Inhomogeneity of the urelements in the usual models of *NFU*

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The simplest typed theory of sets is the multi-sorted first order system TST with equality and membership as primitive predicates and with sorts (types) indexed by the natural numbers. Atomic formulas are well-formed if they are of one of the forms $x^n \in y^{n+1}; x^n = y^n$. The axioms of TST are extensionality (objects of positive type are equal iff they have the same members) and comprehension ("$\{x^n \mid \phi\}^{n+1}$ exists" for any formula ϕ in the language of TST). (this theory has often been incorrectly attributed to Russell, by this author among others: see [17] for a discussion of the actual history of this system).

Quine's New Foundations (*NF*) ([14]) is obtained from TST by abandoning the types but retaining the same axioms. Note that the comprehension axioms of *NF* are not all the axioms "$\{x \mid \phi\}$ exists" for ϕ a formula in the language of *NF*: this would be the inconsistent comprehension axiom of naive set theory. The comprehension axioms of *NF* are those assertions "$\{x \mid \phi\}$ exists" where ϕ can be obtained from a formula of TST by dropping distinctions of type between variables (without creating any additional identifications between variables). Formulas ϕ which are typable in TST in this sense are said to be *stratified*. The extensionality axiom of *NF* is the same as the extensionality axiom of ZFC.

NF does not provide the evil set $\{x \mid x \notin x\}$ because it is impossible to assign types to the occurrences of x in $x \notin x$ in a way consistent with the requirements of TST. So far so good, but *NF* does allow the construction of $\{x \mid x = x\}$ (the universal set) and of other big sets (such as the Frege natural numbers) which are not allowed in ZFC.

The consistency of *NF* remains an open question (which we will not address at all here). Specker showed in 1953 ([16]) that *NF* disproves the Axiom of Choice (this is a very forceful result because *NF* also proves at least the stratified versions of all the standard equivalences between forms of AC.) This immediately reduced the popularity of this approach to foundations. In 1969, however, Jensen showed (in [13]) that if extensionality is weakened to apply only to objects with elements (allowing many elementless *urelements* as well as the empty set), the resulting system NFU is consistent relative to the usual set theory, and is moreover consistent with Infinity, Choice, and further strong axioms (see [10] for an investigation of various strong extensions of NFU). Note that NFU admits the full power of the scheme of stratified comprehension.

It was shown by Maurice Boffa ([2]) that a model of *NFU* in which the universe is no larger than the collection of all sets can be used to construct a model of *NF*. In all known models of *NFU*, there are many more urelements than sets; in fact, we have shown in [4] that *NFU* + *AC* proves that for any concrete natural number n there are at least n cardinals between the cardinality of the collection of all sets and the cardinality of the universe.

The question which this paper addresses was first raised to us by Thomas Forster. Are the urelements all "the same" in some sense? Standard permutation techniques used in *NF* or *NFU* can be adapted to show that for any n-ary *stratified* predicate ϕ involving no parameters other than urelements $u_1, \ldots, u_n$ the truth value of $\phi(u_1, \ldots, u_n)$, where the u_i's are distinct urelements, is the same no matter how the urelements are chosen. For these permutation techniques in a general context, see [6] and [1]; historical references for these methods are [15] and [8]. The general idea is that if f is a set function which is a permutation of the universe fixing all urelements (more general "setlike" permutations of the universe which may not be sets can also be used, with care, as we note below) we can replace the membership relation of our model of *NFU* with the relation $x \in_f y \equiv_{\texttt{def}} x \in f(y)$ and obtain a structure which will satisfy the same stratified sentences (formulas without free variables) as the original structure (and so will still be a model of *NFU*). In spite of the fact that we will make only one small application of the main theorem of permutation methods in this paper, we give a complete if brief account, since two closely analogous proofs will be used here and it is useful to be aware of the analogy. Let J be the operation which sends any set function h to the function $J(h)$ which fixes all urelements and sends each set to its elementwise image h"x under x. Observe that $x \in y$ is equivalent to $h(x) \in J(h)(y)$ for any function h. We define a hierarchy of functions f_n as follows: f_0 is the identity and $f_{n+1}(x) = J^n(f)(f_n(x))$. This definition is chosen so that $x \in_f y$ is equivalent to $f_n(x) \in f_{n+1}(y)$ for each n by an obvious induction: $x \in_f y \equiv x \in f(y) \equiv f_0(x) \in f_1(y)$ and $f_n(x) \in f_{n+1}(y) \equiv J^n(f)(f_n(x)) \in J^{n+1}(f)(f_{n+1}(y)) \equiv f_{n+1}(x) \in f_{n+2}(y)$. Note that if f is a bijection, so is each f_i. Now stratify the formula ϕ (assign a type to each variable occurring in ϕ in such a way that the type restrictions of *TST* are satisfied). Replace each variable x which is assigned type i with $f_i(x)$ to obtain a formula ϕ'. Each equation $x = y$ in ϕ is transformed to the form $f_i(x) = f_i(y)$, clearly still equivalent to $x = y$, and each membership statement $x \in y$ in ϕ assumes a form $f_i(x) \in f_{i+1}(y)$ which is equivalent to $x \in_f y$. Thus the transformed version of ϕ is equivalent to the formula ϕ_f in which $\in$ is replaced by $\in_f$. We can eliminate all applications of f_i to quantified variables in ϕ' because (f_i being a permutation of the universe) $(\forall x.\psi(x)) \equiv (\forall x.\psi(f_i(x)))$ and $(\exists x.\psi(x)) \equiv (\exists x.\psi(f_i(x)))$. Thus ϕ_f is equivalent to the formula ϕ'' with the usual membership relation $\in$ obtained by stratifying ϕ and replacing each free variable x in ϕ assigned type i with $f_i(x)$. For stratified *sentences* ϕ (formulas with no free variables), we obtain $\phi \equiv \phi_f$ (the basic theorem of permutation methods). Since stratified sentences include all the axioms of *NFU*, the structure with $\in_f$ in place of $\in$ is also a model of *NFU*. (Note that it is not strictly necessary in this construction for f to be a set function; note that J makes sense for proper class maps as well, and all that is actually needed is that $J^n(f)$ be total for each n; this is what it means for a map to be "setlike").

To establish the homogeneity of urelements, we employ a variation of the general permutation method described above; the technique is due to Thomas Forster (see [7]; our proof here uses different notation, avoiding a nonce redefinition of J).

Theorem: For any permutation g of the universe which moves only atoms and any stratified formula $\phi(u_1, \ldots, u_n)$ whose only free variables are the u_i's (understood to represent urelements), $\phi(u_1, \ldots, u_n) \equiv \phi(g(u_1), \ldots, g(u_n))$. That is, the urelements are homogeneous with respect to stratified properties.

Proof: Let g be a permutation of the universe which moves only atoms. Define G_0 as g and $G_{n+1}(x)$ as $J^{n+1}(g)(G_n(x))$. It is important to note that $G_i(u) = g(u)$ for any urelement u. We then prove by induction that for each n, $G_n(x) \in G_{n+1}(y) \equiv x \in y$: $G_0(x) \in G_1(y) \equiv g(x) \in J(g)(g(y)) \equiv x \in y$ (this last clause needs to be checked remembering that g moves only urelements and $J(g)$ fixes urelements, so $J(g)(g(y))$ is either an urelement $g(y)$, if y is an urelement, or the set $g``y$, if y is a set) and $G_n(x) \in G_{n+1}(y) \equiv J^{n+1}(g)(G_n(x)) \in J^{n+2}(g)(G_{n+1}(y)) \equiv G_{n+1}(x) \in G_{n+2}(y)$. Let ϕ be a stratified formula whose only parameters are urelements. Assign relative types to each variable in ϕ and replace each variable x of type i with $G_i(x)$. The results we have shown above indicate that the transformed formula has exactly the same truth value, since each $x \in y$ is transformed to $G_i(x) \in G_{i+1}(y)$ and each $x = y$ is transformed to $G_i(x) = G_i(y)$. Further, we can eliminate the G_i's from all and only the bound variables in the transformed ϕ, using the same considerations as in the previous paragraph, so ϕ is equivalent to the formula obtained by replacing each (urelement) parameter u of type i in ϕ with $G_i(u) = g(u)$. So any assertion $\phi(u_1, \ldots, u_n)$ with no parameters other than urelements $u_1, \ldots, u_n$ is equivalent to $\phi(g(u_1), \ldots, g(u_n))$, which establishes the homogeneity of the urelements with respect to stratified predicates.

Forster asked whether this continues to be true when we do not require the predicate $\phi(u_1, \ldots, u_n)$ to be stratified.

It is fairly easy, again using permutation methods, to create models of *NFU* in which some urelement is uniquely specifiable by an unstratified formula. For example, a permutation can be used to introduce a unique urelement u such that there is a set $x = \{u, x\}$. To do this, start with one of the "usual models of *NFU*" described below (which are easily seen to contain no such sets) and use the permutation f of sets which permutes the sets $\emptyset$ and $\{u, \emptyset\}$ (for a fixed urelement u) and fixes all other sets (and all the urelements). The structure with $\in_f$ in place of $\in$ is still a model of *NFU*, in which it is fairly easy to show that the empty set of the original model is a set $x = \{u, x\}$ in the new model, while it is also fairly easy to show that this is the only such set in the new model. The strong homogeneity result for unstratified formulas is then falsified, since the particular urelement u has a unique unstratified property not shared by any other urelement, $(\exists x.x = \{u, x\})$.

We granted Forster this point, but we were certain that in the usual models of *NFU* (which we will shortly describe) the urelements "must" be indistinguishable. Forster was equally certain that the urelements should be distinguishable by suitable unstratified formulas. Surprisingly to us, we have been able to show that Forster is correct: all the models constructed in the usual way have distinguishable urelements.

What we call the "usual" technique for constructing models of *NFU* was first described by Maurice Boffa in [3] (though it is clearly very closely related to Jensen's construction in [13]; see also the more accessible [5]).

Construction: Construct a nonstandard model of "enough" of the usual set theory (bounded Zermelo set theory is sufficient) in which there is an external automorphism j and a rank V_α of the cumulative hierarchy which is moved by j (we say that a rank V_α is moved by j rather than that an ordinal α is moved by j so that we do not have to assume that every ordinal determines a rank). We may assume without loss of generality that $j(\alpha) < \alpha$ (this method has usually been presented with $j(\alpha) > \alpha$, but it is technically more convenient here to take j as acting downward). The domain of the structure we consider is the extension of the set V_α in the nonstandard model of set theory. The membership relation of this structure is $x \in_{\mathtt{NFU}} y \equiv_{\mathtt{def}} j(x) \in y \wedge y \in V_{j(\alpha)+1}$.

Theorem: The structure described in the immediately preceding Construction is a model of *NFU*.

Proof: One can see [5] as well as the following. Note the analogy between this proof and the proofs re permutation methods above.

Let $\phi(x)$ be a stratified formula (interpreted as a formula about *NFU*) in which x is free. Make an assignment of relative types to the variables appearing in ϕ. Let N be an integer greater than any type assigned to a variable in ϕ. Translate ϕ from the language of *NFU* into a formula ϕ_1 in the language of the nonstandard model of the usual set theory with the automorphism j. Now apply suitable powers of j to each side of each atomic formula in ϕ_1 in such a way as to cause each occurrence of a variable y assigned type i to appear in the context $j^{N-i}(y)$; the relations between exponents of j in atomic formulas $x = y$, $j(x) \in y$, and $y \in V_{j(\alpha)+1}$ are such that this is possible. Call the resulting formula ϕ_2. Each bound variable $j^k(y)$ with quantifier restricted to V_α can be replaced with y while changing the set bounding the quantifier to $V_{j^k(\alpha)}$. The resulting formula $\phi_3(j^i(x))$ (where $N - i$ is the type assigned to x) contains j only in parameters. Uniform application of j^{-i} throughout gives an equivalent formula $\phi_4(x)$; since it contains j only in parameters it has an extension in the nonstandard model. The elements of $j(\{x \in V_\alpha \mid \phi_4(x)\})$ in the model of *NFU* (the additional j corrects for the modified definition of membership in *NFU*) are the objects x such that $\phi_1(x)$ in the model of nonstandard set theory, and so the objects x such that $\phi(x)$ in the model of *NFU*.

Definition: A model of *NFU* built as in the Construction is called a *usual model* of *NFU*.

Observation: We use V_α as our universe; its power set $V_{\alpha+1}$ is properly larger than our universe, but we replace all references to $V_{\alpha+1}$ with references to its isomorphic copy $V_{j(\alpha)+1}$ which is properly contained in our universe. Each subset of the universe is coded by its image under j which will be inside the universe, and each object which is not an image under j of such a subset has its extension discarded (the elements of $V_\alpha - V_{j(\alpha)+1}$ become urelements). It should be clear that the urelements are almost the entirety of the universe! Since the extensions of the urelements are discarded, it is unclear how any information distinguishing them from one another survives this construction. It turns out that *all* information about the original nonstandard model of set theory is recoverable (in a first-order definable way), including the extensions of the urelements (whence of course the urelements are not homogeneous)!

We first exhibit an unstratified assertion which is satisfied in any model of *NFU* obtained in this way, with two additional related assertions. This assertion was first discussed by us in [11] with a different application in mind (there is no need to consult that paper to understand anything here). Consider the function in the nonstandard model of set theory which sends each singleton $\{x\}$ for $x \in V_{j(\alpha)}$ to its element x. In the model of *NFU*, the "singleton $\{x\}$" of the original nonstandard model has $j^{-1}(x)$ as its sole member, rather than x, and the function now maps each singleton $\{x\}$ (without qualification) to $j(x)$. Further, this function is actually realized as a function in the internal sense of the model of *NFU* (this is the case for any function in the original nonstandard model whose domain and range are included in V_α); so we have a function in the model of *NFU* (which we will call $\mathbf{j}$) with the following properties: its domain is the set of all singletons, its range is a subcollection of the sets (because any $j(x) \in V_{j(\alpha)} \subseteq V_{j(\alpha)+1}$), and it is an injection.

Of course, these properties in themselves are not very remarkable. The strong property of $\mathbf{j}$ which we now claim and proceed to prove is this (in terms of the model of *NFU*): if y is a set, then

$$\mathbf{j}(\{y\}) = \{\mathbf{j}(\{z\}) \mid z \in y\}.$$

Suppose that y is a set: in terms of the underlying nonstandard model, this means $y \in V_{j(\alpha)+1}$. What we need to show is that for all w, $w \in \mathbf{j}(\{y\}) \equiv (\exists z.w = \mathbf{j}(\{z\}) \wedge z \in y)$. Translate this into terms of the underlying nonstandard model, recalling that $\mathbf{j}(\{x\}) = j(x)$: we want to show (in the nonstandard model) that $j(w) \in j(y) \wedge j(y) \in V_{j(\alpha)+1} \equiv (\exists z.w = j(z) \wedge j(z) \in y \wedge y \in V_{j(\alpha)+1})$. The assumption that y is a set implies that $y \in V_{j(\alpha)+1}$ and $j(y) \in V_{j(\alpha)+1}$ is simply true. So this simplifies to $j(w) \in j(y) \equiv (\exists z.w = j(z) \wedge j(z) \in y)$. Suppose that $j(w) \in j(y)$. This is equivalent to $w \in y$, which entails that there is a z such that $w = j(z)$ (since all elements of y are in $V_{j(\alpha)}$) and indeed $j(z) = w \in y$. Conversely, if there is z such that $w = j(z)$ and $j(z) \in y$, it follows trivially that $w \in y$ from which it follows that $j(w) \in j(y)$. This completes the proof of the claim.

The properties of $\mathbf{j}$ are summed up in the following

Axiom of Endomorphism: There is a function $\mathbf{j}$, an injection from the set of all singletons into the set of all sets, satisfying

$$\mathbf{j}(\{y\}) = \{\mathbf{j}(\{z\}) \mid z \in y\}$$

whenever y is a set.

This axiom will hold in any model of *NFU* constructed in the usual way.

We can use the function $\mathbf{j}$ to recover the membership relation of the original nonstandard model of set theory.

Definition: The relation $x\,E\,y$ is defined as holding exactly when $x \in \mathbf{j}(\{y\})$. Note that E is a set relation (x and y have the same relative type). (Note further that in the usual models of *NFU* this will reduce to $j(x) \in j(y) \wedge j(y) \in V_{j(\alpha)+1}$; the second conjunct is true and j is an automorphism so this is equivalent to $x \in y$ in the sense of the original nonstandard model of set theory).

In the usual models of *NFU* an additional important statement holds, though this is somewhat counterintuitive.

Definition: A relation R is said to be *well-founded* iff any subset A of the field of R (the union of the domain and range of R) has an element whose R-preimage does not meet A.

Axiom of "Foundation": E is a well-founded relation.

This clearly holds in the usual models because the membership relation in the original nonstandard model of set theory is well-founded (at least so far as it can see internally: externally the ordinals $j^n(\alpha)$ (for example) are a descending chain in the membership relation).

We briefly note that we use the usual Kuratowski ordered pair to define relations in *NFU* in this paper; this is somewhat inconvenient because the pair is two types higher than its projections, so relations and functions are three types higher than members of their fields, rather than one, but it is convenient to use the same pair in *NFU* as in the underlying set theory, and it is also convenient not to assume Infinity (as one must to obtain a type-level pair).

Ordinals are traditionally defined in *NFU* as equivalence classes of well-orderings under isomorphism. For any well-ordering W belonging to an ordinal β, there is a well-ordering $W^\iota = \{\langle\{x\},\{y\}\rangle \mid x\,W\,y\}$; the order type $T(\beta)$ of

W^ι does not depend on the choice of W in β. An analogous T operation may be defined for any relation types under isomorphism. It may seem to the mind trained in the usual set theory that the T operation is trivial, but this is demonstrably not the case, though it is the case that all usual operations and relations on ordinals commute with T (in particular, T is an order endomorphism). The order type of the ordinals $\leq \beta$ in the natural order can be shown by transfinite induction to be $T^4(\beta)$ (transfinite induction only applies to stratified properties, and in the assertion that the order type of the ordinals $\leq \beta$ is β the type of the second β is four higher than the type of the first: the use of T^4 repairs this); this means that the order type of all ordinals $\leq \Omega$, where Ω is the order type of *all* the ordinals in the natural order, is $T^4(\Omega)$, and thus it is clear that $T^4(\Omega) < \Omega$ (resolving the Burali-Forti paradox). This is less weird in TST: the two Ω's are of different types, and we show that the length of the ordinals in a given type must be shorter than the length of the ordinals four types up. (Since we have the descending "sequence" $\Omega > T^4(\Omega) > T^8(\Omega)\ldots$ in the ordinals, it is clear that T is not a set function.)

It is also worth observing that in the usual models it is the case that if β is the *NFU* order type of an order W, the *NFU* order type of $j(W)$ will be $T(\beta)$ (we cannot say $T(\beta) = j(\beta)$ because $j(\beta)$ will not be an ordinal in the sense of *NFU* unless $\beta = 0$ (it will not contain *all* well-orderings of order type $T(\beta)$ but only those which are images under j)

The relation E is not only well-founded but extensional, since $\mathbf{j}$ is an injection. Induction on the structure of E shows that each object x (set or urelement) is uniquely determined by the isomorphism type of the restriction of E to the set of all iterated preimages of x under E (including x itself) (this is analogous to the Mostowski Collapsing Lemma). Note that the associated isomorphism type is the isomorphism type of a well-founded extensional relation with a top element. A thorough discussion of the theory of such isomorphism types in *NF* or *NFU* can be found in [9] or [12].

Definition: For any object x, the isomorphism class of the restriction of E to the smallest set containing x and containing any $y\,E\,z$ if it contains z is termed the *associated isomorphism type* of x.

Lemma: The associated isomorphism class of $\mathbf{j}(\{x\})$ is the image under T of the associated isomorphism class of x.

Proof: If one takes the graph of E restricted to iterated preimages of a particular object x and applies the operation $t \mapsto \mathbf{j}(\{t\})$ to each node in this graph, one obtains a graph whose isomorphism type is the image under T of the original isomorphism type ($\mathbf{j}$ is an injective function, so this is isomorphic to the graph obtained by simply replacing nodes with their singletons) and which is further also the downward closed part of the graph of E determined by $\mathbf{j}(\{x\})$. To see this it is sufficient to see that $x\,E\,y \equiv x \in \mathbf{j}(\{y\}) \equiv \mathbf{j}(\{x\}) \in \mathbf{j}(\{\mathbf{j}(\{y\})\}) \equiv \mathbf{j}(\{x\})\,E\,\mathbf{j}(\{y\})$ and also that any preimage under E of any $\mathbf{j}(\{y\})$ is an element of $\mathbf{j}(\{\mathbf{j}(\{y\})\})$ and so is an image under $\mathbf{j}$, since $\mathbf{j}(\{y\})$ is a set and the image of the singleton of a set under $\mathbf{j}$ is a set of images under $\mathbf{j}$.

Each object in the domain of a well-founded relation can be assigned an ordinal rank. We outline how this is done. Let R be a well-founded relation (any subset A of the field of R (the union of its domain and range) has an element x none of whose preimages under R are in A). Construct the set $\mathcal{R}$ of R-ranks as follows: $\emptyset$ is an R-rank; if $A \in \mathcal{R}$, then the set A^+ of all elements of the field of R all of whose R-preimages are in A belongs to $\mathcal{R}$; the union of any subcollection of $\mathcal{R}$ is an element of $\mathcal{R}$. $\mathcal{R}$ can be formally defined as the intersection of all sets with these closure conditions (containing $\emptyset$; containing A^+ if they contain A; closed under unions of subcollections). It is straightforward but tedious to prove that $\mathcal{R}$ is well-ordered by inclusion. Define R_α as the element of $\mathcal{R}$ such that the order type of the elements of $\mathcal{R}$ properly included in it is α (this is defined only for an initial segment of the ordinals, and it is worth noting that the type of α is higher than that of R_α: this indexing is not a set function). We say that an object is of type α iff it belongs to $R_{\alpha+1} - R_\alpha$: note for example that R_0 is empty, but R_1 will contain all objects with empty R-preimage, and these objects will have rank 0. Finally, it is straightforward to prove that the field of R is itself a rank (obviously the last one). (It is further worth noting that there is a theory of rank for isomorphism types of well-founded extensional relations with top; the ranks of the associated isomorphism types of objects are parallel to the ranks of the objects.)

Definition: The ranks under E are denoted by E_β, following the notation above. Note that E_β is the set of objects with E-rank $< \beta$.

Definition: A rank E_β is said to be *complete* if every subset of E_β is the preimage under E of some object.

Observation: It is straightforward to show that completeness of E_β is equivalent to the condition that any associated isomorphism class which would give an object of rank $\leq \beta$ is actually realized by some object. Suppose that some potential associated isomorphism class is not realized; there will be a lowest rank γ at which this occurs. If rank $\beta \geq \gamma$ is complete, this would imply that the supposedly missing isomorphism class was realized by an object at rank $\beta + 1$ – and so of course also existed at rank $\gamma + 1$.

Lemma: The associated isomorphism type of an object of rank $T(\alpha)$ will always be an image under T, if E_α is a complete rank.

Proof: Suppose that α is the lowest rank at which this fails to be true: so there is some y of rank $T(\alpha)$ whose associated isomorphism type is not an image under T, and this does not happen for any $\beta < \alpha$. This means that y cannot be an image under $\mathbf{j}$. However, every element z of $\mathbf{j}(\{y\})$ is of some rank $T(\beta)$ for $\beta < \alpha$ and so has an associated isomorphism class which is an image under T, whose inverse image under T would be associated with an object w of rank $\beta < \alpha$, which will exist if α is a complete rank, and which will satisfy $\mathbf{j}(w) = z$. Now the collection of such objects w, a set of objects all of rank $< \alpha$, cannot be the E-preimage of any x (if it were, we would have $y = \{\mathbf{j}(w) \mid w \in x\} = \mathbf{j}(\{x\})$). So the rank α cannot be complete, nor can any higher rank.

Lemma: The universe V is an E-rank E_α for some index α. with $T(\alpha) < \alpha$.

Proof: The field of E is V and the field of any well-founded relation is the highest rank for that relation as noted above. Note that $\mathbf{j}(\{V\}) = \{\mathbf{j}(\{x\}) \mid x = x\}$. For every x, we have $\mathbf{j}(\{x\}) \in \mathbf{j}(\{V\})$ and thus $\mathbf{j}(\{x\})\, E\, V$, so the rank of V is higher than that of any $\mathbf{j}(\{x\})$. For any $\beta < \alpha$, there is x of rank β, so there is $\mathbf{j}(\{x\})$ of rank $T(\beta)$, so the rank of V is at least $T(\alpha) + 1$ (and in fact is exactly $T(\alpha) + 1$, since certainly no E-member of V is of rank $T(\alpha)$ or higher!). The rank of V, like the rank of any object, is strictly less than α, so $T(\alpha) < \alpha$.

The highest rank E_α is an incomplete rank: for example, there can be no object with the E-preimage E_α (this is obvious because E is well-founded). It is not necessary that there be any other incomplete rank; in fact, the usual models have an additional

Rank Property: Every E-rank is complete except the highest rank E_α.

This property holds in the usual models because there E coincides with the membership relation on the underlying nonstandard rank V_α, which obviously has this property.

We make some side remarks to put the theory *NFU* + Endomorphism + Foundation + Rank Property into context: this theory can be interpreted in *NFU*, using the isomorphism classes of well-founded extensional relations with top, as discussed in [12]. In our paper [11] where the Axiom of Endomorphism was first defined, we proved that the Axiom of Endomorphism is inconsistent with *NF*.

We are now ready to prove the

Main Theorem: The urelements in a usual model of *NFU* are not homogeneous.

This result follows from the Main Lemma, whose proof appears below.

Main Lemma: If there is a relation $\mathbf{j}$ witnessing the truth of the Axioms of Endomorphism and "Foundation" and the Rank Property (as there must be in any usual model), then there can be only one such relation.

Proof of the Main Theorem: From the Main Lemma it follows that the map $\mathbf{j}$ and the relation E are first order definable in a usual model of *NFU* without mentioning parameters; but this means that the membership of the original nonstandard model of set theory is first-order definable without parameters, because it coincides with E. From this it follows that the urelements are discernible in many ways in the usual models (they have "extensions" under E which may or may not include (for example) the empty set, they have rank in terms of the original membership relation, etc.)

Proof of the Main Lemma: We construct a model of *NFU* by the usual method. This model will satisfy the Axioms of Endomorphism and "Foundation" and enjoy the Rank Property.

Suppose that there are two distinct functions $\mathbf{j}$ and $\mathbf{j}'$ witnessing the truth of the Axiom of Endomorphism, Axiom of "Foundation", and the Rank Property. With them are associated relations E and E' defined as above.

Suppose that β is chosen as small as possible so that there is $\mathbf{j}^* \in \{\mathbf{j}, \mathbf{j}'\}$ (with associated relation E^*) such that the rank of x with respect to E^* (all ranks in this paragraph are to be understood to be with respect to E^*) is β and $\mathbf{j}(\{x\}) \neq \mathbf{j}'(\{x\})$. Suppose that x of the form $\mathbf{j}^*(\{y\})$ (and so is a set). So $\mathbf{j}(\{x\}) = \{\mathbf{j}(\{z\}) \mid z \in x\} \neq \{\mathbf{j}'(\{z\}) \mid z \in x\} = \mathbf{j}'(\{x\})$ shows that one of the elements z of x must also be a counterexample. If $x = \mathbf{j}^*(\{y\})$, then the rank of z with respect to E^* is less than the rank of y, which would be $T^{-1}(\beta)$. The rank of z must be greater than or equal to β as well (since it is a counterexample), so we have $\beta < T^{-1}(\beta)$, so $T(\beta) < \beta$. Now we must have $\mathbf{j}(\{\{x\}\}) = \{\mathbf{j}(\{x\})\} \neq \{\mathbf{j}'(\{x\})\} = \mathbf{j}'(\{\{x\}\})$. So $\{x\}$ is also a counterexample. But the rank of $\{x\}$ with respect to E^* is $T(\beta) + 1$ (since the sole E^*-member of this set is $\mathbf{j}^*(\{x\})$, which has rank $T(\beta)$). And we see that $T(\beta) + 1 \leq \beta$, but $T(\beta) + 1 = \beta$ is impossible because the parity of the finite part of an ordinal is preserved by T. This is a contradiction of the minimality of β. So we have succeeded in showing that a minimal rank counterexample cannot be of the form $\mathbf{j}^*(\{y\})$. A minimal rank counterexample (of rank β) must be of rank less or equal to that of its singleton by the same argument given above, whose rank is $T(\beta)+1$, so we must have $\beta \leq T(\beta)$. We claim that in fact any object x not of the form $\mathbf{j}^*(\{y\})$ must have rank $\beta > T(\beta)$. Suppose otherwise: since x is of rank $\leq T(\beta)$, where E^*_β, β being the rank of an object, is not the highest rank and so is complete, its associated isomorphism type must be an image under T. Its inverse image under T, an isomorphism type which would be associated with an object of rank $T^{-1}(\beta) \leq \beta$, cannot thus be realized by any object. This implies that the rank $E^*_{T^{-1}(\beta)}$ is incomplete; but this is impossible by the Rank Property, since $T^{-1}(\beta) \leq \beta$ is not the index of the highest rank.

The proof of the Main Lemma, and so of the Main Theorem, is complete.

A further corollary, perhaps more interesting than the main result (which gets pride of place as the main result because it settles the problem we were trying to solve) is that the construction of the usual models of *NFU* does not in fact eliminate any information at all about the underlying V_α, despite the apparent disposal of the extensions of the urelements. All this information remains first-order definable in the model of *NFU* (though the full definition is quite complicated, since it needs to refer to the unique object witnessing the Axiom of Endomorphism, the Axiom of "Foundation", and the Rank Property). This is rather surprising (in our opinion) but serves as a strong confirmation of our general opinion that *NFU* as a theory is a restatement of the theory of a model of enough of the usual set theory with an external automorphism (or at least of the theory of an initial segment of such a model).

We have a question about the possibility of making the uniqueness result sharper; can it be proved that there is no more than one **j** witnessing Endomorphism and Foundation (leaving out the Rank Property?). If one assumes that $T(\beta) \leq \beta$ for all ordinals β, then the case requiring the Rank Property is neatly eliminated, but this is a not entirely trivial assumption.

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