Leray-Serre) spectral sequence etc..

Remark 1.2. Cup products are great because rings have more properties, and cup products can be used to distinguish between spaces that might have the same cohomology groups. In above example, the torus $T = S^1 \times S^1$ can be distinguished from the the wedge sum of two circles and one spheres $S^1 \vee S^1 \vee S^2$, even though they have the same homology groups, since all cup products in $S^1 \vee S^1 \vee S^2$ are zeros.

1.2. **Triple Massey Products.** If $x \in C^i(X;R)$, the symbol $\overline{x}$ will denote $(-1)^{1+i}x$. We first define the Massey triple product. Let x_1, x_2 and x_3 be cocycle of degrees r_1, r_2 and r_3 with cohomology classes $[x_1] \cup [x_2] = 0$ and $[x_2] \cup [x_3] = 0$. Thus, there are cochains x_{12} of degree $r_1 + r_2 - 1$ and x_{23} of degree $r_2 + r_3 - 1$ such that $dx_{12} = \bar{x}_1 \cup x_2$ and $dx_{23} = \bar{x}_2 \cup x_3$. Define the cochain ω of degree $(r_1 + r_2 + r_3 - 1)$ by

$$\omega = \overline{x}_{12} \cup x_3 + \overline{x}_1 \cup x_{23},$$

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Then ω satisfies

$$\begin{aligned} d(\omega) &= (-1)^{r_1+r_2} dx_{12} \wedge x_3 + (-1)^{r_1} \bar{x}_1 \wedge dx_{23} \\ &= (-1)^{r_1+r_2} \bar{x}_1 \wedge x_2 \wedge x_3 + (-1)^{r_1+r_2+1} \bar{x}_1 \wedge x_2 \wedge x_3 \\ &= 0. \end{aligned}$$

So, ω is in fact a cocycle.

Definition 1.3. A set of all the cohomology classes $[\omega]$ obtained by the above procedure is defined to be the *Massey triple product* $\langle [x_1], [x_2], [x_3] \rangle$ of $[x_1]$, $[x_2]$ and $[x_3]$. Due to the ambiguity of x_{12} and x_{23}, the Massey triple product $\langle [x_1], [x_2], [x_3] \rangle$ is an element of the quotient group

$$H^{r_1+r_2+r_3-1}(X)/([x_1]H^{r_2+r_3-1}(X) + H^{r_1+r_2-1}(X)[x_3]).$$

For example, we consider the Massey products of 1 dimensional classes $u_1, u_2, u_3 \in H^1(X; R)$. Hence, $\langle u_1, u_2, u_3 \rangle \subset H^2(X; R)$, and the *indeterminacy* $In\langle u_1, u_2, u_3 \rangle$ of triple Massey product $\langle u_1, u_2, u_3 \rangle$ is

$$In\langle u_1, u_2, u_3 \rangle = u_1 H^1(X) + H^1(X) u_3 \subset H^2(X; R).$$

$\langle u_1, u_2, u_3 \rangle$ can be seen as an element in $H^2(G)/In\langle u_1, u_2, u_3 \rangle$. We say $\langle u_1, u_2, u_3 \rangle$ is *vanishing* if this element is 0. The Massey product $\langle u_1, u_2, u_3 \rangle$ is said to be *decomposable* if it contains a cohomology class that can be written as a product $\alpha\beta$ of two elements in $H^{>0}(G)$; otherwise, it is called *indecomposable*.

Example 1.4. Let $T = S^1 \times S^1$ be the torus. Then $H^0(T; R) = R$, $H^1(T; R) = R^2$ generated by $\{a, b\}$, $H^2(T; R) = R$ generated by $\{\gamma\}$. Since $a \cup a = 0$, we can define $\langle a, a, a \rangle$. However, it is vanishing by the reason of definition. In fact, all Massey products in T are vanishing.

1.3. **Quadruple Massey Products.** We can generalize the triple Massey products to the 4th order Massey product. Let u, v, w and x be homogeneous elements from $H^*(X; R)$ or degree p, q, r and s respectively. Assume $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ exist and vanish simultaneously, then we can define the fourth order Massey product $\langle u, v, w, x \rangle$.

Let x_1, x_2, x_3, x_4 be the representative cocycles for u, v, w, x respectively. Because of the existence of $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$, we have cochains x_{12}, x_{23}, x_{34} such that

$$dx_{12} = \bar{x}_1 \cup x_2, dx_{23} = \bar{x}_2 \cup x_3, dx_{34} = \bar{x}_3 \cup x_4.$$

Since both $\langle u, v, w \rangle$ and $\langle v, w, x \rangle$ vanish, there exist cochains x_{13} and x_{24} such that

$$d(x_{13}) = \overline{x}_1 \cup x_{23} + \overline{x}_{12} \cup x_3, \quad d(x_{24}) = \overline{x}_2 \cup x_{34} + \overline{x}_{23} \cup x_4.$$

One can show that the cochain

$$\omega = \overline{x}_1 \cup x_{24} + \overline{x}_{12} \cup x_{34} + \overline{x}_{13} \cup x_4$$

is actually a cocycle of degree $p+q+r+s-2$. The quadruple Massey product $\langle u, v, w, x \rangle$ is defined to be the set of all cohomology class $[\omega] \in H^{p+q+r+s-2}(X; R)$.

1.4. **Higher Order Massey Products.** Let's use the following convention to describe higher Massey product.

It is better to look at the data for triple Massey product (and quadruple Massey product) in matrix

$$\begin{pmatrix} 1 & x_1 & x_{12} & * \\ & 1 & x_2 & x_{23} \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & x_1 & x_{12} & x_{13} & * \\ & 1 & x_2 & x_{23} & x_{24} \\ & & 1 & x_3 & x_{34} \\ & & & 1 & x_4 \\ & & & & 1 \end{pmatrix}$$

First, we consider the Massey products of 1 dimensional classes $u_1, u_2, \cdots, u_k \in H^1(X;\mathbb{C})$.

Definition 1.5. Given $u_1, \cdots, u_k \in H^1(G;\mathbb{C})$. We call a *defining system* for the k-fold Massey product $\langle u_1, \cdots, u_k\rangle$ an array of 1-cochains $M = \{m_{i,j} \in C^1(G;\mathbb{C}) \mid 1 \le i < j \le k+1, (i,j) \ne (1,k+1)\}$ such that

(1) $m_{i,i+1}$ is a cocycle representing u_i;

(2) $\delta m_{i,j} = \sum\limits_{s=i+1}^{j-1} m_{i,s} \cup m_{s,j}$.

The value of the product at M, denoted $\langle u_1, \cdots, u_k\rangle_M$ is defined to be the cohomology class in $H^2(G;\mathbb{C})$ represented by the cocycle

$$\omega = \sum_{s=2}^{k} m_{1,s} \cup m_{s,k+1}.$$

The product $\langle u_1, \cdots, u_k\rangle$ is defined only if such an M exists, and is taken to be the subset of $H^2(G;\mathbb{C})$ consisting of all elements of the form $\langle u_1, \cdots, u_k\rangle_M$:

$$\langle u_1, \cdots, u_k\rangle = \{\langle u_1, \cdots, u_k\rangle_M \mid M \text{ a defining system}\}.$$

The *indeterminacy* of Massey product $\langle u_1, \cdots, u_k\rangle$ is

$$In\langle u_1, \cdots, u_k\rangle = \{a - b \mid a, b \in \langle u_1, \cdots, u_k\rangle\} \subset H^2(G;\mathbb{C}).$$

Remark 1.6. There is a unique matrix associated to each defining system M as follows.

$$\begin{pmatrix} 1 & m_{1,1} & m_{1,2} & m_{1,3} & \cdots & m_{1,n-1} & \\ & 1 & m_{2,2} & m_{2,3} & \cdots & m_{2,n-1} & m_{2,n} \\ & & 1 & m_{3,3} & \cdots & m_{3,n-1} & m_{3,n} \\ & & & 1 & \ddots & \vdots & \vdots \\ & & & & & m_{n-1,n-1} & m_{n-1,n} \\ & & & & & 1 & m_{n,n} \\ & & & & & & 1 \end{pmatrix}$$

The index of the input is a little different from the triple and quadruple Massey products. The reason is that people want to use this as a subgroup of matrix group.

Massey product has the following properties. ([7])

Proposition 1.7. *([7]) Let $u_1, u_2, \cdots, u_k \in H^*(X;\mathbb{C})$. Then,*

1.(Linearity) If $k \in R$, then for all $1 \le i \le n$,

$$k\langle u_1, \cdots, u_n\rangle \subseteq \langle u_1, \cdots, ku_i, \cdots, u_n\rangle.$$

2.(Naturality) If $f : X\rangle Y$ is a continuous map, then

$$f^*(\langle u_1, \cdots, u_n\rangle) \subseteq \langle f^*(u_1), \cdots, f^*(u_n)\rangle.$$

3.(Associativity) For $v \in H^*(X)$,

$$\begin{array}{l} \langle u_1, \cdots, u_n\rangle v \subseteq \langle u_1, \cdots, u_n v\rangle, \\ v\langle u_1, \cdots, u_n\rangle \subseteq (-1)^{\deg v}\langle vu_1, \cdots, u_n\rangle \end{array}.$$

4.(Symmetry) If $p_j = \deg u_j$ and $l = \sum_{1\le r<s\le n} p_r p_s + (n-1)\sum_{r=1}^{n} p_r + (n-1)(n-2)/2$, then

$$\langle u_n, \cdots, u_1\rangle = (-1)^l \langle u_1, \cdots, u_n\rangle.$$

The last property in ([7]) maybe not precisely, we will see that in the examples of Borromean ring. I change it by David Kraines's paper "Massey Higher Products"

2. Classical Application: Borromean rings

2.1. **Some Knot Theory.** Let L be an oriented tame ordered link in S^3, with components $L_1, \cdots, L_n$. Its complement, $S^3\setminus\bigcup_i L_i$, has the homotopy type of a connected 2-dimensional finite CW-complex. More precisely, let nS^1 be the disjoint union of n circles $S_1, S_2, \cdots S_n$. A link is a embedding $l : nS^1 \to S^3$. Denote T_i be $S_i \times D^2$, we can extend l to an embedding $T_1 \cup \cdots \cup T_n \to S^3$. Let X be the closure of $S^3 - \bigcup T_i$, a compact 3-manifold with boundary $\bigcup S_i \times S^1$. In $G = \pi_1(X) \cong \pi_1(S^3\setminus L)$, the images of the $z_i \times S^1 (z_i \in S_i)$ are called the *meridians* m_i. The images of the S_i are called the *longitudes* l_i. The *linking number* $l_{ij} = \mathrm{lk}(L_i, L_j)$ is the image of the i-th longitude in the $\mathbb{Z} \cong H_1(X(L_j);\mathbb{Z})$. In particular, $l_{ii} = 0$ and $l_{ij} = l_{ji}$.

If $L = \widehat{\beta}$ is the closure of a pure braid $\beta \in P_n$, then $G = \pi_1(X)$ is a commutator-relator group with presentation $G = \langle x_1, \cdots, x_n \mid \beta(x_i)x_i^{-1} = 1, 1 \le i < n\rangle$. Another efficient way to compute the fundamental group of X is using *Wirtinger presentation*. Select an orientation of L. Label the three arcs at a crossing with gi for distinct is. For convenience, were going to label g_i, g_j, g_k as below. Following the picture, define the relations $g_j = g_k g_i g_k^{-1}$ and $g_j = g_k^{-1} g_i g_k$.

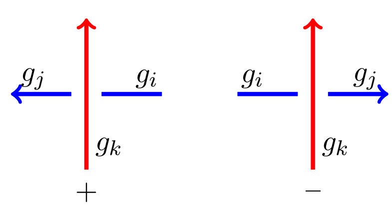

FIGURE 1. Wirtinger relation

Example 2.1. The link group of the Hopf link. We use clockwise orientation. Then the

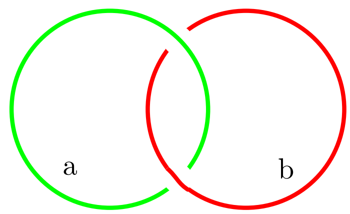

FIGURE 2. Hopf link

link group of Hopf link is $\langle a, b \mid a = bab^{-1} \rangle$.

The homology groups of X are computed by Alexander duality,

$$\widetilde{H}_i(S^3 \setminus L) \cong \widetilde{H}^{3-i-1}(L),$$

that is $\widetilde{H}_0(X) = 0$, $H_1(X) = R^n$, $H_2(X) = R^{n-1}$, $H_{i \geq 3}(X) = 0$.

Let $\{e_1, \cdots, e_n\}$ be the basis for $H^1(X)$ dual to the *meridians* of L. Choose arcs $c_{i,j}$ in X connecting L_i to L_j, and let $\gamma_{i,j} \in H^2(X)$ be their duals. Then $\{\gamma_{1,n}, \cdots, \gamma_{n-1,n}\}$ forms a basis for $H^2(X)$. Let $l_{i,j} = \mathrm{lk}(L_i, L_j)$ be the linking numbers of L. Then the cup product in $H^*(X)$ is given by

$$e_i \cup e_j = l_{i,j} \cdot \gamma_{i,j}$$

Hence, the linking numbers determine and are determined by cup products.

There is an algorithm to compute the linking number of two knots from a link. Label each crossing as positive or negative, according to the following rule. (Right(left) rule).

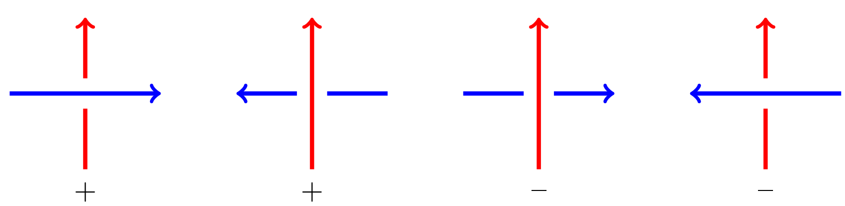

FIGURE 3. Linking numbers

Linking numbers=$(n_1 + n_2 - n_3 - n_4)/2$, where where n_1, n_2, n_3, n_4 represent the number of crossings of each of the four types.

2.2. **Borromean rings.**

Example 2.2. Let X be the complement in S^3 of the 3-component Borromean link L, see Figure 4.

The computation of triple Massey products of Borromean link L is given by Massey in [5]. $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ exists and non-vanishing. Next, we use an alternative method to compute the Massey product.

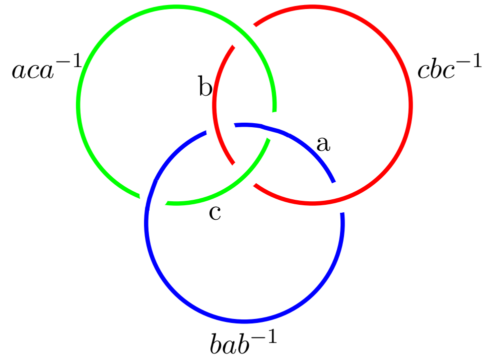

FIGURE 4. Borromean link L

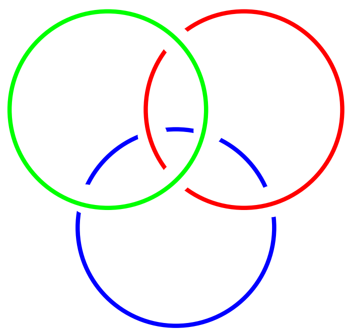

FIGURE 5. 3-component unlink

2.3. **Fox calculus and computation of Massey products.**

Definition 2.3. Let $\mathbb{ZF}$ be the group ring of $\mathbb{F}$, with augmentation map $\epsilon : \mathbb{ZF} \to \mathbb{Z}$ given by $\epsilon(e_i) = 1$. The *Fox derivative* is $\partial_i : \mathbb{ZF} \to \mathbb{ZF}$, given by

$$\begin{cases} \partial_i(1) = 0; \\ \partial_i(e_j) = \delta_{ij}; \\ \partial_i(uv) = \partial_i(u)\epsilon(v) + u\partial_i(v) \end{cases} \tag{1}$$

.

We use $\frac{\partial}{\partial e_i}$ to denote ∂_i sometimes. By the formula (1), we can get $\frac{\partial x^{-1}}{\partial x} = -x^{-1}$.

Lemma 2.4. *We have some computation results:*
(a) $\partial_i[u,v] = (1 - uvu^{-1})\partial_i(u) + (u - [u,v])\partial_i v$.
(b) $\epsilon_I(w) = 0$, *if* $w \in F_q$, *and* $|I| < q$.
(c) $\epsilon_I(uv) = \epsilon_I(u) + \epsilon_I(v)$, *if* $u, v \in F_p$, *and* $|I| = q$.

Let $G = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_m \rangle$ be a commutator-relators group, such that $H_2(G)$ is free abelian with basis $\{\theta_1, \cdots, \theta_m\}$, and their dual $\{\gamma_1 \cdots, \gamma_m\}$ form a basis for $H^2(G)$. Suppose $H^1(G)$ have dual basis $\{e_1, \cdots, e_n\}$.

Proposition 2.5. [6] *Under above assumption, the cup-product map* $\mu : H^1(G) \cup H^1(G) \to H^2(G)$ *is given by*

$$e_i \cup e_j = \mu(e_i \wedge e_j) = \sum_{k=1}^{m} \epsilon_{i,j}(r_k)\gamma_k. \tag{2}$$

Let $\alpha_i = \sum_{j=1}^{n} a_{i,j}e_j$. Let $J = (j_1, \cdots, j_k)$ be a fixed sequence, ∂_J be the iterated Fox derivative $\partial_{j_1} \circ \cdots \circ \partial_{j_k}$ and $\epsilon_J = \epsilon \circ \partial_J$, where $\epsilon : \mathbb{Z}F \to \mathbb{Z}$ is the augmentation, for free group F with generators $x_1, \cdots, x_n$. Suppose r_i satisfies

$$\epsilon_{j_1,\cdots,j_{t-s}}(r_i) = 0, \tag{3}$$

for all $(j_1, \cdots, j_{t-s})$ satisfying

$$1 \leq s < t \leq k+1, (s,t) \neq (1, k+1), a_{s,j_1} \cdots a_{t-1,j_{t-s}} \neq 0.$$

Proposition 2.6. [2] *Suppose* $r_i \in [F,F]$ *is not a proper power,* $J = (j_1, \cdots, j_k)$ *is a fixed sequence and above (3) holds. Then there exists a defining system* M *for the Massey product* $\langle -\alpha_1, -\alpha_2, \cdots, -\alpha_k\rangle$ *and*

$$(\langle -\alpha_1, \cdots, -\alpha_k\rangle_M, [\{r_i\}]) = \sum_J a_J \epsilon_J(r_i)$$

where $a_J = a_{1j_1}a_{2j_2}\cdots a_{kj_k}$.

Example 2.7 (Example 2.2 continued)**.** Go back to our example of Borromean link. A presentation for the fundamental group $G = \pi_1(X)$ can be computed by Wirtinger presentation (see [3] Change a, b, c to x_3, x_2, x_1.)

$$G = \langle x_1, x_2, x_3 \mid [x_2x_3x_2^{-1}, x_1] = [x_3, x_1], [x_3x_1x_3^{-1}, x_2] = [x_1, x_2]\rangle.$$

Using Hall-Witt identities, we can simplify these relations. For the first relation r_1, it is the same as $[[x_1, x_2], x_3]$, and the relation r_2 is the same as $[[x_2, x_3], x_1]$. That is

$$G = \langle x_1, x_2, x_3 \mid [x_1, [x_2, x_3^{-1}]], [x_2, [x_3, x_1^{-1}]], \rangle.$$

Using formula, we have $(\langle e_1, e_2, e_3\rangle, \theta_1) = -\epsilon_{(1,2,3)}(r_1) = 1$ and $(\langle e_1, e_2, e_3\rangle, \theta_2) = -\epsilon_{(1,2,3)}(r_2) = 0$. Hence, $(\langle e_1, e_2, e_3\rangle = \gamma_1$ and using similar computation, we have $(\langle e_2, e_1, e_3\rangle = -\gamma_2$.

	(123)	(231)	(312)
	(312)	(132)	(213)
r_1	-1	1	0
r_2	0	-1	1
r_3	1	0	-1

Remark 2.8. From this example, we could know that $\langle e_1, e_2, e_3\rangle$ may not equal $\langle e_2, e_3, e_1\rangle$. Maybe one exists but the other one even does not, in group

$$G = \langle x_1, x_2, x_3 \mid [x_1, [x_2, x_3^{-1}]], [x_1, x_3]\rangle$$

for example. However, we know that $\langle e_1, e_2, e_3\rangle = \langle e_3, e_2, e_1\rangle$ by symmetry property. (See the property.)

2.4. **Magnus expansion and Milnor invariants.** The free group on n generators will be denoted $F(n)$, or just F when no ambiguity exists. For a group G, $\{\Gamma_k(G)\}_{k\geq 1}$(sometimes denoted by) is the *lower central series* of the group G, which is defined inductively by the formulas:

$$\Gamma_1 G = G;$$
$$\Gamma_{k+1}(G) = [\Gamma_k G, G], k \geq 1.$$

The Lie bracket $[x, y]$ is induced from the group commutator $[x, y] = xyx^{-1}y^{-1}$. Group G is said to be *nilpotent* of class $\leq k$ if $\Gamma_{k+1}(G) = \{1\}$.

Magnus gives us a useful criterion for determining the place of an element in the lower central series filtration of the free group. There is a well-defined ring homomorphism $\delta : \mathbb{Z}F(n) \to P$, called the *Magnus Expansion*, where $P = \mathbb{Z}[[X_1, \cdots, X_n]]$ is the power series ring in n non-commuting variables $X_1, \cdots, X_n$. If $F(n) = F(x_1, \cdots, x_n)$ then

$$\delta(x_i) = 1 + X_i;$$
$$\delta(x_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots .$$

If $y \in F$, denote

$$\delta(y) = 1 + \sum \mu(i_1, \cdots, i_s) X_{i_1} \cdots X_{i_s}.$$

By Magnus, $y \in F_k$ if and only if all coefficient, $\mu(i_1, \cdots, i_s)$, vanish for $s \leq k - 1$.

Now, we review the definition Milnor's $\bar{\mu}$-invariants. For link L with complement X, choose a path to each component L_i from a common basepoint. Combining this with meridianal circles linking each component once defines a map $\tau : \bigvee_n S^1 \to X$. This determines a homomorphism $\tau_* : F(n) \to \pi_1(X)$. The path to the i-th component, together with a null homologous pushoff of the i-th component into the link exterior, determines a longitude $l_i \in \pi_1(X)$. According to Milnor, a collection of meridians determines a presentation for the group

$$\pi_1(X)/\pi_1(X)_k = \langle x_1, \cdots, x_n \mid [x_i, l_i^k], F_k \rangle$$

where F_k is the low central series of free group $F(x_1, \cdots, x_n)$, l_i^k is a word in the x_i's representing the image of the i-th longitude l_i in $\pi_1(X)/\pi_1(X)_k$

Definition 2.9. The integer $\bar{\mu}(i_1, \cdots, i_s, j)$, the Milnor $\bar{\mu}$ invariant of the based link (L, τ), is the coefficient $\mu(i_1, \cdots, i_s)$ in $\delta(l_j^k)$. This will be called a *Milnor invariant of length* $s + 1$.

Milnor shows that a link has vanishing Milnor invariants of length $\leq k$ if and only if the longitudes lie in the k-th term of the lower central series of the fundamental group of the link complement.

Now suppose the longitudes of L are contained in $\pi_1(X)_k$, then all Milnor invariants of length $\leq k$ vanish, and Milnor invariants of length $k + 1$ is well defined. For Milnor invariants of length $\geq k + 2$, they are well defined modulo the GCD(all Milnor invariants $\mu(J)$, where J is obtained from $(i_1, \cdots, i_k, j)$ by removing at least one index). This is similar as the Massey products.

Example 2.10 (Example 2.2 continued)**.**

$$G = \langle x_1, x_2, x_3 \mid [x_1, [x_2, x_3^{-1}]], [x_3, [x_1, x_2^{-1}]] \rangle.$$

$$G/G_q = \langle x_1, x_2, x_3 \mid [x_1, [x_2, x_3^{-1}]], [x_2, [x_3, x_1^{-1}]], [x_3, [x_1, x_2^{-1}]], F_q] \rangle.$$

Hence, $l_3 = [x_1, x_2^{-1}] = x_1 x_2^{-1} x_1^{-1} x_2$, and

$$\delta(l_3) = \delta(x_1)\delta(x_2^{-1})\delta(x_1^{-1})\delta(x_2) = 1 - X_1X_2 + X_2X_1 + \text{higher terms}.$$

So, $\bar{\mu}(1,2,3) = -1$ and $\bar{\mu}(2,1,3) = +1$.

3. Another Application: Spectral Sequences for Twisted Cohomology

3.1. **Twisted Cohomology.** Let M be a smooth compact closed manifold of dimension n, and $\Omega^*(M)$ the space of smooth differential forms over $\mathbb{R}$ on M. We have the de Rham cochain complex $(\Omega^*(M), d)$, where $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is the exterior differentiation, and its cohomology $H^*(M)$ (the de Rham cohomology).

Let H be $\sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$, where H_{2i+1} is a closed $(2i+1)$-form. Then one can define a new operator $D = d + H$ on $\Omega^*(M)$, where H is understood as an operator acting by exterior multiplication (for any differential form w, $H(w) = H \wedge w$). It is easy to show that

$$D^2 = (d + H)^2 = d^2 + dH + Hd + H^2 = 0.$$

However D is not homogeneous on the space of smooth differential forms $\Omega^*(M) = \bigoplus_{i\geq 0} \Omega^i(M)$.

Define $\Omega^*(M)$ a new (mod 2) grading

$$\Omega^*(M) = \Omega^o(M) \oplus \Omega^e(M), \tag{4}$$

where

$$\Omega^o(M) = \bigoplus_{\substack{i\geq 0 \\ \mathrm{i}\equiv 1 \pmod 2}} \Omega^i(M) \quad \text{and} \quad \Omega^e(M) = \bigoplus_{\substack{i\geq 0 \\ \mathrm{i}\equiv 0 \pmod 2}} \Omega^i(M). \tag{5}$$

Then D is homogenous for this new (mod 2) grading:

$$\Omega^e(M) \xrightarrow{D} \Omega^o(M) \xrightarrow{D} \Omega^e(M).$$

Define the twisted de Rham cohomology groups of M:

$$H^o(M, H) = \frac{\ker[D : \Omega^o(M) \to \Omega^e(M)]}{\mathrm{im}[D : \Omega^e(M) \to \Omega^o(M)]} \tag{6}$$

and

$$H^e(M, H) = \frac{\ker[D : \Omega^e(M) \to \Omega^o(M)]}{\mathrm{im}[D : \Omega^o(M) \to \Omega^e(M)]}. \tag{7}$$

3.2. **Spectral Sequence.** As a review, we only give the definition of spectral sequence here, more details in my another note on spectral sequence.

Definition 3.1. A *spectral sequence* is a collection of differential bigraded R-modules $\{E_r^{p,q}, d_r\}$, where $r = 1, 2, \cdots$ and

$$E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}) \cong ker(d_r : E_r^{p,q} \to E_r^{*,*})/im(d_r : E_r^{*,*} \to E_r^{p,q}).$$

In practice, we have the differential d_r of bidegree $(r, 1-r)$ (for a spectral sequence of cohomology type) or $(-r, r-1)$ (for a spectral sequence of homology type).

As in [1], there is a filtration on $(\Omega^*(M), D)$:

$$K_p = F_p(\Omega^*(M)) = \bigoplus_{i \geq p} \Omega^i(M). \tag{8}$$

This filtration gives rise to a spectral sequence

$$\{E_r^{p,q}, d_r\} \tag{9}$$

converging to the twisted de Rham cohomology $H^*(M, H)$ with

$$E_2^{p,q} \cong \begin{cases} H^p(M) & q \text{ is even,} \\ 0 & q \text{ is odd.} \end{cases} \tag{10}$$

That is

$$\bigoplus_{\text{p+q=1}} E_\infty^{p,q} \cong H^o(M, H) \quad \text{and} \quad \bigoplus_{\text{p+q=0}} E_\infty^{p,q} \cong H^e(M, H). \tag{11}$$

Theorem 3.2. [4] *For $H = \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$ and $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \geq 1)$, the differential of the spectral sequence (9), $d_{2t+3} : E_{2t+3}^{p,q} \to E_{2t+3}^{p+2t+3, q-2t-2}$ is given by*

$$d_{2t+3}[x_p]_{2t+3} = (-1)^t [\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p \rangle_A]_{2t+3},$$

and $[\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p \rangle_A]_{2t+3}$ is independent of the choice of the defining system.

Atiyah and Segal in [1] gave the differential expression in terms of Massey products when $H = H_3$, without a detailed proof. The above theorem generalizes the result to a more general case. W.Li, X.Liu and myself give an explicitly proof for the general result in paper [4] (arXiv:0911.1417).

4. **Other Applications**

There are many applications of Massey product in many fields of mathematics. I copy some applications from a talk note of Laurence R. Taylor. See which is your staff.

1. Uehara and Massey used the triple product to settle the signs in the Jacobi identity for Whitehead products. (1956).
2. Massey used the triple product to elucidate some of the cup product structure of sphere bundles. (1958).

3. J.May (1969) described a generalization called Matrix Massey products, which can be used to describe the differentials of the EilenbergMoore spectral sequence.

4. Some Massey triple products of Pontryagin classes of normal bundles to foliations vanish. Shulman (1974).

5. Massey triple products vanish in a compact Kahler manifold. Deligne, Griffiths, Morgan and Sullivan (1975).

6. Symplectic manifolds can have non-trivial rational Massey triple products. Babenko and Tamanov, and Rudyak and Tralle (2000).

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