Cauchy-Hadamard formula

Theorem [Cauchy, 1821] The radius of convergence of the power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ is

$$R = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|}}.$$

Example. For any increasing sequence of natural numbers n_j the radius of convergence of the power series $\sum_{j=1}^{\infty} z^{n_j}$ is $R = 1$.

Proof. Let $R = 1/\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|} \in [0, \infty]$.

If $R < \infty$, choose any $r > R$. Then $\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|} > 1/r$ and so $|c_n| r^n > 1$ for infinitely many indices n. So $c_n(z-z_0)^n$ does not approach 0 for any z with $|z-z_0| = r > R$. So the power series diverges for any z with $|z-z_0| > R$.

If $R > 0$, choose any $0 < r < R$. Then $\overline{\lim}_{n\to\infty} \sqrt[n]{|c_n|} < 1/r$ and so $|c_n| r^n < 1$ for all but finitely many indices n. Hence for any z with $|z-z_0| < r$,

$$\sum_{n=0}^{\infty} |c_n||z-z_0|^n = \sum_{n=0}^{\infty} |c_n| r^n \left|\frac{z-z_0}{r}\right|^n \le M \sum_{n=0}^{\infty} \left|\frac{z-z_0}{r}\right|^n < \infty.$$

So the power series converges for any z with $|z-z_0| < R$.

It follows that the radius of convergence is R. □

Exercises. Find the radius of convergence of each of the following power series.

$$\sum_{n=1}^{\infty} (1+1/n)^n z^n$$

$$\sum_{n=1}^{\infty} (4+i/n)^n z^{2n}$$

$$\sum_{n=1}^{\infty} \frac{z^n}{n^p}$$

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sum_{n=0}^{\infty} (2+i^n)^n z^n$$

$$\sum_{n=1}^{\infty} \sin(n) z^n$$