

SHOT AND THERMAL NOISE

INTRODUCTION

Intrinsic noise, random and uncorrelated fluctuations of signals, is a fundamental ingredient in any measuring process. This experiment is an investigation of two electrical noise phenomena: thermal noise and shot noise. Thermal noise is an energy equilibrium fluctuation phenomenon whereas shot noise involves current fluctuations, which deliver power to the system in question. Both are inherent noise that is always present in a real electrical system and represent fundamental limitations and difficulties in making sensitive electrical measurements.

Thermal noise arises from the thermal fluctuations in the electron density within a conductor. In a formulation due to H. Nyquist in 1928, an idealized resistor is assumed to contain a voltage generator causing a fluctuating *emf* at the terminals, the mean squared value of which is

$$\langle V^2 \rangle = 4RkT\Delta f \quad (\text{Nyquist formula})$$

where R is the resistance of the conductor, k is Boltzmann's constant, T is the absolute temperature, and Δf is the bandwidth of the measuring instrument. See Appendices A and B to this writeup for derivations. In the same year, J.B. Johnson experimentally verified the dependence of the thermal noise voltage on the resistance (thus thermal noise is also referred to as Johnson or Nyquist noise). Thermal noise is independent of the material of the resistor and is constant with frequency ("white" noise) up to microwave frequencies; at higher frequencies the quantum energy hf of the oscillations becomes comparable with kT requiring a modification of the Nyquist formula.

Another random noise signal is due to fluctuations in current. Temperature-limited vacuum diodes, zener diodes, heated resistors, and gas discharge tubes can generate these fluctuations. One of the most common is the temperature-limited vacuum diode, which will be referred to as a noise diode.¹ With no space-charge region around the cathode to smooth out the electron emission, the emission becomes a random statistical process. The instantaneous anode current deviates from the average value due to the

¹ A noise diode is operated with its anode voltage large enough to collect all the electrons emitted by the cathode, hence the name temperature-limited diode. In a diode operated in the space charge limited region, the current of electrons is not completely random, since the motion of each electron is influenced by the electrons already present in the space charge; the shot noise is consequently decreased by a factor, the *space charge depression factor*, which is difficult to calculate. This is why a temperature-limited diode is used in the experiment.

discreteness of the electron's charge, and this fluctuation in the anode current is called *shot noise*. The mean squared value of the noise current is

$$\langle i_{sh}^2 \rangle = 2e i_{dc} \Delta f \quad (\text{Schottky formula})$$

where e is the charge of the electron, i_{dc} is the average dc diode current, and Δf is the bandwidth of the measuring instrument. See Appendix C to this writeup for a derivation. The noise is Gaussian and independent of frequency from a few kHz to several hundred MHz. At low frequencies, $1/f$ (flicker) noise dominates.² In the high-frequency region (frequencies above that corresponding to the transit time of an electron across the gap of the diode), the output noise decreases because the flight of an electron is affected by the charge of other electrons likewise in transit.

The noise sources mentioned above are incoherent and the total noise in a system is the square root of the sum of the squares of all the incoherent noise sources. In addition to these fundamental or intrinsic noise sources, there are a variety of other noise sources across the electromagnetic spectrum. These include noise from the power line frequency (and harmonics), AM broadcast stations, TV and FM broadcasts, microwaves, etc., and we generically call them "interference" or non-essential noise sources because they can be minimized with good laboratory practices. There are many ways in which these noise sources work their way into an experiment and some of the techniques used to reduce or eliminate them are described in the appendix. A thorough discussion is given by Horowitz and Hill (see references).

In this experiment you will be measuring the shot noise generated by a vacuum diode and the thermal noise generated by several carbon resistors. By amplifying the noise voltage generated across a reference resistor through the use of a low noise preamplifier and tunable microvoltmeter, you can examine the shot noise dependence on i_{dc} and the thermal noise dependence on R and T .³ In doing so, you will be able to determine the values of the fundamental constants e and k . Being able to measure the charge of the electron and Boltzmann's constant with a voltmeter, ammeter, and thermometer is neat but, in practice, is not as easy as suggested by the Nyquist and Schottky equations. The experiment will give you a good introduction to the measuring

² This noise arises from resistance fluctuations in a current carrying resistor (or any other electronic component) and the mean squared noise voltage due to $1/f$ noise is given by $\langle v \rangle^2 = AR^2 I^2 \Delta f / f$, where A is a dimensionless constant (10^{-11} for carbon), R is the resistance, I the current, Δf the bandwidth of the detector, and f is the frequency to which the detector is tuned.

³ The shot noise measurement will necessarily also include the thermal noise of the reference resistor, but the two are uncorrelated and the shot noise can be adjusted to be much greater than the thermal noise.

process as well as dealing with other noise sources. If you look at the papers by Hull and Williams (1925) and Stigmark (1952), you will begin to appreciate the difficulty in achieving uncertainties less than 2%.

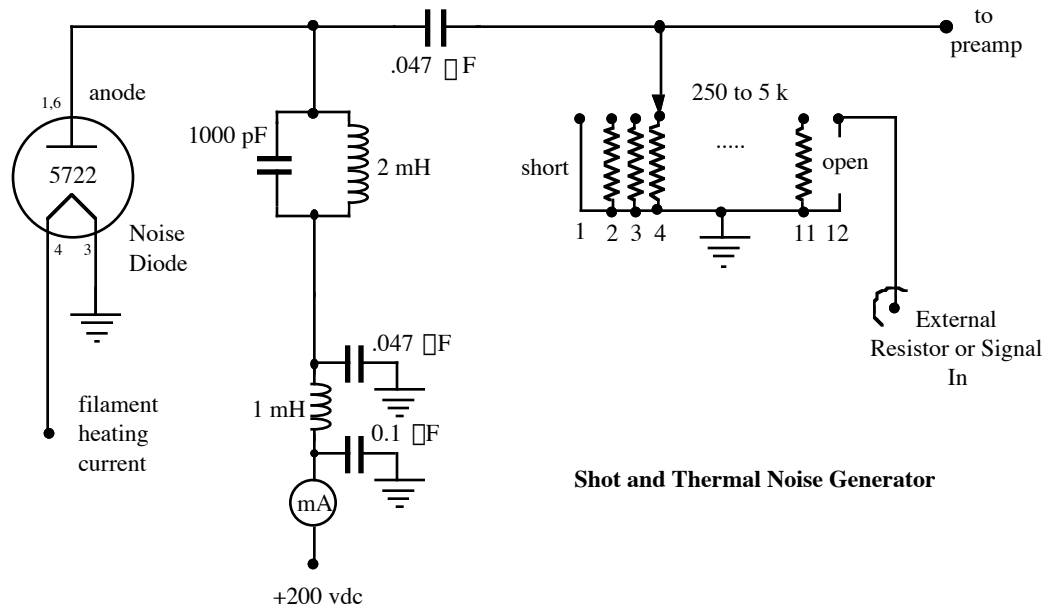
APPARATUS AND INSTRUMENTATION

The apparatus consists of three main units:

- noise generator
- low noise preamplifier (inside noise generator box)
- tunable microvoltmeter (HP model 312B)

And ancillary instrumentation:

- noise diode filament power supply (Lambda LH-121FM)
- noise diode anode (plate) voltage supply (Lambda C-281M)
- RF signal generator (LogiMetrics 925-S125)
- digital multimeter (Keithley 169)
- oscilloscope (Tektronix 2215)
- function generator (Wavetek 132)
- counter (Hewlett Packard 5381A)
- thermometer
- Dewar flask



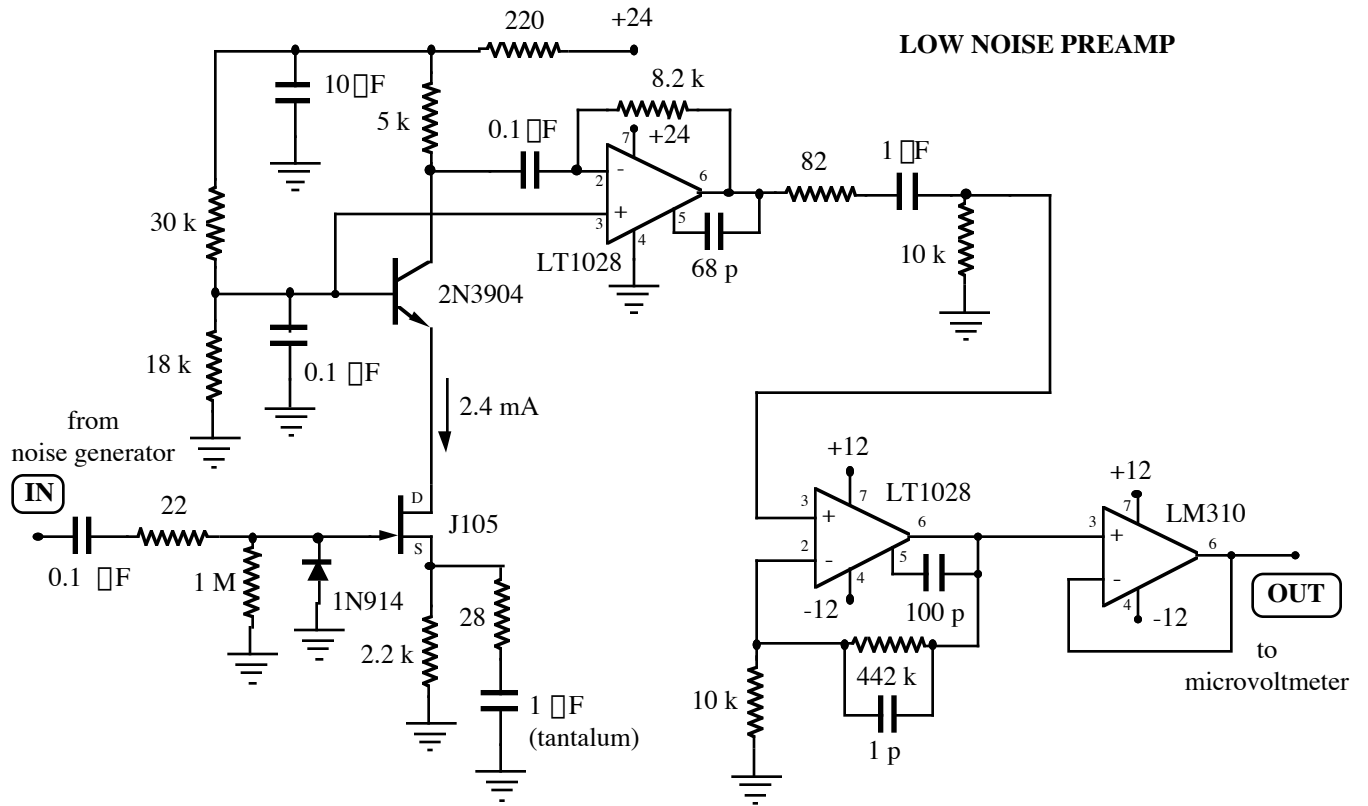
Noise generator: The circuit that produces the thermal and shot noise is diagrammed above. A bank of resistors is used as the source of thermal noise and a type 5722 vacuum diode generates shot noise.

The noise resistors range from 250 to 5.5 k Ω and are selected by means of a twelve position rotary switch; a *short* and *open* position allows for checking the preamp noise and injecting known signals for calibration (gain) and testing purposes as well as external resistances for noise thermometry. The resistor bank is capacitively coupled (isolated) from dc currents, thereby eliminating 1/f noise from fluctuations in the resistance. Standard metal film resistors are used (the resistances should be measured with a DMM). Carbon resistors are known to have noise in excess of the Johnson formula.

In order to make sure that the noise diode operates in the temperature-limited region, it is necessary to apply a fairly high dc voltage (+200V) between the cathode (filament) and anode (plate). To determine the noise current, one converts it to noise voltage by passing it through a resistor and then measures the voltage across this resistor. The resistor has to be large enough for an appreciable (ac) noise voltage to develop across it, but then it would be difficult to maintain the anode at a sufficiently high voltage because the resistor will also be large enough for the (dc) voltage drop across it to be considerable. To get around this difficulty, a tuned circuit is placed in the anode circuit of the diode, and a resistor (used to measure thermal noise) is connected in parallel with it. The tuned circuit has a very low dc resistance but, for a range of frequencies near resonance, it presents an extremely high impedance. Thus, if one measures signals in a frequency band coincident with the resonance frequency of the tuned circuit, one can measure the ac noise signal shunted through the noise resistor. A further advantage of the resonant circuit is that it eliminates the problem of shunt capacitance (detailed by Kittel *et al* in the Am. J. Phys.). Stray capacitance from the diode anode, circuit wiring, the preamp input, and the thermometry probe are all included as part of the tuned circuit capacitance. Finally, the resonant circuit limits sensitivity to stray radiation pickup at frequencies other than that at which noise is observed. The 100 kHz resonance frequency is well below the usual broadcast frequencies but high enough so that 1/f noise is not a concern. The resonant circuit has a $Q = 10$ with the resistor bank switched out.

Separate external power supplies provide the filament and anode voltages for the noise diode. **Be sure to observe polarities on the connectors.** The anode current is controlled by varying the temperature of the filament (cathode). Since the electron emission current from the filament is an exponential function of filament temperature, it is important that the filament heating current be well regulated. A dc milliammeter in series with the tuned circuit monitors the anode current. **Do not exceed 10 mA (filament heating current is indicated by the power supply meter and will be less than 1.5 A).**

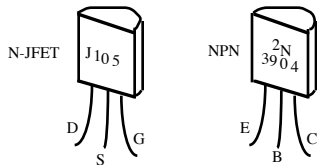
Make sure the filament current is turned to zero before switching the power supply on or off.



NOTES: (1) gain = 2700 into 500 Ω @ 110 kHz

(2) all ± 12 volt and +24 volt pins on IC's are bypassed to ground by 15 MFD tantalum and 0.1 MFD capacitors (in parallel)

(3) LT1028 is an ultra-low noise, high speed op amp
LM310 is a voltage follower; there is NC to pin 2 - it is internally connected to pin 6



Low noise preamplifier:

To minimize pickup of noise other than shot and thermal, the battery-powered preamplifier is contained in the same chassis box as the noise generator.

Tunable microvoltmeter:

Basically, this instrument is a radio receiver with adjustable bandwidth and a signal strength meter (a little fancier, but much like a "VU" or "S" meter on a radio).

Indeed, if you wish, you can even listen to AM radio stations with the headphones. The tunable microvoltmeter has a large gain (about 10^6) over a relatively narrow frequency range (100, 1000, or 3100 Hz selectable). It is calibrated to read rms voltage for sine wave inputs within the selected band at the frequency to which it is tuned. This behavior can be verified using the RF function generator. It is important to note input and output impedances of the various instruments and understand how matches (and mismatches) affect calibration.

EXPERIMENTAL PROCEDURE

Ultimately, the aim of your measurements is to investigate the shot and thermal noise as predicted by the Schottky and Nyquist formulas and determine the values of e and k in the process. If time permits, you can try your hand at noise thermometry (thermal noise temperature dependence) by immersing external resistors in liquid nitrogen and other known temperature baths.

Use the RF signal generator, oscilloscope, etc. to calibrate and become familiar with the measurement apparatus. Refer to the operating manuals of the various instruments for details in their use and precautions. On the HP 312B, note that there are three controls that influence the sensitivity of the instrument: impedance, reference level, and amplitude range. Set the input impedance to maximum (600Ω). Use the highest reference level that will result in a useable meter reading. Also note that the lighted window to the left of the analog meter indicates the applicable voltage multiplier for the '50 volts calibrated' impedance setting only. Since you are using the 600Ω setting, you must read the analog meter on the next *higher* scale than that indicated by the lighted window. Begin measurements with the sensitivity set low (high voltage scales) so you don't inadvertently blow out the meter.

Measure thermal noise first. Begin by inspecting and identifying the insides of the noise box. Be sure the cover is replaced and securely fastened before starting any readings (it shields the apparatus). Extract a value for Boltzmann's constant from your data. The shot noise measurement must be performed at the resonant frequency of the tank circuit. Measure the frequency response of the tank circuit (you will have to isolate it (disconnect it) from the rest of the circuit to do this). Measure shot noise as a function of diode plate current. Extract a value for the electron charge from your data.

The uncertainties and correction factors in the parameters you measure (resistance, bandwidth, voltage, temperature) need to be scrutinized. The following points are just some of many considerations to think about.

- The noise resistors (in the resistor bank) can be measured quite accurately with a DMM, but the effects of other resistances in parallel should be evaluated. For example, the tuned circuit ideally has an infinite impedance at resonance, but this is not an ideal circuit. What effect does it have on your measurements?
- The noise bandwidth is defined differently from signal bandwidth and, for accurate measurements, you will need to determine the equivalent noise bandwidth (ENBW).⁴ Also note the effect of measuring time on accuracy - for the same level of accuracy, narrowband measurements require a longer averaging time than do wideband measurements. As a rule of thumb, use the widest possible bandwidth for shortest reading times and greatest accuracy. One further comment on the bandwidth: if the determination of the bandwidth turns out to be your largest source of uncertainty, you can eliminate it entirely at the price of just measuring the ratio, e/kT . The trick is the following. A plot of the square of the shot noise voltage⁵ vs diode current and the square of the thermal noise voltage vs resistance should both give you straight lines. The ratio of the slopes of these lines will yield e/kT , with Δf canceling out. You still need to know R precisely, which means taking into account the various resistances in parallel with the noise resistor.
- The meter measures rms (as you will discover in your calibration). The average value of a full-wave rectified sine wave is 0.636 of the peak amplitude; the rms is 0.707 of the peak. To indicate the rms value of a sine wave the meter has a multiplying scale correction factor of $0.707/0.636 = 1.11$. In other words, 1.11 times the average value gives rms of a sine wave. A problem arises when the input signal is noise, and therefore not a sine wave. Gaussian noise has an average value of 0.798 of rms. Since an averaging meter indicates 1.11 times higher, it reads $1.11 \times 0.798 = 0.885$ of the true rms value - the averaging meter reads too low. Thus, when reading noise on an average responding meter, multiply the reading by 1.13 to obtain the correct value.⁶ Other characteristics you may

⁴ See Kittel, Hackelman and Donnelly, for example.

⁵ The assumption is that the shot noise is much greater than the thermal noise.

⁶ See Kittel, Hackelman and Donnelly, for example.

wish to consider are the bandwidth and crest factor of the meter. All of these considerations (and more) are fully discussed in the literature.

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APPENDIX A: A Derivation of Nyquist's Theorem

P. Pershan

10/98

Write the instantaneous current as the Fourier integral

$$I(t) = \int_{-\infty}^{\infty} I(\omega) \exp(i\omega t) d\omega$$

Take the statistical average

$$\langle I(t)I(t+\tau) \rangle = \int \int d\omega d\omega' \langle I(\omega)I(\omega') \rangle \exp(i(\omega+\omega')t) \exp(i(\omega-\omega')\tau)$$

and recognize that for a "stationary random process"

$$\langle I(t)I(t+\tau) \rangle = \langle I(t')I(t'+\tau) \rangle \text{ for all } t \text{ and } t'.$$

This is only possible if $\langle I(\omega)I(\omega') \rangle = \delta(\omega+\omega')$.

We define the current-current spectral density $S_{II}(\omega)\delta(\omega+\omega') = \langle I(\omega)I(\omega') \rangle$

Note the units for the various quantities:

$$[I(\omega)] = [I(t)/\omega]$$

$$[\langle I(\omega)I(\omega') \rangle] = [I(t)^2/\omega^2]$$

$$[\delta(\omega+\omega')] = [1/\omega]$$

$$[S_{II}(\omega)] = [I(t)^2/\omega]$$

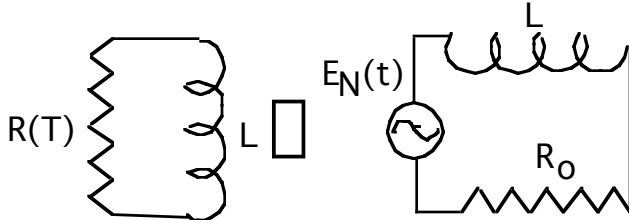
It follows that

$$\langle I(t)I(t+\tau) \rangle = \int_{-\infty}^{\infty} S_{II}(\omega) \exp(-i\omega\tau) d\omega$$

but since $\langle I(t)I(t+\tau) \rangle = \langle I(t-\tau)I(t) \rangle$ this can also be written as

$$\langle I(t)I(t+\tau) \rangle = \int_{-\infty}^{\infty} S_{II}(\omega) \exp(+i\omega\tau) d\omega$$

Consider now an R-L circuit. The R can be written as a noise generator $e(t)$ and an ideal resistor with no noise.



The current in the loop can be written as

$$I(\omega) = \frac{e(\omega)}{R+i\omega L} \text{ from which the spectral density can be written as}$$

$$S_{II}(\omega) = \frac{S_{ee}(\omega)}{R^2+\omega^2 L^2} \cdot \text{where } S_{ee}(\omega) = \text{Constant}$$

It follows that

$$\langle I(t)I(t+\tau) \rangle = \int_{-\infty}^{\infty} \frac{S_{ee}(\omega)}{R^2+\omega^2 L^2} \exp(i\omega\tau) d\omega$$

or

$$\langle I(t)I(t) \rangle = \frac{\int S_{ee}(\omega) d\omega}{RL}.$$

We know from the equipartition theorem that in this circuit

$$\frac{1}{2} L \langle I(t)I(t) \rangle = \frac{k_B T}{2} \quad \text{or} \quad \langle I(t)I(t) \rangle = \frac{k_B T}{L}.$$

It follows that the voltage-voltage spectral density from the noise source of a resistor

$$S_{ee}(\omega) = \frac{Rk_B T}{\omega}.$$

The spectral density can also be written as a function of $f = \omega/2\pi$.

$$S_{ee}(\omega) d\omega = S_{ee}(f)df \quad S_{ee}(f) = 2\pi S_{ee}(\omega) = 2Rk_B T$$

Now we ask for the mean square voltage at the output of some narrow band amplified with unit gain at the center frequency when a resistor R is connected to the input of the amplifier.

If we could assume that the amplifier had unit gain over a band width of $2\Delta f$ and zero gain elsewhere we would get the result given in the laboratory notes:

$$\langle V^2 \rangle = \int_{-\Delta f}^{+\Delta f} df S_{ee}(f) = 4Rk_B T \Delta f \quad \text{Eq 1}$$

On the other hand if the amplifier had a more realistic gain vs f profile such as

$$G(f) = \frac{\Delta f^2}{\Delta f^2 + (f-f_0)^2} \frac{\Delta f^2}{\Delta f^2 + (f+f_0)^2}$$

the output voltage

$$\langle V^2 \rangle = 2Rk_B T \int_{-\infty}^{+\infty} df \frac{\Delta f^2}{\Delta f^2 + (f-f_0)^2} \frac{\Delta f^2}{\Delta f^2 + (f+f_0)^2}$$

$$\langle V^2 \rangle = 4\pi Rk_B T \Delta f \quad \text{Eq 2}$$

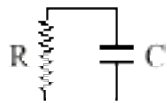
APPENDIX B: An Approximate Derivation of Nyquist's Theorem

R. Walsworth

09/02

Random statistical fluctuations of electrons at finite temperature in a conductor of finite resistance R imply the existence of transient differences in electron density at the two ends of the resistor. This is the source of the fluctuating voltage difference across the resistor.

Consider an equivalent circuit for the resistor, including a shunt capacitance, which is a non-trivial effect at most frequencies.



Energy stored in the electric field in the capacitor due to instantaneous voltage fluctuations, V is defined as $E = \frac{1}{2} CV^2$.

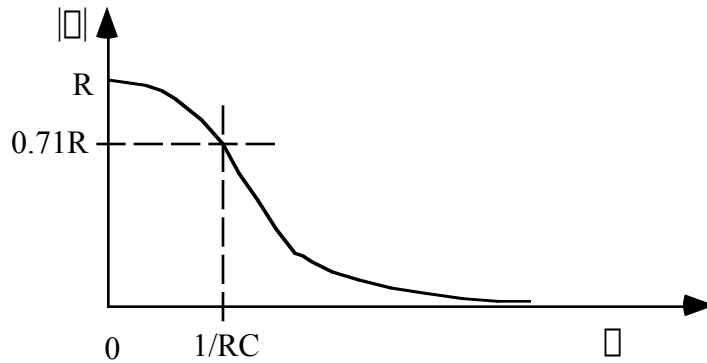
The equipartition theorem implies that $\frac{1}{2} k_B T$ is the thermal energy stored in all degrees of freedom, such as the electric field in the shunt capacitor. Therefore,

$$\frac{1}{2} CV^2 = \frac{1}{2} k_B T,$$

where $\overline{V^2}$ refers to time average. Therefore

$$\overline{V^2} = \frac{k_B T}{C}.$$

Now, the bandwidth of a parallel RC circuit is from DC out to the half-power point, i.e., the frequency at which the magnitude of the total impedance is reduced by 3dB ($\sqrt{2}$). Refer to the figure at right. Adding the impedances in parallel,



$$\frac{1}{Z} = \frac{1}{R} + j\omega C.$$

Thus,

$$|Z| = \frac{R}{\sqrt{1 + \omega^2 C^2 R^2}}.$$

Therefore, $|Z| = R$ at $\omega = 0$ and $|Z| = R/\sqrt{2}$ at $\omega = \frac{1}{RC}$. Thus the bandwidth in Hertz is

$$\omega f = \frac{1}{2RC}.$$

So, from the above expression for $\overline{V^2}$, we have

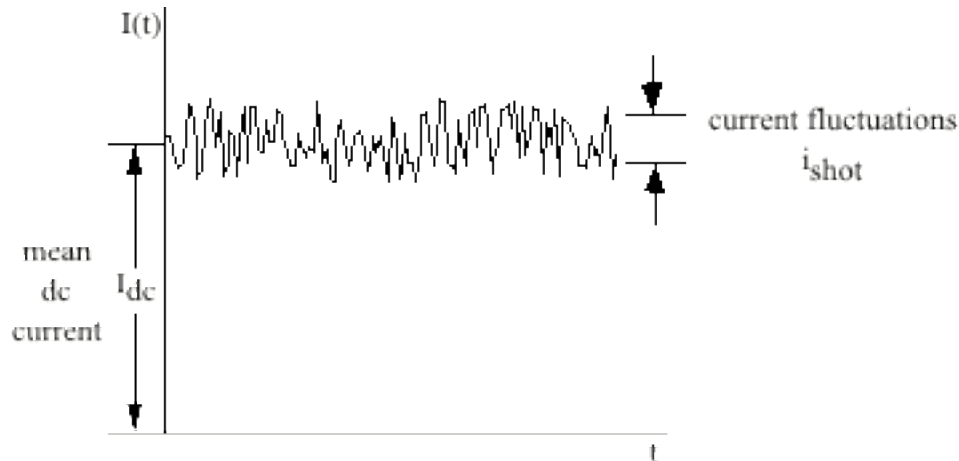
$$\overline{V^2} = 2\omega k_B T R \omega f.$$

APPENDIX C: An Intuitive Derivation of the Shot Noise Formula

R. Walsworth

09/02

Devices that emit discrete charged particles (e.g., electrons) at an average rate proportional to I_{dc} , but with each emission event being statistically independent, will exhibit shot noise in the current, as shown in the graph:



The shot noise is defined as

$$i_{shot} \equiv \sqrt{\overline{(I(t) - I_{dc})^2}}$$

where the bar implies a time average. During a time interval Δt , the dc current is

$$I_{dc} = \frac{e\bar{n}}{\Delta t}$$

where \bar{n} is the mean number of charge carriers emitted in time Δt and e is the charge of the particles. Similarly,

$$I(t) = \frac{en(t)}{\Delta t}$$

where $n(t)$ is the instantaneous number of charged particles emitted in time Δt . (Δt here is assumed to be the shortest time interval over which the behavior of the current is equivalent to that in the long time limit.)

Independent emission events with a constant mean \bar{n} and variation Δn^2 imply that $n(t)$ is generally well described by a Gaussian (i.e., normal) distribution of probabilities:

$$P(n(t)) = \frac{1}{\sqrt{2\Delta n^2}} e^{-\frac{(n(t) - \bar{n})^2}{2\Delta n^2}}$$

If we also assume the common situation where $\bar{n} \gg 1$, but where most of the time during Δt there is no charge carrier reaching the detector, then the above probability distribution becomes a special case known as a Poisson distribution, characterized by

$$\Delta n^2 = \bar{n}.$$

One can then use the probability distribution

$$P(n(t)) = \frac{1}{\sqrt{2\bar{n}}} e^{-\frac{(n(t) - \bar{n})^2}{2\bar{n}}}$$

to calculate

$$\overline{(n(t) - \bar{n})^2} = 2\bar{n}.$$

(Here it is assumed that a time average and an ensemble average are equivalent; this ergodic assumption is worth learning more about – it is usually valid as well.)

Therefore,

$$\begin{aligned}
 i_{shot} &= \sqrt{(I(t) \Delta I_{dc})^2} \\
 &= \frac{e}{\Delta t} \sqrt{(n(t) \Delta \bar{n})^2} \\
 &= \frac{e}{\Delta t} \sqrt{2\bar{n}}
 \end{aligned}$$

within the approximations and assumptions already discussed. Substituting from $I_{dc} = \frac{e\bar{n}}{\Delta t}$ gives

$$i_{shot} = \sqrt{\frac{2I_{dc}e}{\Delta t}}.$$

Taking the bandwidth to be $\Delta f = \frac{1}{\Delta t}$, i.e., the fastest rate at which the detector and current source can respond with the same average behavior, gives the final result

$$i_{shot} = \sqrt{2eI_{dc}\Delta f}.$$