Three Dimensional Narayana and Schröder Numbers

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Abstract

Consider the 3-dimensional lattice paths running from $(0,0,0)$ to (n,n,n), constrained to the region $\{(x,y,z) : 0 \leq x \leq y \leq z\}$, and using various step sets. With $\mathcal{C}(3,n)$ denoting the set of constrained paths using the steps $X := (1,0,0)$, $Y := (0,1,0)$, and $Z := (0,0,1)$, we consider the statistic $des(P) := |\{i : p_i p_{i+1} \in \{YX, ZX, ZY\}, 1 \leq i \leq 3n-1\}|$, where $P = p_1 p_2 \ldots p_{3n} \in \mathcal{C}(3,n)$. A combinatorial cancellation argument and a result of MacMahon yield a formula for the *3-Narayana number*, $N(3,n,k) := |\{P \in \mathcal{C}(3,n) : des(P) = k+2\}|$. We define other statistics distributed by the 3-Narayana number and show that $4\sum_k 2^k N(3,n,k)$ yields the n-th *large 3-Schröder number* which counts the constrained paths using the seven positive steps of the form (ξ_1, ξ_2, ξ_3), $\xi_i \in \{0,1\}$.

1 Introduction

We begin by reviewing the usual Narayana and Schröder numbers. Let $\mathcal{C}(2,n)$ denote the set of planar lattice paths using the unit steps $E := (1,0)$ and $N := (0,1)$, running from $(0,0)$ to (n,n), and lying in the wedge $\{(x,y) : 0 \leq x \leq y\}$. Common, and essentially equivalent, statistics on $\mathcal{C}(2,n)$ are the number of peaks and the number of valleys. These are defined as the number of NE pairs (EN pairs, respectively) on P, which we denote by $peaks(P)$ ($vals(P)$, respectively). If we define the Narayana number

$$N(n,k) := |\{P \in \mathcal{C}(2,n) : vals(P) = k\}| = |\{P \in \mathcal{C}(2,n) : peaks(P) = k+1\}|,$$

for $0 \leq k \leq n-1$, it is well known that

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1} = det \begin{bmatrix} \binom{n}{k+1} & \binom{n-1}{k+1} \\ \binom{n+1}{k+1} & \binom{n}{k+1} \end{bmatrix}. \tag{1}$$

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(See Section 6 and sequence A001263 in [12].)

We define the Narayana polynomial so

$$N_{2,n}(t) := \sum_k N(n,k)t^k = \sum_{P\in\mathcal{C}(2,n)} t^{vals(P)} = \sum_{P\in\mathcal{C}(2,n)} t^{peaks(P)-1}.$$

Note that $(|\mathcal{C}(2,n)|)_{n\geq 0} = (N_{2,n}(1))_{n\geq 0}$ is the sequence of Catalan numbers. When $t = 2$, consider counting the set of all copies of the paths of $\mathcal{C}(2,n)$ on which the peaks are independently colored blue or red. If on each copy we change each blue peak into a diagonal step of the form $(1,1)$ and discard the red color, then the copies are transformed into Schröder paths. Indeed $(2N_{2,n}(2))_{n\geq 1}$ (with initial term for $n = 0$ equal 1) is the sequence of large Schröder numbers (sequence A006318 in [12]).

For 3-dimensional results, let $\mathcal{C}(3,n)$ denote the set of paths using the steps $X := (1,0,0)$, $Y := (0,1,0)$, and $Z := (0,0,1)$, running from $(0,0,0)$ to (n,n,n), and lying in the chamber $\{(x,y,z) : 0 \leq x \leq y \leq z\}$. On $\mathcal{C}(3,n)$ we define the statistics, the *number of descents* (briefly *des*) and the *number of ascents* (briefly *ascs*) so that for any path $P := p_1p_2\ldots p_{3n}$,

$$\begin{aligned} des(P) &:= |\{i : p_ip_{i+1} \in \{YX, ZX, ZY\}, 1 \leq i \leq 3n-1\}|, \\ ascs(P) &:= |\{i : p_ip_{i+1} \in \{XY, XZ, YZ\}, 1 \leq i \leq 3n-1\}|. \end{aligned}$$

$P \in \mathcal{C}(3,2)$	$des(P)$	$ascs(P)$	$des(P) - ascs(P)$
$ZZYYXX$	2	0	2
$ZZYXYX$	3	1	2
$ZYZYXX$	3	1	2
$ZYZXYX$	3	2	1
$ZYXZYX$	4	1	3

Table 1: The statistical values for paths in $\mathcal{C}(3,2)$.

For $0 \leq k \leq 2n-2$, we define the *3-Narayana numbers* and the *3-Narayana polynomial*, respectively, to be

$$\begin{aligned} N(3,n,k) &:= |\{P \in \mathcal{C}(3,n) : ascs(P) = k\}|, \\ N_{3,n}(t) &:= \sum_k N(3,n,k)t^k. \end{aligned}$$

We find that (see sequence A087647 in [12]; [17])

$$\begin{gathered} N_{3,1}(t) = 1 \\ N_{3,2}(t) = 1 + 3t + t^2 \\ N_{3,3}(t) = 1 + 10t + 20t^2 + 10t^3 + t^4 \\ N_{3,4}(t) = 1 + 22t + 113t^2 + 190t^3 + 113t^4 + 22t^5 + t^6 \\ N_{3,5}(t) = 1 + 40t + 400t^2 + 1456t^3 + 2212t^4 + 1456t^5 + 400t^6 + 40t^7 + t^8 \end{gathered}$$

When $t=1$ and $n \geq 0$, as seen in Section 2.1,

$$|\mathcal{C}(3,n)| = N_{3,n}(1) = \frac{2(3n)!}{n!(n+1)!(n+2)!} \tag{2}$$

which are known as the *3-dimensional Catalan numbers* ([8, p.133],[18]. Sequence A005789 in [12]). As we will see in Section 5 where we count paths with steps being the edges and diagonals of the unit cube, we should take the sequence

$$(4N_{3,n}(2))_{n\geq 1} = (4, 44, 788, 18372, 505156, 15553372, 520065572, 18518471492, \ldots)$$

to be the 3-dimensional analogue of the large Schröder numbers. (Sequences A088594 in [12]).

In section 2 we will use a combinatorial cancellation argument and a result of MacMahon to prove

Proposition 1 *If $M(n,h,i,j)$ denotes the matrix*

$$\begin{bmatrix} \binom{n}{i}\binom{n-i}{j} & \binom{n-1}{i}\binom{n-1-i}{j} & \binom{n-2}{i}\binom{n-2-i}{j} \\ \binom{n+1}{h-i-j}\binom{n+1+i}{i+j} & \binom{n}{h-i-j}\binom{n+i}{i+j} & \binom{n-1}{h-i-j}\binom{n-1+i}{i+j} \\ \binom{n+2}{h-j} & \binom{n+1}{h-j} & \binom{n}{h-j} \end{bmatrix},$$

then, for $2 \leq h \leq 2n$,

$$|\{P \in \mathcal{C}(3,n) \ : \ des(P) = h\}| \ = \sum_{i=0}^{h}\sum_{j=0}^{h} det(M(n,h,i,j)). \tag{3}$$

In section 3 we will establish a bijection showing

Proposition 2 *The statistics ascs and des $-\,2$ are equi-distributed, i.e., for $0 \leq k \leq 2n-2$,*

$$|\{P \in \mathcal{C}(3,n) \ : \ ascs(P) = k\}| = |\{P \in \mathcal{C}(3,n) \ : \ des(P) = k+2\}|.$$

In section 4 we will indicate how the following result can be routinely deduced from more general results of MacMahon [8]:

Proposition 3 *For $0 \leq k \leq 2n-2$,*

$$N(3,n,k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{3n+1}{k-j}\prod_{i=0}^{2}\binom{n+i+j}{n}\binom{n+i}{n}^{-1}. \tag{4}$$

Moreover, the 3-Narayana polynomial $N_{3,n}(t)$ is a reciprocal polynomial (i.e., the sequence of its coefficients is palindromic) of degree $2n-2$.

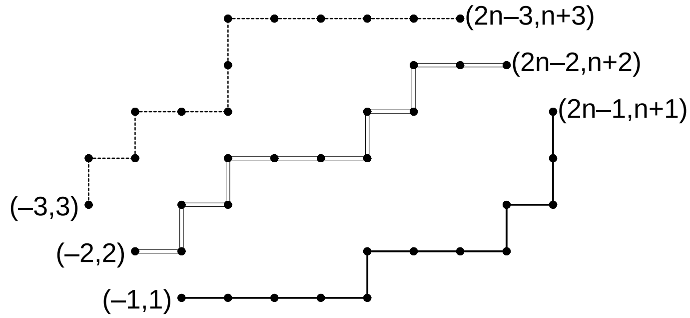

Figure 1: The triple of paths (R, R', R'') belonging to $\mathcal{N}(4) \subset \mathcal{T}(4, \sigma_1)$. This triple corresponds to the path $ZYZYXZZYXYXX \in \mathcal{C}(3, 4)$.

This paper emphasizes the derivation of formula (3). Upon reviewing the methods used to derive (3) and (4), it does not appear that either method can be modified so that formula (3) can be derived directly with respect to the number of ascents or that formula (4) can be derived directly with respect to the number of descents. In [17] the author studies (4), showing it to be much more tractable than (3) for establishing the reciprocity of the 3-Narayana polynomials, for finding recurrence relations, and for generalizing results to higher dimensions.

2 Derivation of the determinantal formula (3)

In subsection 2.1 we will "twist" the combinatorial-cancellation scheme of the Gessel-Viennot method [3, 4] to enumerate *unconstrained* paths with respect to the *number of descents*. We do so by first mapping the paths of $\mathcal{C}(3, n)$ to certain nonintersecting triples of planar lattice paths, which are a special case of "vicious walkers" introduced by Fisher [2] (See also [4, 5].). We will then do our counting using these triples.

In subsection 2.2 we will complete the derivation of formula (3) by using a result of MacMahon regarding the equi-distribution of the *number of descents* and the *number of excedances* to enumerate the unconstrained paths using the steps X, Y, and Z. The extension of the techniques of this section to higher dimensions does not appear to yield a neat general formula.

2.1 Combinatorial cancellation with a twist

Let $\mathcal{S}_3$ denote the set of the permutations on $\{1, 2, 3\}$. Using the notation, $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix}$, we designate the permutations as $\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, and $\sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Observe that their signs satisfy $sgn(\sigma_1) = sgn(\sigma_4) = sgn(\sigma_5) = 1$

while $sgn(\sigma_2) = sgn(\sigma_3) = sgn(\sigma_6) = -1$.

For $\sigma \in \mathcal{S}_3$ and $n \geq 1$, let $\mathcal{T}(n,\sigma)$ denote the set of all ordered triples of 2-dimensional lattice paths (R, R', R'') where

(i) the paths $R = r_1 r_2 \ldots r_{3n}$, $R' = r'_1 r'_2 \ldots r'_{3n}$, and $R'' = r''_1 r''_2 \ldots r''_{3n}$ use the steps $E = (1,0)$ and $N = (0,1)$,

(ii) R runs from $(-\sigma(1), \sigma(1))$ to $(2n-1, n+1)$,

(iii) R' runs from $(-\sigma(2), \sigma(2))$ to $(2n-2, n+2)$,

(iv) R'' runs from $(-\sigma(3), \sigma(3))$ to $(2n-3, n+3)$,

(v) for each i, $1 \leq i \leq 3n$, exactly one of r_i, r'_i, and r''_i is an N step.

Figures 1 and 3 illustrate triples belonging to $\mathcal{T}(4,\sigma_1)$, $\mathcal{T}(7,\sigma_1)$, and $\mathcal{T}(7,\sigma_2)$. Further, let $\mathcal{T}(n,\sigma,XY)$ denote that subset of triples (R,R',R'') in $\mathcal{T}(n,\sigma)$ where R and R' have the most north-eastern point of intersection. Likewise, let $\mathcal{T}(n,\sigma,YZ)$ denote that subset of triples (R,R',R'') in $\mathcal{T}(n,\sigma)$ where R' and R'' have the most north-eastern point of intersection. Condition (v) guarantees that there is no triple where R and R'' have the most north-eastern point of intersection. Let $\mathcal{N}(n)$ denote that subset of $\mathcal{T}(n,\sigma_1)$ containing no intersecting paths.

Let $\mathcal{L}(n_1,n_2,n_3)$ denote the set of unconstrained lattice paths running from $(0,0,0)$ to (n_1,n_2,n_3) using the unit steps X, Y, and Z. We define a bijection ν with domain $\bigcup_{\sigma\in\mathcal{S}_3} \mathcal{T}(n,\sigma)$ so that, for each $\sigma \in \mathcal{S}_3$,

$$\nu : \mathcal{T}(n,\sigma) \to \mathcal{L}(n+1-\sigma(1), n+2-\sigma(2), n+3-\sigma(3))$$

and, if $\nu((R,R',R'')) = p_1 p_2 \ldots p_{3n}$, then, for $1 \leq i \leq 3n$,

$$\begin{aligned}
(r_i, r'_i, r''_i) &= (N,E,E) \quad \text{if and only if} \quad p_i = X \\
(r_i, r'_i, r''_i) &= (E,N,E) \quad \text{if and only if} \quad p_i = Y \\
(r_i, r'_i, r''_i) &= (E,E,N) \quad \text{if and only if} \quad p_i = Z.
\end{aligned}$$

We now define an involution μ on $\bigcup_{\sigma\in\mathcal{S}_3} \mathcal{T}(n,\sigma)$, which is sign reversing with respect to the signs of the permutations, by the following three cases. Figures 2 and 3 illustrate this involution.

Case (i): For $(R,R',R'') \in \mathcal{N}(n)$, $\mu((R,R',R'')) = (R,R',R'')$.

Case (ii): For $(R,R',R'') \in \mathcal{T}(n,\sigma,XY)$, factor R'' as $R'' = R''_1 R''_2 R''_3 R''_4$ where

$$R''_1 = r''_1 \ldots r''_j, \quad R''_2 = r''_{j+1} \ldots r''_k, \quad R''_3 = r''_{k+1} \ldots r''_\ell, \quad R''_4 = r''_{\ell+1} \ldots r''_{3n}$$

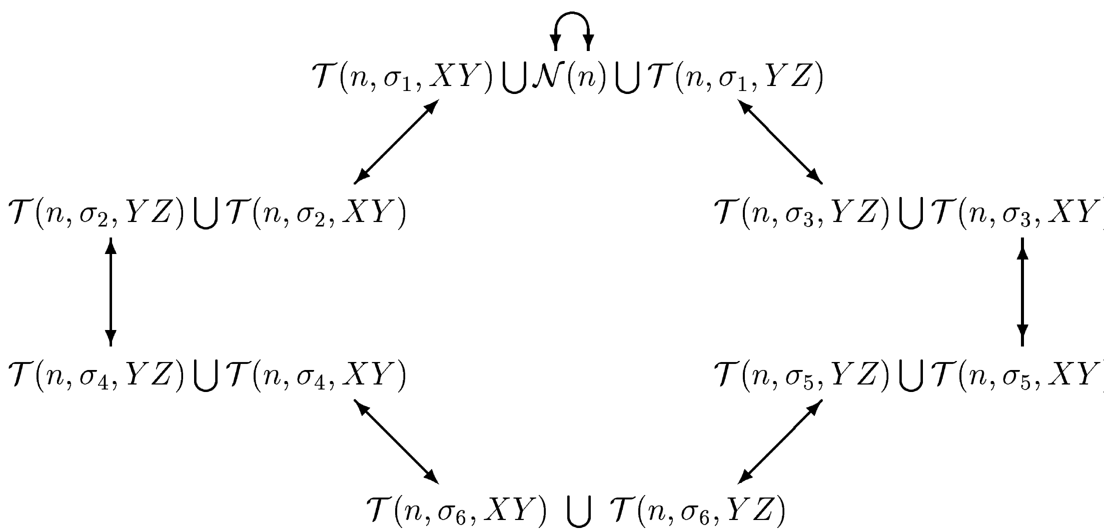

Figure 2: The action of the involution μ on $\bigcup_{\sigma\in\mathcal{S}_3}\mathcal{T}(n,\sigma)$.

so that the initial points of the steps $r_{\ell+1}$ and $r'_{\ell+1}$ are the most north-eastern point of intersection of R and R', and so that R''_1 and R''_3 are maximal subpaths (perhaps empty) consisting only of N steps. Using the j, k, and ℓ so determined, let

$$\begin{array}{llll} R_1 = r_1\ldots r_j, & R_2 = r_{j+1}\ldots r_k, & R_3 = r_{k+1}\ldots r_\ell, & R_4 = r_{\ell+1}\ldots r_{3n} \\ R'_1 = r'_1\ldots r'_j, & R'_2 = r'_{j+1}\ldots r'_k, & R'_3 = r'_{k+1}\ldots r'_\ell, & R'_4 = r'_{\ell+1}\ldots r'_{3n} \end{array}$$

For any path $Q = q_1\ldots q_m$, let $\overline{Q}$ denote $q_m\ldots q_1$; this is the "twist" of Q. We then define

$$\mu((R,R',R'')) = (R_1\overline{R'_2}R_3R_4, R'_1\overline{R_2}R'_3R'_4, R''_1\overline{R''_2}R''_3R''_4).$$

Case (iii): Similarly for $(R,R',R'')\in\mathcal{T}(n,\sigma,YZ)$, factor R as $R = R_1R_2R_3R_4$ where

$$R_1 = r_1\ldots r_j,\quad R_2 = r_{j+1}\ldots r_k,\quad R_3 = r_{k+1}\ldots r_\ell,\quad R_4 = r_{\ell+1}\ldots r_{3n}$$

so that the initial points of the steps $r'_{\ell+1}$ and $r''_{\ell+1}$ are the most north-eastern point of intersection of R' and R'', and so that R_1 and R_3 are maximal subpaths (perhaps empty) consisting only of N steps. Using the j, k, and ℓ so determined, let

$$\begin{array}{llll} R'_1 = r'_1\ldots r'_j, & R'_2 = r'_{j+1}\ldots r'_k, & R'_3 = r'_{k+1}\ldots r'_\ell, & R'_4 = r'_{\ell+1}\ldots r'_{3n} \\ R''_1 = r''_1\ldots r''_j, & R''_2 = r''_{j+1}\ldots r''_k, & R''_3 = r''_{k+1}\ldots r''_\ell, & R''_4 = r''_{\ell+1}\ldots r''_{3n} \end{array}$$

We then define

$$\mu((R,R',R'')) = (R_1\overline{R_2}R_3R_4, R'_1\overline{R''_2}R'_3R'_4, R''_1\overline{R'_2}R''_3R''_4).$$

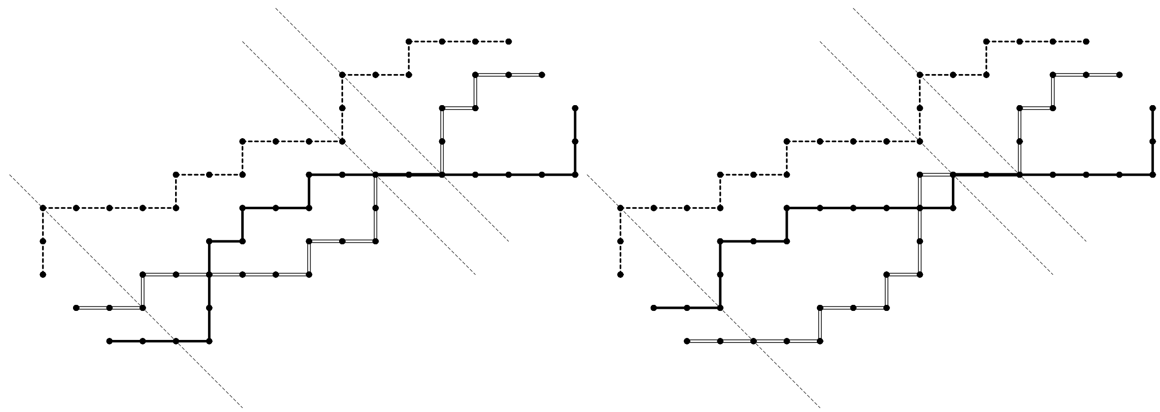

Figure 3: The first triple belongs to $\mathcal{T}(7, \sigma_1, XY)$ and corresponds to the path $ZZYXXXZXYZXYYZZYYZXX \in \mathcal{L}(7,7,7)$. The second triple is its image under μ. It belongs to $\mathcal{T}(7, \sigma_2, XY)$ and corresponds to the path $ZZXXYZXYZYYYXZZYYZYXX \in \mathcal{L}(6,8,7)$.

In essence, μ acts by cutting out a section (R_2, R_2', R_2''), twisting (i.e., rotating) each subpath of this section 180 degrees, and then re-attaching each subpath according to the relevant permutation. We observe that, except on $\mathcal{N}(n)$, μ is sign reversing with respect to the permutations defining the set of triples. Importantly, one can routinely check that the twisting scheme yields

$$des(\nu(\mu(R, R', R''))) = des(\nu(R, R', R'')).$$

for all $(R, R', R'') \in \bigcup_{\sigma \in \mathcal{S}_3} \mathcal{T}(n, \sigma)$. Hence, by the cancellation summarized in Figure 2, we have

Lemma 1 *For $n \geq 1$,*

$$\sum_{(R,R',R'') \in \mathcal{N}(n)} t^{des(\nu(R,R',R''))} = \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \sum_{(R,R',R'') \in \mathcal{T}(n,\sigma)} t^{des(\nu(R,R',R''))}$$

or equivalently,

$$\sum_{P \in \mathcal{C}(3,n)} t^{des(P)} = \sum_{\sigma \in \mathcal{S}_3} sgn(\sigma) \sum_{P \in \mathcal{L}(n+1-\sigma(1), n+2-\sigma(2), n+3-\sigma(3))} t^{des(P)}.$$

In the case that $t = 1$, we immediately obtain the formula for the three dimensional Catalan numbers. Since the cardinality of $\mathcal{L}(n_1, n_2, n_3)$ is $(n_1+n_2+n_3)!/(n_1!n_2!n_3!)$, Lemma 1 implies the following, which simplifies to (2):

$$|\mathcal{C}(3,n)| = (3n)! \det \begin{bmatrix} \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} \\ \frac{1}{(n+2)!} & \frac{1}{(n+1)!} & \frac{1}{n!} \end{bmatrix}.$$

2.2 Two statistics for unconstrained paths

With $\mathcal{L}(n_1, n_2, n_3)$ previously defined, we now seek a formula for $\sum_{P \in \mathcal{L}(n_1,n_2,n_3)} t^{des(P)}$. Order the steps so that $X < Y < Z$. We recall two statistics considered by MacMahon [8, arts. 149-151]: the *number of descents* (called *number of major contacts* in [8]) and *number of excedances*. For any path $P = p_1 p_2 \ldots p_m$ in $\mathcal{L}(n_1, n_2, n_3)$, define

- $des(P) = |\{i \ : \ p_i > p_{i+1}\}| \ = \$ the number of ZY, ZX, or YX pairs on P.
- $exced(P) = |\{i \ : \ p_i > q_i\}|$ where $q_1 q_2 \ldots q_m$ is that path in $\mathcal{L}(n_1, n_2, n_3)$ for which $q_i \le q_{i+1}$ for $1 \le i < m$.

  Observe that $exced(P)$ is the number of Z steps in first $n_1 + n_2$ positions of P plus the number of Y steps in first n_1 positions of P.

We will use a result of MacMahon [7, 8] which is considered bijectively by Foata, as recorded in [6] and [7, pp. 455-6], and which in our case reduces to the following:

Proposition 4 *The statistics des and exced are identically distributed on* $\mathcal{L}(n_1, n_2, n_3)$.

Hence, using this proposition and counting the paths having i Z's and j Y's in the first n_1 positions and k Z's and ℓ Y's in the next n_2 positions, we have

$$\begin{aligned} \sum_{P \in \mathcal{L}(n_1,n_2,n_3)} t^{des(P)} &= \sum_h |\{P \ : \ des(P) = h\}| t^h \\ &= \sum_h |\{P \ : \ exced(P) = h\}| t^h \\ &= \sum_{i,j,k,\ell} g(n_1, n_2, n_3, i, j, k, \ell) t^{i+j+k} \end{aligned}$$

where $g(n_1, n_2, n_3, i, j, k, \ell)$ denotes

$$\frac{n_1!}{i!j!(n_1-i-j)!} \frac{n_2!}{k!\ell!(n_2-k-\ell)!} \frac{n_3!}{(n_3-i-k)!(n_2-j-\ell)!(i+j+k+\ell-n_3)!}.$$

Consequently, upon applying the Chu-Vandermonde convolution, we find

$$\sum_{P \in \mathcal{L}(n_1,n_2,n_3)} t^{des(P)} =$$

$$\sum_i \sum_j \sum_k \binom{n_1}{i} \binom{n_1 - i}{j} \binom{n_2}{k} \binom{n_2 + i}{i + j} \binom{n_3}{i + k} t^{i+j+k} \tag{5}$$

We remark that MacMahon [8, art. 151] derives an equivalent formula. Finally, Lemma 1 and (5), with $k = h - i - j$, yield formula (3).

$M_D := \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 1&1&0 \end{bmatrix}$	$M_c := \begin{bmatrix} 1&0&1 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$	$M_e := \begin{bmatrix} 0&0&0 \\ 1&1&0 \\ 0&1&0 \end{bmatrix}$	$M_f := \begin{bmatrix} 0&1&0 \\ 1&1&0 \\ 0&0&0 \end{bmatrix}$
$M_a := \begin{bmatrix} 0&0&0 \\ 0&1&1 \\ 0&0&1 \end{bmatrix}$	$VM_c := \begin{bmatrix} 0&0&0 \\ 1&0&1 \\ 1&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0 \\ 0&0&1 \\ 1&0&1 \end{bmatrix}$	$\begin{bmatrix} 1&0&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$
$VM_a := \begin{bmatrix} 0&1&1 \\ 0&0&0 \\ 0&1&0 \end{bmatrix}$	$M_d := \begin{bmatrix} 0&0&0 \\ 0&1&0 \\ 0&1&1 \end{bmatrix}$	$\begin{bmatrix} 1&0&1 \\ 1&0&0 \\ 0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 1&0&1 \end{bmatrix}$
$M_b := \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 1&0&1 \end{bmatrix}$	$M_A := \begin{bmatrix} 0&1&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&1&0 \\ 0&1&1 \\ 0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&0 \\ 0&1&1 \\ 0&1&0 \end{bmatrix}$
$\begin{bmatrix} 0&0&1 \\ 1&0&1 \\ 0&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&0&0 \\ 1&1&0 \\ 0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 0&1&1 \end{bmatrix}$	$\begin{bmatrix} 0&1&1 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$
$\begin{bmatrix} 1&1&0 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}$	$\begin{bmatrix} 0&1&0 \\ 0&0&0 \\ 1&1&0 \end{bmatrix}$	$\begin{bmatrix} 1&1&0 \\ 0&0&0 \\ 1&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 1&1&0 \end{bmatrix}$

Table 2: The candidate matrices

3 Other statistics having the 3-Narayana distribution

Our main intention is obtain a bijective proof of Proposition 2. To do so, we will consider 24 statistics for $\mathcal{C}(3,n)$, each of which is encoded in terms of a 3 by 3 0-1 matrix M. With $X_1 := X$, $X_2 := Y$, $X_3 := Y$, with $P := p_1p_2\ldots p_{3n} \in \mathcal{C}(3,n)$, and with $(M)_{j\ell}$ denoting the entry in row j and column ℓ of M, we let Θ_M denote a statistic such that

$$\Theta_M(P) := \sum_{j=1}^{3}\sum_{\ell=1}^{3}(M)_{j\ell}|\{i : p_ip_{i+1} = X_jX_\ell, 1 \le i \le 3n-1\}|.$$

For example, the statistic *ascs* corresponds to the matrix $M_A := \begin{bmatrix} 0&1&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$, since

$$ascs(P) = |\{i : p_ip_{i+1} \in \{X_1X_2, X_1X_3, X_2X_3\}\}|.$$

Similarly, the statistic *des* corresponds to the matrix $M_D := \begin{bmatrix} 0&0&0 \\ 1&0&0 \\ 1&1&0 \end{bmatrix}$.

For small values of n, a simple search over $\mathcal{C}(3,n)$ shows that, if a statistic has the prescribed form Θ_M and is distributed by $N(3,n,k-c)$, for some $c \in \{0,1,2\}$, then it must correspond to one of the 24 matrices of Table 2. A series of lemmas will establish

Proposition 5 *For $n \ge 1$, each matrix M in the first two columns of Table 2 yields a statistic $\Theta_M - (M)_{21} - (M)_{32}$ having the 3-Narayana distribution. In particular, ascs and des -2 are equi-distributed. (The sum $(M)_{21} + (M)_{32}$ adjusts the statistic so $|\{P \in \mathcal{C}(3,n) : \Theta_M(P) - (M)_{21} - (M)_{32} = k\}| = N(3,n,k)$.)*

P	$\Theta_{M_D}(P)$	$\Theta_{M_A}(P)$	$\Theta_{M_e}(P)$	$\Theta_{M_f}(P)$	$hdes(P)$
$ZZYYXX$	2	0	1	1	2
$ZZYXYX$	3	1	1	2	1
$ZYZYXX$	3	1	1	0	1
$ZYZXYX$	3	2	0	1	1
$ZYXZYX$	4	1	2	1	0

Table 3: Statistical values

Conjecture 1 *For $n \geq 1$, each matrix M in the last two columns of Table 2 yields a statistic $\Theta_M - (M)_{21} - (M)_{32}$ having the 3-Narayana distribution. Consequently, the statistic $des - ascs - 1$ (which was introduced in Table 1) has the 3-Narayana distribution in agreement with $M_b + M_A + M_e = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right]$.*

If one considers any two statistics Θ_1 and Θ_2 on $\mathcal{C}(3,n)$ to be equivalent when either $\Theta_1 + \Theta_2$ or $\Theta_1 - \Theta_2$ is a constant statistic for each n, then Table 3 shows the non-equivalency of Θ_{M_D}, Θ_{M_A}, Θ_{M_e} and Θ_{M_f}. The statistic *hdes* counting the *high descents*, considered at the end of this section, requires $n = 3$ to see that it is not equivalent to the others. Lemma 2, below, shows that each column of Table 2 corresponds to an equivalence class.

For each matrix M being considered, we define the *horizontal complement*, HM, and the *vertical complement*, VM, to be matrices defined so

$$(HM)_{j\ell} := \begin{cases} 0 & \text{if } j \text{ is a zero row of } M \\ 1 - (M)_{j\ell} & \text{if otherwise,} \end{cases}$$

$$(VM)_{j\ell} := \begin{cases} 0 & \text{if } \ell \text{ is a zero column of } M \\ 1 - (M)_{j\ell} & \text{if otherwise.} \end{cases}$$

(E.g., see Table 2, where $M_a = HM_D$. See also Figure 4.)

Lemma 2 *For any M in Table 2 and for any $P \in \mathcal{C}(3,n)$,*

$$\Theta_M(P) + \Theta_{HM}(P) = \begin{cases} 2n & \text{if row 1 of } M \text{ is a zero row} \\ 2n-1 & \text{if otherwise,} \end{cases}$$

$$\Theta_M(P) + \Theta_{VM}(P) = \begin{cases} 2n & \text{if column 1 of } M \text{ is a zero column} \\ 2n-1 & \text{if otherwise.} \end{cases}$$

Proof. We note that each path begins with Z, ends with X, and has a total of $3n-1$ consecutive step pairs. If row 1 of M is a zero row, then the $n-1$ non-final X steps, all of which immediately precede some other step on P, do not contribute to $\Theta_M(P) + \Theta_{HM}(P)$. Hence, $\Theta_M(P) + \Theta_{HM}(P) = (3n-1) - (n-1)$. If row 2 of M is a zero row, then only

$$M_D \xrightarrow{H} M_a \xrightarrow{V} VM_a \xrightarrow{H} M_b \xrightarrow{T} M_c \xrightarrow{V} VM_c \xrightarrow{H} M_d \xrightarrow{V} M_A$$

Figure 4: The schema for proving that $des - 2$ and $ascs$ are equi-distributed.

the n Y steps, which must immediately precede some other step on P, do not contribute to $\Theta_M(P) + \Theta_{HM}(P) = (3n-1) - n$. Similarly, the other instances of the lemma are valid. □

Lemma 3 *For any M in Table 2 and for any $P \in \mathcal{C}(3,n)$,*

$$\begin{aligned} \Theta_M(P) - (M)_{23} - (M)_{32} + \Theta_{HM}(P) - (HM)_{23} - (HM)_{32} &= 2n-2. \\ \Theta_M(P) - (M)_{23} - (M)_{32} + \Theta_{VM}(P) - (VM)_{23} - (VM)_{32} &= 2n-2. \end{aligned}$$

Proof. This is an easily checked consequence of Lemma 2. □

Lemma 4 *Suppose that Θ_1 is distributed by a reciprocal polynomial of degree $2n-2$ on $\mathcal{C}(3,n)$. If $\Theta_1(P) + \Theta_2(P) = 2n-2$ for all $P \in \mathcal{C}(3,n)$, then Θ_1 and Θ_2 are equi-distributed.*

Proof.

$$\sum_{P \in \mathcal{C}(3,n)} t^{\Theta_2(P)} = \sum_{P \in \mathcal{C}(3,n)} t^{2n-2-\Theta_1(P)} = \sum_{P \in \mathcal{C}(3,n)} t^{\Theta_1(P)}. \quad \Box$$

Lemma 5 *For M_b and M_c defined in Table 2 and for $1 \le k < 2n-2$, there is an explicit bijection*

$$\beta : \{P \in \mathcal{C}(3,n) : \Theta_{M_b}(P) = k\} \to \{P \in \mathcal{C}(3,n) : \Theta_{M_c}(P) = k\},$$

and hence Θ_{M_b} and Θ_{M_c} are equi-distributed.

Proof. For any $P \in \mathcal{C}(3,n)$, we split P into maximal blocks (i.e., maximal subpaths) which either contain only Y steps or contain no Y step. In each block of the second type, we exchange its initial maximal subblock (perhaps empty) of X steps with its final maximal subblock (perhaps empty) of X steps. $\beta(P)$ is the resulting path. We note that $\beta(P) \in \mathcal{C}(3,n)$ since the condition $0 \le x \le y \le z$ for any point (x,y,z) on a path holds during the exchanges. The action of β leaves the number of XX and ZZ pairs fixed and transforms the number of ZX pairs to the number of XZ pairs. Since $M_c = TM_b$, where T denotes the usual transpose operator, the proof is complete. □

Proof of Proposition 5: This is a consequence of Lemmas 3, 4, 5, and the reciprocity of the 3-Narayana polynomials, which is considered in Section 4. See Figure 4 where T denotes the transpose operator.

In particular by Lemma 2, for any $P \in \mathcal{C}(3,n)$, the following identities

$$\begin{array}{ll} \Theta_{M_D}(P) + \Theta_{M_a}(P) = 2n, & \Theta_{M_a}(P) + \Theta_{VM_a}(P) = 2n-1, \\ \Theta_{VM_a}(P) + \Theta_{M_b}(P) = 2n-1, & \Theta_{M_c}(\beta(P)) + \Theta_{VM_c}(\beta(P)) = 2n-1, \\ \Theta_{VM_c}(\beta(P)) + \Theta_{M_d}(\beta(P)) = 2n, & \Theta_{M_d}(\beta(P)) + \Theta_{M_A}(\beta(P)) = 2n-1, \end{array}$$

together with $\Theta_{M_b}(P) = \Theta_{M_c}(\beta(P))$, yield bijectively that $\Theta_{M_D}(P) - 2 = \Theta_{M_A}(\beta(P))$. $\square$

High descents: On any planar path a *high peak* is any YX pair whose intermediate vertex (x,y) satisfies $y - x > 1$. Deutsch [1] introduced the statistic $hpeaks(P)$, which gives the number of high peaks on the path P, and showed that $hpeaks$ has the Narayana distribution on $\mathcal{C}(2,n)$.

Now, for any path $P = p_1p_2\dots p_{3n} \in \mathcal{C}(3,n)$, call any step pair p_ip_{i+1} a *high descent* if $p_ip_{i+1} = X_jX_\ell$ for $j > \ell$ and its intermediate vertex (x_1,x_2,x_3) satisfies $x_j - x_\ell > 1$. With $hdes(P)$ denoting the number of high descents on the path P, we show in [17] that $|\{P \in \mathcal{C}(3,n) : hdes(P) = k\}| = N(3,n,h)$.

4 Another formula for the 3-Narayana numbers

Here we briefly indicate how Proposition 3 can be obtained by specializing and, then translating into our notation, q-results of MacMahon [8], Articles 436–498. One can also obtain formula (4) either from a fundamental theorem on order polynomials on posets developed by Stanley [13][14, Theorem 4.5.14] or from an alternative proof of the author [17].

In 1910 MacMahon [7, 8] introduced the *sub-lattice function*, which is a q-analogue of a "d-dimensional Narayana number". Instead of the steps X, Y, and Z, MacMahon uses the symbols γ, β, and α, respectively. He uses "lattice permutation" for "constrained path in $\mathcal{C}(3,n)$." Hence, for example, "the constrained path $ZZYXYX$ with one ascent at XY" corresponds to his "lattice permutation $\alpha\alpha\beta\gamma\beta\gamma$ with one major contact at $\gamma\beta$." Specific to the 3-dimensional case (and in our notation), the "sub-lattice function of order k", denoted by $L_k(n,3;\infty)$, is defined to satisfy

$$\sum_k L_k(n,3;\infty)t^k = \sum_{P\in\mathcal{C}(3,n)} t^{ascs(P)}q^{\iota(P)}$$

where (in our notation) $\iota(P)$ denotes the lessor index of P, i.e., $\iota(p_1p_2p_3\cdots p_i\cdots p_{3n})$ denotes the sum of the indices i where p_ip_{i+1} is an ascent. Hence, when $q \to 1$,

$$L_k(n,3;\infty) = N(3,n,k).$$

In [8, art. 429] MacMahon uses $GF(n;d;j)$ to denote the generating function, with respect to the sum of the parts with size marked by q, of the plane partitions having at most n

columns, at most d rows, and part size bounded by j. In [8, art. 443] he records the bottom formula of page 197, which reduces to

$$\sum_j GF(n;3;j)g^j = \frac{L_0(n,3;\infty) + L_1(n,3;\infty)g + \cdots + L_{2n-2}(n,3;\infty)g^{2n-2}}{(1-g)(1-gq)\cdots(1-gq^{3n})} \tag{6}$$

In [8, art. 495] he obtains a formula for $GF(n;3;j)$ which becomes, for $q \to 1$,

$$GF(n;3;j) = \prod_{i=0}^{2} \binom{j+n+i}{n} \binom{n+i}{n}^{-1}.$$

With $q \to 1$, this and (6) yield

$$\sum_k N(3,n,k)g^k = (1-g)^{3n+1} \sum_j \prod_{i=0}^{2} \binom{j+n+i}{n} \binom{n+i}{n}^{-1}.$$

Since $(1-g)^{3n+1} = \sum_j (-1g)^j \binom{3n+1}{j}$, formula (4) follows by forming the convolution.

MacMahon gave a proof for the reciprocity of the polynomial $N_{3,n}(t)$ in [7, art. 29]; his argument in [8, art. 449] seems incomplete. The reciprocity can also be derived either from the results of [13, sect. 18][14, sect. 4.5] or by two different methods in [17]. The degree of this polynomial is considered in [8, art. 445], in [14, sect. 4.5], and in [17].

5 Three dimensional Schröder paths

Now we count constrained paths using the seven steps corresponding to the edges and the diagonals of the unit cube. Let $\mathcal{D}(n)$ denote the set of paths running from $(0,0,0)$ to (n,n,n), lying in $\{(x,y,z) : 0 \leq x \leq y \leq z\}$, and using the nonzero steps of the form (ξ_1, ξ_2, ξ_3) where $\xi_i \in \{0,1\}$ for $1 \leq i \leq 3$. One might call each path in $\mathcal{D}(n)$, a *3-Schröder path.*

Proposition 6 *For $n \geq 1$, $\mathcal{D}(n)$ has cardinality equal to $4N_{3,n}(2)$.*

Proof. Let $\mathcal{C}'(3,n)$ denote the set of all possible paths formed by independently coloring blue or red the intermediate vertices of the YX, ZX, and ZY pairs on copies of the paths of $\mathcal{C}(3,n)$. Hence, $\mathcal{C}'(3,n)$ has cardinality $\sum_h 2^h |\{P \in \mathcal{C}(3,n) : des(P) = h\}| = 4N_{3,n}(2)$.

Next we define a bijection

$$\mu : \mathcal{C}'(3,n) \longrightarrow \mathcal{D}(n)$$

so that, for each $P \in \mathcal{C}'(3,n)$, $\mu(P)$ is obtained by first replacing sequentially each maximal factor of steps having consecutively blue, B, intermediate vertices as follows:

$$\begin{array}{lcl} YBX & \longrightarrow & (1,1,0) \\ ZBX & \longrightarrow & (1,0,1) \\ ZBY & \longrightarrow & (0,1,1) \\ ZBYBX & \longrightarrow & (1,1,1) \end{array}$$

On the resulting path keep the remaining nondiagonal unit steps and remove the color red. Hence the cardinalities of $\mathcal{C}'(3,n)$ and $\mathcal{D}(n)$ agree. □

6 Notes

In MacMahon's text [8], one finds the formula for the Narayana numbers as a special case of the 5th formula of article 495. In 1955, using different terminology, Narayana [9, 10], introduced the number $\frac{1}{n}\binom{n}{h}\binom{n}{h-1}$ to count, in essence, pairs of nonintersecting paths using the steps E and N, where the lower path runs from $(1,0)$ to $(h, n-h)$ and the upper path runs from $(0,1)$ to $(h-1, n-h+1)$. His interest was the derivation of distributions related to statistical tests of the Kolmogorov-Smirnov type. Implicit in [10] (explicit in [16, Sect. 2]) is a bijection between these pairs of such nonintersecting paths and paths in $\mathcal{C}(2,n)$ having h peaks. The papers [15, 16] study the Narayana polynomial.

We are appreciative of the references on counting paths in higher dimensions given us by Heinrich Niederhausen, especially those listed in his paper [11], which studies enumerating constrained walks in terms of diffusion walks where particles from different sources with opposite charges cancel upon meeting. His cancellation scheme differs from reflection or that of this note.

We thank Christian Krattenthaler for refreshing our memory about bijection ν and the result of Proposition 4. His recent paper [5] includes a summary of the literature on the enumeration of general higher-dimensional walks in regions bounded by hyperplanes and the associated enumeration of n-tuples of nonintersecting walks in the plane.

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