Math for Poets and Drummers

The Mathematics of Rhythm

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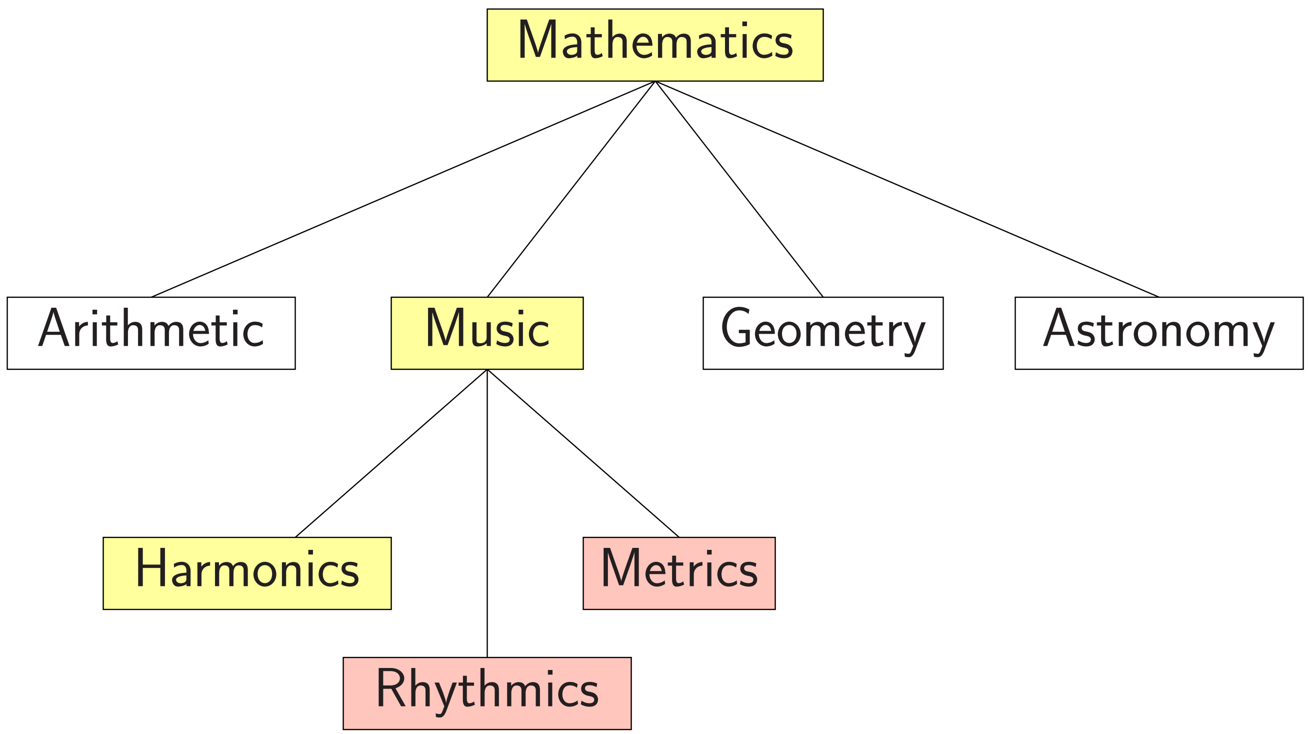
Listen

Cassiodorus (6th century)



Meter in Poetry

Much poetry follows one of a set of rhythmic rules, called *meters*.

In English and Spanish, the meter of a poem is determined by its pattern of stressed and unstressed syllables.

In Sanskrit (and many other languages), meter is determined by the pattern of long and short syllables. There are dozens of meters; some are determined by the number of syllables in a line, and some by the total length of the syllables in a line.

Names of meters and numbers of syllables

ukta	1	*pratistha*	4	*usnik*	7	*pankti*	10
atyukta	2	*supratistha*	5	*anustubh*	8	*tristubh*	11
madhya	3	*gayatri*	6	*brhati*	9	$\cdots$	$\cdots$

Pingala's Chandahsutra (c. 200 B.C.)

Syllables are short or long; in length

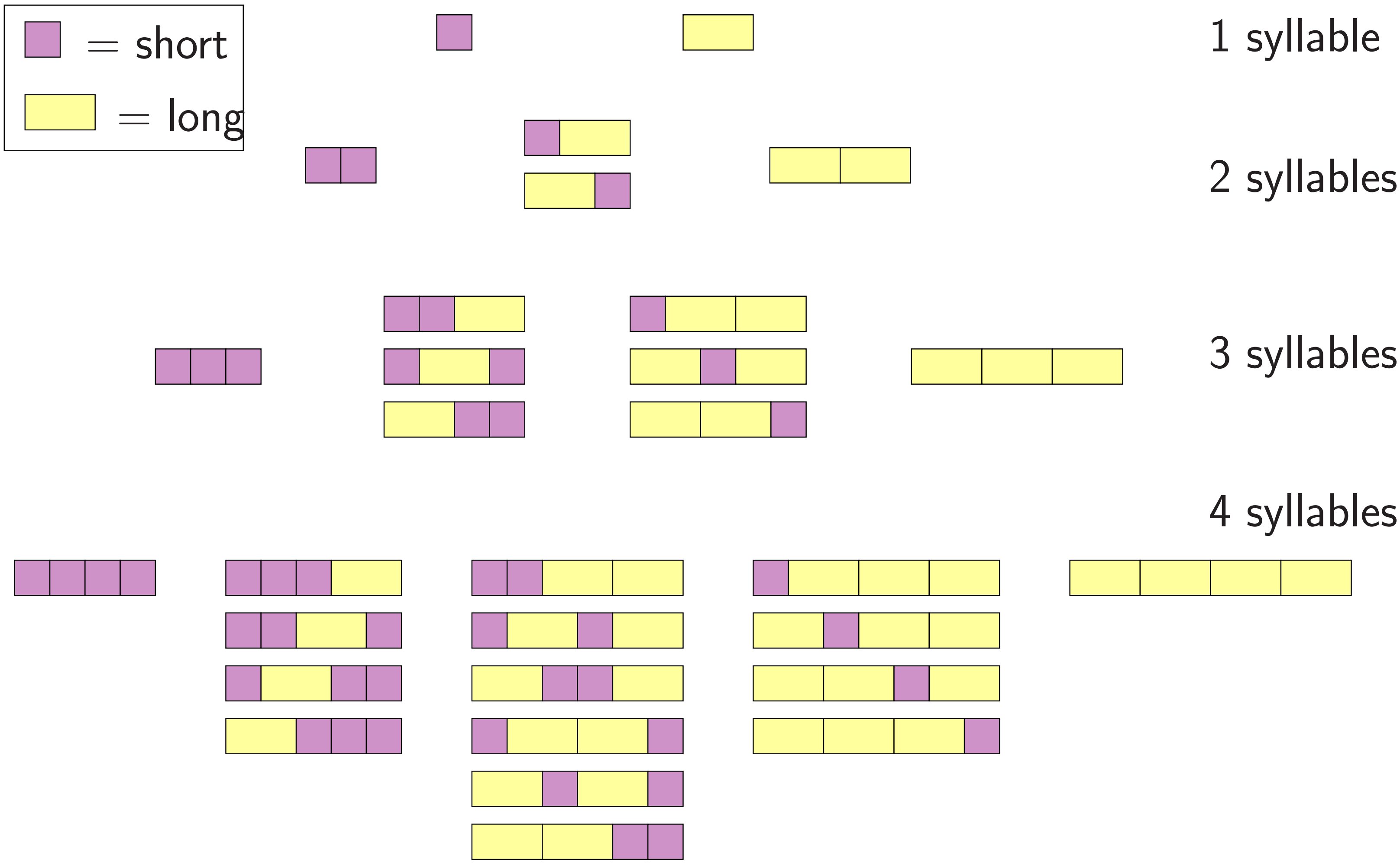
$$1 \text{ long} = 2 \text{ short}$$

Pingala (Bag, 1966) classified the 16 different meters of four syllables like this:

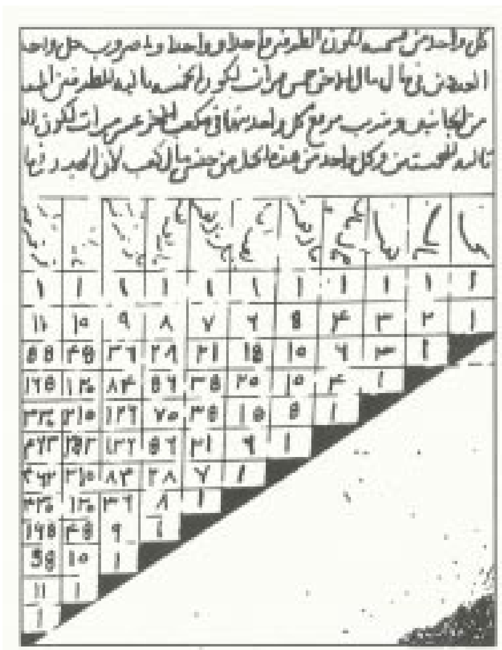
1	meter of four short syllables	SSSS
4	meters of three shorts	SSSL, SSLS, SLSS, LSSS
6	meters of two shorts	LLSS, LSSL, SSLL, SLLS, LSLS, SLSL
4	meters of one short	SLLL, LSLL, LLSL, LLLS
1	meter of no shorts	LLLL

He described a very interesting pattern. . .

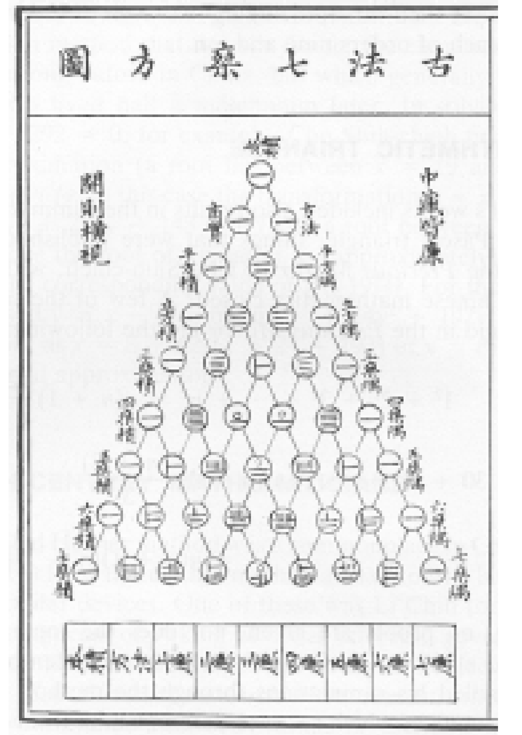
The Pattern. . .



Pascal's Triangle



North Africa, c. 1150
As Samaw'al ibn Yahya
al-Maghribi

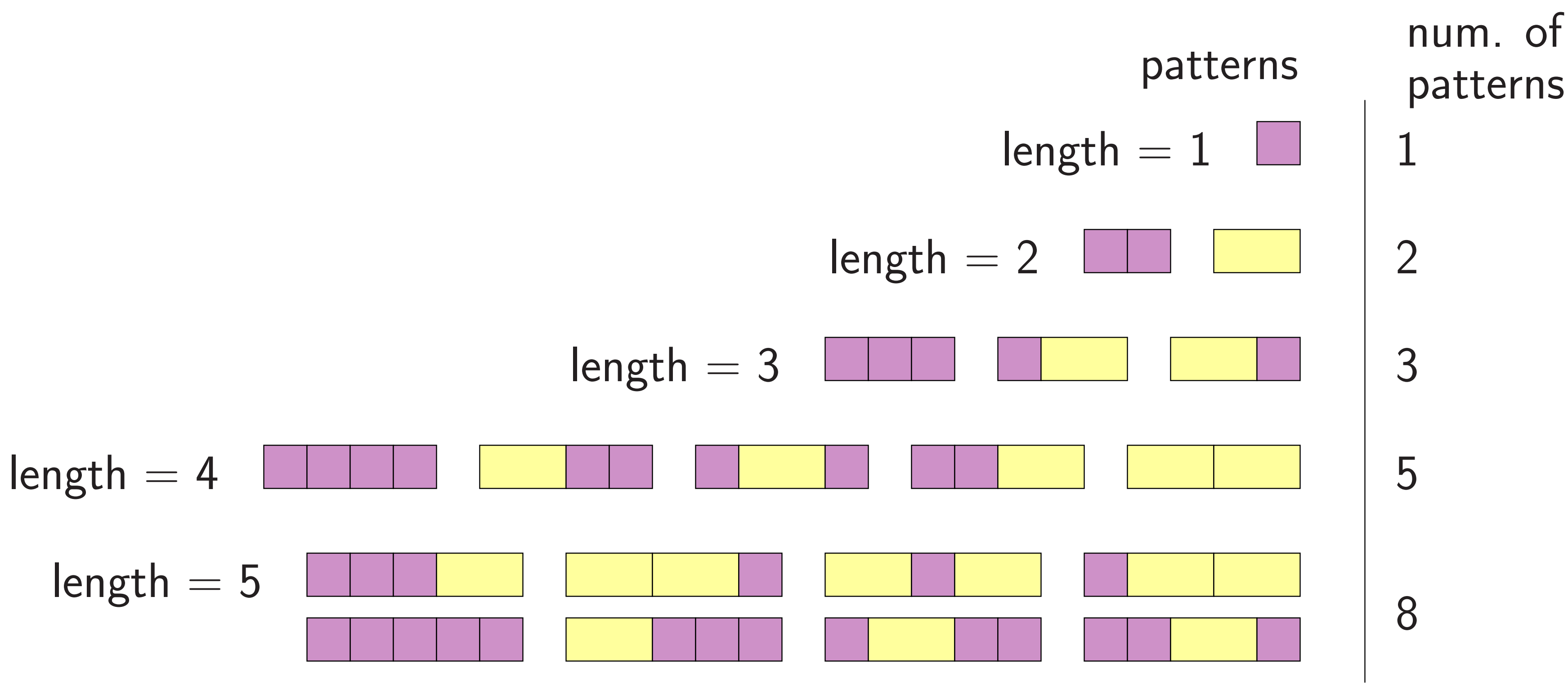


China, 1303
Chu Shih-Chieh

Germany, 1527
Petrus Apianus

Ācārya Hemacandra (c. 1150 AD)

Hemacandra enumerated meters in which the length of a line is fixed, but the number of syllables is not (Singh, 1986). Remember 1 long = 2 short. We'll use squares for short syllables and dominoes for long syllables. What's the pattern? Make a conjecture!



The Fibonacci Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 . . .

The first Fibonacci number is 1. After that, each number is the sum of the two previous numbers. A mathematician would express this relationship as

$$F_N = F_{N-1} + F_{N-2}$$

where F_N is the Nth Fibonacci number.

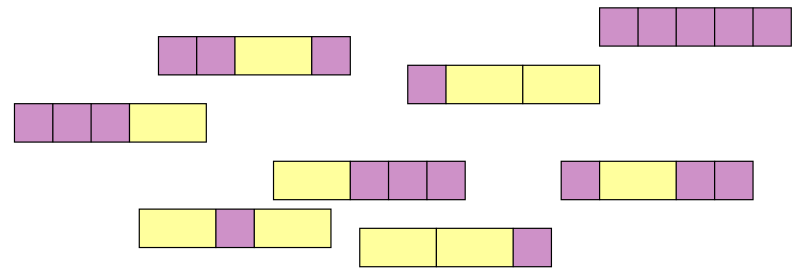
These numbers have an *amazing* variety of applications, not only to poetry, but also to nature, art, music, science, and many areas of mathematics.

See Ron Knott's Fibonacci web site:

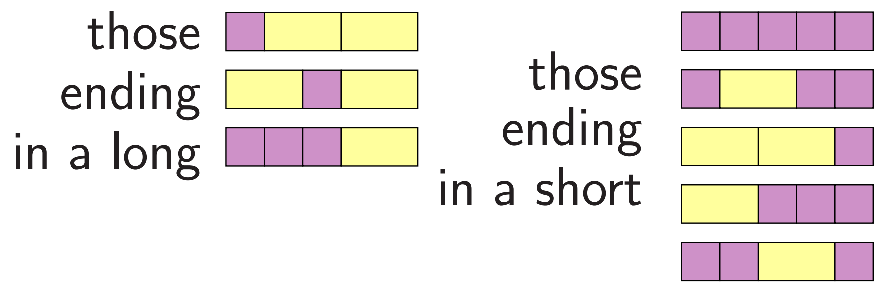
http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html

Got Proof?

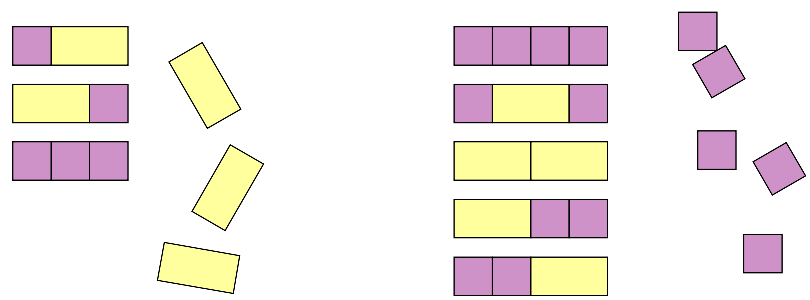
Start with the patterns of length N in one big pile



Separate them into two smaller piles:



OFF WITH the last syllables!



Drum Roll, Please...

There is 1 meter of length 1.
There are 2 meters of length 2.

For $N = 3, 4, 5, \ldots$, the number of meters of length N is equal to

(number of meters of length $N - 2$) + (number of meters of length $N - 1$).

This is precisely the Fibonacci relationship!

Musical Rhythm

Rhythms in music are patterns of note onsets or accents. A *note* is the interval between successive attacks. It plays the role of a syllable in poetry.

Rhythms of 1 and 2 beat notes

Merengue bell part (Dominican Rep.)

Cumbia bell part (Columbia)

Mambo bell part (Cuba)

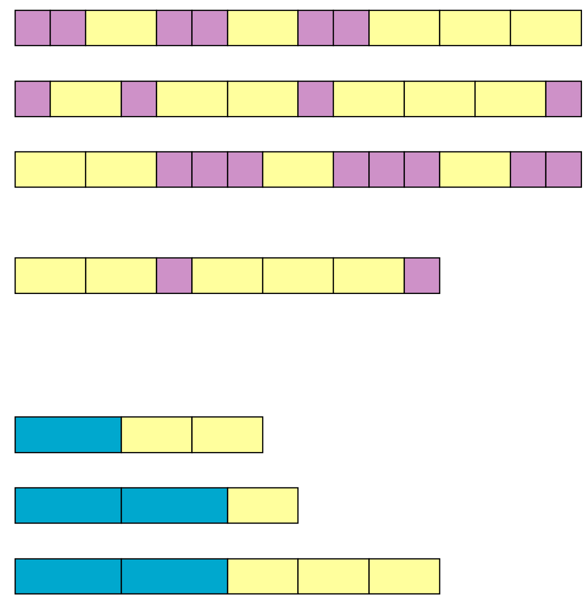
Bintin bell pattern (Ghana)
also Bembe Shango (Afro-Cuban)

Rhythms of 2 and 3 beat notes

Lesnoto (Bulgaria)

Bomba (Puerto Rico)

Guajira (Spain)



Rhythms on SongTrellis: http://www.songtrellis.com/rhythmPage

Challenge

In Eastern European music, rhythms are composed of long and short notes, where a long note is 3 unit pulses and a short note is 2 unit pulses. For example, the Bulgarian *lesnoto* rhythm is a 7-beat rhythm of a long note followed by 2 short notes, written $3+2+2$, while the *rachenitsa* rhythm is $2+2+3$.

What sequence of numbers gives us the number of rhythms of length N?

Challenge

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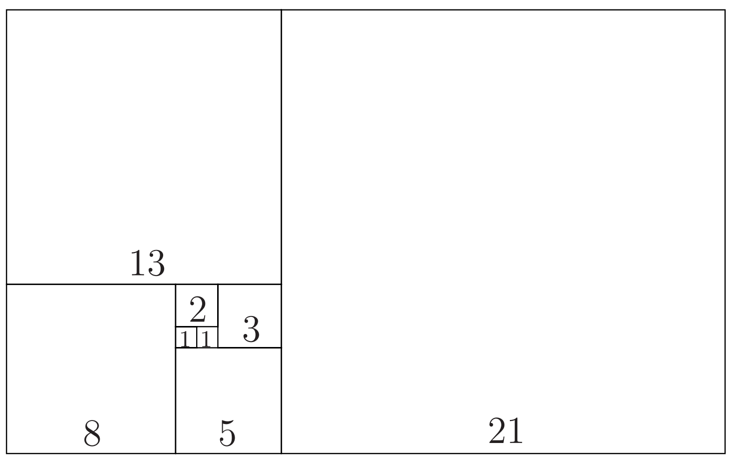
What sequence of numbers gives us the number of rhythms of length N?

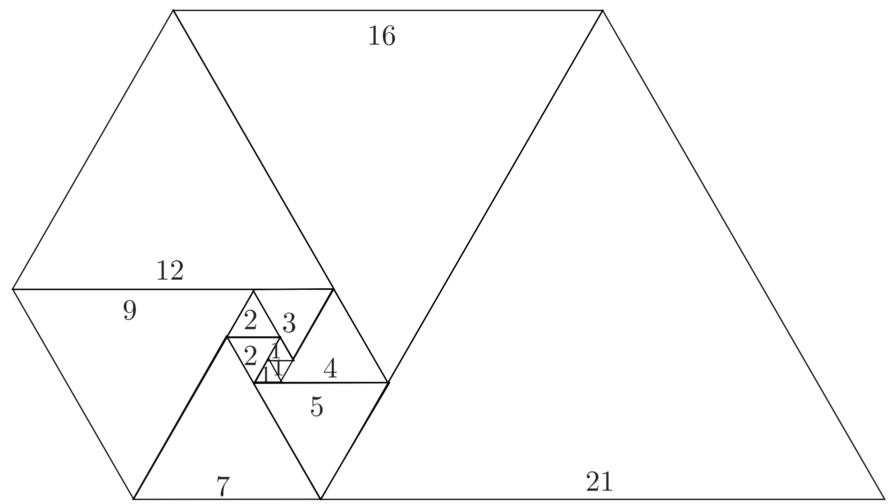
length (N)	1	2	3	4	5	6	7	8	9	10	11	12
number of patterns (R_N)	0	1	1	1	2	2	3	4	5	7	9	12

What's the next number? What's the rule? Can you prove it?

Padovan Numbers (Stewart, 2004)

The Padovan sequence has the same relationship to a spiral of equilateral triangles that the Fibonacci sequence has to a spiral of squares.



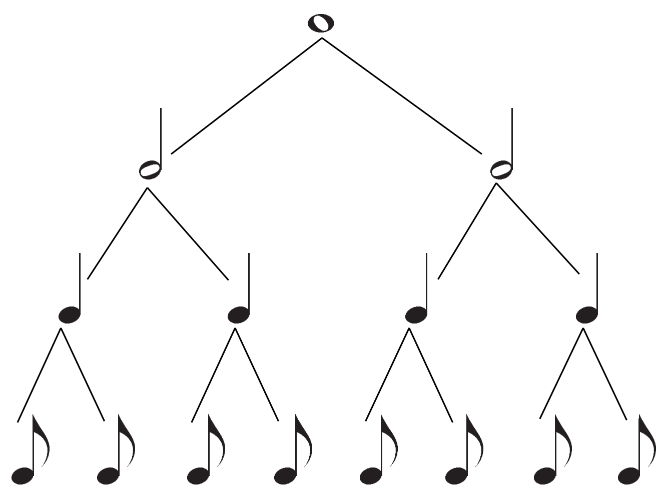


The limit of the ratios of successive terms in the Padovan sequence is the real solution to $r^3 - r - 1 = 0$, or approximately $r = 1.3247179572447\ldots$, the *plastic number*.

Divisive Rhythm

Divisive rhythm occurs when the basic unit of time (a measure) is subdivided into a number of equal notes, each of which may be further subdivided.

Here's the classic picture. In this case, we're only dividing notes into halves—we'll call this *binary division type*. Listen



Syncopation

The bomba rhythm () is perceived as a deviation from the expected pattern of accents. This deviation creates a rhythmic tension known as *syncopation*. Syncopated rhythm is common in African-American and Latin American music. It is also highly developed in Renaissance music.

Examples: (1) E. T. Mensah, Ghanaian pop song Listen
(2) Cuarteto Patria, "Chan-chan" Listen

How many syncopated rhythm patterns of k notes are there?

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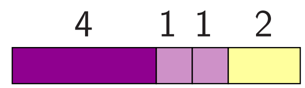
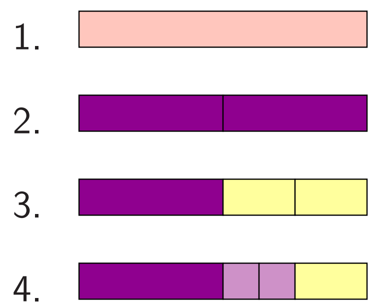
The difficulty lies in defining syncopation mathematically.
Perhaps we can define "not syncopated"....

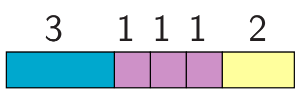
Uniform Rhythm Patterns

Uniform rhythms are definitely **not** syncopated.
Example: (1) The Beatles, "Free as a Bird" Listen (2) MIDI Listen

Let a *uniform rhythm* be any rhythm pattern that results from a sequence of equal divisions of notes. For example, 

is a uniform rhythm. The corresponding sequence would be



However, 

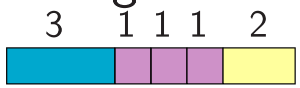
is not uniform, since the first half of the measure has been divided in the ratio $3:1$.

The Catalan numbers got rhythm!

Theorem 1 (Hall and Klingsberg) *The number of uniform rhythm patterns of binary division type with k notes equals the Catalan number $C(k-1) = (2k-2)!/(k!(k-1)!)$.*

Catalan numbers: 1, 1, 2, 5, 14, 42, 132, 429, 1430,...

Are “Non-uniform” and “Syncopated” Equivalent?

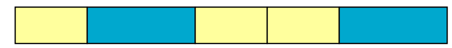
All uniform rhythms are non-syncopated. However, a musician would probably not agree that all non-syncopated rhythms are uniform. For example, 

does not sound syncopated.

According to the *New Harvard Dictionary of Music*, syncopation is “. . . a momentary contradiction of the prevailing meter or pulse. This may take the form of a temporary transformation of the fundamental character of the meter . . . or it may be simply the contradiction of the regular succession of strong and weak beats within a measure or group of measures (Randel, 1986).”

So this example is not syncopated because long notes correspond to strong beats.

Asymmetric Rhythms

Paul Klingsberg and I have studied asymmetric rhythms, which are commonly found in African drumming. They are rhythms that are "maximally syncopated" in that, even when delayed, they never are aligned with the primary divisions in a measure. Example: 

Examples: (1) Aka Pygmies, "Bobangi" Listen (2) MIDI Listen

Our results had surprising applications to the study of rhythmic canons. . .

Listen

Listen

For more information, see
http://www.sju.edu/~rhall/research.html

In the classroom

Multicultural Math

http://www.sju.edu/~rhall/Multi

Abcdrums

http://www.sju.edu/~rhall/Multi/drums.html

Students

Mateo Quiñones

Listen

Brian Centanni

Listen

Kate McCormick

Listen

References

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Don Randel, editor. *The new Harvard dictionary of music*. Belknap Press, Cambridge, MA, 1986.

Parmanand Singh. Ācāraya Hemacandra and the (so-called) Fibonacci numbers. *Math. Ed. (Siwan)*, 20(1):28–30, 1986. ISSN 0047-6269.

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