A PRIMER ON SPECTRAL SEQUENCES

Contents

1. Definitions 1
2. Exact Couples 3
3. Filtered Complexes 5
4. Products 6
5. The Serre spectral sequence 8
6. The comparison theorem 12
7. Convergence proofs 12
References 15

This chapter contains those results about spectral sequences that we used earlier in the book, incorporated into a brief background compendium of the very minimum that anybody interested in algebraic topology needs to know about spectral sequences. Introductory books on algebraic topology usually focus on the different kinds of chain and cochain complexes that can be used to define ordinary homology and cohomology. It is a well kept secret that the further one goes into the subject, the less one uses such complexes for actual calculation. Rather, one starts with a few spaces whose homology and cohomology groups can be computed by hand, using explicit chain complexes. One then bootstraps up such calculations to the vast array of currently known computations using a variety of spectral sequences. McCleary's book [3] is a good encyclopedic reference for the various spectral sequences in current use. Other introductions can be found in many texts in algebraic topology and homological algebra [1, 4, 5]. However, the truth is that the only way to master the use of spectral sequences is to work out many examples in detail.

All modules are over a commutative ring R, and understood to be graded, whether or not the grading is mentioned explicitly or denoted. In general, we leave the gradings implicit for readability. The preliminaries on tensor product and Hom functors of Section **??** remain in force in this chapter.

1. Definitions

While spectral sequences arise with different patterns of gradings, the most commonly encountered homologically and cohomologically graded spectral sequences fit into the patterns given in the following pair of definitions.

Definition 1.1. A homologically graded spectral sequence $E = \{E^r\}$ consists of a sequence of $\mathbb{Z}$-bigraded R modules $E^r = \{E^r_{p,q}\}_{r\geq 1}$ together with differentials

$$d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $E^{r+1} \cong H_*(E^r)$. A morphism $f\colon E \to E'$ of spectral sequences is a family of morphisms of complexes $f^r\colon E^r \to E'^r$ such that f^{r+1} is the morphism $H_*(f^r)$ induced by f^r.

Definition 1.2. A cohomologically graded spectral sequence $E = \{E_r\}$ consists of $\mathbb{Z}$-bigraded R-modules $E_r = \{E_r^{p,q}\}_{r\geq 1}$ together with differentials

$$d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $E_{r+1} \cong H_*(E_r)$. We can regrade E_r homologically by setting $E_r^{p,q} = E^r_{-p,-q}$, so in principle the two grading conventions define the same concept.

Let $E = \{E^r\}$ be a spectral sequence. Let Z^1, the cycles, be the kernel of d^1 and B^1, the boundaries, be the image of d^1. Then, under the identification of $H_*(E^1)$ with E^2, d^2 is a map

$$Z^1/B^1 \to Z^1/B^1$$

Continuing this identification, E^r is identified with Z^{r-1}/B^{r-1} and the map

$$d^r\colon Z^{r-1}/B^{r-1} \to Z^{r-1}/B^{r-1}$$

has kernel Z^r/B^{r-1} and image B^r/B^{r-1}. These identifications give a sequence of submodules

$$0 = B^0 \subset B^1 \subset \ldots \subset Z^2 \subset Z^1 \subset Z^0 = E^1.$$

Define $Z^\infty = \cap_{r=1}^\infty Z^r$, $B^\infty = \cup_{r=1}^\infty B^r$, and $E^\infty_{p,q} = Z^\infty_{p,q}/B^\infty_{p,q}$, writing $E^\infty = \{E^\infty_{p,q}\}$.

We say that E is a first quadrant spectral sequence if $E^r_{p,q} = 0$ for $p < 0$ or $q < 0$. In a first quadrant spectral sequence the terms $\{E^r_{p,0}\}$ are called the base terms and the terms $\{E^r_{0,q}\}$ are called the fiber terms. Note that elements of $E^r_{p,0}$ cannot be boundaries for $r \geq 2$ since the differential

$$d^r\colon E^r_{p+r,-r+1} \longrightarrow E^r_{p,0}$$

has domain the 0 group. Thus

$$E^{r+1}_{p,0} = \operatorname{Ker}(d^r\colon E^r_{p,0} \to E^r_{p-r,r-1})$$

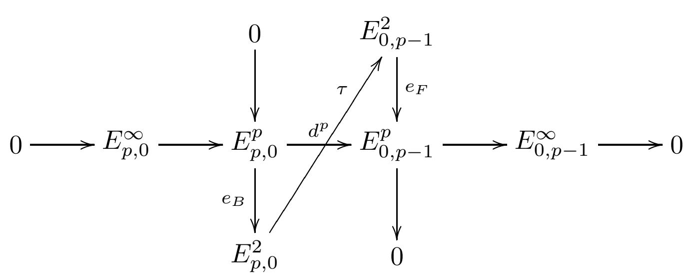
and there is a sequence of monomorphisms

$$e_B\colon E^\infty_{p,0} = E^{p+1}_{p,0} \to E^p_{p,0} \to \ldots \to E^3_{p,0} \to E^2_{p,0}.$$

Similarly, for $r \geq 1$, $E^r_{0,q}$ consists only of cycles and so there are epimorphisms

$$e_F\colon E^2_{0,q} \to E^3_{0,q} \to \ldots \to E^{q+2}_{0,q} = E^\infty_{0,q}.$$

The maps e_B and e_F are called edge homomorphisms. From these maps we define a "map" $\tau = e_F^{-1} d^p e_B^{-1}\colon E^2_{p,0} \to E^2_{0,p-1}$ as in the following diagram.



This map is called the transgression. It is an additive relation [1, II.6] from a submodule of $E^2_{p,0}$ to a quotient module of $E^2_{0,p-1}$.

A cohomologically graded first quadrant spectral sequence E is also defined to have $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$. However, when regraded homologically it becomes a third quadrant spectral sequence. Again, its base terms have $q = 0$ and its fiber terms have $p = 0$. It has edge homomorphisms

$$e_B \colon E_2^{p,0} \to E_3^{p,0} \to \ldots \to E_p^{p,0} \to E_{p+1}^{p,0} = E_\infty^{p,0}$$

(which are epimorphisms) and

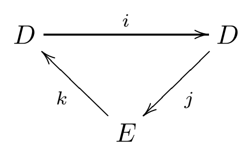
$$e_F \colon E_\infty^{0,q} = E_{q+2}^{0,q} \longrightarrow E_{q+1}^{0,q} \longrightarrow \ldots \longrightarrow E_3^{0,q} \to E_2^{0,q}$$

(which are monomorphisms). Its transgression $\tau = e_B^{-1} d_p e_F^{-1}$ is induced by the differential $d_p \colon E_p^{0,p-1} \to E_p^{p,0}$. It is an additive relation from a submodule of $E_2^{0,p-1}$ to a quotient module of $E_2^{p,0}$.

2. Exact Couples

Exact couples provide an especially useful and general source of spectral sequences. We first define them in general, with unspecified gradings. This leads to the most elementary example, called the Bockstein spectral sequence. We then describe the gradings that usually appear in practice.

Definition 2.1. Let D and E be modules. An exact couple $\mathscr{C} = \langle D, E; i, j, k \rangle$ is a diagram



in which Ker $j = \operatorname{Im} i$, Ker $k = \operatorname{Im} j$, and Ker $i = \operatorname{Im} k$.

If $d = jk \colon E \to E$, then $d \circ d = jkjk = 0$. Construct $\mathscr{C}' = \langle D', E'; i', j', k' \rangle$ by letting

$$D' = i(D) \quad \text{and} \quad E' = H_*(E; d),$$

and, writing overlines to denote passage to homology classes,

$$i' = i|_{i(D)}, \quad j'(i(x)) = \overline{j(x)} = j(x) + jk(E), \quad \text{and} \quad k'(\bar{y}) = k'(y + jk(E)) = k(y).$$

That is, i' is a restriction of i and j' and k' are induced from j and k by passing to homology on targets and sources. We easily check that j' and k' are well-defined and the following result holds.

Lemma 2.2. *$\mathscr{C}'$ is an exact couple.*

Starting with $\mathscr{C} = \mathscr{C}^{(1)}$, we can iterate the construction to form

$$\mathscr{C}^{(r)} = \langle D^r, E^r, i^r, j^r, k^r \rangle.$$

(The notation might be confusing since the maps i^r, j^r, and k^r given by the construction are not iterated composites). Then, with $d^r = j^r k^r$, $\{E^r\}$ is a spectral sequence. It can be graded differently than in the previous section since we have not specified conditions on the grading of D and E. The Bockstein spectral sequence in the proof of **??** comes from a particularly simple exact couple and is singly graded rather than bigraded.

Example 2.3. Let C be a torsion free chain complex over $\mathbb{Z}$. From the short exact sequence of groups

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

we obtain a short exact sequence of chain complexes

$$0 \longrightarrow C \longrightarrow C \longrightarrow C \otimes \mathbb{Z}/p\mathbb{Z} \longrightarrow 0\ .$$

The induced long exact homology sequence is an exact couple

$$\begin{array}{ccc} H_*(C) & \longrightarrow & H_*(C)\ . \\ & \nwarrow \quad \swarrow & \\ & H_*(C \otimes \mathbb{Z}/p\mathbb{Z}) & \end{array}$$

The resulting spectral sequence is called the mod p Bockstein spectral sequence. Here $d^r\colon E^r_n \to E^r_{n-1}$ for all $r \geq 1$ and all n, and we have short exact sequences

$$0 \longrightarrow (p^{r-1}H_n(C)) \otimes \mathbb{Z}/p\mathbb{Z} \longrightarrow E^r_n \longrightarrow \mathrm{Tor}(p^{r-1}H_{n-1}(C), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0\ .$$

When $r = 1$, this is the universal coefficient exact sequence for calculating $H_n(C; \mathbb{F}_p)$, and we may view it as a higher universal coefficient exact sequence in general.

We can describe this spectral sequence in very elementary terms. Let Σ^n be the functor on graded Abelian groups given by $(\Sigma^n A)_{q+n} = A_q$. For a cyclic Abelian group π, we have a $\mathbb{Z}$-free resolution $C(\pi)$ given by $\mathbb{Z}$ in degree 0 if $\pi = \mathbb{Z}$ and by copies of $\mathbb{Z}$ in degrees 0 and 1 with differential q^s if $\pi = \mathbb{Z}/q^s$. Assume that $H_*(C)$ is of finite type and write $H_n(C)$ as a direct sum of cyclic groups. For each cyclic summand, choose a representative cycle x and, if $\pi = \mathbb{Z}/q^s$, a chain y such that $d(y) = q^s x$. For each cyclic summand π, these choices determine a chain map $\Sigma^n C(\pi) \longrightarrow C$. Summing over the cyclic summands and over n, we obtain a chain complex C' and a chain map $C' \longrightarrow C$ that induces an isomorphism on homology and on Bockstein spectral sequences.

The Bockstein spectral sequences $\{E^r\}$ of the $\Sigma^n C(\pi)$ are trivial to compute. When $\pi = \mathbb{Z}$, $E^r_n = \mathbb{Z}$ and $E^r_m = 0$ for $m \neq n$ for all r. When $\pi = \mathbb{Z}/q^s$ for $q \neq p$, $E^r_n = 0$ for all n and r. When $\pi = \mathbb{Z}/p^s$, $E^1 = E^s$ is $\mathbb{F}_p$ in degrees n and $n+1$, $d^s\colon E^s_{n+1} \longrightarrow E^s_n$ is an isomorphism, and $E^r = 0$ for $r > s$. Returning to C, we see that $E^\infty \cong (H_*(C)/TH_*(C)) \otimes \mathbb{F}_p$, where $T\pi$ denotes the torsion subgroup of a finitely generated Abelian group π. Moreover, there is one summand $\mathbb{Z}/p^s$ in $H_*(C)$ for each summand $\mathbb{F}_p$ in the vector space $d^s E^s$. The higher universal coefficient exact sequences are easy to see from this perspective.

We conclude that complete knowledge of the Bockstein spectral sequences of C for all primes p allows a complete description of $H_*(C)$ as a graded Abelian group.

The previous example shows that if X is a space whose homology is of finite type and if one can compute $H_*(X; \mathbb{Q})$ and $H_*(X; \mathbb{F}_p)$ together with the mod p Bockstein spectral sequences for all primes p, then one can read off $H_*(X; \mathbb{Z})$. For this reason, among others, algebraic topologists rarely concern themselves with integral homology but rather focus on homology with field coefficients. This is one explanation for the focus of this book on rationalization and completion at primes.

This is just one particularly elementary example of an exact couple. More typically, D and E are $\mathbb{Z}$-bigraded and, with homological grading, we have

$$\deg i = (1,-1), \quad \deg j = (0,0), \quad \text{and} \quad \deg k = (-1,0).$$

This implies that

$$\deg i^r = (1,-1), \quad \deg j^r = (-(r-1), r-1), \quad \text{and} \quad \deg k^r = (-1,0).$$

Since $d^r = j^r k^r$, we then have

$$d^r \colon E^r_{p,q} \to E^r_{p-r,q+r-1},$$

as in our original definition of a spectral sequence.

3. Filtered Complexes

Filtered chain complexes give rise to exact couples and therefore to spectral sequences. This is one of the most basic sources of spectral sequences. The Serre spectral sequence, which we describe in Section 5, could be obtained as an example, although we shall construct it differently.

Let A be a $\mathbb{Z}$-graded complex of modules. An (increasing) filtration of A is a sequence of subcomplexes

$$\ldots \subset F_{p-1}A \subset F_pA \subset F_{p+1}A \subset \ldots$$

of A. The associated graded complex E^0A is the bigraded complex defined by

$$E^0_{p,q}A = (F_pA/F_{p-1}A)_{p+q},$$

with differential d^0 induced by that of A. The homology $H_*(A)$ is filtered by

$$F_pH_*(A) = \operatorname{Im}(H_*(F_pA) \to H_*(A)),$$

and thus $E^0H_*(A)$ is defined.

Let $A_{p,q} = (F_pA)_{p+q}$. The inclusion $F_{p-1}A \subset F_pA$ restricts to inclusions $i\colon A_{p-1,q+1} \to A_{p,q}$ and induces quotient maps $j\colon A_{p,q} \to E^0_{p,q}$. The short exact sequence

$$0 \longrightarrow F_{p-1}A \xrightarrow{i} F_pA \xrightarrow{j} E^0_pA \longrightarrow 0 \tag{3.1}$$

of chain complexes induces a long exact sequence

$$\cdots \longrightarrow H_n(F_{p-1}A) \xrightarrow{i_*} H_n(F_pA) \xrightarrow{j_*} H_n(E^0_pA) \xrightarrow{k_*} H_{n-1}(F_{p-1}A) \longrightarrow \cdots$$

Let $D^1_{p,q} = H_{p+q}(F_pA)$ and $E^1_{p,q} = H_{p+q}(E^0_p)$. Then

$$\langle D^1, E^1; i_*, j_*, k_* \rangle$$

is an exact couple. It gives rise to a spectral sequence $\{E^rA\}$, which is functorial on the category of filtered complexes.

Theorem 3.2. *If $A = \cup_p F_pA$ and for each n there exists $s(n)$ such that $F_{s(n)}A_n = 0$, then $E^\infty_{p,q}A = E^0_{p,q}H_*(A)$.*

The proof is tedious, but elementary. We give it in the last section of the chapter for illustrative purposes. The conclusion of the theorem, $E^\infty_{p,q}A \cong E^0_{p,q}H_*(A)$, is often written

$$E^2_{p,q}A \Rightarrow H_{p+q}(A),$$

and E^r is said to converge to $H_*(A)$.

The filtration of A is said to be canonically bounded if $F_{-1}A = 0$ and $F_nA_n = A_n$ for all n, and in this case E^r certainly converges to $H_*(A)$.

Dually, cohomology spectral sequences arise naturally from decreasing filtrations of complexes. Regrading complexes cohomologically, so that the differentials are maps $\delta\colon A^n \longrightarrow A^{n+1}$, a decreasing filtration is a sequence

$$\ldots \supset F^pA \supset F^{p+1}A \supset \ldots.$$

If we rewrite A as a complex, $A_n = A^{-n}$, and define $F_pA = F^{-p}A$, then our construction of homology spectral sequences immediately gives a cohomology spectral sequence $\{E_rA\}$. With evident changes of notation, Theorem 3.2 takes the following cohomological form.

Theorem 3.3. *If $A = \cup_p F^pA$ and for each n there exists $s(n)$ such that $F^{s(n)}A^n = 0$, then $E_\infty^{p,q}A = E_0^{p,q}H^*(A)$.*

A decreasing filtration is canonically bounded if $F^0A = A$ and $F^{n+1}A^n = 0$ for all n, and in this case E_r certainly converges to $H^*(A)$.

In practice, we often start with a homological filtered complex and dualize it to obtain a cohomological one, setting $A^* = \mathrm{Hom}(A, R)$ and filtering it by

$$F^pA^* = \mathrm{Hom}(A/F_{p-1}A, R)$$

At least when R is a field, the resulting cohomology spectral sequence is dual to the homology spectral sequence.

4. Products

A differential graded algebra (DGA) A over R is a graded algebra with a product that is a map of chain complexes, so that the Leibnitz formula

$$d(xy) = d(x)y + (-1)^{degx}xd(y)$$

is satisfied. When suitably filtered, A often gives rise to a spectral sequence of DGA's, meaning that each term E^r is a DGA. It is no exaggeration to say that the calculational utility of spectral sequences largely stems from such multiplicative structure. We give a brief description of how such structure arises in this section. We work more generally with exact couples rather than filtered chain complexes, since our preferred construction of the Serre spectral sequence will largely avoid the use of chains and cochains.

Let $\mathscr{C}_1 = \langle D_1, E_1; i_1, j_1, k_1\rangle$, $\mathscr{C}_2 = \langle D_2, E_2; i_2, j_2, k_2\rangle$, and $\mathscr{C} = \langle D, E, i, j, k\rangle$ be exact couples. A pairing

$$\phi\colon E_1 \otimes E_2 \to E$$

is said to satisfy the condition μ_n if for any $x \in E_1$, $y \in E_2$, $a \in D_1$ and $b \in D_2$ such that $k_1(x) = i_1^n(a)$ and $k_2(y) = i_2^n(b)$ there exists $c \in D$ such that

$$k(xy) = i^n(c)$$

and

$$j(c) = j_1(a)y + (-1)^{\deg x}xj_2(b).$$

We write the pairing by concatenation rather than using ϕ to minimize notation. By convention, we set $i_1^0 = \mathrm{id}$ and $i_2^0 = \mathrm{id}$. Then the only possible choices are $a = k_1(x)$, $b = k_2(y)$, and $c = k(xy)$, so that μ_0 is the assertion that

$$jk(xy) = j_1k_1(x)y + (-1)^{\deg x}xj_2k_2(y).$$

Since the differential on E is jk, and similarly for E_1 and E_2, μ_0 is precisely the assertion that ϕ is a map of chain complexes, and it then induces

$$\phi' \colon E_1' \otimes E_2' \longrightarrow E'.$$

We say that ϕ satisfies the condition μ if ϕ satisfies μ_n for all $n \geq 0$.

Proposition 4.1. *Assume that ϕ satisfies μ_0. Then ϕ satisfies μ_n if and only if $\phi' \colon E_1' \otimes E_2' \to E'$ satisfies μ_{n-1}.*

Proof. Suppose that ϕ satisfies μ_n. Let $x' \in E_1'$, $y' \in E_2'$, $a' \in D_1'$, $b' \in D_2'$ satisfy $k_1'(x') = {i'}_1^{n-1}(a')$ and $k_2'(y') = {i'}_2^{n-1}(b')$. If $x' = \bar{x}$, $y' = \bar{y}$, $a' = i_1(a)$, and $b' = i_2(b)$, we find that

$$k_1(x) = i_1^n(a) \quad \text{and} \quad k_2(y) = i_2^n(b).$$

It follows that there exits $c \in D$ such that

$$k(xy) = i^n(c) \quad \text{and} \quad j(c) = j_1(a)y + (-1)^{\deg x} x j_2(b).$$

Taking $c' = i(c)$, we find that

$$k'(\overline{xy}) = k'(\bar{x}\bar{y}) = {i'}^{n-1}(c')$$

and

$$j'(c') = j_1'(a')y' + (-1)^{\deg x'} x' j_2'(b').$$

The converse is proven similarly. □

Corollary 4.2. *If ϕ satisfies μ, then so does ϕ', and therefore so do all successive*

$$\phi^r \colon E_1^r \otimes E_2^r \to E^r,$$

$r \geq 1$, where ϕ^{r+1} is the composite

$$H_*(E_1^r) \otimes H_*(E_2^r) \to H_*(E_1^r \otimes E_2^r) \longrightarrow H_*(E^r)$$

of the Künneth map and $H_*(\phi^r)$. Thus each ϕ^r is a map of chain complexes.

The point is that it is usually quite easy to see explicitly that ϕ satisfies μ, and we are entitled to conclude that the induced pairing of E^r terms satisfies the Leibnitz formula for each $r \geq 1$.

Example 4.3. The cup product in the singular cochains $C^*(X)$ gives rise to the product

$$\phi \colon H^*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \to H^*(X; \mathbb{F}_p).$$

Regarding $H^*(X; \mathbb{F}_p)$ as the E_1 term of the Bockstein spectral sequence of the cochain complex $C^*(X)$, we find that ϕ satisfies μ. Therefore each E^r in the mod p cohomology Bockstein spectral sequence of X is a DGA, and $E^{r+1} = H^*(E^r)$ as an algebra.

Now let A, B, and C be filtered complexes. Filter $A \otimes B$ by

$$F_p(A \otimes B) = \sum_{i+j=p} F_i A \otimes F_j B.$$

Suppose $\phi \colon A \otimes B \to C$ is a morphism of filtered complexes, so that

$$F_p A \cdot F_q B \subset F_{p+q} C.$$

Then ϕ induces a morphism of spectral sequences

$$E^r(A \otimes B) \to E^r(C).$$

Since $E^rA \otimes E^rB$ is a complex, we have a Künneth map

$$E^rA \otimes E^rB \to E^r(A \otimes B),$$

and its composite with $H_*(\phi)$ defines a pairing

$$E^rA \otimes E^rB \to E^rC.$$

This is a morphism of complexes since an easy verification shows that

$$\phi_*\colon E^1A \otimes E^1B \to E^1C$$

satisfies the condition μ. If R is a field, or more generally if our Künneth map is an isomorphism, then $\{E^rA \otimes E^rB\}$ is a spectral sequence isomorphic to $\{E^r(A \otimes B)\}$ and the product is actually a morphism of spectral sequences. In general, however, we are concluding the Leibnitz formula even when the Künneth map of E^r terms does not induce an isomorphism on homology.

If, further, each of of the filtered complexes A, B and C satisfies the hypothesis of the convergence theorem, Theorem 3.2, then inspection of its proof shows that the product

$$E^\infty A \otimes E^\infty B \to E^\infty C$$

agrees with the product

$$E^0H_*(A) \otimes E^0H_*(B) \longrightarrow E^0H_*(C)$$

induced by passage to quotients from the induced pairing

$$H_*(A) \otimes H_*(B) \to H_*(C).$$

5. The Serre spectral sequence

We give what we feel is perhaps the quickest construction of the Serre spectral sequence, but, since we do not want to go into details of local coefficients, we leave full verifications of its properties, in particular the identification of the E_2 term, to the reader. In applications, the important thing is to understand what the properties say. Their proofs generally play no role. In fact, this is true of most spectral sequences in algebraic topology. It is usual to construct the Serre spectral using the singular (or, in Serre's original work, cubical) chains of all spaces in sight. We give a more direct homotopical construction that has the advantage that it generalizes effortlessly to a construction of the Serre spectral sequence in generalized homology and cohomology theories.

For definiteness, we take $R = \mathbb{Z}$ here, and we fix an Abelian group π of coefficients. We could just as well replace $\mathbb{Z}$ by any commutative ring R and π by any R-module. Let $p\colon E \to B$ be a Serre fibration with fiber F and connected base space B. It is usual to assume that F too is connected, but that is not really necessary. Fixing a basepoint $b \in B$, we may take $F = p^{-1}(b)$, and that fixes an inclusion $i\colon F \longrightarrow E$. Using [2, p. 48], we may as well replace p by a Hurewicz fibration. This is convenient since it allows us to exploit a relationship between cofibrations and fibrations that does not hold for Serre fibrations. Using [2, p. 75], we may choose a based weak equivalence f from a CW complex with a single vertex to B. Pulling back p along f, we may as well replace p by a Hurewicz fibration whose base space is a CW complex B with a single vertex b. Having a CW base space gives a geometric filtration with which to work, and having a single vertex fixes a canonical basepoint and thus a canonical fiber.

Give B its skeletal filtration, $F_pB = B^p$, and define $F_pE = p^{-1}(F_pB)$. Observe that $F_0E = F$. By **??**, the inclusions $F_{p-1}E \subset F_pE$ are cofibrations. They give long exact sequences of pairs on homology with coefficients in any fixed Abelian group π. We set

$$D^1_{p,q} = H_{p+q}(F_pE;\pi) \text{ and } E^1_{p,q} = H_{p+q}(F_pE, F_{p-1}E;\pi),$$

and we may identify $E^1_{p,q}$ with $\tilde{H}_{p+q}(F_pE/F_{p-1}E;\pi)$. The cited long exact sequences are given by maps

$$i^1\colon H_{p+q}(F_{p-1}E;\pi) \longrightarrow H_{p+q}(F_pE;\pi)$$

and

$$j^1\colon H_{p+q}(F_pE;\pi) \longrightarrow H_{p+q}(F_pE, F_{p-1}E;\pi)$$

induced by the inclusions $i\colon F_{p-1}E \subset F_pE$ and $j\colon (F_pE,\emptyset) \subset (F_pE, F_{p-1}E)$ and by connecting homomorphisms

$$k^1\colon H_{p+q}(F_pE, F_{p-1}E;\pi) \longrightarrow H_{p+q-1}(F_{p-1}E;\pi).$$

We have an exact couple and therefore a spectral sequence. Let $C_*(B)$ denote the *cellular* chains of the CW complex B. Filter $H_*(E;\pi)$ by the images of the $H_*(F_pE;\pi)$.

Theorem 5.1 (Homology Serre spectral sequence)**.** *There is a first quadrant homological spectral sequence* $\{E^r, d^r\}$*, with*

$$E^1_{p,q} \cong C_p(B;\mathscr{H}_q(F;\pi)) \text{ and } E^2_{p,q} \cong H_p(B;\mathscr{H}_q(F;\pi))$$

that converges to $H_*(E;\pi)$*. It is natural with respect to maps*

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

of fibrations. Assuming that F *is connected, the composite*

$$H_p(E;\pi) = F_pH_p(E;\pi) \longrightarrow F_pH_p(E;\pi)/F_{p-1}H_p(E;\pi) = E^\infty_{p,0} \xrightarrow{e_B} E^2_{p,0} = H_p(B;\pi)$$

is the map induced by $p\colon E \to B$*. The composite*

$$H_q(F;\pi) = H_0(B;H_q(F;\pi)) = E^2_{0,q} \xrightarrow{e_F} E^\infty_{0,q} = F_0H_q(E;\pi) \subset H_q(E;\pi)$$

is the map induced by $i\colon F \subset E$*. The transgression* $\tau\colon H_p(B;\pi) \to H_{p-1}(F;\pi)$ *is the inverse additive relation to the suspension* $\sigma_*\colon H_{p-1}(F;\pi) \longrightarrow H_p(B;\pi)$*.*

Sketch proof. Consider the set of p-cells

$$e\colon (D^p, S^{p-1}) \longrightarrow (B^p, B^{p-1}).$$

When we pull the fibration p back along e, we obtain a trivial fibration since D^p is contractible. That is, $p^{-1}(D^p) \simeq D^p \times F$. Implicitly, since $F = p^{-1}(b)$ is fixed, we are using a path from b to a basepoint in D^p when specifying this equivalence, and it is here that the local coefficient systems $\mathscr{H}_q(F)$ enter into the picture. These groups depend on the action of $\pi_1(B,b)$ on F. We prefer not to go into the details of this since, in most of the usual applications, $\pi_1(B,b)$ acts trivially on F and $\mathscr{H}_q(F)$ is just the ordinary homology group $H_q(F)$, so that

$$E^2_{p,q} = H_p(B;H_q(F)).$$

For $p = 0$, the local coefficients have no effect and we may use ordinary homology, as we have done when describing the fiber edge homomorphism.

Of course, $F_pB/F_{p-1}B$ is the wedge over the maps e of the spheres $D^p/S^{p-1} \cong S^p$. We conclude that $F_pE/F_{p-1}E$ is homotopy equivalent to the wedge over e of copies of $S^p \wedge F_+$. Therefore, as an Abelian group, $H_{p+q}(F_pE, F_{p-1}E; \pi)$ is the direct sum over e of copies of $H_q(F)$. This group can be identified with $C_p(B) \otimes H_q(F)$. Using the precise description of cellular chains in terms of cofiber sequences given in [2, pp 96-97], we can compare the cofiber sequences of the filtration of E with those of the filtration of B to check that $E^1_{*,q}$ is isomorphic as a chain complex to $C_*(B; \mathscr{H}_q(F))$. This is straightforward when $\pi_1(B)$ acts trivially on F, and only requires more definitional details in general. The identification of E^2 follows. We shall return to the proof of convergence in Section 5. The naturality is clear. The statements about the edge homomorphisms can be seen by applying naturality to the maps of fibrations

$$\begin{array}{ccccc} F & \xrightarrow{i} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle =} \\ \{b\} & \longrightarrow & B & \xrightarrow{=} & B. \end{array}$$

The additive relation $\sigma_* \colon H_{p-1}(F; \pi) \longrightarrow H_p(B; \pi)$, $p \geq 1$, admits several equivalent descriptions. The most convenient one here is in terms of the following diagram.

$$\begin{array}{ccccccc} H_p(E;\pi) & \xrightarrow{j_*} & H_p(E,F;\pi) & \xrightarrow{\partial} & H_{p-1}(F;\pi) & \xrightarrow{i_*} & H_{p-1}(E;\pi) \\ & & \downarrow{\scriptstyle p_*} & \swarrow{\scriptstyle \sigma_*} & & & \\ & & H_p(B,b;\pi) & & & & \end{array}$$

The additive relation σ_* is defined on $\operatorname{Ker} i_*$ and takes values in $\operatorname{Coker} p_*j_*$. If $i_*(x) = 0$, there exists y such that $\partial(y) = x$, and $\sigma_*(x) = p_*(y)$. Thinking in terms of a relative spectral sequence or using $(F_pE, F_0E) \subset (E, F)$, we see that $d^r(p_*(y)) = 0$ for $r < p$, so that the transgression $\tau(p_*(y)) = d^p(p_*(y))$ is defined. Since $i_*(x) = 0$, x cannot survive the spectral sequence. A check from the definition of the differentials in terms of our exact couple shows that $d^p(p_*(y)) = x$. □

There is also a cohomological Serre spectral sequence. When $\pi = R$ is a commutative ring, this is a spectral sequence of DGA's by an application of Corollary 4.2. To construct this variant, we use the cohomological exact couple obtained from the long exact sequences in cohomology of the pairs $(F_pE, F_{p-1}E)$. The diagonal map gives a map of fibrations

$$\begin{array}{ccc} E & \xrightarrow{\Delta} & E \times E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p\times p} \\ B & \xrightarrow[\Delta]{} & B \times B \end{array}$$

and therefore gives a map of cohomological spectral sequences.

Theorem 5.2 (Cohomology Serre spectral sequence)**.** *There is a first quadrant cohomological spectral sequence* $\{E_r, d_r\}$, *with*

$$E_1^{p,q} \cong C^p(B; \mathscr{H}^q(F;\pi)) \quad \text{and} \quad E_2^{p,q} \cong H^p(B; \mathscr{H}^q(F;\pi))$$

that converges to $H^*(E;\pi)$*. It is natural with respect to maps of fibrations. Assuming that* F *is connected, the composite*

$$H^p(B;\pi) = H^p(B;H^0(F;\pi)) = E_2^{p,0} \overset{e_B}{\to} E_\infty^{p,0} \to H^p(E;\pi)$$

is the map induced by $p\colon E \to B$*. The composite*

$$H^q(E;\pi) \longrightarrow E_\infty^{0,q} \xrightarrow{e_F} E_2^{0,q} = H^0(B;H^q(F;\pi)) = H^q(F;\pi)$$

is the map induced by $i\colon F \subset E$*. The transgression* $\tau\colon H^{p-1}(F;\pi) \to H^p(B;\pi)$ *is the inverse additive relation to the suspension* $\sigma^*\colon H^p(B;\pi) \longrightarrow H^{p-1}(F;\pi)$*. If* $\pi = R$ *is a commutative ring, then* $\{E_r\}$ *is a spectral sequence of DGA's such that* $E_2 = H^*(B;\mathscr{H}^*(F;R))$ *as an* R*-algebra and* $E_\infty = E^0H^*(E;R)$ *as* R*-algebras.*

Sketch proof. Up to the last statement, the proof is the same as in homology. For the products, we already have the map of spectral sequences induced by Δ, so it suffices to work externally, in the spectral sequence of $E \times E$. Since we are using cellular chains, we have a canonical isomorphism of chain complexes $C_*(B) \otimes C_*(B) \cong C_*(B \times B)$ [2, p. 99]. Using this, it is not difficult to define a pairing of E_1 terms

$$\phi\colon C^*(E;\mathscr{H}^*(F)) \otimes C^*(E;\mathscr{H}^*(F)) \longrightarrow C^*(E;\mathscr{H}^*(F))$$

that satisfies μ. Then the last statement follows from Corollary 4.2. □

A short exact sequence

$$1 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 1$$

of (discrete) groups gives a fibration sequence

$$K(G',1) \longrightarrow K(G,1) \longrightarrow K(G'',1),$$

and there result Serre spectral sequences in homology and cohomology. Focusing on cohomology for definiteness, it takes the following form. This spectral sequence can also be constructed purely algebraically, and it is then sometimes called the Lyndon spectral sequence. It is an example where local coefficients are essential.

Proposition 5.3 (Lyndon-Hochschild-Serre spectral sequence)**.** *Let* G' *be a normal subgroup of a group* G *with quotient group* G'' *and let* π *be a* G*-module. Then there is a spectral sequence with*

$$E_2^{p,q} \cong H^p(G'';H^q(G';\pi))$$

that converges to $H^*(G;A)$*.*

Proof. The point that needs verification in a topological proof is that the E_2 term of the Serre spectral sequence agrees with the displayed algebraic E_2 term. The latter is shortened notation for

$$\mathrm{Ext}^p_{\mathbb{Z}[G'']}(\mathbb{Z}, \mathrm{Ext}^q_{\mathbb{Z}[G']}(\mathbb{Z},\pi)),$$

where the group actions on $\mathbb{Z}$ are trivial. The algebraic action of G'' on G' coming from the short exact sequence agrees with the topologically defined action of the fundamental group of $\pi_1(K(G'',1))$ on $\pi_1(K(G',1))$. We can take account of the G''-action on π when defining the local cohomology groups $\mathscr{H}^*(K(G',1);\pi)$ and identifying E_2, and then the point is to identify the displayed Ext groups with

$$H^p(K(G'',1);\mathscr{H}^*(K(G',1);\pi)).$$

The details are elaborations of those needed to work [2, Ex's 2, pp 127, 141]. □

6. The comparison theorem

We have had several occasions to use the following standard result. We state it in homological terms, but it has an evident cohomological analogue.

Theorem 6.1 (Comparison Theorem, [1, XI.11.1])**.** *Let $f\colon E \longrightarrow {}'E$ be a homomorphism of first quadrant spectral sequences of modules over a commutative ring. Assume that E_2 and E_2' admit universal coefficient exact sequences as displayed in the following diagram, and that, on the E_2 level, f is given by a map of short exact sequences as displayed.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^2_{p,0}\otimes E^2_{0,q} & \longrightarrow & E^2_{p,q} & \longrightarrow & \mathrm{Tor}_1(E^2_{p-1,0},E^2_{0,q}) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f\otimes f} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathrm{Tor}(f,f)} & & \\
0 & \longrightarrow & {}'E^2_{p,0}\otimes {}'E^2_{0,q} & \longrightarrow & {}'E^2_{p,q} & \longrightarrow & \mathrm{Tor}_1({}'E^2_{p-1,0},{}'E^2_{0,q}) & \longrightarrow & 0
\end{array}$$

Write $f^r_{p,q}\colon E^r_{p,q} \longrightarrow {}'E^r_{p,q}$. Then any two of the following imply the third.

(i) $f^2_{p,0}\colon E^2_{p,0} \longrightarrow {}'E^2_{p,0}$ *is an isomorphism for all* $p \geq 0$.
(ii) $f^2_{0,q}\colon E^2_{0,q} \longrightarrow {}'E^2_{0,q}$ *is an isomorphism for all* $q \geq 0$.
(iii) $f^\infty_{p,q}\colon E^\infty_{p,q} \longrightarrow {}'E^\infty_{p,q}$ *is an isomorphism for all* p *and* q.

Details can be found in [1, XI.11]. They amount to well arranged induction arguments. The comparison theorem is particularly useful for the Serre spectral sequence when the base and fiber are connected and the fundamental group of the base acts trivially on the homology of the fiber. The required conditions on the E^2 terms are then always satisfied.

7. Convergence proofs

To give a little more insight into the inner workings of spectral sequences, we give the proof of Theorem 3.2 in detail. In fact, the convergence proof for filtered complexes will give us an alternative description of the entire spectral sequence that avoids explicit use of exact couples. If we had given a chain level construction of the Serre spectral sequence, its convergence would be a special case. The proof of convergence with the more topological construction that we have given is parallel, but simpler, as we explain at the end of the section.

We begin with a description of the E^∞-term of the spectral sequence of an arbitrary exact couple $\langle D, E; i, j, k\rangle$. Recall that, for any (homological) spectral sequence, we obtain a sequence of inclusions

$$0 = B^0 \subset B^1 \subset \ldots \subset Z^2 \subset Z^1 \subset Z^0 = E^1$$

such that $E^{r+1} \cong Z^r/B^r$ for $r \geq 1$ by setting $Z^r = \mathrm{Ker}(d^r)$ and $B^r = \mathrm{Im}(d^r)$.

When $\{E^r\}$ arises from an exact couple, $d^r = j^r k^r$. Here $Z^r = k^{-1}(\mathrm{Im}\, i^r)$. Indeed, $j^r k^r(z) = 0$ if and only if $k^r(z) \in \mathrm{Ker}\ j^r = \mathrm{Im}\, i^r$. Since k^r is the map induced on homology by k, $z \in k^{-1}(\mathrm{Im}\, i^r)$. Similarly, $B^r = j(\mathrm{Ker}\ i^r)$. Indeed, $b = j^r k^r(c)$ for some $c \in C^{r-1}$ if and only if $b \in j^r(\mathrm{Im}\, k^r) = j^r(\mathrm{Ker}\ i^r)$. Since j^r is induced from j acting on D, $b \in j(\mathrm{Ker}\ i^r)$. Applying this to the calculation of E^r rather than E^{r+1}, we obtain

(7.1) $$E^r = Z^{r-1}/B^{r-1} = k^{-1}(\mathrm{Im}\, i^{r-1})/j(\mathrm{Ker}\ i^{r-1})$$

and therefore

$$E^{\infty} = k^{-1}D^{\infty}/jD^0, \tag{7.2}$$

where $D^{\infty} = \cap_{r\geq 1}\mathrm{Im}\, i^r$ and $D^0 = \cup_{r\geq 1}\,\mathrm{Ker}\; i^r$.

Now let A be a filtered complex. Define a (shifted) analogue C^r of Z^r by

$$C^r_{p,q} = \{a | a \in F_pA_{p+q} \ \text{ and } \ d(a) \in F_{p-r}A_{p+q-1}\}$$

These are the cycles up to filtration r. We shall prove shortly that

$$E^r_{p,q}A = (C^r_{p,q} + F_{p-1}A_{p+q})/(d(C^{r-1}_{p+r-1,q-r+2}) + F_{p-1}A_{p+q}) \tag{7.3}$$

for $r \geq 1$ and therefore

$$E^{\infty}_{p,q}A = (C^{\infty}_{p,q} + F_{p-1}A_{p+q})/(d(C^{\infty})_{p,q}) + F_{p-1}A_{p+q}), \tag{7.4}$$

where $C^{\infty}_{p,q} = \cap_{r\geq 1}C^r_{p,q}$ and $d(C^{\infty})_{p,q} = \cup_{r\geq 1}d(C^{r-1}_{p+r-1,q-r+2})$.

Recall that j denotes the quotient map $F_pA \longrightarrow F_pA/F_{p-1}A = E^0_pA$, which fits into the exact sequence (3.1). Formula (7.3) rigorizes the intuition that an element $x \in E^r_{p,q}$ can be represented as $j(a)$ for some cycle up to filtration r, say $a \in C^r_{p,q}$, and that if $d(a) = b \in F_{p-r}A$, then $j(b)$ represents $d^r(x)$ in $E^r_{p-r,q+r-1}$. The formula can be turned around to give a construction of $\{E^rA\}$ that avoids the use of exact couples. Historically, the alternative construction came first. Assuming this formula for the moment, we complete the proof of Theorem 3.2 as follows.

Proof of Theorem 3.2. We are assuming that $A = \cup F_pA$. and, for each n, there exists $s(n)$ such that $F_{s(n)}A_n = 0$. Give the cycles and boundaries of A the induced filtrations

$$F_pZ_{p+q} = Z^{p+q}(A) \cap F_pA \ \text{ and } \ F_pB_{p+q} = B^{p+q}(A) \cap F_pA.$$

Then $F_pB \subset F_pZ$ and $H(F_pA) = F_pZ/F_pB$. Since $F_pH(A)$ is the image of $H(F_pA)$ in $H(A)$, we have

$$F_pH(A) = (F_pZ + B)/B \ \text{ and } \ E^0_{p,*}H(A) = F_pH_*(A)/F_{p-1}H_*(A).$$

With a little check for the third equality, this implies

$$\begin{aligned} E^0_{p,*}H_*(A) &= (F_pZ + B)/(F_{p-1}Z + B) \\ &= (F_pZ)/(F_pZ \cap (F_{p-1}Z + B)) \\ &= (F_pZ)/(F_pZ \cap (F_{p-1}A + F_pB)) \\ &= (F_pZ + F_{p-1}A)/(F_pB + F_{p-1}A). \end{aligned}$$

For each q and for sufficiently large r, namely $r \geq p - s(p+q-1)$, we have

$$F_pZ_{p+q} + F_{p-1}A_{p+q} = C^r_{p,q} + F_{p-1}A_{p+q} = C^{\infty}_{p,q} + F_{p-1}A_{p+q}.$$

Therefore

$$F_pZ + F_{p-1}A = C^{\infty}_{p,*} + F_{p-1}A.$$

If $b \in F_pB_{p+q}$, then $b = d(a)$ for some $a \in A_{p+q+1}$. By assumption, $a \in F_tA$ for some t, and then, by definition, $a \in C^{t-p}_{t,p+q+1-t} = C^{r-1}_{p+r-1,q-r+2}$, where $r = t+1-p$. Therefore

$$F_pB + F_{p-1}A = d(C^{\infty}_{p,*}) + F_{p-1}A.$$

By (7.4), we conclude that $E^0H(A) = E^{\infty}A$. □

Proof of (7.3). To see the starting point, observe that $j\colon F_pA \longrightarrow E^0_pA$ carries $C^1_{p,q}$ onto the cycles of $E^0_{p,q}A$ and carries $d(C^0_{p,q+1})$ onto the boundaries of $E^0_{p,q}A$. The proof of (7.3) has four steps. We show first that j induces a map

$$\bar{j}\colon C^r_{p,q} + F_{p-1}A_{p+q} \longrightarrow Z^{r-1}_{p,q}.$$

We show next that $\bar{j}$ is surjective. We then observe that

$$\bar{j}(d(C^{r-1}_{p+r,p-q+1}) + F_{p-1}A_{p+q}) \subset B^{r-1}_{p,q}.$$

Finally, we show that the inverse image of $B^{r-1}_{p,q}$ is exactly $d(C^{r-1}_{p+r,p-q+1}) + F_{p-1}A_{p+q}$. These statements directly imply (7.3).

Let $x \in C^r_{p,q}$ and let $y = d(x) \in F_{p-r}A$. Note that y is a cycle, but not generally a boundary, in the chain complex $F_{p-r}A$ and continue to write y for its homology class. Note too that $y \in C^\infty_{p-r,q+r-1}$ since $d(y) = 0$. Write $\bar{x}$ for the element of $E^1_{p,q}A$ represented by $j(x)$. The connecting homomorphism

$$k_*\colon E^1_{p,q} = H_{p+q}(E^0_pA) \longrightarrow H_{p+q-1}(F_{p-1}A) = D^1_{p-1,q}$$

takes $\bar{x}$ to $i^{r-1}_*(y)$. Therefore $\bar{x} \in Z^{r-1}_{p,*}$ and we can set $\bar{j}(x) = \bar{x}$.

To see the surjectivity, consider an element $w \in Z^{r-1}_{p,q} \subset E^1_{p,q}$. We have $k_*(w) = i^{r-1}_*(y)$ for some $y \in H_{p+q-1}(F_{p-r}A)$, and we again also write y for a representative cycle. Let w be represented by $j(x')$, where $x' \in F_pA$. Then $k_*(w)$ is represented by $d(x') \in F_{p-1}A$, and $d(x')$ must be homologous to y in $F_{p-1}A$, say $d(x'') = d(x') - y$. Let $x = x' - x''$. Then $d(x) = y$ and $j(x) = j(x')$ since $x'' \in F_{p-1}A$. Therefore $\bar{j}(x) = w$ and $\bar{j}$ is surjective.

Now let $v \in d(C^{r-1}_{p+r-1,q-r+2}) \subset C^r_{p,q}$, say $v = d(u)$, where $u \in F_{p+r-1}A$. Again, v is a cycle but not necessarily a boundary in F_pA, and we continue to write v for its homology class. Since v becomes a boundary when included into $F_{p+r-1}A$, $i^{r-1}_*(v) = 0$. Thus the class $\bar{v}$ represented by $j(v)$ is in $j_*(\text{Ker } i^{r-1}_*) = B^{r-1}$.

Conversely, suppose that $\bar{j}(x) \in B^{r-1}_{p,q}$, where $x \in C^r_{p,q}$. This means that $\bar{j}(x) = j_*(v)$ for some $v \in \text{Ker } i^{r-1}_*$. Then $j(x)$ is a chain, also denoted v, such that $v = d(u)$ for some chain $u \in F_{p+r-1}A$. Since $j(x - d(u)) = 0$, $x - d(u) \in F_{p-1}A$. Thus $x = d(u) + (x - d(u))$ is an element of $d(C^{r-1}_{p+r,p-q+1}) + F_{p-1}A_{p+q}$. □

Proof of convergence of the Serre spectral sequence. For a large enough r, we have $E^\infty_{p,q} = E^r_{p,q}$.

Precisely, $E^r_{p,q}$ consists of permanent cycles when $r > p$ and it consists of non-bounding elements when $r > q+1$, since the relevant differentials land in or come from zero groups. Fix $r > \max(p, q+1)$ and consider the description of $E^r_{p,q}$ given in (7.1). We take notations as in Section 3, writing i_* etc. Omitting the coefficient group π from the notation, we have the exact sequence

$$\cdots \longrightarrow H_{p+q}(F_{p-1}E) \xrightarrow{i_*} H_{p+q}(F_pE) \xrightarrow{j_*}$$

$$H_{p+q}(F_pE, F_{p-1}E) \xrightarrow{k_*} H_{p+q-1}(F_{p-1}E) \longrightarrow \cdots$$

With $D^1_{p,q} = H_{p+q}(F_pE)$ and $E^1_{p,q} = H_{p+q}(F_pE, F_{p-1}E)$, this displays our exact couple. Consider $Z^{r-1}_{p,q}$, which is $k^{-1}_*(\text{Im}\, i^{r-1}_*)$. The domain of i^{r-1}_* is zero with our choice of r, so that

$$Z^{r-1}_{p,q} = \text{Im}\, j_*.$$

Similarly, consider $B^{r-1}_{p,q}$, which is $j_*(\text{Ker } i^{r-1}_*)$. With our choice of r, Ker i^{r-1}_* is the kernel of the map

$$i^\infty_{p,*}\colon H_{p+q}(F_pE) \longrightarrow H_{p+q}(E),$$

so that

$$B^{r-1}_{p,q} = j_*(\operatorname{Im}(\partial\colon H_{p+q+1}(E, F_pE) \longrightarrow H_{p+q}(F_pE))).$$

Recall that $F_pH_{p+q}(E) = \operatorname{Im} i^\infty_*$ and define

$$\bar{j}\colon F_pH_{p+q}(E) \longrightarrow Z^{r-1}_{p,q}/B^{r-1}_{p,q} = E^\infty_{p,q}$$

by

$$\bar{j}(i^\infty_*(x)) = j_*(x).$$

This is well-defined since Ker $i^\infty_* = \operatorname{Im}\partial$, and it is clearly surjective. Its kernel is $F_{p-1}H^*_{p+q}(E) = \operatorname{Im} i^\infty_{p-1,*}$ since $i^\infty_{p-1,*} = i^\infty_{p,*} \circ i_*$ and Ker $j_* = \operatorname{Im} i_*$. □

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