Al-Kāshī's constant

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Here I have collected some facts about *al-Kāshi's constant* that was used by the Persian mathematician Jamshid al-Kāshī in 1424. He calculated the number to 16 decimals and several computer programs still use similar precision for numerical calculations involving al-Kāshi's constant. On this page I will denote al-Kāshi's constant by τ (tau), but for those readers who do not like τ as notation for al-Kāshī's constant, I suggest to use ϖ (varpi).

1 Definition of Al-Kashī's constant

Definition *Al-Kāshi's constant* τ is defined as the circumference of a unit circle.

As a consequence al-Kāshī's constant equals the circumference divided the *radius* for any circle. Since *Archimedes' constant* π equals the circumference divided by the *diameter* we have that $\tau = 2\pi$.

2 Angle measurements

When angles are measured in radians, τ corresponds to *one turn* (or 360°). Similarly $\tau/2$ corresponds to a *halfturn* (straight angle, 180°), and $\tau/4$ corresponds to a *quaterturn* (right angle, 90°). For circular sectors we have that a semi-circle corresponds to $\tau/2$, a quadrant corresponds to $\tau/4$, a quintant correspomnds to $\tau/5$, a sextant corresponds to $\tau/6$, and a octant corresponds to $\tau/8$. Thus, when using τ there is no dichotomy between measuring angles in radians, measuring angles in turns, and the names of circular sectors.

A turn can be subdivided in *centiturns* (cτ) and *milliturns* (mτ). A centiturn corresponds to 3.6°, which can also be written as $3°36'$. A milliturn corresponding to an angle of 0.36°, which can also be written as 21'36". Pie charts illustrate proportions of a whole as fractions of a turn. Each one percent is shown as an angle of one centiturn. Angles can be measured in centiturns by use of a centiturn protractor. Like degrees centiturns cannot be constructed by ruler and compass. A *deciturn* can be constructed because a pentagon is constructable and the angles can be divided in two.

Binary subdivisions of a turn are often used. Thus, $^1/_{32}$ turn is called a *point* and has been popular in navigation. In programming one often uses $^1/_{256} = 2^{-8}$ turn *(a binary radian)* or $^1/_{65\,336} = 2^{-16}$ turn as units. In this way an angle can be stored as integers in one or two bytes, and addition of angles is simplifyed because overflow does not create any problems. Binary subdivisions are constructable by ruler and compass.

3 Decimal expansion

The first part of the digital expansion of al-Kāshī's constant is

$$\begin{aligned}\tau = 6.&2831853071\,7958647692\,5286766559\,0057683943\,3879875021\\ &1641949889\,1846156328\,1257241799\,7256069650\,6842341359\\ &6429617302\,6564613294\,1876892191\,0116446345\,0718816256\end{aligned}$$

It is belived that τ is a normal number where any digit or string of digits asymptotically appear with the same frequency. The Feynman point of τ begins at digit 761 and consist of a sequence of seven 9s. The similar Feynman point of π has only six 9s.

4 Continued fraction

Like Archimedes' constant π one can expand al-Kāshī's constant τ as a continued fraction. It is

$$\tau = 6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{2 + \cfrac{1}{146 + \cfrac{1}{3 + \cfrac{1}{6 + \cdots}}}}}}}}$$

This leads to the following rational approximations of al-Kāshī's constant.

Approximation	Described by	Year	Dec. exp.
6	The Bible	1st millenium BC	6
$6\,\frac{1}{3}$			6.33
$6\,\frac{1}{4}$	Babylonian math	c. 1 600 BC	6.250
$6\,\frac{2}{7}$	Archimedes	c. 250 BC	6.2857
$6\,\frac{15}{53}$			6.28302
$6\,\frac{32}{113}$	Zǔ Chōngzhī	5th century AD	6.28318 584
$6\,\frac{4687}{16551}$			6.28318 53060
$6\,\frac{14093}{49766}$			6.28318 53072 37
$6\,\frac{89245}{315147}$			6.28318 53071 741

5 Formulas

A lot of formulas simplify by using al-Kāshī's constant rather than Archimedes' constant. A few formulas should simplify by changing to the circle constant $\eta = \tau/4$. Below I have listed some important formulas. I have used a check mark if a formula has simplified and a sad smiley if a formula has become more complicated. No smiley means that there have not been any significant change in complexity. I have not included a list of integrals since all kinds of factors appear giving no clear preference to any special choice of circle constant. One should be aware that each formula is a result of a derivation and sometimes a more complicated formula may may better reflect how the formula is derived. Some formulas would simplify if other of the involved constants were redefined.

5.1 Geometry

Circumference of a circle

$$\tau R. \checkmark$$

Area of disc

$$\frac{1}{2}\tau R^2$$

become a special case of the formula for the area of a circular sector $\frac{1}{2}\theta R^2$.
Surface area of sphere

$$2\tau R^2.$$

Volume of ball

$$\frac{2}{3}\tau R^3.$$

Surface of a spherical segment

$$\tau h R. \checkmark$$

Volume of spherical sector

$$\frac{\tau h R^2}{3}. \checkmark$$

Volume of spherical segment

$$\frac{\tau}{6}h^2(3R-h). ☹$$

Surface area of torus

$$\tau^2 R_1 R_2. \checkmark$$

Volume of torus

$$\frac{\tau^2}{2}R_1 R_2^2.$$

Volume of d dimensional unit ball when d is even

$$\frac{\tau^{d/2}}{2\cdot 4\ldots d}. \checkmark$$

Volume of d dimensional unit ball when d is odd

$$\frac{2\tau^{\lfloor d/2\rfloor}}{1\cdot 3\ldots d}. \checkmark$$

Gauss-Bonnet formula

$$\int_M K\,\mathrm{d}A = \tau\cdot\chi(M). \checkmark$$

What is normally called the 2π theorem states: A Dehn filling of M with each filling slope greater than τ results in a 3-manifold with a complete metric of negative sectional curvature. ✓

5.2 Complex numbers and complex analysis

Euler's formula

$$\mathrm{e}^{\tau\mathrm{i}} = 1. \checkmark$$

The solutions to the equation $z^n = 1$ are

$$z = \mathrm{e}^{k\tau\mathrm{i}/n},\ k = 1, 2, \cdots, n. \checkmark$$

Logarithms in the complex plane

$$\ln z = \ln r + (\theta + n\tau)\,\mathrm{i},\ \ n\in\mathbb{Z}. \checkmark$$

Cauchy's integral formula

$$f(z_0) = \frac{1}{\tau\mathrm{i}}\oint\frac{f(z)}{z-z_0}\,\mathrm{d}z, \checkmark$$

$$f^{(n)}(z_0) = \frac{n!}{\tau\mathrm{i}}\oint\frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z. \checkmark$$

Residues

$$\frac{1}{\tau\mathrm{i}}\oint f(z)\,\mathrm{d}z = \sum_{k=1}^{n}\operatorname*{Res}_{\mathrm{z}=\mathrm{a_k}} f(z). \checkmark$$

Laurant series

$$
\begin{aligned}
f(z) &= \sum_{n=-\infty}^{\infty} c_n (z-a)^n , \\
c_n &= \frac{1}{\tau \mathrm{i}} \oint \frac{f(z)}{(z-z_0)^{n+1}} \,\mathrm{d}z. \checkmark
\end{aligned}
$$

5.3 Trigonometric function, oscilations, and harmonic analysis

Periodicity of trigonometric functions

$$
\begin{gathered}
\sin(x+\tau) = \sin(x) \ , \\
\cos(x+\tau) = \cos(x) \ , \checkmark \\
\tan\left(x+\frac{\tau}{2}\right) = \tan(x) \ .
\end{gathered}
$$

Derived functions

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) &= \sin\left(x+\frac{\tau}{4}\right), \\
\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) &= \cos\left(x+\frac{\tau}{4}\right).
\end{aligned}
$$

Harmonic oscillator

$$\omega = \frac{\tau}{T}. \checkmark$$

Kepler's third law constant, relating the orbital period (T) and the semi-major axis (a) to the masses (M and m) of two co-orbiting bodies

$$\left(\frac{\tau}{T}\right)^2 a^3 = \omega^2 a^3 = G(M+m). \checkmark$$

The circle group

$$\mathbb{T} = \mathbb{R}/\tau\mathbb{Z}. \checkmark$$

Fourier series

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}, \\
c_n &= \frac{1}{\tau}\int_0^{\tau} f(x)\,\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x \,.
\end{aligned}
\checkmark
$$

Fourier integrals

$$
\begin{aligned}
f(x) &= \frac{1}{\tau}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \\
F(x) &= \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.
\end{aligned}
\checkmark
$$

5.4 Probability and statistics

Density of Gaussian distribution

$$\frac{1}{\tau^{1/2}} \exp\left(-\frac{x^2}{2}\right). \checkmark$$

The error function

$$\frac{1}{\tau^{1/2}} \int_0^x \exp\left(-\frac{t^2}{2}\right) \mathrm{d}t. \checkmark$$

Density of inverse Gaussian distribution

$$\frac{1}{\left(\tau x^3\right)^{1/2}} \exp\left(-\frac{(x-\mu)^2}{2\mu^2 x}\right). \checkmark$$

Density of lognormal distribution

$$\frac{1}{x \cdot \tau^{1/2}} \exp\left(-\frac{(\ln x)^2}{2}\right). \checkmark$$

Cauchy distribution

$$\frac{2}{\tau} \frac{1}{x^2+1}. ☹$$

Density of von Mises distribution

$$\frac{1}{\tau I_0(\kappa)} \exp(\kappa \cos(x)). \checkmark$$

Buffon needle experiment gives probability

$$\frac{4}{\tau}.$$

Asymptotic minimax redundancy of a d-dimensional random variable with distribution P_θ

$$\frac{d}{2} \ln\left(\frac{n}{\tau}\right) + \ln \int \left(\det I(\theta)^{1/2} \, \mathrm{d}\theta\right). \checkmark$$

Entropy power in terms of entropy

$$N(X) = \frac{\exp(2 \cdot h(X))}{\tau \cdot \mathrm{e}}, \checkmark$$

5.5 Physics

The approximate length of a simple pendulum with small amplitude is

$$L = g\frac{A^2}{\tau^2}. \checkmark$$

Planck's constant

$$\hbar = \frac{h}{\tau}. ✓$$

Einstein's field equation

$$R_{ik} - \frac{g_{ik}R}{2} + \Lambda g_{ik} = \tau \frac{4G}{c^4} T_{ik},$$

where the cosmological constant Λ is given by

$$\Lambda = \tau \rho_{\text{vac}} 4G.$$

The reactance of an inductor is

$$\tau f L. ✓$$

The susceptance of a capacitor is

$$\tau f C. ✓$$

Coulomb's constant where ϵ_0 is the vacuum permittivity of free space

$$F = \frac{1}{2\tau\varepsilon_0}.$$

Magnetic permeability of free space relates the production of a magnetic field in a vacuum by an electric current in units of Newtons (N) and Amperes (A):

$$\mu_0 = 2\tau \cdot 10^{-7}\,\mathrm{N/A^2}.$$

5.6 Miscellaneous

Stirling's approximation

$$n! \approx (n\tau \mathrm{e})^{1/2} n^n \mathrm{e}^{-n}. ✓$$

The Riemann ζ-function evaluated for a positive even integer n

$$\zeta(n) = \frac{\tau^n}{2n!} B_{n/2}. ✓$$

Functional equation for the Riemann function

$$\zeta(1-z) = \frac{2}{\tau^z} \cos\left(\frac{\tau \cdot z}{4}\right) \Gamma(z) \zeta(z). ✓$$

Euler's reflection formula (where crd is the cord function)

$$\Gamma(z)\Gamma(1-z) = \frac{\tau}{\operatorname{crd}(\tau z)}.$$

Jacobi's identities for the theta function involves in its normal formulation both π and τ, that in this case denotes the half period ratio. ☹

Weyl's law

$$\lim_{x\to\infty} \frac{N(x)}{x^{d/2}} = \tau^{-d} \omega(d)\, Vol(\Omega). ✓$$

6 Programming languages

Many computer programming languages have al-Kāshi's constant as a built in constant.

Program	Name	Value
Python	`tau`	6.28318 5... to the available precision
Processing	`TAU`	6.28318 55
Perl 6	`tau` or τ	6.28318 53071 79586
Google	`tau` or τ	6.28318 53071 79586
Nim	`TAU`	6.28318 53071 79586
Java	`TWOPI`	6.28318 53071 79586
Pascal	`TwoPI`	6.28318 53071 79586
Wiring	`TWO_PI`	6.28318 53071 79586 47693
OpenFOAM	`twoPi()`	6.28318 53071 79586 47693
Extreme Optimization Libraries	`TwoPi`	6.28318 53071 79586 47692 52867 66558
Calculators HP 39gII and HP Prime	`tr`	

In the Haskell programming language there is a module that defines the constant τ.

7 The symbol τ

The symbol "τ" is the 19th letter in the Greek alphabet and denotes the 't'-sound. It is not an ASCII character, but is available in most modern text processing systems.

Typesetting system	code
LaTeX	`\tau`
HTML entity	`τ`
HTML decimal	`τ`
HTML Hex	`τ`
Unicode	`U+03C4` *or* `U+F074`

In MSWord and OpenOffice "τ" can be inserted by choosing the font "Symbol" and typing "t".

8 Other applications of the symbol τ

Like π and all other letters in the Latin and Greek alphabet the letter τ is used in different ways in different parts of science. If the context is so different that no confusion is possible or if $\tau$is used in a combinations with other symbols in a way that no confusion is possible the application is marked with a check mark ✓. If an application could require modified notation or some extra clarifying text to ensure that no confusion occur the application is marked by a sad smiley ☹ .

8.1 Mathematics

- Ramanujan's tau function in number theory. ✓
- $\tau(n)$ can denote the number of divisors in n. ✓
- Kendall's tau rank correlation coefficient as a non-parametric correlation measure in statistics. ✓
- Torsion of a space curve. ☹
- The half-period ratio in the theory of elliptic functions. ☹

8.2 Physics

- The tau lepton in particle physics. The tau leptons are denoted τ^- and τ^+. ✓
- Torque, the rotational force in mechanics. Also called moment and denoted M. Torque and moment are vectors so the symbols are normally equipped with a vector arrow or typesetted in bold face ($\boldsymbol{\tau}$, $\overrightarrow{\tau}$, or $\mathbf{M}$, *or* $\overrightarrow{M}$). ☹
- Shear stress in continuum mechanics. This is a tensor denoted τ_{xy} when it measures stress along the xy plane. ✓
- In the physical sciences, τ is sometimes used as time variable, to avoid confusing with t as temperature. Examples are: The lifetime of a spontaneous emission process; the time constant of any device, such as an RC circuit; proper time in relativity. It is used to denote 'a specific time' or 'a second timescale'. It substitutes for t' and T when these have already been used. ☹
- Tau in astronomy is a measure of opacity, or how much sunlight cannot penetrate the atmosphere. ✓
- The prefix of many stars, via the Bayer stellar designation system. (Tau Ceti is such a star.) ✓

8.3 Other sciences

- The specific tax amount. ✓
- The symbol for tortuosity in hydro-geology. There are several competing definitions of the concept of tortuosity.
- The dose interval in pharmacokinetics. ✓

- The expressed period of the free-running rhythm of an animal (circadian rhythm terminology), i.e., the length of the daily cycle of an animal when kept in constant light or constant darkness. ☹

- In General Tau Theory $\tau(x,t)$ denotes the perceived motion-gap (a psychological principle of perception). Often it is denoted tau rather than τ. ✓

- Tau in biochemistry is a protein associated with microtubules and is implicated in certain neurodegenerative diseases. ✓

9 Alternative symbols

Several symbols have been proposed to denote al-Kāshī's constant.

In German speaking languages there have been some attempts to introduce "pla" (from Latin *plenus angulus*) as an abbreviation for a turn, but at present the word "Vollwinkel" is used without abbreviation and without SI prefixes.

In 1994 Dhananjay Ostawal proposed to use Ω as symbol for al-Kāshī's constant. This proposal was presented at the 63rd annual conference of the Indian Mathematical Society in 1997

Robert Palais wrote an article in 2001 entitled "Pi is wrong!" [Pal01] where he proposed to have a symbol and suggested to use the symbol $\pi\mskip-7.8mu\pi$, which is generated by the LaTeX code \pi\mskip-7.8mu\pi . This symbol did not gain popularity because it is only possible to write it using LaTeX. Therefore several other symbols have been proposed.

The symbols ϖ (varpi) and ϡ (sampi) have been proposed. Both of these symbols are variations of π and are seldomly used so they will not create any notational conflicts. Varpi is available in most programs that can write Greek letters. Sampi can be written in LaTeX using the code \textgreek{\char27}.

The symbol for registrered trademark ® has been proposed because it contains both a circle and a R that may refer to the radius, and similarly ⊚ has been proposed.

V. Marciante has been proposed to use circ. to N. Wilderberger.

The Greek letter τ was proposed independently by several people. In 2010 Michael Hartl launced a *Tau Manifesto* where he advocated for using τ and declared June 28th (6.28) as tau-day [Har10].

Thomas Cool has published two textbooks on geometry that used al-Kāshī's constant [Col08, Col11]. As symbol for al-Kāshī's constant he used Θ.

10 Various circle constants

During the history of mathematics various circle constants have been used.

Babylonian constant The old Babylonians used the circumference of a regular hexagon divided by the circumference of the circumscribed circle and this gives $^{6}/_{\tau}$. They used the approximate value $^{57}/_{60} + ^{36}/_{60^2}$.

Archimedes' constant This constant is defined as the circumference of a circle divided by its diameter. Since ancient times round objects have been characterized by their diameter. At an early time it was realized that the circumference could be calculated by multiplying the diamter by a certain number. Archimedes proved that it is the same constant one need to use for calculating the area of a circle. Before that different constants were used to calculate the area and circumference of a circle. The first use of π on its own with its present meaning was by William Jones in 1706. Jones introduces π as

$$\frac{1}{2}\,\text{Periphery}\,(\pi)$$

and used the following formulas for circumference and area of a circle

$$\begin{aligned} c &= \pi d \\ \alpha &= \frac{1}{4}\pi d^2 = c^2 \div 4\pi \end{aligned}$$

where α denoted the area.

Al-Bīrūnī's constant The Islamic scholar Abur-Raiḥān al-Bīrūnī (973-1048) [Sch27, Kap. V] calculated the diameter of a circle divided by its circumference and got the result

$$\frac{diameter}{circumference} = \frac{114}{360} + \frac{954312306}{1628681471 \cdot 360}\,.$$

Later in 1647 William Oughtred calculated the same constant and denoted it by $^{\delta}/_{\pi}$. Oughtred used this 'fraction' as one symbol rather than a numerator divided by a denomination.

Al-Kāshī's constant Ramshīd Al-Kāshī used the circumference of a circle divided by its radius as circle constant in *Treatise on the Circumference of the Circle* published in 1424 A.D. The idea of using this constant rather than Archimedes' constant may have been used earlier among Islamic mathematicians than al-Kāshī, but al-Kāshī calculated it with higher precision than any previous mathematician.

D. Gregory seems to be the first in Europe to use al-Kāshi's constant.

Eagle's constant Albert Eagle published a book on elliptic functions that introduced a lot of non-standard notation [Eag58]. For instance he used "τ" as notation for $^{\pi}/_{2}$, but it should be noted that π still appear in many of his formulas. Eagle's notational proposals have never been adapted. Recently, David Butler has proposed to use the symbol η to denote Eagle's constant. In geometry the idea of using a right angle as unit dates back to Euclid. In Germany and Schwitzerland the symbol $\llcorner$ has been used to denote a right angle and it was officially recognized as a unit for angle measurements in the period 1970-1996.

11 History

The Persian mathematician Jamshīd al-Kāshī seems to have been the first to use the circumference divided by the radius as circle constant rather than Archimedes' constant. In *Treatise on the Circumference* published 1424 he calculated the circumference of a unit circle to 10 significant sexagesimal figures as 6;16,59,28,1,34,51,46,14,50. He then converted that into 17 significant decimal figures 6.2831853071795865 [Luc53]. It took about 200 years before a more precise circle constant was calculated by Ludolph van Ceulen. For comparison, the most precise measurements in physics are about 16 significant figures and appear in the field of quantum electro dynamics. Calculations with 16 significant decimals are nowadays called double precision calculations.

Before al-Kāshī in 1262-1263 the mathematician Naṣīr al-Dīn al-Ṭūsī presented an example where he calculates the circumference of a circle with radius is 1,00 in sexagemial ciffres [vL08]. Therefore the circumference of his circle is 60τ and based on earlier calculations by Abul-Wafā' he calculates the circumference. "So if we make the diameter 120, the perimeter [i.e. of the circle] is 376 parts and a fraction that is greater than 0;59,10,59,0 and less than 0;59,23,54,12 , and if we change the two to the measure that Archimedes mentioned, the perimeter is greater than 3 times the diameter plus somthing that is greater than $\frac{10}{70;38,41,21}$, and less than $\frac{10}{70;38,14,29}$." Although the calculation of Naṣīr al-Dīn al-Ṭūsī may have inspired al-Kāshī to base his circle constant on a unit circle, the way Naṣīr al-Dīn al-Ṭūsī wrote the result of his calculation demonstrates that he just calculated a circumference of a specific circle and he did not calculate a circle constant with the falue 6.283... Naṣīr al-Dīn al-Ṭūsī also writes "If the perimeter of the circle is $3^1/_7$ times the diameter, and this is an approximate ratio used by the survayors..." [vL08], which indicates that at time using Archimedes' constant was thre standard.

In 1697 David Gregory used $^\pi/_\rho$ to denote the circumference of a circle divided by its radius, and he used this 'fraction' as one symbol rather than a numerator divided by a denomination. The first use of π on its own with its present meaning was by William Jones in 1706. It took almost 100 years before the notation π became standard notation. For instance M. Nicole [Nic47] did not use any special symbol for the circle constant but made tables of the circumference of inscribes and circumscribed polygons of the unit circle. This gave 6.28318 53070 31961 6 and 6.28318 53072 67891 2 as lower and upper bounds on the circumference. Leonard Euler first used π in 1737 [Eul37, Thm. 3, p. 165], but in his papers π denoted perimeter. In some problems he used π to denote al-Kāshī's constant and in other problems he used it to denote Archimedes' constant or Eagle's constant. At a place in a very influential book on analysis by Euler [Eul48, Chapter VII] π was used to denote Archimedes' constant and because of Euler's prestige in general, mathematicians have π to denote Archimedes' constant ever since. For instance T. Bugge in 1797 [Bug95, p. 237-239] describes the idea of finding the value of the circumference divided by the radius by inscribing and circumscribing a regular polygons leading to the value

6.283185307. As a consequence, he writes, the circumference divided by the diameter is equal to 3.141592653. Bugge then explains that this number was studied in more detail by L. Euler in 1737 and is denoted as π. After π had become standard notation, some mathematicians have used 2π as if it was one symbol. For instance H. Laurant always wrote $^{2\pi}/_4$ rather than $^{\pi}/_2$ [Lau89].

The idea of using centiturns and milliturns as units was introduced by the British astronomer and science writer Sir Fred Hoyle [Hoy62].

The idea of using τ as symbol for al-Kāshi's constant was first discussed in an unpublished manuscript by Joseph Lindenberg in 1992 [Lin11]. Dhananjay Ostawal from Pune, India claims that he sought a copyright for 'omega' to denote al-Kāshi's constant in 1997 and the idea was presented in a paper at the annual conference of Indian Mathematical Society in 1997. Robert Palais wrote an article in 2001 entitled "Pi is wrong!" [Pal01] where he proposed to have a symbol and suggested to use a 'three-legged pi' $\pi\mskip-7.8mu\pi$ with the LaTeX code \pi\mskip-7.8mu\pi . This symbol did not get any popularity because it is only possible to write it using LaTeX. After the paper of Palais a number of people have proposed to replace the three legged pi by another more convenient symbols as for instance ϖ, or τ that has been proposed independently by several people including the author of this page [Fre07, wee10]. In 2010 Michael Hartl launced a *Tau Manifesto* where he advocated for using τ [Har10] and declared June 28th to be Tauday. In 2008 Thomas Colignatus published the first textbook on geometry using al-Kashi's constant [Col08, Col11].

The Greek character τ comes from a similar character in the Phoenician alphabet and is derived from a cross. The Greeks aslo took over the name "tau" from the Phoenicians and at that time the original meaning of the word was already forgotten.

The word *turn* originates from Old English *tyrnan* and *turnian*. It comes from Medieval Latin *tornare*, from Latin, to turn on a lathe, from Greek $\tau\acute{o}\rho\nu o\sigma$ a 'lathe'. The word was influenced by Anglo-French *turner, tourner* to turn, from Medieval Latin *tornare*, akin to Latin *tenere* to rub. The geometric notion of a turn has its origin in the sailors terminology of knots where a turn means one round of rope on a pin or cleat, or one round of a coil. For knots the English terms of single turn, round turn and double round turn do not translate directly into the geometric notion of turn, but in German the correspondence is exact.

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12 Link collection

Below are a number of links related to the use of al-Kāshī's constant.

12.1 Instructional material

Animation of cosine labelled by τ.

Animation of sine labelled by τ.

Sams Teach Yourself Windows 8 A textbook that uses τ in its examples.

Tau for Trigonometry A website for learning trigonometry with τ instead of π.

Using Tau

Textbooks College Trigonometry with extensive use of the tau transcendental

Trigonometry with tau as circle constant.

Faith In Tau Animations and figures using τ.

6.283 A song on how to use τ.

Trig rerigged A trigonometry book by Thomas Colignatus where Θ is used to denote al-Kāshī's constant.

Conquest of the Plane Yet another trigonometry book by Thomas Colignatus where Θ is used to denote al-Kāshī's constant.

12.2 Scientific use

Theorem of the day Here τ is used in place of π in all theorems.

Online Encyclopedia of Integer Sequences This site has various links related to al-Kāshī's constant.

Treatise on the Circumference of the Circle by al-Kāshī 1426. This is the Arabic text. It has ben tranlated into German.

Information Divergence is more χ^2-distributed than the χ^2-statistic ISIT 2012.

Extendable MDL ISIT 2013.

Mutual information of Contingency Tables and Related Inequalities ISIT 2014.

Bounds on tail probabilities for negative binomial distributions Kybernetika 2016.

Sharp Bounds on Tail Probabilities for Poisson Random Variables WITMSE 2017.

Statistical Inference and Exact Saddle Point Approximations ISIT 2018.

12.3 Debate

The idea of introducing a symbol has been debated surprisingly much.

12.3.1 Proposals

Here are links to places where people have made new proposals regarding notation related to al-Kāshī's constant.

Tau before it was Cool Joseph Lindenberg describes that he proposed to use τ to denote al-Kāshī's constant allready in 1991.

Pune researcher claims he derived 'tau' constant first Researcher Dhananjay Ostawal from Pune, India said that he sought a copyright for 'omega' as a constant derivative for circle in 1997 and that he has presented a paper at the annual conference of Indian Mathematical Society in 1997.

Pi is wrong! Page edited by Robert Palais.

Tau Manifesto Page edited by Michael Hartl.

Square CircleZ The author of this page proposes to use the symbol for registered trademark ® as symbol for al-Kāshī's constant.

Tau Day 6.28 Page where it is proposed to use the symbol sampi to denote al-Kāshī's constant.

TeamLiquid Discussion forum where julianto proposes to use the symbol $\circledcirc$.

Pi may be wrong, but so is Tau! Video by David Butler, where he argues that it is more reasonable to use $\eta = \tau/4$ (Eagle's constant) as circle constant.

12.3.2 Discussions

The idea of using τ to denote al-Kāshi's constant is discussed at numerous math oriented news groups and we pages. Only a few are listed here.

Mathematical constant Archimedes $\Theta = 2\pi = 6.2831853\ldots$ Blog by Thomas Colignatus.

http://www.math.utah.edu/~palais/cose.html

The Last Stoic

Tau and dozenal notation

The way of the tau at hexnet.

Happy Tau (6.28) day by Alex Masterley.

The Pi Manifesto Page edited by Michael Cavers, where he argues in favor of π.

Pi is wrong! Long live Tau! Page edited by Dimitri Brant.

Radian Measurement: What It Is and How to Calculate It More Easily Using τ Instead of π. Paper on angle measurements by Stanley Max.

Circle constant is a turn by Lulzim Gjyrgjialli.

The Circle Constant at Scienceline.

2 Pi or Not 2 Pi? Blog by Giorgia Fortuna where she carefully discuss to what extend formulas become simpler or more complicated by switching from π to τ.

The Tao of Tau by Elizabeth Landau in Scientific American 2017.

Is Pi wrong? Is Tau the correct circle constant? Stated as a question at Quora.com with a lot of people stating their opinion.

Spiked Math Forums

https://matheducators.stackexchange.com/questions/530/pi-or-tau-how-should-the-circle-constant-be-taught/1694#1694 Discussion at Mathematics Educators.

PI should be 6.283185... A Facebook group.

Tau (The True Circle Constant) Yet another Facebook group.

Tau Day - June 28 A facebook profile of Tau Day.

Half-tau Day The day where $\tau/2$ is celebrated.

Stop celebrating Pi Day, and embrace Tau as the true circle constant by Chaim Gartenberg in The Verge 2018.

12.3.3 Videos and animations

Fun video by Vi Hart describing the problem of cutting a pie when using pi.

Pi is wrong! Here comes Tau day. Video by Kevin Houston.

Tau versus Pi by Khan Academy.

Pi ain't all that by Robert Dixon.

Protesting Pi on 1/2 Tau Day

Pi vs Tau

Tau (6,28...=2PIi) in Pascal's Triangle Tau en el Triángulo de Pascal

Tau=6,28... and has Perfect Numbers Tau tiene Números Perfectos

Is Tau Better Than Pi? Discussion about the circle constant at 14. mar. 2014

How pi was almost 6.283185... Explanation of how π became the standard notation for the circle constant $3.14\ldots$

12.4 Articles in news medias

The use of τ has got a lot of attention in news medias.

Pi's nemesis: Mathematics is better with tau Interview with M. Hartl in New Scientist.

Why we have to get rid of pi for the sake of good math Interview with M. Hartl at i09 webpage.

On Pi-Day, 'pi' is under attack Article by CNN.

Mathematics Upstarts Look to Replace Pi With New Circle Constant Article at Daily Tech.

Life of pi over? 'Tau' may set calculations aright Article in the Times of India.

Tau Day: An Even More Fundamental Holiday Than Pi Day by Alessondra Springmann.

'Tau day' marked by opponents of maths constant pi BBC News.

Forget Pi, Here Comes Tau Using a new constant would simplify things, say experts. Article by Evann Gastaldo from the Newser Staff.

Happy Tau Day, everybody! Article at CBSNews.

Happy Tau Day! Article in International Business Times.

Tau Day today: Mathematicians show their work Article in Oregon Live.

On National Tau Day, Pi Under Attack Fox News.

Push to roll Pi Discussion at ABC in Australia.

Down with ugly pi, long live elegant Tau, physicist urges Article in the Star.

Pi's 4,000-yr reign faces Tau challenge Article in Deccan Chronicle.

Mathematicians Want to Say Goodbye to Pi Yahoo News.

Second Annual Tau Day: Interview and Ideas!

What is Tau Day?

Your number's up: Why mathematicians are campaigning for pi to be replaced with alternate value tau

Tau Day Generates Controversy Among Math Scholars

Math wars: Debate sparks anti-pi day

Bye Bye Pi: Mathematic Scholars Want To Replace The Circle Constant

Moves to replace Pi with Tau

Pi Is Wrong! Mathematicians Declare Today 'Tau Day'

Life of pi in no danger Experts cold-shoulder campaign to replace with tau

Pi is wrong McGill Daily.

Why we have to get rid of pi for the sake of good math

A new place for activists: math

The Circle Constant Or what's the matter with pi by Greg Uyeno 2016.

12.5 Various links

http://breadpig.com/tees/tau/ Here you can buy a τ-shirt.

http://www.zazzle.com/tau_day_shirt-235815786260277849 Here you can buy a τ-day shirt.

Yet another tau-shirt page