

Convolution Theorems for Quaternion Fourier Transform: Properties and Applications

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Abstract

General convolution theorems for two-dimensional quaternion Fourier transforms(QFTs) are presented. It is shown that these theorems are not only valid for real-valued functions, but also for quaternion-valued functions. We describe some useful properties of generalized convolutions and compare with the convolution theorems of the classical Fourier transform. We finally apply the obtained results to study hypoellipticity and to solve the heat equation in quaternion algebra framework.

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1 Introduction

Convolution is a mathematical operation with several applications in pure and applied mathematics such as numerical analysis, numerical linear algebra and the design and implementation of finite impulse response filters in signal processing. In [12, 19, 22], the authors introduced the Clifford convolution. It is found that some properties of convolution, when generalized to the Clifford Fourier transform (CFT), are very similar to the classical ones.

On the other hand, the quaternion Fourier transform (QFT) is a nontrivial generalization of the classical Fourier transform (FT) using quaternion algebra. The QFT has been shown to relate to the other quaternion signal analysis tools such as quaternion wavelet transform, fractional quaternion Fourier transform, quaternionic windowed Fourier transform and quaternion Wigner transform [6, 16, 5, 3, 2, 27]. A number of already known and useful properties of this extended transform are generalizations of the corresponding properties of the FT with some modifications, but the generalization of convolution theorems of the QFT is still an open problem. In the recent past, several authors [10, 13, 17, 4] tried to formulate

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convolution theorems for the QFT. But they only treated them for real valued functions which is quite similar to the classical case. In [1], the authors briefly introduced, without proof, the QFT of the convolution of two-dimensional quaternion signals.

In this paper, we establish general convolutions for QFT. Because quaternion multiplication is not commutative, we find new properties of the QFT of convolution of two quaternion-valued functions. These properties describe closely the relationship between the quaternion convolution and its QFT. The generalization of the convolution theorems of the QFT is mainly motivated by the Clifford convolution of general geometric Fourier transform, which has been recently studied in [7, 8]. We further establish the inverse QFT of the product of the QFT, which is very useful in solving partial differential equations in quaternion algebra framework.

This paper consists of the following sections. Section 2 deals with some results on the real quaternion algebra and the definition of the QFT and its basic properties. We also review some basic properties of QFT, which will be necessary in the next section. Section 3 establishes convolution theorems of QFT and some of their consequences. Section 4 presents an application of QFT to study hypoellipticity and to solve the heat equation in quaternion algebra. Some conclusions are drawn in Section 5.

2 Quaternion algebra

For convenience, we specify the notation used in this paper. The quaternion algebra over \mathbb{R} , denoted by \mathbb{H} , is an associative non-commutative four-dimensional algebra,

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 ; q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \quad (1)$$

which obeys the following multiplication rules:

$$\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}, \mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}, \mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1. \quad (2)$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is called the *scalar* part of q denoted by $Sc(q)$ and $\mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is called the *vector* (or *pure*) part of q . The vector part of q is conventionally denoted by \mathbf{q} . Let $p, q \in \mathbb{H}$ and \mathbf{p}, \mathbf{q} be their vector parts, respectively. It is common to write for short

$$\begin{aligned} \mathbf{q} \cdot \mathbf{p} &= q_1p_1 + q_2p_2 + q_3p_3, \\ \mathbf{q} \times \mathbf{p} &= \mathbf{i}(q_2p_3 - q_3p_2) + \mathbf{j}(q_3p_1 - q_1p_3) + \mathbf{k}(q_1p_2 - q_2p_1). \end{aligned}$$

Then, equation (2) yields the quaternionic multiplication qp as

$$qp = q_0p_0 - \mathbf{q} \cdot \mathbf{p} + q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{q} \times \mathbf{p}. \quad (3)$$

The quaternion conjugate of q , given by

$$\bar{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \quad q_0, q_1, q_2, q_3 \in \mathbb{R}, \quad (4)$$

is an anti-involution, that is,

$$\overline{qp} = \bar{p}\bar{q}. \quad (5)$$

From (4) we obtain the norm or modulus of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \quad (6)$$

It is not difficult to see that

$$|qp| = |q||p|, \quad \forall p, q \in \mathbb{H}. \quad (7)$$

Using the conjugate (4) and the modulus of q , we can define the inverse of $q \in \mathbb{H} \setminus \{0\}$ as

$$q^{-1} = \frac{\bar{q}}{|q|^2}, \quad (8)$$

which shows that \mathbb{H} is a normed division algebra. As in the algebra of complex numbers, we can define three nontrivial quaternion involutions [10]:

$$\begin{aligned} \alpha(q) &= -\mathbf{i}q\mathbf{i} = -\mathbf{i}(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)\mathbf{i} = q_0 + \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3, \\ \beta(q) &= -\mathbf{j}q\mathbf{j} = -\mathbf{j}(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)\mathbf{j} = q_0 - \mathbf{i}q_1 + \mathbf{j}q_2 - \mathbf{k}q_3, \\ \gamma(q) &= -\mathbf{k}q\mathbf{k} = -\mathbf{k}(q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3)\mathbf{k} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 + \mathbf{k}q_3. \end{aligned} \quad (9)$$

Hereinafter, besides the quaternion units \mathbf{i} , \mathbf{j} , \mathbf{k} , and the vector part \mathbf{q} of a quaternion $q \in \mathbb{H}$, we shall use the real vector notation:

$$\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad |\mathbf{x}|^2 = x_1^2 + x_2^2, \quad \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2, \quad f(\mathbf{x}) = f(x_1, x_2),$$

and so on when there is no confusion. This gives the following definition.

Definition 2.1. (See [10].) *A function $f : \mathbb{R}^2 \rightarrow \mathbb{H}$ is called quaternionic Hermitian if, for the involutions α and β ,*

$$f(-x_1, x_2) = \beta(f(\mathbf{x})) \quad \text{and} \quad f(x_1, -x_2) = \alpha(f(\mathbf{x})), \quad (10)$$

for each $\mathbf{x} \in \mathbb{R}^2$.

For any unit quaternion

$$q = q_0 + \mathbf{q} = \cos(\theta/2) + \sin(\theta/2),$$

and for any vector $\mathbf{v} \in \mathbb{R}^3$ the action of the operator

$$L_q(\mathbf{v}) = q\mathbf{v}\bar{q},$$

on \mathbf{v} is equivalent to a rotation of the vector \mathbf{v} through an angle θ about \mathbf{u} as the axis of rotation.

It is convenient to introduce an inner product for two functions $f, g : \mathbb{R}^2 \rightarrow \mathbb{H}$ as follows:

$$(f, g)_{L^2(\mathbb{R}^2; \mathbb{H})} = \int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})} d^2\mathbf{x}, \quad (11)$$

In particular, for $f = g$, we obtain the scalar product of the above inner product (11) given by

$$\langle f, f \rangle = \|f\|_{L^2(\mathbb{R}^2; \mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d^2\mathbf{x} \right)^{1/2}. \quad (12)$$

2.1 Multi-indices and derivatives

A couple $\alpha = (\alpha_1, \alpha_2)$ of nonnegative integers is called a *multi-index*. We denote

$$|\alpha| = \alpha_1 + \alpha_2, \quad \alpha! = \alpha_1! \alpha_2!,$$

and for $\mathbf{x} \in \mathbb{R}^2$,

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}.$$

Derivatives are conveniently expressed by multi-indices:

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

Denote by $\{\mathbf{e}_1, \mathbf{e}_2\}$ the standard basis of \mathbb{R}^2 . The *vector differential* $\mathbf{a} \cdot \nabla$ along the direction \mathbf{a} is defined by

$$\mathbf{a} \cdot \nabla = a_1 \partial_1 + a_2 \partial_2, \quad (13)$$

where $\nabla = \mathbf{e}_1 \partial_1 + \mathbf{e}_2 \partial_2$.

2.2 QFT and its properties

Definition 2.2. The QFT of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2, \mathbb{H})$ given by the integral

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x}) e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x}, \quad (14)$$

where \mathcal{F}_q is called the quaternion Fourier transform operator or the quaternion Fourier transformation.

Using the Euler formula for the quaternion Fourier kernel $e^{-\mathbf{i}\omega_1 x_1} e^{-\mathbf{j}\omega_2 x_2}$, we can rewrite (14) in the following form

$$\begin{aligned} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\mathbf{x}) \cos(\omega_1 x_1) \cos(\omega_2 x_2) d\mathbf{x} - \int_{\mathbb{R}^2} \mathbf{i}f(\mathbf{x}) \sin(\omega_1 x_1) \cos(\omega_2 x_2) d\mathbf{x} \\ &\quad - \int_{\mathbb{R}^2} f(\mathbf{x}) \mathbf{j} \cos(\omega_1 x_1) \sin(\omega_2 x_2) d\mathbf{x} + \int_{\mathbb{R}^2} \mathbf{i}f(\mathbf{x}) \mathbf{j} \sin(\omega_1 x_1) \sin(\omega_2 x_2) d\mathbf{x}. \end{aligned} \quad (15)$$

Definition 2.3. The inverse QFT of $g \in L^2(\mathbb{R}^2; \mathbb{H})$ is the transform $\mathcal{F}_q^{-1}\{g\} \in L^2(\mathbb{R}^2, \mathbb{H})$ given by the integral

$$\mathcal{F}_q^{-1}[g(\boldsymbol{\omega})](\mathbf{x}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\omega_1 x_1} g(\boldsymbol{\omega}) e^{\mathbf{j}\omega_2 x_2} d\boldsymbol{\omega}. \quad (16)$$

Some important properties of the QFT are stated in the following lemmas proved in [17, 18].

Lemma 2.1. Let $f \in L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H})$. If $\mathcal{F}_q\{\partial^\alpha f\} \in L^1(\mathbb{R}^2; \mathbb{H})$, then

$$\mathcal{F}_q\{\partial^\alpha f\}(\boldsymbol{\omega}) = (\mathbf{i}\omega_1)^{\alpha_1} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) (\mathbf{j}\omega_2)^{\alpha_2}. \quad (17)$$

In particular, if $\mathcal{F}_q\{\partial^{(2,0)} f\} \in L^1(\mathbb{R}^2; \mathbb{H})$, then

$$\mathcal{F}_q\{\partial^{(2,0)} f\}(\boldsymbol{\omega}) = (\mathbf{i}\omega_1)^2 \mathcal{F}_q\{f\}(\boldsymbol{\omega}), \quad (18)$$

and if $\mathcal{F}_q\{\partial^{(0,2)} f\} \in L^1(\mathbb{R}^2; \mathbb{H})$, then

$$\mathcal{F}_q\{\partial^{(0,2)} f\}(\boldsymbol{\omega}) = \mathcal{F}_q\{f\}(\boldsymbol{\omega}) (\mathbf{j}\omega_2)^2. \quad (19)$$

Lemma 2.2 (Scalar QFT Parseval). *The scalar product of $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ and its QFT is related by*

$$\langle f, g \rangle_{L^2(\mathbb{R}^2; \mathbb{H})} = \frac{1}{(2\pi)^2} \langle \mathcal{F}_q\{f\}, \mathcal{F}_q\{g\} \rangle_{L^2(\mathbb{R}^2; \mathbb{H})}. \quad (20)$$

and in particular, with $f = g$, the Plancherel theorem indicates that

$$\|f\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2 = \frac{1}{(2\pi)^2} \|\mathcal{F}_q\{f\}\|_{L^2(\mathbb{R}^2; \mathbb{H})}^2. \quad (21)$$

This shows that the total signal energy computed in the spatial domain is equal to the total signal energy computed in the quaternion domain.

3 Convolution of QFT

In this section, we establish the quaternion convolution of the QFT which extends the classical convolution to quaternion fields. Let us first define the convolution of two quaternion-valued functions.

Definition 3.1. *The convolution of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $g \in L^2(\mathbb{R}^2; \mathbb{H})$, denoted $f \star g$, is defined by*

$$(f \star g)(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{y})g(\mathbf{x} - \mathbf{y}) d\mathbf{y}. \quad (22)$$

Example 3.1. *To illustrate the general non-commutativity $(f \star g) \neq (g \star f)$, let us compute the convolution of $f(\mathbf{x}) = \mathbf{k}e^{-|\mathbf{x}|^2}$ and $g(\mathbf{x}) = (\mathbf{i} + \mathbf{j})e^{i\omega_1 x_1}e^{j\omega_2 x_2}$. Although, $g \in L^\infty(\mathbb{R}^2; \mathbb{H})$, we can still define the convolution of f and g , because f decays rapidly at infinity. A simple calculation gives*

$$\begin{aligned} (f \star g)(\mathbf{x}) &= \int_{\mathbb{R}^2} \mathbf{k}e^{-|\mathbf{y}|^2}(\mathbf{i} + \mathbf{j})e^{i\omega_1(x_1 - y_1)}e^{j\omega_2(x_2 - y_2)} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \mathbf{k}(\mathbf{i} + \mathbf{j})e^{i\omega_1 x_1}e^{-i\omega_1 y_1}e^{-|\mathbf{y}|^2}e^{-j\omega_2 y_2} d\mathbf{y} e^{j\omega_2 x_2} \\ &= \mathbf{k}(\mathbf{i} + \mathbf{j})e^{i\omega_1 x_1}e^{-|\omega|^2/4}e^{j\omega_2 x_2} \\ &= (\mathbf{j} - \mathbf{i})e^{i\omega_1 x_1}e^{-|\omega|^2/4}e^{j\omega_2 x_2}. \end{aligned} \quad (23)$$

On the other hand, we have

$$\begin{aligned} (g \star f)(\mathbf{x}) &= \int_{\mathbb{R}^2} (\mathbf{i} + \mathbf{j})e^{i\omega_1 y_1}e^{j\omega_2 y_2} \mathbf{k}e^{-|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} (\mathbf{i} + \mathbf{j})\mathbf{k}e^{-i\omega_1 y_1}e^{-|\mathbf{x} - \mathbf{y}|^2}e^{-j\omega_2 y_2} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} (\mathbf{i} + \mathbf{j})\mathbf{k}e^{-i\omega_1 y_1}e^{-|\mathbf{y} - \mathbf{x}|^2}e^{-j\omega_2 y_2} d\mathbf{y} \\ &= (\mathbf{i} - \mathbf{j})e^{-i\omega_1 x_1}e^{-|\omega|^2/4}e^{-j\omega_2 x_2}. \end{aligned} \quad (24)$$

In the following, we summarize the elementary properties of quaternion convolution as shown in Table 1 (compare to Folland [15]).

Lemma 3.1 (Linearity). *For quaternion functions f, g and h and quaternion constants κ_1 and κ_2 we get*

$$(\kappa_1 f + \kappa_2 g) \star h = \kappa_1 (f \star h) + \kappa_2 (g \star h). \quad (25)$$

We also get for real constants κ_1 and κ_2

$$h \star (\kappa_1 f + \kappa_2 g) = \kappa_1 (h \star f) + \kappa_2 (h \star g). \quad (26)$$

Lemma 3.2 (Shifting). *Given a quaternion function $f \in L^2(\mathbb{R}^2; \mathbb{H})$, let $\tau_{\mathbf{a}} f(\mathbf{x})$ denote the shifted (translated) function defined by $\tau_{\mathbf{a}} f(\mathbf{x}) = f(\mathbf{x} - \mathbf{a})$, where $\mathbf{a} \in \mathbb{R}^2$. Then we get*

$$(\tau_{\mathbf{a}} f \star g)(\mathbf{x}) = \tau_{\mathbf{a}} (f \star g)(\mathbf{x}), \quad (27)$$

and

$$(f \star \tau_{\mathbf{a}} g)(\mathbf{x}) = \tau_{\mathbf{a}} (f \star g)(\mathbf{x}). \quad (28)$$

Proof. For (27), a direct calculation gives

$$\begin{aligned} (\tau_{\mathbf{a}} f \star g)(\mathbf{x}) &= \int_{\mathbb{R}^2} \tau_{\mathbf{a}} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^2} f(\mathbf{y} - \mathbf{a}) g(\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^2} f(\mathbf{y} - \mathbf{a}) g((\mathbf{x} - \mathbf{a}) - (\mathbf{y} - \mathbf{a})) d\mathbf{y} \\ &= \int_{\mathbb{R}^2} f(\mathbf{s}) g((\mathbf{x} - \mathbf{a}) - \mathbf{s}) d\mathbf{s} \\ &= (f \star g)(\mathbf{x} - \mathbf{a}) \\ &= \tau_{\mathbf{a}} (f \star g)(\mathbf{x}), \end{aligned} \quad (29)$$

which finishes the proof. \square

Remark 3.1. *From equations (27) and (28), it is not difficult to see that $(\tau_{\mathbf{a}} f \star g)(\mathbf{x}) = (f \star \tau_{\mathbf{a}} g)(\mathbf{x})$ and $(\tau_{\mathbf{a}} g \star f)(\mathbf{x}) = (g \star \tau_{\mathbf{a}} f)(\mathbf{x})$.*

Lemma 3.3 (Conjugation). *For all quaternion functions $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ we have*

$$\overline{(f \star g)}(\mathbf{x}) = (\bar{g} \star \bar{f})(\mathbf{x}). \quad (30)$$

Proof. A straightforward computation gives

$$\begin{aligned} \overline{(f \star g)}(\mathbf{x}) &= \int_{\mathbb{R}^2} \overline{f(\mathbf{y}) g(\mathbf{x} - \mathbf{y})} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \bar{g}((\mathbf{x} - \mathbf{y})) \bar{f}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \bar{g}(\mathbf{z}) \bar{f}((\mathbf{x} - \mathbf{z})) d\mathbf{z} \\ &= (\bar{g} \star \bar{f})(\mathbf{x}). \end{aligned}$$

This finishes the proof. \square

Table 1: Basic properties of quaternion convolution.

Basic property	Quaternion convolution
Linearity	$(\kappa_1 f + \kappa_2 g) \star h = \kappa_1(f \star h) + \kappa_2(f \star h), \quad \kappa_1, \kappa_2 \in \mathbb{H}$ $h \star (\kappa_1 f + \kappa_2 g) = \kappa_1(h \star f) + \kappa_2(h \star g), \quad \kappa_1, \kappa_2 \in \mathbb{R}$
Shifting	$(\tau_{\mathbf{a}} f \star g) = \tau_{\mathbf{a}}(f \star g)$ $(f \star \tau_{\mathbf{a}} g) = \tau_{\mathbf{a}}(f \star g)$
Conjugation	$\overline{(f \star g)} = (\bar{g} \star \bar{f})$
Associativity	$(f \star g) \star h = f \star (g \star h)$
Distributivity	$f \star (g + h) = (f \star g) + (f \star h)$
Vector differential	$\mathbf{a} \cdot \nabla(f \star g) = (\mathbf{a} \cdot \nabla f) \star g = f \star (\mathbf{a} \cdot \nabla g)$
Impulse convolution	$f \star \delta = f$

Lemma 3.4 (Vector differential). (See [19, 22].) *For all quaternion functions $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ we have*

$$\mathbf{a} \cdot \nabla(f \star g) = (\mathbf{a} \cdot \nabla f) \star g = f \star (\mathbf{a} \cdot \nabla g). \quad (31)$$

Ell and Sangwine [14] distinguish between right and left discrete quaternion convolution due to the non-commutative property of the quaternion multiplication. Here, we only consider one kind of quaternion convolutions. We come now to the main theorem² of this paper. This theorem describes the relationship between the convolution of two quaternion functions and its QFT.

Theorem 3.5. *Let $f(\mathbf{x}) = f_0(\mathbf{x}) + \mathbf{i}f_1(\mathbf{x}) + \mathbf{j}f_2(\mathbf{x}) + \mathbf{k}f_3(\mathbf{x})$ and $g(\mathbf{x}) = g_0(\mathbf{x}) + \mathbf{i}g_1(\mathbf{x}) + \mathbf{j}g_2(\mathbf{x}) + \mathbf{k}g_3(\mathbf{x})$ be two quaternion-valued functions, then the QFT of the convolution of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $g \in L^2(\mathbb{R}^2; \mathbb{H})$ is given by*

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= (\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}))(\mathcal{F}_q\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathcal{F}_q\{g_2\}(-\omega_1, \omega_2)) \\ &\quad + (\mathcal{F}_q\{f_0\}(\omega_1, -\omega_2) + \mathbf{i}\mathcal{F}_q\{f_1\}(\omega_1, -\omega_2))(\mathbf{i}\mathcal{F}_q\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{g_3\}(-\omega_1, \omega_2)) \\ &\quad + (\mathbf{j}\mathcal{F}_q\{f_2\}(-\omega_1, \omega_2) + \mathbf{k}\mathcal{F}_q\{f_3\}(-\omega_1, \omega_2))(\mathcal{F}_q\{g_0\}(-\omega_1, \omega_2) + \mathbf{j}\mathcal{F}_q\{g_2\}(\boldsymbol{\omega})) \\ &\quad + (\mathbf{j}\mathcal{F}_q\{f_2\}(-\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{f_3\}(-\boldsymbol{\omega}))(\mathbf{i}\mathcal{F}_q\{g_1\}(-\omega_1, \omega_2) + \mathbf{k}\mathcal{F}_q\{g_3\}(\boldsymbol{\omega})). \end{aligned} \quad (32)$$

Proof. In this proof we shall use the decomposition of quaternion functions and their QFT. Let $\mathcal{F}_q\{f\}(\boldsymbol{\omega})$ and $\mathcal{F}_q\{g\}(\boldsymbol{\omega})$ denote the QFT of $f \in L^2(\mathbb{R}^2; \mathbb{H})$ and $g \in L^2(\mathbb{R}^2; \mathbb{H})$, respectively. Expanding the QFT of the left-hand side of (32), we immediately get

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &\stackrel{(14)}{=} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} (f \star g)(\mathbf{x}) e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \\ &\stackrel{(22)}{=} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} \left[\int_{\mathbb{R}^2} f(\mathbf{y})g(\mathbf{x} - \mathbf{y})d\mathbf{y} \right] e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \end{aligned}$$

¹Due to the non-commutativity of the quaternion multiplication, equation (26) does not hold for quaternion constants κ_1 and κ_2 .

²Generalization of the QFT of the quaternion convolution in general geometric Fourier transform is investigated in [7]. It can easily be seen that the result is closely related to equation (4.30) of [7].

$$= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{y}) \left[\int_{\mathbb{R}^2} g(\mathbf{x} - \mathbf{y}) e^{-\mathbf{j}\omega_2 x_2} d\mathbf{x} \right] d\mathbf{y}. \quad (33)$$

By the change of variables $\mathbf{z} = \mathbf{x} - \mathbf{y}$, the above transform can be written as

$$\begin{aligned} & \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} f(\mathbf{y}) g(\mathbf{z}) e^{-\mathbf{j}\omega_2(y_2+z_2)} d\mathbf{z} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (\{f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})\} + \mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) \\ & \quad \times \int_{\mathbb{R}^2} (\{g_0(\mathbf{z}) + \mathbf{j}g_2(\mathbf{z})\} + \mathbf{i}g_1(\mathbf{z}) + \mathbf{k}g_3(\mathbf{z})) g(\mathbf{z}) e^{-\mathbf{j}\omega_2(y_2+z_2)} d\mathbf{z} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})) (g_0(\mathbf{z}) + \mathbf{j}g_2(\mathbf{z})) e^{-\mathbf{j}\omega_2(y_2+z_2)} d\mathbf{z} d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})) (\mathbf{i}g_1(\mathbf{z}) + \mathbf{k}g_3(\mathbf{z})) e^{-\mathbf{j}\omega_2(y_2+z_2)} d\mathbf{z} d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) (g_0(\mathbf{z}) + \mathbf{j}g_2(\mathbf{z})) e^{-\mathbf{j}\omega_2(y_2+z_2)} d\mathbf{z} d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) (\mathbf{i}g_1(\mathbf{z}) + \mathbf{k}g_3(\mathbf{z})) e^{-\mathbf{j}\omega_2(y_2+z_2)} d\mathbf{z} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})) (\mathcal{F}_q\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathcal{F}_q\{g_2\}(-\omega_1, \omega_2)) \\ & \quad \times e^{-\mathbf{j}\omega_2 y_2} d\mathbf{z} d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})) (\mathbf{i}\mathcal{F}_q\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{g_3\}(-\omega_1, \omega_2)) \\ & \quad \times e^{-\mathbf{j}\omega_2 y_2} d\mathbf{z} d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) (\mathcal{F}_q\{g_0\}(-\omega_1, \omega_2) + \mathbf{j}\mathcal{F}_q\{g_2\}(\boldsymbol{\omega})) \\ & \quad \times e^{-\mathbf{j}\omega_2 y_2} d\mathbf{z} d\mathbf{y} \\ & \quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) (\mathbf{i}\mathcal{F}_q\{g_1\}(-\omega_1, \omega_2) + \mathbf{k}\mathcal{F}_q\{g_3\}(\boldsymbol{\omega})) \\ & \quad \times e^{-\mathbf{j}\omega_2 y_2} d\mathbf{z} d\mathbf{y} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})) e^{-\mathbf{j}\omega_2 y_2} d\mathbf{y} (\mathcal{F}_q\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathcal{F}_q\{g_2\}(-\omega_1, \omega_2)) \\ & \quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})) e^{\mathbf{j}\omega_2 y_2} d\mathbf{y} (\mathbf{i}\mathcal{F}_q\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{g_3\}(-\omega_1, \omega_2)) \\ & \quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) e^{-\mathbf{j}\omega_2 y_2} d\mathbf{y} (\mathcal{F}_q\{g_0\}(-\omega_1, \omega_2) + \mathbf{j}\mathcal{F}_q\{g_2\}(\boldsymbol{\omega})) \\ & \quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) e^{\mathbf{j}\omega_2 y_2} d\mathbf{y} (\mathbf{i}\mathcal{F}_q\{g_1\}(-\omega_1, \omega_2) + \mathbf{k}\mathcal{F}_q\{g_3\}(\boldsymbol{\omega})), \end{aligned}$$

where the assumption $\mathcal{F}_q\{g_i\} \in L^2(\mathbb{R}^2; \mathbb{R})$ for $i = 1, 2, 3$ is used in the fourth line. This gives the desired result. \square

The following lemmas are special cases of Theorem 3.5.

Lemma 3.6. Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$, where

$$f = f_0 + \mathbf{i}f_1 + \mathbf{j}f_2 + \mathbf{k}f_3, \quad g = g_0 + \mathbf{i}g_1 + \mathbf{j}g_2 + \mathbf{k}g_3.$$

If $\mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$, then equation (32) takes the form

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= (\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}))\mathcal{F}_q\{g\}(\boldsymbol{\omega}) \\ &\quad + (\mathbf{j}\mathcal{F}_q\{f_2\}(-\omega_1, \omega_2) + \mathbf{k}\mathcal{F}_q\{f_3\}(-\omega_1, \omega_2))\mathcal{F}_q\{g\}(-\omega_1, \omega_2). \end{aligned} \quad (34)$$

On the other hand, if $\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2; \mathbb{R})$, then

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \mathcal{F}_q\{f\}(\boldsymbol{\omega})(\mathcal{F}_q\{g_0\}(\boldsymbol{\omega}) + \mathbf{j}\mathcal{F}_q\{g_2\}(-\omega_1, \omega_2)) \\ &\quad + \mathcal{F}_q\{f\}(\omega_1, -\omega_2)(\mathbf{i}\mathcal{F}_q\{g_1\}(\boldsymbol{\omega}) + \mathbf{k}\mathcal{F}_q\{g_3\}(-\omega_1, \omega_2)). \end{aligned} \quad (35)$$

Proof. We only prove expression (34) of Lemma 3.6, the other being similar. Following the steps of (33) we immediately get

$$\begin{aligned} &\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} f(\mathbf{y})g(\mathbf{z}) e^{-\mathbf{j}\omega_2(y_2+z_2)} dz d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (\{f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y})\} + \mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) \\ &\quad \times g(\mathbf{z})e^{-\mathbf{j}\omega_2(y_2+z_2)} dz d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y}))g(\mathbf{z}) e^{-\mathbf{j}\omega_2(y_2+z_2)} dz d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1(y_1+z_1)} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) g(\mathbf{z}) e^{-\mathbf{j}\omega_2(y_2+z_2)} dz d\mathbf{y} \\ &= \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (f_0(\mathbf{y}) + \mathbf{i}f_1(\mathbf{y}))\mathcal{F}_q\{g\}(\boldsymbol{\omega}) e^{-\mathbf{j}\omega_2 y_2} d\mathbf{y} \\ &\quad + \int_{\mathbb{R}^2} e^{-\mathbf{i}\omega_1 y_1} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y})) \mathcal{F}_q\{g\}(-\omega_1, \omega_2) e^{-\mathbf{j}\omega_2 y_2} d\mathbf{y} \\ &= (\mathcal{F}_q\{f_0\}(\boldsymbol{\omega}) + \mathbf{i}\mathcal{F}_q\{f_1\}(\boldsymbol{\omega}))\mathcal{F}_q\{g\}(\boldsymbol{\omega}) \\ &\quad + (\mathbf{j}\mathcal{F}_q\{f_2\}(-\omega_1, \omega_2) + \mathbf{k}\mathcal{F}_q\{f_3\}(-\omega_1, \omega_2))\mathcal{F}_q\{g\}(-\omega_1, \omega_2), \end{aligned} \quad (36)$$

which was to be proved. \square

Lemma 3.7. Let $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$, where

$$f = f_0 + \mathbf{i}f_1, \quad g = g_0 + \mathbf{i}g_1 + \mathbf{j}g_2 + \mathbf{k}g_3.$$

If $\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$, then

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \mathcal{F}_q\{f\}(\boldsymbol{\omega})\mathcal{F}_q\{g\}(\boldsymbol{\omega}) \\ &= \mathcal{F}_q\{g\}(\boldsymbol{\omega})\mathcal{F}_q\{f\}(\boldsymbol{\omega}), \end{aligned} \quad (37)$$

which is of the same form as a convolution of the classical Fourier transform [21].

Table 2: Comparison of convolution theorems of the QFT and classical FT for $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$.

Assumptions on quaternion functions	QFT of convolution
$\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{H})$	QFT \neq classical FT
$\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2; \mathbb{R})$ and $\mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{H})$	QFT \neq classical FT
$\mathcal{F}_q\{f\} \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$	QFT \neq classical FT
$f = f_0 + \mathbf{i}f_1$ and $\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$	QFT = classical FT
$f = \mathbf{j}f_2 + \mathbf{k}f_3$ and $\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$	QFT = classical FT

Remark 3.2. It is important to notice that if $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$, where

$$f = \mathbf{j}f_2 + \mathbf{k}f_3, \quad g = g_0 + \mathbf{i}g_1 + \mathbf{j}g_2 + \mathbf{k}g_3,$$

then Lemma 3.7 reduces to

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) &= \mathcal{F}_q\{f\}(\boldsymbol{\omega})\mathcal{F}_q\{g\}(-\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \\ &= \mathcal{F}_q\{g\}(-\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)\mathcal{F}_q\{f\}(\boldsymbol{\omega}), \end{aligned} \quad (38)$$

where $\mathcal{F}_q\{f\}, \mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$.

Table 2 compares convolution theorems of the QFT and classical FT for $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$.

The following theorem is useful for solving the heat equation in quaternion algebra.

Theorem 3.8. If $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$, then

$$\begin{aligned} \mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}\mathcal{F}_q\{g\}](\boldsymbol{x}) &= (f_0 \star g)(\boldsymbol{x}) + (\mathbf{i}f_1 \star g)(\boldsymbol{x}) \\ &\quad + (\mathbf{j}f_2 \star g)(-x_1, x_2) + (\mathbf{k}f_3 \star g)(-x_1, x_2). \end{aligned} \quad (39)$$

Proof. By the QFT inversion, we get, after some simplification,

$$\begin{aligned} &\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}\mathcal{F}_q\{g\}](\boldsymbol{x}) \\ &\stackrel{(16)}{=} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\boldsymbol{\omega}_1 x_1} e^{-\mathbf{i}\boldsymbol{\omega}_1 y_1} f(\boldsymbol{y}) e^{-\mathbf{j}\boldsymbol{\omega}_2 y_2} d\boldsymbol{y} \mathcal{F}_q\{g\}(\boldsymbol{\omega}) e^{\mathbf{j}\boldsymbol{\omega}_2 x_2} d\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\boldsymbol{\omega}_1(x_1 - y_1)} f(\boldsymbol{y}) \mathcal{F}_q\{g\}(\boldsymbol{\omega}) e^{\mathbf{j}\boldsymbol{\omega}_2(x_2 - y_2)} d\boldsymbol{y} d\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\boldsymbol{\omega}_1(x_1 - y_1)} (f_0(\boldsymbol{y}) + \mathbf{i}f_1(\boldsymbol{y}) + \mathbf{j}f_2(\boldsymbol{y}) + \mathbf{k}f_3(\boldsymbol{y})) \\ &\quad \times \mathcal{F}_q\{g\}(\boldsymbol{\omega}) e^{\mathbf{j}\boldsymbol{\omega}_2(x_2 - y_2)} d\boldsymbol{y} d\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{\mathbf{i}\boldsymbol{\omega}_1(x_1 - y_1)} (f_0(\boldsymbol{y}) + \mathbf{i}f_1(\boldsymbol{y})) + e^{\mathbf{i}\boldsymbol{\omega}_1(x_1 - y_1)} (\mathbf{j}f_2(\boldsymbol{y}) + \mathbf{k}f_3(\boldsymbol{y})) \\ &\quad \times \mathcal{F}_q\{g\}(\boldsymbol{\omega}) e^{\mathbf{j}\boldsymbol{\omega}_2(x_2 - y_2)} d\boldsymbol{y} d\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^2} (f_0(\boldsymbol{y}) + \mathbf{i}f_1(\boldsymbol{y})) g(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y} \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^2} (\mathbf{j}f_2(\mathbf{y}) + \mathbf{k}f_3(\mathbf{y}))g(-(x_1 - y_1), x_2 - y_2) d\mathbf{y} \\
& = (f_0 \star g)(\mathbf{x}) + (\mathbf{i}f_1 \star g)(\mathbf{x}) + (\mathbf{j}f_2 \star g)(-x_1, x_2) + (\mathbf{k}f_3 \star g)(-x_1, x_2), \tag{40}
\end{aligned}$$

where, in the second line, we have used the assumption $\mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$. This completes the proof of (39). \square

As an immediate consequence of Theorem 3.8, we get the following corollaries.

Corollary 3.9. *If $f, g \in L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_q\{g\} \in L^2(\mathbb{R}^2; \mathbb{R})$, where*

$$f = f_0 + \mathbf{i}f_1, \quad g = g_0 + \mathbf{i}g_1 + \mathbf{j}g_2 + \mathbf{k}g_3,$$

then (39) reduces to

$$\mathcal{F}_q^{-1}[\mathcal{F}_q\{f\}\mathcal{F}_q\{g\}](\mathbf{x}) = (f \star g)(\mathbf{x}). \tag{41}$$

Corollary 3.10. *Let*

$$f(\mathbf{x}) = \begin{cases} e^{-(x_1+x_2)}, & \text{if } x_1 > 0 \text{ and } x_2 > 0, \\ 0, & \text{otherwise,} \end{cases} \tag{42}$$

and consider the quaternionic Gabor filter

$$g(\mathbf{x}) = e^{\mathbf{i}u_0x_1} e^{\mathbf{j}v_0x_2} e^{-\frac{1}{2}|\mathbf{x}|^2}. \tag{43}$$

Then,

$$\mathcal{F}_q\{f \star g\}(\boldsymbol{\omega}) = \frac{e^{-\frac{1}{2}((\omega_1-u_0)^2+(\omega_2-v_0)^2)}(1 - \mathbf{i}\omega_1 - \mathbf{j}\omega_2 - \mathbf{k}\omega_1\omega_2)}{(2\pi)^2(1 + \omega_1^2 + \omega_2^2 + \omega_1^2\omega_2^2)}. \tag{44}$$

Proof. The QFT of f is given by

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \frac{1 - \mathbf{i}\omega_1 - \mathbf{j}\omega_2 - \mathbf{k}\omega_1\omega_2}{(2\pi)^2(1 + \omega_1^2 + \omega_2^2 + \omega_1^2\omega_2^2)}, \tag{45}$$

and the QFT of g is given by

$$\mathcal{F}_q\{g\}(\boldsymbol{\omega}) = e^{-\frac{1}{2}((\omega_1-u_0)^2+(\omega_2-v_0)^2)}. \tag{46}$$

Therefore, using Corollary 3.9, we obtain (44). \square

4 Applications of QFT

In [14], the authors proposed to use quaternions in order to define a Fourier transform applicable to color images. Their framework makes it possible to compute a single, holistic, Fourier transform which treats a color image as a vector field. In image processing, taking a given image as the initial value, the forward solution to the heat equation, or a diffusion equation in general, produces blurred images and the backward solution produces sharpen images, for example, see [24], pages 342–350.

In this section, we present two applications of QFT to partial differential equations in quaternion algebra.

4.1 Hypoellipticity

In this paper, since we only deal with QFT in the $L^2(\mathbb{R}^2; \mathbb{H})$ framework, we shall discuss the hypoellipticity in this framework, that is, we shall only deal with $L^2(\mathbb{R}^2; \mathbb{H})$ solutions for linear partial differential operators with constant quaternion coefficients:

$$P(\partial) = \sum_{0 \leq |\alpha| \leq n} a_\alpha \partial^\alpha b_\alpha, \quad a_\alpha, b_\alpha \in \mathbb{H}. \quad (47)$$

The non-commutativity of quaternion gives different aspects of $P(\partial)$ with constant complex coefficients $a_\alpha, b_\alpha \in \mathbb{C}$.

Example 4.1. Let $P(\partial) = \partial_{x_1} b$, $b \in \mathbb{H}$ and $f(\mathbf{x}), g(\mathbf{x}) \in C^1(\mathbb{R}^2; \mathbb{H})$.

(i) Since $\partial_{x_1}(f(\mathbf{x})g(\mathbf{x})) = (\partial_{x_1}f(\mathbf{x}))g(\mathbf{x}) + f(\mathbf{x})(\partial_{x_1}g(\mathbf{x}))$, we have $P(\partial)(f(\mathbf{x})g(\mathbf{x})) = (P(\partial)f(\mathbf{x}))g(\mathbf{x}) + f(\mathbf{x})(P(\partial)g(\mathbf{x}))$ when $b \in \mathbb{C}$. But, when $b \in \mathbb{H} \setminus \mathbb{C}$, as $bf(\mathbf{x}) \neq f(\mathbf{x})b$ in general, we cannot have $P(\partial)(f(\mathbf{x})g(\mathbf{x})) = (P(\partial)f(\mathbf{x}))g(\mathbf{x}) + f(\mathbf{x})(P(\partial)g(\mathbf{x}))$ in general.

(ii) We have $P(\partial)f \star g = f \star P(\partial)g$ when $b \in \mathbb{C}$. But, by the same reason as (i), when $b \in \mathbb{H} \setminus \mathbb{C}$, we cannot have $P(\partial)f \star g = f \star P(\partial)g$ in general.

Let us start with the definition of our $L^2(\mathbb{R}^2; \mathbb{H})$ version of hypoellipticity (compare to [20], page 110).

Definition 4.1. The linear partial differential operator $P(\partial)$ in \mathbb{R}^2 is said to be $L^2(\mathbb{R}^2; \mathbb{H})$ -hypoelliptic if, given any subset U of \mathbb{R}^2 and any solution u in $L^2(\mathbb{R}^2; \mathbb{H})$ such that $P(\partial)u$ is a C^∞ function in U , then all its components $u(u_i, i = 0, 1, 2, 3)$ is a C^∞ function in U .

Definition 4.2. Given a linear partial differential operator $P(\partial)$ of (47) with the quaternion constant coefficients. We say that a solution $E(\mathbf{x})$ of $P(\partial)u = \delta$, where δ is the delta function, is called a fundamental solution of $P(\partial)$.

Let A and B be subsets of \mathbb{R}^2 . Define the sum $A + B$ by $A + B = \{\mathbf{x} + \mathbf{y} \in \mathbb{R}^2; \mathbf{x} \in A, \mathbf{y} \in B\}$.

Theorem 4.1. Assume that there is one fundamental solution $E(\mathbf{x})$ of $P(\partial)$ which is a C^∞ function in $\mathbb{R}^2 \setminus \{0\}$, and the identities

$$P(\partial)(f \star g) = P(\partial)f \star g = f \star P(\partial)g \quad (48)$$

are satisfied for arbitrary sufficiently smooth quaternion-valued functions f and g such that gf is a compactly supported C^∞ quaternion function with a_α of $P(\partial)$ are quaternion constant coefficients and b_α of $P(\partial)$ is real constant coefficients. Then, the linear partial differential operator $P(\partial)$ is $L^2(\mathbb{R}^2; \mathbb{H})$ -hypoelliptic in \mathbb{R}^2 .

Proof. Firstly, let U be an arbitrary open subset of \mathbb{R}^2 , u a solution in U with values in \mathbb{H} such that $f = P(\partial)u$ is a C^∞ function in U . Let \mathbf{x}_0 be an arbitrary point in U . It will suffice to show that u is a C^∞ function in some open neighborhood of \mathbf{x}_0 . Take an open disc $D_\eta(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^2; |\mathbf{x} - \mathbf{x}_0| < \eta\}$ such that $D_\eta(\mathbf{x}_0) \subset U$. There exists a function $g \in C^\infty(U; \mathbb{R})$ such that $\text{supp } g \subset U$ and $g = 1$ in $D_\eta(\mathbf{x}_0)$. Then, we have

$$P(\partial)(gu) = gP(\partial)u + v = gf + v,$$

where every term of v contains a derivative of g of non-zero order, therefore $v = 0$ where the derivatives of g vanish, especially in $D_\eta(\mathbf{x}_0)$ and outside of $\text{supp } g$. For the fundamental solution $E(\mathbf{x})$, the hypothesis (48) implies

$$E \star P(\partial)(gu) = \{P(\partial)E\} \star (gu) = gu,$$

hence

$$gu = E \star (gf) + E \star v.$$

But gf is a compactly supported C^∞ function and the convolution of any function with any compactly supported C^∞ function is a C^∞ function. Therefore, it suffices to show that $E \star v$ is a C^∞ function in an open neighborhood of \mathbf{x}_0 , because gu is also a C^∞ function in an open neighborhood of \mathbf{x}_0 and $gu = u$ in $D_\eta(\mathbf{x}_0)$.

Finally, we shall show that $E \star v$ is a C^∞ function in an open neighborhood of \mathbf{x}_0 . Let us select $\varepsilon > 0$ such that $\varepsilon < 1/2\eta$. Then, the open disc $D_\varepsilon(\mathbf{x}_0)$ is a neighborhood of \mathbf{x}_0 . Let $\zeta_\varepsilon(\mathbf{x}) \in C^\infty(\mathbb{R}^2; \mathbb{R})$, another cutoff function, be equal to one for $|\mathbf{x}| < \varepsilon/2$ and to zero for $|\mathbf{x} - \mathbf{x}_0| > \varepsilon$. We have

$$E \star v = (\zeta_\varepsilon E) \star v + \{(1 - \zeta_\varepsilon)E\} \star v.$$

The hypothesis implies that $(1 - \zeta_\varepsilon)E \in C^\infty(\mathbb{R}^2; \mathbb{H})$, and therefore $(1 - \zeta_\varepsilon)E \star v \in C^\infty(\mathbb{R}^2; \mathbb{H})$. Since

$$\text{supp}\{(\zeta_\varepsilon E) \star v\} \subset \text{supp}(\zeta_\varepsilon E) + \text{supp } v,$$

$\text{supp}\{(\zeta_\varepsilon E) \star v\}$ is contained in the ε -neighborhood of $\text{supp } v$. We have already seen that $v = 0 \in D_\eta(\mathbf{x}_0)$. Hence, $(\zeta_\varepsilon E) \star v$ vanishes in $D_\varepsilon(\mathbf{x}_0)$ and, therefore, $E \star v$ is a C^∞ function in $D_\varepsilon(\mathbf{x}_0)$. \square

4.2 Parabolic initial value problem

Let us consider the parabolic initial value problem

$$\frac{\partial}{\partial t} u - \nabla^2 u = 0 \quad \text{on } \mathbb{R}^2 \times (0, \infty), \quad (49)$$

with

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \quad f \in \mathcal{S}(\mathbb{R}^2; \mathbb{H}), \quad (50)$$

where $\mathcal{S}(\mathbb{R}^2; \mathbb{H})$ is the quaternion Schwartz space. Applying the QFT, we easily obtain

$$\begin{aligned} \mathcal{F}_q\{u_t\} &= (\mathbf{i}\omega_1)^2 \mathcal{F}_q\{u\}(\boldsymbol{\omega}) + \mathcal{F}_q\{u\}(\boldsymbol{\omega}) (\mathbf{j}\omega_2)^2 \\ &= -|\boldsymbol{\omega}|^2 \mathcal{F}_q\{u\}(\boldsymbol{\omega}). \end{aligned} \quad (51)$$

The general solution of (51) is given by

$$\mathcal{F}_q\{u\}(\boldsymbol{\omega}, t) = C e^{-|\boldsymbol{\omega}|^2 t}, \quad (52)$$

where C is a quaternion constant. We impose the initial condition $\mathcal{F}_q\{u\}(\boldsymbol{\omega}, 0) = \mathcal{F}_q\{f\}(\boldsymbol{\omega})$ to obtain

$$\mathcal{F}_q\{u\}(\boldsymbol{\omega}, t) = e^{-|\boldsymbol{\omega}|^2 t} \mathcal{F}_q\{f\}(\boldsymbol{\omega}). \quad (53)$$

Notice that the QFT of a Gaussian quaternion function is also a Gaussian quaternion function (compare to Mawardi, Hitzer, Hayashi, and Ashino [4]). Hence

$$\frac{1}{4\pi t} \mathcal{F}_q\{(e^{-|\mathbf{x}|^2/(4t)})\} = e^{-|\boldsymbol{\omega}|^2 t}. \quad (54)$$

Applying the inverse QFT, we have

$$\begin{aligned} u(\mathbf{x}, t) &= \mathcal{F}_q^{-1} \left[e^{-|\boldsymbol{\omega}|^2 t} \mathcal{F}_q \{f\} \right] (\mathbf{x}) \\ &= \frac{1}{4\pi t} \mathcal{F}_q^{-1} \left[\mathcal{F}_q \{f\} \mathcal{F}_q \{e^{-|\mathbf{x}|^2/(4t)}\} \right] (\mathbf{x}). \end{aligned} \quad (55)$$

Since

$$\mathcal{F}_q \{e^{-|\mathbf{x}|^2/(4t)}\}(\boldsymbol{\omega}) = 4\pi t e^{-|\boldsymbol{\omega}|^2 t} \in L^2(\mathbb{R}^2; \mathbb{R}),$$

then we can apply the convolution theorem of equation (39) to get

$$u(\mathbf{x}, t) = (f_0 \star K_t)(\mathbf{x}) + (\mathbf{i}f_1 \star K_t)(\mathbf{x}) + (\mathbf{j}f_2 \star K_t)(-x_1, x_2) + (\mathbf{k}f_3 \star K_t)(-x_1, x_2), \quad (56)$$

where $K_t = \frac{1}{4\pi t} e^{-|\mathbf{x}|^2/(4t)}$, and $f_i \in L^2(\mathbb{R}^2; \mathbb{R})$, $i = 0, 1, 2, 3$. By Definition (3.1) of the convolution, we finally obtain

$$\begin{aligned} u(\mathbf{x}, t) &= \frac{1}{4\pi t} \int_{\mathbb{R}^2} f_0(\mathbf{y}) e^{-((x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y} \\ &\quad + \frac{\mathbf{i}}{4\pi t} \int_{\mathbb{R}^2} f_1(\mathbf{y}) e^{-((x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y} \\ &\quad + \frac{\mathbf{j}}{4\pi t} \int_{\mathbb{R}^2} f_2(\mathbf{y}) e^{-((-x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y} \\ &\quad + \frac{\mathbf{k}}{4\pi t} \int_{\mathbb{R}^2} f_3(\mathbf{y}) e^{-((-x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y}. \end{aligned} \quad (57)$$

In an actual application, one often takes the quaternionic Gabor filter (see [10, 5]) as

$$f(\mathbf{x}) = e^{\mathbf{i}u_0x_1} e^{-(x_1^2+x_2^2)} e^{\mathbf{j}v_0x_2}.$$

Therefore, the above identity will reduce to

$$\begin{aligned} u(\mathbf{x}, t) &= \frac{1}{4\pi t} \int_{\mathbb{R}^2} \cos(u_0x_1) \cos(v_0x_2) e^{-(x_1^2+x_2^2)} e^{-((x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y}, \\ &\quad + \frac{\mathbf{i}}{4\pi t} \int_{\mathbb{R}^2} \sin(u_0x_1) \cos(v_0x_2) e^{-(x_1^2+x_2^2)} e^{-((x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y}, \\ &\quad + \frac{\mathbf{j}}{4\pi t} \int_{\mathbb{R}^2} \cos(u_0x_1) \sin(v_0x_2) e^{-(x_1^2+x_2^2)} e^{-((-x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y}, \\ &\quad + \frac{\mathbf{k}}{4\pi t} \int_{\mathbb{R}^2} \sin(u_0x_1) \sin(v_0x_2) e^{-(x_1^2+x_2^2)} e^{-((-x_1-y_1)^2+(x_2-y_2)^2)/(4t)} d\mathbf{y}. \end{aligned} \quad (58)$$

5 Conclusion

Due to the non-commutative property of quaternion multiplication, there are three different types of two-dimensional QFTs. These three QFTs are so-called a left-sided QFT, a right-sided QFT, and a double-sided QFT, respectively. In this work, we have established convolution theorem of the double-sided QFT applied to real fields $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and quaternion fields $f : \mathbb{R}^2 \rightarrow \mathbb{H}$. Some important properties of the QFT convolution are investigated. We have shown that the QFT convolution is useful to study hypoellipticity and to solve

the heat equation in quaternion algebra framework. It easily can be seen that the solution of generalized heat equation is extension of solution of the classical heat equation.

The future work will establish the convolution theorems of the right-sided QFT. We compare some properties of the convolution theorems of the two types QFTs. We will apply the properties to find the solution of partial differential equations in quaternion algebra framework. The solutions of generalized partial differential equations using the properties of the three types of two-dimensional QFTs will be compared too.

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