Section 4.2 Injections, Surjections, and Bijections

Purpose of Section: To introduce three important types of functions: **injections** (one-to-one), **surjections** (onto), and **bijections** (both **one-to-one and onto**). These properties relate to important concepts such as the **inverse** of a function and **isomorphisms** between sets.

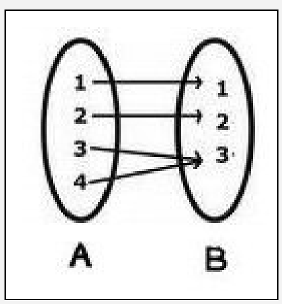
Injections, Surjections, and Bijections

As in any important collection of objects, functions have been classified in many subcategories (differentiable, continuous, increasing, integrable, measurable, ⋯) depending on a wide variety of properties, One basic classification, which relates to how a function maps points from its domain into its codomain is that of **injections** (one-to-one), **surjections** (onto), and **bijections** (one-to-one and onto).

Injection, Surjection and Bijection

Three Important Classifications of Functions

- **Surjection (onto):** Let $f : A \to B$. If the set of images $f(A)$ is equal to the codomain B (i.e. $f(A) = B$) the function f is called a **surjection** (or simply **onto**) and we say "f is from A *onto* B." In other words, for each $y \in B$ there exists an $x \in A$ (called a **pre-image** of y) such that $f(x) = y$

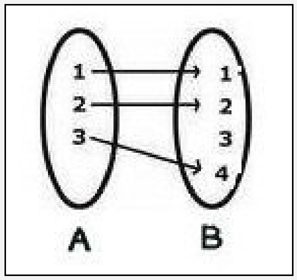


- **Injection (1-1):** Let $f : A \to B$. If for each $y \in f(A)$ there exists a *unique* (pre-image) $x \in A$ such that $f(x) = y$, the function f is called an **injection** (or simply **one-to-one**[1]). More operational forms of this definition are a function is $1-1$ if and only if:

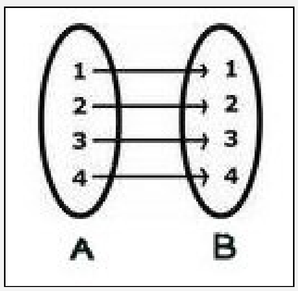
[1] Often one simply writes 1-1 to denote a one-to-one function.

direct form of 1–1: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

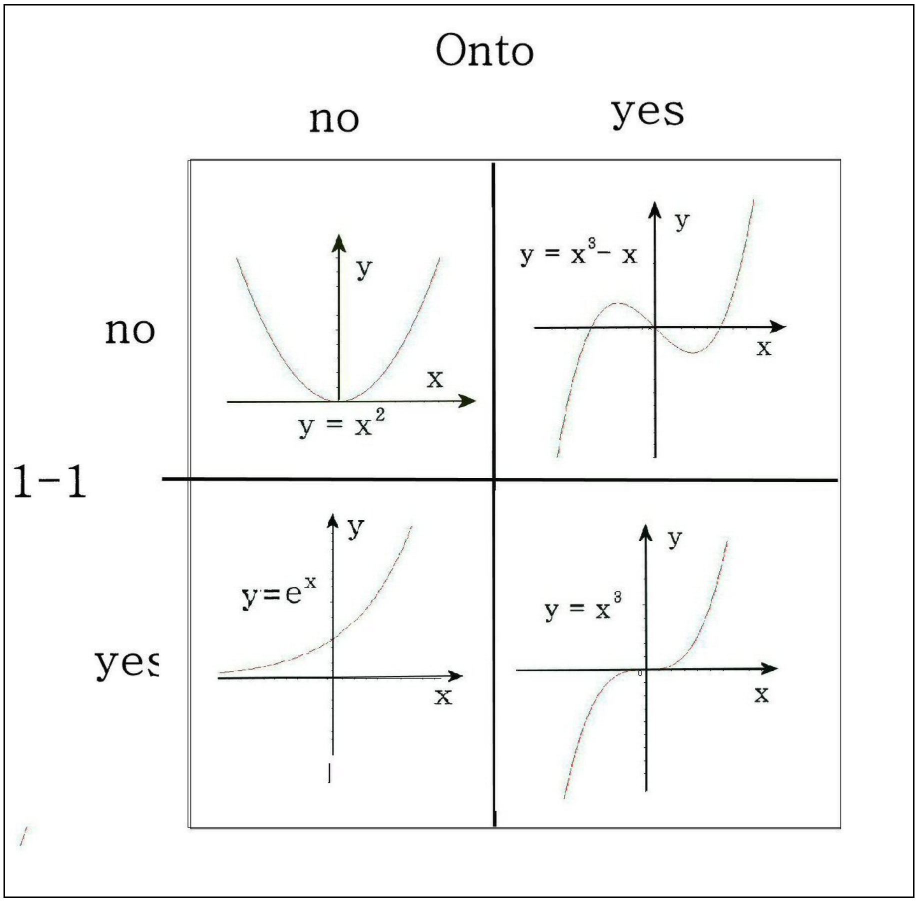
contrapositive form of 1–1: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$



- **Bijection (1-1 correspondence):** If $f : A \to B$ is both an injection and surjection, then f is called a **bijection** (or **one-to-one correspondence**).



Example 1 (Injections, Surjections, and Bijections) The graphs in Figure 1 illustrate typical surjections, injections, and bijections from $\mathbb{R}$ to $\mathbb{R}$. Note that the graph of 1–1 functions intersect any horizontal line at most once.



Types of Functions
Figure 1

Example 2 (Injection)

Show $f:\mathbb{N}\to\mathbb{N}$ defined by $f(n)=n^2$ is an injection (i.e. a one-to-one mapping).

Proof:

Some people would say simply lining up the natural numbers against their squares, as is done in Table 1 is a pretty convincing argument that f is a 1-1 mapping, but we present a more formal proof.

n	1	2	3	4	$\cdots$	n	$\cdots$
$f(n)=n^2$	1	4	9	16	$\cdots$	n^2	$\cdots$

Table 1

We show $m \neq n \Rightarrow f(m) \neq f(n)$ by proving its contrapositive $f(m)=f(n) \Rightarrow m=n$. We write

$$\begin{aligned} f(m) = f(n) &\Rightarrow m^2 = n^2 \\ &\Rightarrow m^2 - n^2 = 0 \\ &\Rightarrow (m-n)(m+n) = 0 \\ &\Rightarrow m = n \text{ or } m = -n \end{aligned}$$

But $m = -n$ is not possible since we are assuming m, n are positive numbers. Hence, we conclude $m = n$ and so f is $1-1$ on $\mathbb{N}$. ∎

Example 3 (Surjection)

Determine if the functions $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 1$ is a surjection.

Proof: We show for any $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that $y = x^3 + 1$. Solving for x gives $x = \sqrt[3]{y-1}$. Hence, for any value of y this value of x maps into y. Hence, f maps $\mathbb{R}$ **onto** $\mathbb{R}$. ∎

Example 4 (Counterexample)

Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $y = x^2 + 2x$ a surjection?

Solution:

No. The number $y = -2$ has no preimage since you can easily show $x^2 + 2x = -2$ has only complex solutions. ∎

Example 5 (Map from $\mathbb{R}^2$ onto $\mathbb{R}^2$) Show that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

is a bijection of the plane onto the plane.

Solution

Surjection: Letting

$$\begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= x_1 - x_2 \end{aligned}$$

we prove there exists a point $(x_1, x_2) \in \mathbb{R}^2$ which maps onto (y_1, y_2) by showing the system of equations

$$x_1 + x_2 = y_1$$
$$x_1 - x_2 = y_2$$

has a solution x_1, x_2 for arbitrary y_1, y_2. Solving these equations, we get

$$x_1 = \frac{1}{2}(y_1 + y_2)$$
$$x_2 = \frac{1}{2}(y_1 - y_2)$$

Hence T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Injection: To show T is an injection, we set the images

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} x_1' + x_2' \\ x_1' - x_2' \end{pmatrix}$$

equal and show their respective preimages (x_1, x_2) and (x_1', x_2') are equal (this is the contrapositive form of $1-1$). Doing this we get

$$x_1 + x_2 = x_1' + x_2'$$
$$x_1 - x_2 = x_1' - x_2'$$

or

$$(x_1 - x_1') - (x_2 - x_2') = 0$$
$$(x_1 - x_1') + (x_2 - x_2') = 0$$

which reduces to $x_1 = x_1',\ x_2 = x_2'$. Hence T is an injection. Since T is both an injection and surjection, it is a bijection. ∎

Inverse Functions

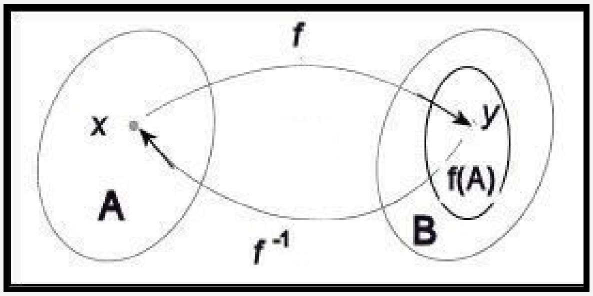
In arithmetic some numbers have inverses. For example -3 is the additive inverse of $+3$ since $3 + (-3) = 0$. Some functions also have inverses in the sense that the inverse of a function "undoes" the operation of the function.

Definition: Function Inverse

If $f : A \to B$ is an injection, then for each $y \in f(A)$ in the range of A, the equation $f(x) = y$ has a *unique* solution $x \in A$. This yields a new function $f^{-1} : f(A) \to A$, defined by

$$x = f^{-1}(y).$$

for all $y \in f(A)$. This function is called the **inverse** of f on $f(A)$.



If f is *both* an injection *and* onto the codomain B, then the inverse function maps $f^{-1} : B \to A$.

In the language of relations, we would define the inverse function of $f \in A \times B$ by $f^{-1} = \{(y, x) \in B \times A : (x, y) \in f\}$.

Example 6 (Inverse Function)

The function $f : [0, \infty) \to [1, \infty)$ defined by

$$f(x) = 1 + x^2, \quad x \geq 0$$

is both 1-1 and onto and hence has an inverse $f^{-1} : [1, \infty) \to [0, \infty)$. Find and draw the graph of this inverse.

Solution Solving the equation

$$y = 1 + x^2$$

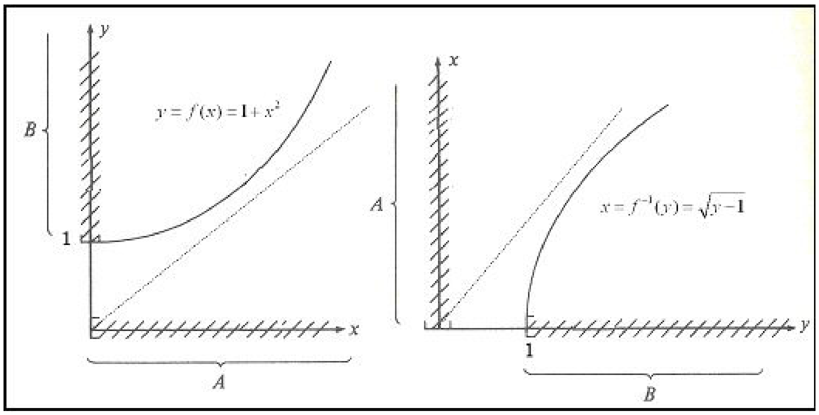
for $x \geq 0$ in terms of y, we find the unique value

$$x = +\sqrt{y-1}\ ,\ y \geq 1$$

thus we would write

$$f^{-1}(y) = \sqrt{y-1}\ ,\ \ y \geq 1 .$$

The graphs of f and f^{-1} are drawn in Figure 2. Note that the graph of f^{-1} is the reflection of the graph of f through the 45 degree line $y = x$.



Function and Its Inverse
Figure 2

Historical Note: The study of functions changed qualitatively with the ideas of Italian mathematician Vito Volterra (1860-1940) who introduced the idea of **functions of functions** (i.e. functions whose arguments were themselves functions). Later French mathematician Jacques Hadamard (1865-1963) named these types of functions **functionals**, and Paul Le´vy (1886-1971) gave the name **functional analysis** to the study of functions interpreted as points in a space, not unlike points in the plane.

One of the most important inverse functions is the inverse of the Fourier transform.

Function $f(x)$	Inverse $f^{-1}(y)$	Comments

$x+a$	$y-a$	
mx	y/m	$m \neq 0$
$1/x$	$1/y$	
x^2	$\sqrt{y}$	$x, y \geq 0$
x^3	$\sqrt[3]{y}$	no restriction on x, y
e^x	$\ln y$	$y > 0$
a^x	$\log_a y$	$a > 0, y > 0$
$x^{a/b}$	$x^{b/a}$	$x \geq 0$
$\tan x$	$\tan^{-1} y$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Common Inverses
Table 2

Example 7 (Inverse Fourier Transform)

The Fourier transform $\Im : x \to X$ maps integrable[2] function $x = x(t)$ of t (normally time) to a function $X = X(\omega)$ of frequency ω by the equation

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

called the Fourier transform of $x(t)$. One can verify that the inverse[3] of the transform is

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega$$

If the transform were followed by the inverse transform, one would have

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \right] e^{i\omega t} d\omega \ .$$

[2] The precise domain of the Fourier transform is all *Lebesgue* integrable functions. Many readers of this book will be introduced to the Lebesgue integral later in a course in real analysis.

[3] The Fourier transform maps functions into functions. Functions are *equivalent* if they differ on a set of "measure zero" and functions in different equivalence classes are considered different functions..

Problems

1. Given the following three pairs of sets

- $\{1,2,3\}$ and $\{4,5,6\}$
- $\mathbb{N}$ and $\mathbb{N}$
- $\mathbb{R}$ and $\mathbb{R}$

For each of the above pairs of functions find four functions f_1, f_2, f_3, f_4 from the first set to the second set such that

a) f_1 is neither 1-1 or onto.
b) f_2 is 1-1 but not onto.
c) f_1 is onto but not 1-1.
d) f_1 is both 1-1 and onto.

2. Find examples of the following functions f.

a) f maps $\mathbb{R}$ to $\{1,2,3\}$
b) f maps $\mathbb{N}$ to $\mathbb{R}$
c) f maps $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$
d) f maps $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$
e) f maps $\{a,b,c\}$ to $[0,1]$

3. **(Injections, Surjections, and Bijections)** Which of the following functions are intective, surjective, and bijective on their respective domains. Take the domains of the functions as those values of x for which the function is well-defined.

a) $f(x) = x^3 - 2x + 1$

b) $f(x) = \dfrac{x+1}{x-1}$

c) $f(x) = \begin{cases} x^2 & x \le 0 \\ x+1 & x > 0 \end{cases}$

d) $f(x) = \dfrac{1}{x^2+1}$

4. **(Inverse Function)** For the function

$$f:(1,\infty)\to(0,1)$$

defined by

$$f(x)=\frac{x-1}{x+1},\ x>1$$

a) Draw the graph of the function
b) Prove that the function is 1-1.
c) Find the inverse of the function.
d) Find the domain and range of the inverse function.
e) Draw the graph of the inverse function.

5. **(Function as Ordered Pairs)** Given the function $f:\{1,2,3\}\to\mathbb{N}$ defined by $f=\{(1,3),(2,5),(3,1)\}$:

a) Is f 1-1?
b) Is f onto?
c) What is the range of f?

6. **(Trick Question)** If f maps $\mathbb{R}$ **onto** $\mathbb{R}$ and if $f(1)=2$ then find

a) $f^{-1}(2)$
b) $f^{-1}(f(3))$
c) $f(f^{-1}(8))$

7. **(Hmmmmmmmmm)** For what value of the exponent $n\in\mathbb{N}$ is the function $f(x)=x^n$ an injection?

8. **(Well-Defined Functions)** Let a **and** b denote real numbers. Do the following defined well-defined functions?

a) $f(a+b)=b$
b) $f(a+b)=\sin(a+b)$
c) $f(a+b)=a+2b$
d) $f(a+b)=ab$

9. **(Finding Domains)** Given the function $f(x)=x^3-x$ from the reals to the reals, find a domain A so that f is a bijection.

10. **(Prove or Find a Counterexample)** Is it true that if a function is 1-1, then its inverse is also 1-1? If so prove it, if not find a counterexample.