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FROM CLASSICAL TO RELATIVISTIC MECHANICS: ELECTROMAGNETIC MODELS OF THE ELECTRON

1. INTRODUCTION

"Special relativity killed the classical dream of using the energy-momentum-velocity relations as a means of probing the dynamical origins of [the mass of the electron]. The relations are purely kinematical" (Pais, 1982, 159). This perceptive comment comes from a section on the pre-relativistic notion of electromagnetic mass in '*Subtle is the Lord* ...', Abraham Pais' highly acclaimed biography of Albert Einstein. 'Kinematical' in this context means 'independent of the details of the dynamics'. In this paper we examine the classical dream referred to by Pais from the vantage point of relativistic continuum mechanics.

There were actually two such dreams in the years surrounding the advent of special relativity. Like Einstein's theory, both dreams originated in the electrodynamics of moving bodies developed in the 1890s by the Dutch physicist Hendrik Antoon Lorentz. Both took the form of concrete models of the electron. Even these models were similar. Yet they were part of fundamentally different programs competing with one another in the years around 1905. One model, due to the German theoretician Max Abraham (1902a), was part of a revolutionary effort to substitute the laws of electrodynamics for those of Newtonian mechanics as the fundamental laws of physics. The other model, adapted from Abraham's by Lorentz (1904b) and fixed up by the French mathematician Henri Poincaré (1906), was part of the attempt to provide a general explanation for the absence of any sign of the earth's motion through the ether, the elusive 19th-century medium thought to carry light waves and electromagnetic fields. A choice had to be made between the objectives of Lorentz and Abraham. One could not eliminate all signs of ether drift and reduce all physics to electrodynamics at the same time. Special relativity was initially conflated with Lorentz's theory because it too seemed to focus on the undetectability of motion at the expense of electromagnetic purity. The theories of Lorentz and Einstein agreed in all their empirical predictions, including those for the velocity-dependence of electron mass, even though special relativity was not wedded to any particular model of the electron. For a while there was a third electron model, a variant on Lorentz's proposed independently by Alfred Bucherer (1904, 57–60; 1905) and Paul Langevin (1905). At the time, the acknowledged arbiter between these models and the broader theories (perceived to be) attached to them was a series of experiments by Walter Kaufmann and others on the deflection of high-speed

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electrons in β-radiation and cathode rays by electric and magnetic fields for the purpose of determining the velocity-dependence of their mass.[1]

As appropriate for reveries, neither Lorentz's nor Abraham's dream about the nature and structure of the electron lasted long. They started to fade a few years after Einstein's formulation of special relativity, even though the visions that inspired them lingered on for quite a while. Lorentz went to his grave in 1928 clinging to the notion of an ether hidden from view by the Lorentz-invariant laws governing the phenomena. Abraham's electromagnetic vision was pursued well into the 1920s by kindred spirits such as Gustav Mie (1912a, 1912b, 1913). By then mainstream physics had long moved on. The two dreams, however, did not evaporate without a trace. They played a decisive role in the development of relativistic mechanics.[2] It is no coincidence therefore that relativistic (continuum) mechanics will be central to our analysis in this paper. The development of the new mechanics effectively began with the non-Newtonian transformation laws for force and mass introduced by Lorentz (1895, 1899). It continued with the introduction of electromagnetic momentum and electromagnetic mass by Abraham (1902a, 1902b, 1903, 1904a, 1905, 1909) in the wake of the proclamation of the electromagnetic view of nature by Willy Wien (1900). Einstein (1907b), Max Planck (1906a, 1908), Hermann Minkowski (1908), Arnold Sommerfeld (1910a, 1910b), and Gustav Herglotz (1910, 1911)—the last three champions of the electromagnetic program[3]—all contributed to its further development in a proper relativistic setting. These efforts culminated in a seminal paper by Max Laue (1911a) and were enshrined in the first textbook on relativity published later that year (Laue, 1911b).

There already exists a voluminous literature on the various aspects of this story.[4] We shall freely draw and build on that literature. One of us has written extensively on the development of Lorentz's research program in the electrodynamics of moving bodies (Janssen, 1995, 2002b; Janssen and Stachel, 2004).[5] The canonical source for the electromagnetic view of nature is still (McCormmach, 1970), despite its focus on Lorentz whose attitude toward the electromagnetic program was ambivalent (cf. Lorentz, 1900; 1905, 93–101; 1915, secs. 178–186). His work formed its starting point and he was sympathetic to the program, but never a strong advocate of it. (Goldberg, 1970) puts the spotlight on the program's undisputed leader, Max Abraham. (Pauli, 1921, Ch. 5) is a good source for the degenerative phase of the electromagnetic program in the 1910s.[6] For a concise overview of the rise and fall of the electromagnetic program, see Ch. 8 of (Kragh, 1999), aptly titled "A Revolution that Failed."

Another important source for the electromagnetic program is Ch. 5 in (Pyenson, 1985), which discusses a seminar on electron theory held in Göttingen in the summer semester of 1905. Minkowski was one of four instructors of this course. The other three were Herglotz, David Hilbert, and Emil Wiechert. Max Laue audited the seminar as a postdoc. Among the students was Max Born, whose later work on the problem of rigid bodies in special relativity (Born, 1909a, 1909b, 1910) was inspired by the seminar.[7] The syllabus for the seminar lists papers by Lorentz (1904a, 1904b), Abraham (1903), Karl Schwarzschild (1903a, 1903b, 1903c), and Sommerfeld (1904a, 1904b, 1905a). This seminar gives a good indication of how active and cutting edge this research area was at the time. Further evidence of this vitality is provided by debates in the literature of the day over various points concerning these electron models such as those between Wien (1904a, 1904b, 1904c, 1904d) and Abraham (1904b, 1904c),[8] Bucherer (1907, 1908a, 1908b) and Ebenezer Cunningham (1907, 1908),[9] and Einstein (1907a) and Paul Ehrenfest (1906, 1907). The roll call of researchers active in this area also included the Italian mathematician Tullio Levi-Civita (1907, 1909).[10] One may even get the impression that in the early 1900s the journals were flooded with papers on electron models. We wonder, for instance, whether the book by Bucherer (1904) was not originally written as a long journal article, which was rejected, given its similarity to earlier articles by Abraham, Lorentz, Schwarzschild, and Sommerfeld.

The saga of the Abraham, Lorentz, and Bucherer-Langevin electron models and their changing fortunes in the laboratories of Kaufmann, Bucherer, and others has been told admirably by Arthur I. Miller (1981, secs. 1.8–1.14, 7.4, and 12.4). Miller (1973) is also responsible for a detailed analysis of the classic paper by Poincaré (1906) that introduced what came to be known as "Poincaré pressure" to stabilize Lorentz's purely electromagnetic electron.[11] The model has been discussed extensively in the physics literature, by Fritz Rohrlich and by such luminaries as Enrico Fermi, Paul Dirac, and Julian Schwinger.[12] It is also covered elegantly in volume two of the Feynman lectures (Feynman et al., 1964, Ch. 28). (Pais, 1972) and (Rohrlich, 1973) combine discussions of physics and history in an informative way.

Given how extensively this episode has been discussed in the historical literature, the number of sources covering its denouement with the formulation of Laue's relativistic continuum mechanics is surprisingly low. Max Jammer does not discuss relativistic continuum mechanics at all in his classic monograph on the development of the concept of mass (cf. Jammer, 1997, Chs. 11–13). Miller prominently discusses Laue's work, both in (Miller, 1973, sec. 7.5) and in the concluding section of his book (Miller, 1981, sec.

12.5.8), but does not give it the central place that in our opinion it deserves. To bring out the importance of Laue's work, we show right from the start how the kind of spatially extended systems studied by Abraham, Lorentz, and Poincaré can be dealt with in special relativity. We use modern notation and modern units throughout and give self-contained derivations of almost all results. Our treatment of these electron models follows the analysis of the experiments of Trouton and Noble in (Janssen, 1995, 2002b, 2003), which was inspired in part by the discussion in (Norton, 1992) of the importance of Laue's relativistic mechanics for the development of Gunnar Nordström's special-relativistic theory of gravity. The focus on the conceptual changes in mechanics that accompanied the transition from classical to relativistic kinematics was inspired in part by the work of Jürgen Renn and his collaborators on pre-classical mechanics (Damerow et al., 2004). Ultimately, our story is part of a larger tale about shifts in such concepts as mass, energy, momentum, and stresses and the relations between them in the transition from Newtonian mechanics and the electrodynamics of Maxwell and Lorentz to special relativity.

2. ENERGY-MOMENTUM-MASS-VELOCITY RELATIONS

2.1. Special relativity

In special relativity, the relations between energy, momentum, mass, and velocity of a system are encoded in the transformation properties of its four-momentum. This quantity combines the energy U and the three components of the ordinary momentum $\mathbf{P}$:[13]

(4) $$P^\mu = \left(\frac{U}{c}, \mathbf{P}\right)$$

(where c is the velocity of light). In the system's rest frame, with coordinates $x_0^\mu = (ct_0, x_0, y_0, z_0)$, the four-momentum reduces to:

(5) $$P_0^\mu = \left(\frac{U_0}{c}, 0, 0, 0\right),$$

i.e., $\mathbf{P}_0 = 0$. The system's rest mass is defined as $m_0 \equiv U_0/c^2$.

We transform P_0^μ from the x_0^μ-frame to some new x^μ-frame, assuming for the moment that P_0^μ always transforms as a four-vector under Lorentz transformations. Let the two frames be related by the Lorentz transformation $x^\mu = \Lambda^\mu{}_\nu x_0^\nu$, where the transformation matrices $\Lambda^\mu{}_\nu$ satisfy $\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}$, the defining equation for Lorentz transformations, with $\eta^{\mu\nu} \equiv \text{diag}(1, -1, -1, -1)$ the standard diagonal Minkowski metric. Here and in the rest of the paper summation over repeated indices is implied. We follow the convention that

Greek indices run from 0 to 3 and Latin ones from 1 to 3. Since, in general, the four-momentum does *not* transform as a four-vector, the Lorentz transform of P_0^μ will, in general, not be the four-momentum in the x^μ-frame. We therefore cautiously write the result of the transformation with an asterisk:

$$P^{*\mu} = \Lambda^\mu{}_\nu P_0^\nu. \tag{6}$$

Without loss of generality we can focus on the special case in which the motion of the x^μ-frame with respect to the x_0^μ-frame is with velocity v in the x-direction. The matrix for this transformation is:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

with $\gamma \equiv 1/\sqrt{1-\beta^2}$ and $\beta \equiv v/c$. In that case,

$$P^{*\mu} = \left(\gamma\frac{U_0}{c}, \gamma\beta\frac{U_0}{c}, 0, 0\right) = (\gamma m_0 c, \gamma m_0 \mathbf{v}). \tag{8}$$

If the four-momentum of the system *does* transform as a four-vector, $P^{*\mu}$ in eq. 8 is equal to P^μ in eq. 4 and we can read off the following relations between energy, momentum, mass, and velocity from these two equations:

$$U = \gamma U_0 = \gamma m_0 c^2, \quad \mathbf{P} = \gamma m_0 \mathbf{v}. \tag{9}$$

Eqs. 9 hold for a relativistic point particle with rest mass m_0. Its four-momentum is given by

$$P^\mu = m_0 u^\mu = m_0 \frac{dx^\mu}{d\tau} = \gamma m_0 \frac{dx^\mu}{dt}. \tag{10}$$

Since $u^\mu \equiv dx^\mu/d\tau$ is the four-velocity, this is clearly a four-vector. The relation between proper time τ, arc length s, and coordinate time t is given by $d\tau = ds/c = dt/\gamma$.[14] If the particle is moving with velocity $\mathbf{v}$, $dx^\mu/dt = (c, \mathbf{v})$ and eq. 10 becomes:

$$P^\mu = (\gamma m_0 c, \gamma m_0 \mathbf{v}). \tag{11}$$

Eqs. 9 also hold for spatially extended *closed* systems, i.e., systems described by an energy-momentum tensor $T^{\mu\nu}$ with a vanishing four-divergence, i.e., $\partial_\nu T^{\mu\nu} = 0$ (where ∂_ν stands for $\partial/\partial x^\nu$). The energy-momentum tensor brings together the following quantities. The component T^{00} gives the energy density; T^{i0}/c the components of the momentum density; cT^{0i} the components of the energy flow density;[15] and T^{ij} the components of the momentum flow density, or, equivalently, the stresses.[16] The standard definition of the four-momentum of a spatially extended (not necessarily closed) system described by the (not necessarily divergence-free) energy-momentum tensor $T^{\mu\nu}$ is:

$$P^{\mu} \equiv \frac{1}{c}\int T^{\mu 0} d^3x. \tag{12}$$

Before the advent of relativity, this equation was written as a pair of separate equations:

$$U = \int u d^3x, \quad \mathbf{P} = \int \mathbf{p} d^3x, \tag{13}$$

where u and $\mathbf{p}$ are the energy density and the momentum density, respectively. Definition 12 is clearly not manifestly Lorentz invariant. The space integrals of $T^{\mu 0}$ in the x^{μ}-frame are integrals in space-time over a three-dimensional hyperplane of simultaneity in that frame. A Lorentz transformation does not change the hyperplane over which the integration is to be carried out. A hyperplane of simultaneity in the x^{μ}-frame is *not* a hyperplane of simultaneity in any frame moving with respect to it. From these last three observations, it follows that the Lorentz transforms of the space integrals in eq. 12 will not be space integrals in the new frame. But then how can these Lorentz transforms ever be the four-momentum in the new frame? The answer to this question is that *if the system is closed* (i.e., if $\partial_{\nu} T^{\mu\nu} = 0$), it does not matter over which hyperplane the integration is done. The integrals of the relevant components of $T^{\mu\nu}$ over any hyperplane extending to infinity will all give the same values. So for closed systems a Lorentz transformation does map the four-momentum in one frame to a quantity that is equal to the four-momentum in the new frame even though these two quantities are defined as integrals over different hyperplanes.[17]

The standard definition of four-momentum can be replaced by a manifestly Lorentz-invariant one. First note that the space integrals of $T^{\mu 0}$ in the x^{μ}-frame can be written in a manifestly covariant form as[18]

$$P^{\mu} = \frac{1}{c}\int \delta\left(\eta_{\rho\sigma} x^{\rho} n^{\sigma}\right) T^{\mu\nu} n_{\nu} d^4x, \tag{14}$$

where $\delta(x)$ is the Dirac delta function, defined through $\int f(x)\delta(x-a)dx = f(a)$, and n^{μ} is a unit vector in the time direction in the x^{μ}-frame. In that frame n^{μ} has components $(1,0,0,0)$. The delta function picks out hyperplanes of simultaneity in the x^{μ}-frame. The standard definition 12 of four-momentum can, of course, be written in the form of eq. 14 in any frame, but that requires a different choice of n^{μ} in each one. This is just a different way of saying what we said before: under the standard definition 12, the result of transforming P^μ in the x^μ-frame to some new frame will *not* be the four-momentum in the new frame unless the system is closed. If, however, we take the unit vector n^μ in eq. 14 to be some *fixed* timelike vector—typically the unit vector in the time direction in the system's rest frame[19]—and take eq. 14 with that fixed vector n^μ as our new definition of four-momentum, the problem disappears.

Eq. 14 with a fixed timelike unit vector n^μ provides an alternative manifestly Lorentz-invariant definition of four-momentum. Under this new definition—which was proposed by, among others, Fermi (1922)[20] and Fritz Rohrlich (1960, 1965)[21]—the four-momentum of a spatially extended system transforms as a four-vector under Lorentz transformations no matter whether the system is open or closed. The definitions 12 and 14 are equivalent to one another for closed systems, but only coincide for open systems in the frame of reference in which n^μ has components $(1,0,0,0)$. In this paper, we shall use the admittedly less elegant definition 12, simply because either it or its decomposition into eqs. 13 were the definitions used in the period of interest. Part of the problem encountered by our protagonists simply disappears by switching to the alternative definition 14. With this definition energy and momentum always obey the familiar relativistic transformation rules, regardless of whether we are dealing with closed systems or with their open components. As one would expect, however, a mere change of definition does not take care of the main problem that troubled the likes of Lorentz, Poincaré, and Abraham. That is the problem of the stability of a spatially extended electromagnetic electron.

2.2. Pre-relativistic theory

The analogues of relations 9 between energy, mass, momentum, and velocity in Newtonian mechanics are the basic formulae for kinetic energy and momentum:

$$U_{\text{kin}} = \frac{1}{2}mv^2, \quad \mathbf{p} = m\mathbf{v} \tag{15}$$

In the years before the advent of special relativity, electrodynamics was a hybrid theory in which Galilean-invariant Newtonian mechanics was supposed to govern matter while Maxwell's equations, which are inherently Lorentz invariant, governed the electromagnetic fields. This hybrid theory already harbored the relativistic energy-momentum-velocity relations.

Initially, the starting point of physicists working in this area had unquestionably been Newton's second law, $\mathbf{F} = m\mathbf{a}$, force equals mass times acceleration. Electrodynamics merely supplied the Lorentz force for the left-hand side of this equation. Eventually, however, some of the leading practitioners were leaning toward the view that matter does not have any Newtonian mass at all and that its inertia is just a manifestation of the interaction of electric charge distributions with their self-fields. Lorentz was reluctantly driven to this conclusion because, as we shall see in secs. 3 and 4, it would help explain the absence of any signs of ether drift. Abraham enthusiastically embraced it because it opened up the prospect of a purely electromagnetic basis for all of physics. With $\mathbf{F} = m\mathbf{a}$ reduced to $\mathbf{F} = 0$, Newton's second law only nominally retained its lofty position as the fundamental equation of motion. All real work was done by electrodynamics. Writing $\mathbf{F} = 0$ as $d\mathbf{P}_{\text{tot}}/dt = 0$, one can read it as expressing momentum conservation. Momentum does not need to be mechanical. Abraham introduced the concept of electromagnetic momentum.[22] Lorentz was happy to leave Newtonian royalty its ceremonial role. Abraham, of a more regicidal temperament, sought to replace $\mathbf{F} = m\mathbf{a}$ by a new purely electrodynamic equation that would explain why Newton's law had appeared to be the rule of the land for so long.

Despite their different motivations, Lorentz and Abraham agreed that the effective equation of motion for an electron in some external field is[23]

$$\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}} = 0, \tag{16}$$

with $\mathbf{F}_{\text{ext}}$ the Lorentz force coming from the external field and $\mathbf{F}_{\text{self}}$ the Lorentz force coming from the self-field of the electron. The key experiments to which eq. 16 was applied were the experiments of Kaufmann and others on the deflection of fast electrons by electric and magnetic fields. Both Lorentz and Abraham conceived of the electron as a spherical surface charge distribution. They disagreed about whether the electron's shape would depend on its velocity with respect to the ether, more specifically about whether it would be subject to a microscopic version of the Lorentz-FitzGerald contraction. Lorentz believed it would, Abraham believed it would not.

The Lorentz force that an electron moving through the ether at velocity $\mathbf{v}$ experiences from its self-field can be written as minus the time derivative of the quantity that Abraham proposed to call the electromagnetic momentum:

$$\mathbf{F}_{\text{self}} = \int \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) d^3x = -\frac{d\mathbf{P}_{\text{EM}}}{dt}. \tag{17}$$

In this expression ρ is the density of the electron's charge distribution, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field produced by this charge distribution. The electromagnetic momentum of these fields is defined as

$$\mathbf{P}_{\text{EM}} \equiv \int \varepsilon_0 \mathbf{E} \times \mathbf{B} d^3x, \tag{18}$$

and doubles as the electromagnetic momentum of the electron itself. In general there will be an extra term on the right-hand side of eq. 17. In general, the components of $\mathbf{F}_{\text{self}}$ are given by:

$$F^i_{\text{self}} = -\frac{dP^i_{\text{EM}}}{dt} + \int \partial_j T^{ij}_{\text{Maxwell}} d^3x, \tag{19}$$

where

$$T^{ij}_{\text{Maxwell}} \equiv \varepsilon_0 \left(E^i E^j - \frac{1}{2}\delta^{ij} E^2 \right) + \mu_0^{-1} \left(B^i B^j - \frac{1}{2}\delta^{ij} B^2 \right) \tag{20}$$

is the Maxwell stress tensor (the Kronecker delta δ^{ij} is defined as follows: $\delta^{ij} = 1$ for $i = j$ and 0 otherwise). Gauss's theorem tells us that this additional term vanishes as long as T^{ij}_{Maxwell} drops off faster than $1/r^2$ as $\mathbf{x}$ goes to infinity. Simple derivations of these results can be found in many sources, old and new.[24] With the help of eq. 17 the electromagnetic equation of motion 16 can be written in the form of the Newtonian equation $\mathbf{F} = d\mathbf{p}/dt$ with Abraham's electromagnetic momentum replacing ordinary momentum:

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}_{\text{EM}}}{dt}. \tag{21}$$

Like Newton's second law, which can be written either as $\mathbf{F} = m\mathbf{a}$ or as $\mathbf{F} = d\mathbf{p}/dt$, this new law can, under special circumstances, be written as the product of mass and acceleration. Assume that the momentum is in the direction of motion,[25] i.e., that $\mathbf{P}_{\text{EM}} = (P_{\text{EM}}/v)\mathbf{v}$. We then have

$$\frac{d\mathbf{P}_{\text{EM}}}{dt} = \frac{dP_{\text{EM}}}{dt}\frac{\mathbf{v}}{v} + P_{\text{EM}}\frac{d}{dt}\left(\frac{\mathbf{v}}{v}\right). \tag{22}$$

The first term on the right-hand side can be written as

$$\frac{dP_{\text{EM}}}{dt}\frac{\mathbf{v}}{v} = \frac{dP_{\text{EM}}}{dv}\frac{dv}{dt}\frac{\mathbf{v}}{v} = \frac{dP_{\text{EM}}}{dv}\mathbf{a}_{//}, \tag{23}$$

where $\mathbf{a}_{//}$ is the longitudinal acceleration, i.e., the acceleration in the direction of motion. The second term can be written as

$$P_{\text{EM}}\frac{d}{dt}\left(\frac{\mathbf{v}}{v}\right) = \frac{P_{\text{EM}}}{v}\mathbf{a}_{\perp}, \tag{24}$$

where $\mathbf{a}_{\perp}$ is the transverse acceleration, i.e., the acceleration perpendicular to the direction of motion. The factors multiplying these two components of the acceleration are called the *longitudinal (electromagnetic) mass*, $m_{//}$, and the *transverse (electromagnetic) mass*, $m_{\perp}$, respectively. This terminology is due to Abraham (1903, 150–151). Eq. 22 can thus be written as

$$\frac{d\mathbf{P}_{\text{EM}}}{dt} = m_{//}\mathbf{a}_{//} + m_{\perp}\mathbf{a}_{\perp}, \tag{25}$$

with[26]

$$m_{//} = \frac{dP_{\text{EM}}}{dv}, \;\; m_{\perp} = \frac{P_{\text{EM}}}{v}. \tag{26}$$

The effective equation of motion 21 becomes:

$$\mathbf{F}_{\text{ext}} = m_{//}\mathbf{a}_{//} + m_{\perp}\mathbf{a}_{\perp}. \tag{27}$$

We shall see that, for $v = 0$ (in which case the electron models of Abraham and Lorentz coincide), $m_{//} = m_{\perp} = m_0$, and that, for $v \neq 0$, $m_{//}$ and $m_{\perp}$ differ from m_0 only by terms of order v^2/c^2. For velocities $v \ll c$, eq. 27 thus reduces to:

$$\mathbf{F}_{\text{ext}} \approx m_0(\mathbf{a}_{//} + \mathbf{a}_{\perp}) = m_0\mathbf{a}. \tag{28}$$

Proponents of the electromagnetic view of nature took eq. 21 to be the fundamental equation of motion and derived Newton's law from it by identifying the ordinary Newtonian mass with the electromagnetic mass m_0 of the relevant system at rest in the ether.

Eq. 26 defines the longitudinal mass $m_{//}$ of the electron in terms of its electromagnetic momentum. It can also be defined in terms of the electron's electromagnetic energy. Consider the work done as an electron is moving in the x-direction in the absence of an external field. The work expended goes into the internal energy of the electron, $dU = -dW$. According to eq. 16, the work is done by $\mathbf{F}_{\text{self}}$.[27] The internal energy is identified with the electromagnetic energy U_{EM}:

$$dU_{\text{EM}} = -dW = -\mathbf{F}_{\text{self}} \cdot d\mathbf{x}. \tag{29}$$

Using eqs. 17 and 25, we can write this as

$$dU_{\text{EM}} = \frac{d\mathbf{P}_{\text{EM}}}{dt} \cdot d\mathbf{x} = m_{//}\mathbf{a}_{//} \cdot d\mathbf{x} = m_{//}\frac{dv}{dt}dx = m_{//}v\,dv. \tag{30}$$

It follows that[28]

$$m_{//} = \frac{1}{v}\frac{dU_{\text{EM}}}{dv}. \tag{31}$$

As we shall see in sec. 4, given the standard definitions 13 of electromagnetic energy and momentum, the neglect of non-electromagnetic stabilizing forces in the derivation of eqs. 26 and 31 leads to an ambiguity in the expression for the longitudinal mass of Lorentz's electron.

If the combination of the energy U (divided by c), and the momentum $\mathbf{P}$ for any system, electromagnetic or otherwise, transforms as a four-vector under Lorentz transformations, then $m_{//}$ calculated from eq. 26 (with P substituted for P_{EM}) is equal to $m_{//}$ calculated from eq. 31 (with U substituted for U_{EM}).[29] Consider the transformation from a rest frame with coordinates x_0^μ to the x^μ-frame. In that case (see eqs. 4–8):

$$P^\mu = \left(\frac{U}{c}, \mathbf{P}\right) = (\gamma m_0 c, \gamma m_0 \mathbf{v}). \tag{32}$$

The energy U gives the longitudinal mass (see eq. 31)

$$m_{//} = \frac{1}{v}\frac{dU}{dv} = \frac{1}{v}\frac{d}{dv}(\gamma m_0 c^2) = \frac{m_0 c^2}{v}\frac{d\gamma}{dv}. \tag{33}$$

The momentum $\mathbf{P}$ gives the longitudinal mass (eq. 26):

$$m_{//} = \frac{dP}{dv} = \frac{d}{dv}(\gamma m_0 v) = m_0 \frac{d(\gamma v)}{dv}. \tag{34}$$

Noting that[30]

$$\frac{d\gamma}{dv} = \gamma^3 \frac{v}{c^2}, \quad \frac{d(\gamma v)}{dv} = \gamma^3, \tag{35}$$

we find that eqs. 33 and 34 do indeed give the same result:

$$m_{//} = \frac{1}{v}\frac{dU}{dv} = \frac{dP}{dv} = \gamma^3 m_0. \tag{36}$$

The momentum $\mathbf{P}$ in eq. 32 gives the transverse mass (eq. 26):

$$m_\perp = \frac{P}{v} = \gamma m_0. \tag{37}$$

Eqs. 36 and 37 give mass-velocity relations that hold for any relativistic particle. These equations thus have much broader applicability than their origin in electrodynamics suggests. This is exactly what killed the dreams of Abraham and Lorentz of using these relations to draw conclusions about the nature and shape of the electron.

3. LORENTZ'S THEOREM OF CORRESPONDING STATES, THE GENERALIZED CONTRACTION HYPOTHESIS, AND THE VELOCITY DEPENDENCE OF ELECTRON MASS

Lorentz had already published the relativistic eqs. 36 and 37 for longitudinal and transverse mass, up to an undetermined factor l, in 1899. To understand how Lorentz originally arrived at these equations we need to take a look at his general approach to problems in the electrodynamics of moving bodies.[31] The basic problem that Lorentz was facing was that Maxwell's equations are not invariant under Galilean transformations, which relate frames in relative motion to one another in Lorentz's classical Newtonian space-time. Lorentz thus labored under the impression that Maxwell's equations only hold in frames at rest in the ether and not in the terrestrial lab frames in which all our experiments are done.

Consider an ether frame with space-time coordinates $(t_0, \mathbf{x}_0)$ and a lab frame with space-time coordinates $(t, \mathbf{x})$ related to one another via the Galilean transformation

$$\begin{gathered} t = t_0, \;\; x = x_0 - vt_0, \;\; y = y_0, \;\; z = z_0, \\ \mathbf{E} = \mathbf{E}_0, \;\; \mathbf{B} = \mathbf{B}_0, \;\; \rho = \rho_0. \end{gathered} \tag{38}$$

The second line of this equation expresses that the electric field, the magnetic field, and the charge density remain the same even though after the transformation they are thought of as functions of $(t, \mathbf{x})$ rather than as functions of $(t_0, \mathbf{x}_0)$.

The equations for the fields produced by a charge distribution static in the lab frame as functions of the space-time coordinates $(t, \mathbf{x})$ are obtained by writing down Maxwell's equations for the relevant quantities in the lab frame, adding the current $\mu_0 \rho \mathbf{v}$[32] and replacing time derivatives by the operator $\partial/\partial t - v\partial/\partial x$.[33] We thus arrive at:

$$\begin{gathered} \operatorname{div} \mathbf{E} = \rho/\varepsilon_0, \quad \operatorname{curl} \mathbf{B} = \mu_0 \rho \mathbf{v} + \frac{1}{c^2}\left(\frac{\partial \mathbf{E}}{\partial t} - v\frac{\partial \mathbf{E}}{\partial x}\right), \\ \operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + v\frac{\partial \mathbf{B}}{\partial x}. \end{gathered} \tag{39}$$

Lorentz now replaced the space-time coordinates $(t, \mathbf{x})$, the fields $\mathbf{E}$ and $\mathbf{B}$, and the charge density ρ by auxiliary variables defined as:

$$\begin{gathered} \mathbf{x}' = l \; \mathrm{diag}(\gamma, 1, 1)\mathbf{x}, \;\; t' = l\left(\frac{t}{\gamma} - \gamma\left(\frac{v}{c^2}\right)x\right), \\ \rho' = \frac{\rho}{\gamma l^3}, \\ \mathbf{E}' = \frac{1}{l^2}\mathrm{diag}(1, \gamma, \gamma)(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \\ \mathbf{B}' = \frac{1}{l^2}\mathrm{diag}(1, \gamma, \gamma)\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right), \end{gathered} \tag{40}$$

where l is an undetermined factor that is assumed to be equal to one to first order in v/c. Since the auxiliary time variable depends on position, it is called *local time*. Lorentz showed that the auxiliary fields $\mathbf{E}'$ and $\mathbf{B}'$ and the auxiliary charge density ρ' written as functions of the auxiliary space-time coordinates $(t', \mathbf{x}')$ satisfy Maxwell's equations:

$$\text{div}'\,\mathbf{E}' = \rho'/\varepsilon_0, \quad \text{curl}'\,\mathbf{B}' = \frac{1}{c^2}\frac{\partial \mathbf{E}'}{\partial t'},$$

(41)

$$\text{div}'\,\mathbf{B}' = 0, \quad \text{curl}'\,\mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}.$$

When the factor l is set equal to one, what Lorentz showed, at least for static charge densities,[34] is that Maxwell's equations are invariant under what Poincaré (1906, 495) proposed to call textitLorentz transformations. For $l = 1$, the transformation formulae in eq. 40 for the fields $\mathbf{E}$ and $\mathbf{B}$ and for a static charge density ρ look exactly the same as in special relativity. The transformation formulae for the space-time coordinates do not. Bear in mind, however, that Lorentz did the transformation in two steps, given by eqs. 38 and 40, respectively. Schematically, we have:

(42) $$(t_0, \mathbf{x}_0, \mathbf{E}_0, \mathbf{B}_0, \rho_0) \rightarrow (t, \mathbf{x}, \mathbf{E}, \mathbf{B}, \rho) \rightarrow (t', \mathbf{x}', \mathbf{E}', \mathbf{B}', \rho')_{l=1}.$$

Combining these two steps, we recover the familiar Lorentz transformation formulae. For the fields and the charge density, this is just a matter of replacing $(\mathbf{E}, \mathbf{B}, \rho)$ in eq. 40 by $(\mathbf{E}_0, \mathbf{B}_0, \rho_0)$ and setting $l = 1$. For the space-time coordinates, it takes only a minimal amount of algebra:

$$x' = \gamma x = \gamma(x_0 - vt_0), \;\; y' = y = y_0, \;\; z' = z = z_0,$$

(43)

$$t' = \frac{t}{\gamma} - \gamma\frac{v}{c^2}x = \frac{t_0}{\gamma} - \gamma\frac{v}{c^2}(x_0 - vt_0) = \gamma\left(t_0 - \frac{v}{c^2}x_0\right),$$

where in the second line we used that $1/\gamma + \gamma(v^2/c^2) = \gamma(1/\gamma^2 + \beta^2) = \gamma$.

The inverse of the transformation $(t_0, \mathbf{x}_0, \mathbf{E}_0, \mathbf{B}_0) \rightarrow (t', \mathbf{x}', \mathbf{E}', \mathbf{B}')$ for $l = 1$ is found by interchanging $(t_0, \mathbf{x}_0, \mathbf{E}_0, \mathbf{B}_0)$ and $(t', \mathbf{x}', \mathbf{E}', \mathbf{B}')$ and changing $\mathbf{v}$ to $-\mathbf{v}$. Doing the inversion for $l \neq 1$ also requires changing l to l^{-1}. The inverse of the transformation $(\mathbf{E}_0, \mathbf{B}_0) \rightarrow (\mathbf{E}', \mathbf{B}')$ for $l \neq 1$, for instance, is given by

$$\mathbf{E}_0 = \mathbf{E} = l^2 \text{diag}(1, \gamma, \gamma)(\mathbf{E}' - \mathbf{v} \times \mathbf{B}'),$$

(44)

$$\mathbf{B}_0 = \mathbf{B} = l^2 \text{diag}(1, \gamma, \gamma)\left(\mathbf{B}' + \frac{1}{c^2}\mathbf{v} \times \mathbf{E}'\right).$$

The transformation is symmetric only for $l = 1$. Unlike Lorentz before 1905, Poincaré and Einstein both looked upon the primed quantities as the quantities measured by the observer in the lab frame. In special relativity, the ether frame is just another inertial frame on a par with the lab frame. The situation for observers in these two frames will be fully symmetric only if $l = 1$. This was essentially the argument for both Poincaré and Einstein to set $l = 1$. As we shall see in the next section, Lorentz also ended up setting $l = 1$ but on the basis of a roundabout dynamical argument. For our purposes it is important that we leave the factor l undetermined for the time being.

The invariance of Maxwell's equations under the combination of transformations 38 and 40 allowed Lorentz to formulate what he called the *theorem of corresponding states*. This theorem says that for any field configuration in a frame at rest in the ether there is a corresponding field configuration in a frame moving through the ether such that the auxiliary fields $\mathbf{E}'$ and $\mathbf{B}'$ in the moving frame are the same functions of the auxiliary space and time coordinates $(t', \mathbf{x}')$ as the real fields $\mathbf{E}_0$ and $\mathbf{B}_0$ in the frame at rest of the real space and time coordinates $(t_0, \mathbf{x}_0)$. Lorentz was particularly interested in free field configurations (for which $\rho = 0$) describing patterns of light and darkness. Most experiments in optics eventually boil down to the observation of such patterns.

To describe a pattern of light and darkness it suffices to specify where the fields averaged over times that are long compared to the period of the light used vanish and where these averages are large. $\mathbf{E}'$ and $\mathbf{B}'$ are linear combinations of $\mathbf{E}$ and $\mathbf{B}$ (see eq. 40). They are large (small) when- and wherever $\mathbf{E}$ and $\mathbf{B}$ are. Since patterns of light and darkness by their very nature are effectively static, no complications arise from the x-dependence of local time. If it is light (dark) simultaneously at two points with coordinates $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{x}_0 = \mathbf{b}$ in some field configuration in a frame at rest in the ether, it will be light (dark) simultaneously at the corresponding points $\mathbf{x}' = \mathbf{a}$ and $\mathbf{x}' = \mathbf{b}$ in the corresponding state in a frame moving through the ether. In terms of the real coordinates these are the points $\mathbf{x} = (1/l)\mathrm{diag}(1/\gamma, 1, 1)\mathbf{a}$ and $\mathbf{x} = (1/l)\mathrm{diag}(1/\gamma, 1, 1)\mathbf{b}$. The pattern of light and darkness in a moving frame is thus obtained from its corresponding pattern in a frame at rest in the ether by contracting the latter by a factor γl in the direction of motion and a factor l in the directions perpendicular to the direction of motion. Examining the formula for the local time in eq. 40, one likewise sees that the periods of light waves in a moving frame are obtained by multiplying the periods of the light waves in the corresponding state at rest in the ether by a factor γ/l.

To account for the fact that these length-contraction and time-dilation effects in electromagnetic field configurations were never detected, Lorentz (1899) assumed that matter interacting with the fields (e.g., the optical components producing patterns of light and darkness) experiences these same effects. Lorentz thereby added a far-reaching physical assumption to his purely mathematical theorem of corresponding states. Elsewhere one of us has dubbed this assumption the *generalized contraction hypothesis* (Janssen, 1995, sec. 3.3; 2002b; Janssen and Stachel, 2004). It was through this hypothesis that Lorentz decreed a number of exemptions of the Newtonian laws that had jurisdiction over matter in his theory. The length-contraction and time-dilation rules to which matter and field alike had to be subject to account for the absence of any signs of ether drift are examples of such exemptions. The velocity dependence of mass is another (Janssen, 1995, sec. 3.3.6). This is the one that is important for our purposes.

Suppose an oscillating electron in a light source at rest in S_0 satisfies $\mathbf{F}_0 = m_0\mathbf{a}_0$. In the corresponding state in S the corresponding electron will then satisfy the same equation in terms of the auxiliary quantities, i.e.,

$$\mathbf{F}' = m_0\mathbf{a}', \tag{45}$$

where $\mathbf{F}'$ is the same function of $(t', \mathbf{x}')$ as $\mathbf{F}_0$ is of $(t_0, \mathbf{x}_0)$, and where $\mathbf{a}' = d^2\mathbf{x}'/dt'^2$ and $\mathbf{a}_0 = d^2\mathbf{x}_0/dt_0{}^2$ are always the same at corresponding points in S and S_0. Lorentz assumed that motion through the ether affects all forces on the electron the same way it affects Coulomb forces[35]

$$\mathbf{F}' = \frac{1}{l^2}\mathrm{diag}(1, \gamma, \gamma)\mathbf{F}. \tag{46}$$

For the relation between the acceleration $\mathbf{a}'$ in terms of the auxiliary space and time coordinates and the real acceleration $\mathbf{a}$, Lorentz used the relation

$$\mathbf{a}' = \frac{1}{l}\mathrm{diag}(\gamma^3, \gamma^2, \gamma^2)\mathbf{a}. \tag{47}$$

In general, this relation is more complicated, but when the velocity $d\mathbf{x}_0/dt_0$ with which the electron is oscillating in S_0 is small, $d\mathbf{x}'/dt'$ (equal to $d\mathbf{x}_0/dt_0$ at the corresponding point in S_0) can be neglected and eq. 47 holds. A derivation of the general relation between $\mathbf{a}'$ and $\mathbf{a}$ was given by Planck (1906a) in the context of his derivation of the relativistic generalization of Newton's second law, a derivation mathematically essentially equivalent to Lorentz's 1899 derivation of the velocity dependence of mass, except that Planck only had to consider the special case $l = 1$.[36]

Lorentz probably arrived at eq. 47 through the following crude argument. If an electron oscillates around a fixed point in S with a low velocity and a small amplitude, the x-dependent term in the expression for local time can be ignored. In that case, we only need to take into account that $\mathbf{x}'$ differs from $\mathbf{x}$ by $l\mathrm{diag}(\gamma, 1, 1)$ and that t' differs from t by l/γ (see eq. 40). This gives a quick and dirty derivation of eq. 47:

$$\mathbf{a}' = \frac{d^2\mathbf{x}'}{dt'^2} = \left(\frac{\gamma}{l}\right)^2 l\ \mathrm{diag}(\gamma, 1, 1)\frac{d^2\mathbf{x}}{dt^2} = \frac{1}{l}\mathrm{diag}\left(\gamma^3, \gamma^2, \gamma^2\right)\mathbf{a}. \tag{48}$$

Inserting eqs. 46 and 47 into eq. 45, we find

$$\frac{1}{l^2}\mathrm{diag}(l,\gamma,\gamma)\mathbf{F} = \frac{1}{l}\mathrm{diag}\left(\gamma^3,\gamma^2,\gamma^2\right) m_0\mathbf{a}. \tag{49}$$

This can be rewritten as

$$\mathbf{F} = l\ \mathrm{diag}\left(\gamma^3,\gamma,\gamma\right) m_0\mathbf{a}. \tag{50}$$

From this equation it follows that the oscillation of an electron in the moving source can only satisfy Newton's second law if the mass m of an electron with velocity $\mathbf{v}$ with respect to the ether (remember that the velocity of the oscillation itself was assumed to be negligible) differs from the mass m_0 of an electron at rest in the ether in precisely the following way:

$$m_{//} = l\gamma^3 m_0, \ \ m_\perp = l\gamma m_0. \tag{51}$$

If $l = 1$, these are just the relativistic eqs. 36 and 37. It was Planck who showed in the paper mentioned above that these relations also obtain in special relativity.[37] Planck's interpretation of these relations was very different from Lorentz's. For Planck, as for Einstein, the velocity dependence of mass was part of a new relativistic mechanics replacing classical Newtonian mechanics. Lorentz wanted to retain Newtonian mechanics, even after he accepted in 1904 that there are no Galilean-invariant Newtonian masses or forces in nature. Consequently, he had to provide an explanation for the peculiar velocity-dependence of electron mass he needed to account for the absence of any detectable ether drift. In 1904, adapting Abraham's electron model, Lorentz provided such an explanation in the form of a specific model of the electron that exhibited exactly the velocity dependence of eq. 51 for $l = 1$.

4. ELECTROMAGNETIC ENERGY, MOMENTUM, AND MASS OF A MOVING ELECTRON

In this section we use Lorentz's theorem of corresponding states—or, in modern terms, the Lorentz invariance of Maxwell's equations—to calculate the energy, the momentum, and the Lagrangian for the field of a moving electron, conceived of as nothing but a surface charge distribution and its electromagnetic field. We then compute the longitudinal and the transverse mass of the electron.

We distinguish three different models. In all three the electron at rest in the ether is spherical. In Abraham's model it remains spherical when it is set in motion; in Lorentz's model it contracts by a factor γ in the direction of motion; and in the Bucherer-Langevin model it contracts by a factor $\gamma^{2/3}$ in the direction of motion but expands by a factor $\gamma^{1/3}$ in the directions

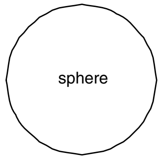
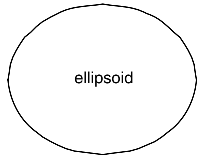
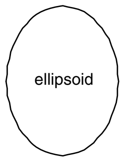
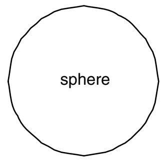
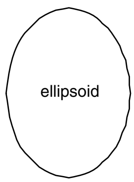
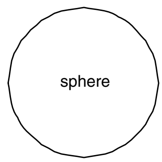
	moving electron	**corresponding state** stretch dimensions of moving system by $\mathrm{diag}(\gamma l, l, l)$
The rigid electron of Abraham (l arbitrary)	 (R, R, R)	 $(\gamma l R, l R, l R)$
The contractile electron of Lorentz and Poincaré ($l = 1$)	 $(R/\gamma, R, R)$	 (R, R, R)
The contractile electron of constant volume of Bucherer and Langevin ($l = \gamma^{-1/3}$)	 $\left(R/\gamma^{2/3}, \gamma^{1/3} R, \gamma^{1/3} R\right)$	 (R, R, R)

FIGURE 1. A moving electron according to the models of Abraham, Lorentz, and Bucherer-Langevin, and the corresponding states at rest in the ether.

perpendicular to the direction of motion so that its volume remains constant. Fig. 1 shows a moving electron according to these three models along with the corresponding states in a frame at rest in the ether. For Abraham's rigid electron the corresponding state is an ellipsoid; for the contractile electrons of Lorentz and Bucherer-Langevin it is a sphere.

In the corresponding state of a moving electron (in relativistic terms: in the electron's rest frame) there is no magnetic field. Hence $\mathbf{B}' = 0$ and eq. 44

gives:

$$\mathbf{E} = l^2 \left(E'_x, \gamma E'_y, \gamma E'_z\right), \quad \mathbf{B} = \frac{\gamma l^2 v}{c^2} \left(0, -E'_z, E'_y\right). \tag{52}$$

4.1. *Energy*

The energy of the electric and magnetic field is defined as

$$U_{\text{EM}} = \int \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0^{-1} B^2\right) d^3x. \tag{53}$$

For the field in eq. 52 with $B_x = 0$, it is given by

$$U_{\text{EM}} = \int \frac{1}{2}\varepsilon_0 E_x^2 d^3x + \int \frac{1}{2}\varepsilon_0 \left(E_y^2 + E_z^2\right) d^3x + \int \frac{1}{2}\mu_0^{-1} \left(B_y^2 + B_z^2\right) d^3x. \tag{54}$$

Following Poincaré (1906, 523), we call these three terms A, B, and C. Using eq. 52 and $d^3x = d^3x'/\gamma l^3$, we find

$$A = \frac{l}{\gamma} \int \frac{1}{2}\varepsilon_0 E'^2_x d^3x' = \frac{l}{\gamma} A',$$

$$B = l\gamma \int \frac{1}{2}\varepsilon_0 \left(E'^2_y + E'^2_z\right) d^3x' = l\gamma B', \tag{55}$$

$$C = \frac{\mu_0^{-1}\gamma l v^2}{c^4} \int \frac{1}{2} \left(E'^2_y + E'^2_z\right) d^3x' = l\gamma\beta^2 B',$$

where in the last step we used $c^2 = 1/\varepsilon_0\mu_0$. If the corresponding state is spherical,

$$B' = 2A' = \frac{2}{3} U'_{\text{EM}}. \tag{56}$$

It follows that for the models of Lorentz and Bucherer-Langevin:

$$U_{\text{EM}} = l\gamma \left(\frac{1}{\gamma^2} + 2 + 2\beta^2\right) A' = l\gamma \left(1 + \frac{1}{3}\beta^2\right) U'_{\text{EM}}, \tag{57}$$

where we used that $\gamma^{-2} = 1 - \beta^2$ and that $3A' = U'_{\text{EM}}$. Eq. 57 can also be written as

$$U_{\text{EM}} = l\gamma \left(\frac{4}{3} - \frac{1}{3}\left(1 - \beta^2\right)\right) U'_{\text{EM}} = l \left(\frac{4\gamma}{3} - \frac{1}{3\gamma}\right) U'_{\text{EM}}. \tag{58}$$

4.2. *Lagrangian*

The Lagrangian can be computed the same way. We start from

(59) $$L_{\text{EM}} = \int \mathcal{L}_{\text{EM}} d^3x,$$

where $\mathcal{L}_{\text{EM}}$ is the Lagrange density defined as (note the sign)

(60) $$\mathcal{L}_{\text{EM}} \equiv \frac{1}{2}\mu_0^{-1}B^2 - \frac{1}{2}\varepsilon_0 E^2.$$

This quantity transforms as a scalar under Lorentz transformations as can be seen from its definition in manifestly Lorentz-invariant form:[38]

(61) $$\mathcal{L}_{\text{EM}} \equiv \frac{1}{4}\mu_0^{-1}F_{\mu\nu}F^{\mu\nu}.$$

It follows that $\mathcal{L}_{\text{EM}} = l^4 \mathcal{L}'_{\text{EM}}$ with $\mathcal{L}'_{\text{EM}} = -(1/2)\varepsilon_0 E'^2$. Eq. 59 thus gives:

(62) $$L_{\text{EM}} = \int l^4 \mathcal{L}'_{\text{EM}} \frac{d^3x'}{\gamma l^3} = -\frac{l}{\gamma}U'_{\text{EM}}.$$

4.3. *Momentum*

The electromagnetic momentum can also be computed in this way. For the field of the electron, the electromagnetic momentum density (see eq. 18) is:

(63) $$\mathbf{p}_{\text{EM}} = \varepsilon_0 \begin{pmatrix} E_yB_z - E_zB_y \\ -E_xB_z \\ E_xB_y \end{pmatrix} = \varepsilon_0\gamma l^4 \frac{v}{c^2} \begin{pmatrix} \gamma(E_y'^2 + E_z'^2) \\ E_x'E_y' \\ E_x'E_z' \end{pmatrix}.$$

Because of symmetry (in all three models)

(64) $$\int p_{y_{\text{EM}}} d^3x = \int p_{z_{\text{EM}}} d^3x = 0.$$

For the x-component, we find

(65) $$P_{x_{\text{EM}}} = \frac{1}{\gamma l^3}\int p_{x_{\text{EM}}} d^3x' = \gamma l \frac{v}{c^2}\int \varepsilon_0 \left(E_y'^2 + E_z'^2\right) d^3x' = \gamma l \frac{v}{c^2} 2B'$$

(see eq. 55). For the contractile electron (Lorentz and Bucherer-Langevin), $B' = (2/3)U'_{\text{EM}}$ (see eq. 56). In that case

(66) $$\mathbf{P}_{\text{EM}} = \frac{4}{3}\gamma l \left(\frac{U'_{\text{EM}}}{c^2}\right)\mathbf{v}.$$

This pre-relativistic equation will immediately strike anyone familiar with the basic formulae of special relativity as odd. Remember that from a relativistic point of view the energy U'_{EM} of the moving electron's corresponding state at rest in the ether is nothing but the energy $U_{0_{\text{EM}}}$ of the electron in its rest frame. Comparison of eq. 66 with $l = 1$ to $\mathbf{P} = \gamma m_0 \mathbf{v}$ (eq. 9) suggests that the rest mass of the electron is $m_0 = \frac{4}{3}U_{0_{\rm EM}}/c^2$. This seems to be in blatant contradiction to the equation everybody knows, $E = mc^2$. This is the notorious "4/3 puzzle" of the energy-mass relation of the classical electron. The origin of the problem is that the system we are considering, the self-field of the electron, is not closed and that its four-momentum consequently does not transform as a four-vector, at least not under the standard definition 12 of four-momentum. The solution to the puzzle is either to add another piece to the system so that the composite system is closed or to adopt the alternative Fermi-Rohrlich definition 14 (with a fixed unit vector n^μ) of the four-momentum of spatially extended systems.[39] As we shall see, the "4/3 puzzle" had already reared its ugly head before the advent of special relativity, albeit in a different guise.

4.4. Longitudinal and transverse mass

Substituting eqs. 58 and 66 for the energy and momentum of the field of a moving contractile electron into the expressions 26 and 31 for the electron's transverse and longitudinal mass, we find:

$$m_{//} = \frac{dP_{\rm EM}}{dv} = \frac{d(\gamma l v)}{dv}\frac{4}{3}\frac{U'_{\rm EM}}{c^2}, \tag{67}$$

$$m_{\perp} = \frac{P_{\rm EM}}{v} = \gamma l \frac{4}{3}\frac{U'_{\rm EM}}{c^2}, \tag{68}$$

$$m_{//} = \frac{1}{v}\frac{dU_{\rm EM}}{dv} = \frac{1}{v}\frac{d}{dv}\left(\frac{4\gamma l}{3} - \frac{l}{3\gamma}\right)U'_{\rm EM}. \tag{69}$$

Several conclusions can be drawn from these equations. First, it turns out that eq. 67 only gives the velocity dependence of the longitudinal mass required by Lorentz's generalized contraction hypothesis for $l = 1$. Unfortunately, for $l = 1$, eq. 69 does not give the same longitudinal mass as eq. 67. One only obtains the same result for $l = \gamma^{-1/3}$. This is the value for the Bucherer-Langevin constant-volume contractile electron.

It is easy to prove these claims. Using eq. 35, we can write eq. 67 as

$$m_{//} = \frac{dP_{\rm EM}}{dv} = \left(\gamma^3 l + \gamma v \frac{dl}{dv}\right)\frac{4}{3}\frac{U'_{\rm EM}}{c^2}. \tag{70}$$

From eqs. 68 and 70 it follows that the only way to ensure that $m_{//} = l\gamma^3 m_0$ and $m_\perp = l\gamma m_0$, as required by the generalized contraction hypothesis (see eq. 51), is to set the Newtonian mass equal to zero, to set $l = 1$, and to define the mass of the electron at rest in the ether as

$$m_0 = \frac{4}{3}\frac{U'_{\rm EM}}{c^2} \tag{71}$$

(which, from a relativistic point of view, amounts to the odd equation $E = \frac{3}{4}mc^2$). Eqs. 70 and 68 then reduce to

$$m_{//} = \gamma^3 m_0, \quad m_\perp = \gamma m_0, \tag{72}$$

in accordance with eq. 51.

Lorentz (1904) had thus found a concrete model for the electron with a mass exhibiting exactly the velocity dependence that he had found in 1899. This could hardly be a coincidence. Lorentz concluded[40] that the electron was indeed nothing but a small spherical surface charge distribution, subject to a microscopic version of the Lorentz-FitzGerald contraction when set in motion, and that its mass was purely electromagnetic, i.e., the result of interaction with its self-field. This is Lorentz's version of the classical dream referred to by Pais in the passage we quoted in the introduction. The mass-velocity relations for Lorentz's electron model are just the relativistic relations 36–37. So it is indeed no coincidence that Lorentz found these same relations twice, first, in 1899, as a necessary condition for rendering ether drift unobservable (see eqs. 45–51) and then, in 1904, as the mass-velocity relations for a concrete Lorentz-invariant model of the electron. But the explanation is not, as Lorentz thought, that his model provides an accurate representation of the real electron; it is simply that the mass of *any* Lorentz-invariant model of *any* particle—whatever its nature and whatever its shape—will exhibit the exact same velocity dependence. This was first shown (for static systems) by Laue (1911a) and, to use Pais' imagery again, it killed Lorentz's dream.

Quite independently of Laue's later analysis, Lorentz's electron model appeared to be dead on arrival. The model as it stands is inconsistent. One way to show this is to compare expression 72 for the longitudinal mass $m_{//}$ derived from the electron's electromagnetic momentum to the expression for $m_{//}$ derived from its electromagnetic energy. These two calculations, it turns out, do not give the same result (Abraham, 1905, 188, 204).[41] Setting $l = 1$ in eq. 69 and using eq. 35, we find

$$m_{//} = \frac{1}{v}\frac{4}{3}\frac{d\gamma}{dv}U'_{\text{EM}} - \frac{1}{3v}\frac{d}{dv}\left(\frac{1}{\gamma}\right)U'_{\text{EM}} = \gamma^3\frac{4}{3}\frac{U'_{\text{EM}}}{c^2} - \frac{1}{3v}\frac{d}{dv}\left(\frac{1}{\gamma}\right)U'_{\text{EM}}. \tag{73}$$

The first term in the last expression is equal to $m_{//}$ in eq. 70 for $l = 1$. Without even working out the second term, we thus see that momentum and energy lead to different expressions for the longitudinal mass of Lorentz's electron.

For the Bucherer-Langevin electron there is no ambiguity in the formula for its longitudinal mass. Inserting $l = \gamma^{-1/3}$ and eq. 71 into eq. 70, we find that the electromagnetic momentum of the Bucherer-Langevin electron gives:

$$m_{//} = \frac{dP_{\rm EM}}{dv} = \left(\gamma^{8/3} + \gamma v \frac{d\gamma^{-1/3}}{dv}\right) m_0 = \gamma^{8/3}\left(1 - \frac{1}{3}\beta^2\right) m_0, \tag{74}$$

where in the last step we used eq. 35 in conjunction with

$$\frac{d\gamma^{-1/3}}{dv} = -\frac{1}{3}\gamma^{-4/3}\frac{d\gamma}{dv} = -\frac{1}{3}\gamma^{5/3}\frac{v}{c^2}. \tag{75}$$

Inserting $l = \gamma^{-1/3}$ and eq. 71 into eq. 69, we find that its electromagnetic energy gives:

$$m_{//} = \frac{1}{v}\frac{dU_{\rm EM}}{dv} = \frac{c^2}{v}\frac{d}{d\gamma}\left(\gamma^{2/3} - \frac{1}{4}\gamma^{-4/3}\right)\frac{d\gamma}{dv} m_0. \tag{76}$$

Some simple gamma gymnastics establishes that eq. 76 reproduces eq. 74:[42]

$$m_{//} = \frac{1}{v}\frac{dU_{\rm EM}}{dv} = \gamma^{8/3}\left(1 - \frac{1}{3}\beta^2\right) m_0. \tag{77}$$

So energy and momentum of the Bucherer-Langevin electron do indeed give the same longitudinal mass. The same is true for the Abraham electron, although the calculation is more involved and unimportant for our purposes.

One feature that the Abraham model and the Bucherer-Langevin model have in common and that distinguishes both models from Lorentz's is that the volume of the electron is constant. Hence, whatever forces are responsible for stabilizing the electron never do any work and can safely be ignored, as was done in the derivation of the basic equations 26 and 31 for longitudinal mass (see eqs. 16 and 29 and notes 23 and 27). This does not mean that no such forces are needed. In all three models, one is faced with the problem of the electron's stability. Abraham, however, argued that whereas Lorentz's contractile electron called for the explicit addition of non-electromagnetic stabilizing forces, he, Abraham, could simply take the rigidity of his own spherical electron as a given and proceed from there without ever running into trouble.

In the introduction of the 1903 exposition of his electron dynamics, Abraham (1903, 108–109) devoted two long paragraphs to the justification of this crucial assumption. He distinguished three sets of equations for the dynamics of the electron. We already encountered two of these, the "field equations" determining the self-field of the electron and the "fundamental dynamical equations" determining the motion of the electron in an external field. Logically prior to these, however, is what Abraham called the "basic kinematical equation," which "limits the freedom of motion of the electron."

This is the assumption that the electron always retains its spherical shape. Abraham tried to preempt the criticism he anticipated on this score:

> This basic kinematical hypothesis may strike many as arbitrary; invoking the analogy with ordinary electrically charged solid bodies, many would subscribe to the view that the truly enormous field strengths at the surface of the electron—field strengths a trillion times larger than those amenable to measurement—are capable of deforming the electron; that electrical and elastic forces on a spherical electron would be in equilibrium as long as the electron is at rest; but that the motion of the electron would change the forces of the electromagnetic field, and thereby the shape of the equilibrium state of the electron. This is not the view that has led to agreement with experiment. It also seemed to me that the assumption of a deformable electron is not allowed on fundamental grounds. The assumption leads to the conclusion that work is done either by or against the electromagnetic forces when a change of shape takes place, which means that in addition to the electromagnetic energy an internal potential energy of the electron needs to be introduced. If this were really necessary, it would immediately make an electromagnetic foundation of the theory of cathode and Becquerel rays, purely electric phenomena, impossible: one would have to give up on an electromagnetic foundation of mechanics right from the start. It is our goal, however, to provide a purely electromagnetic foundation for the dynamics of the electron. For that reason we are no more entitled to ascribe elasticity to the electron than we are to ascribe material mass to it. On the contrary, our hope is to learn to understand the elasticity of matter on the basis of the electromagnetic conception (Abraham, 1903, 108–109).

The suggestion that experimental data, presumably those of Kaufmann, supported his kinematics was wishful thinking on Abraham's part (cf. Miller, 1981, secs. 1.9 and 1.11). In support of his more general considerations—as an argument it is a textbook example of the genetic fallacy—Abraham proceeded to appeal to no less an authority than Heinrich Hertz:

> Hertz has convincingly shown that one is allowed to talk about rigid connections before one has talked about forces. Our dynamics of the electron does not talk about forces trying to deform the electron at all. It only talks about "external forces,"

> which try to give [the electron] a velocity or an angular velocity, and about "internal forces", which stem from the [self-] field of the electron and which balance these external forces. Even these "forces" and "torques" are only auxiliary quantities defined in terms of the fundamental kinematic and electromagnetic concepts. The same holds for terms like "work," "energy," and "momentum." The guiding principle in choosing these terms, however, was to bring out clearly the analogy between electromagnetic mechanics and the ordinary mechanics of material bodies (Abraham, 1903, 109).

Abraham submitted this paper in October 1902, almost three years before the publication of Einstein's first paper on relativity. He can thus hardly be faulted for basing his new electromagnetic mechanics on the old Newtonian kinematics. Minkowski would sneer a few years later that "approaching Maxwell's equation with the concept of a rigid electron seems to me the same thing as going to a concert with your ears stopped up with cotton wool" (quoted in Miller, 1981, sec. 12.4.5, 330). He made this snide comment during the 80th *Versammlung Deutscher Naturforscher und Ärzte* in Cologne in September 1908, the same conference where he gave his now famous talk "Space and Time" (Minkowski, 1909). His veritable diatribe against the rigid electron, which he called a "monster" and "no working hypothesis but a working hindrance," came during the discussion following a talk by Bucherer (1908c), who presented data that seemed to contradict Abraham's predictions for the velocity dependence of electron mass and support what was by then no longer just Lorentz's prediction but Einstein's as well. It was only decades later that these data were also shown to be inconclusive (Zahn and Spees, 1938; quoted in Miller, 1981, 331).

Minkowski's comment suggests that we run Abraham's argument about the kinematics of the electron in Minkowski rather than in Newtonian space-time. We would then take it as a given that the electron has the shape of a sphere in its rest frame, which implies that it will have the shape of a sphere contracted in the direction of motion in any frame in which it is moving. This, of course, is exactly Lorentz's model. This gives rise to a little puzzle. The point of Abraham's argument in the passage we just quoted was that by adopting rigid kinematical constraints we can safely ignore stabilizing forces. His objection to Lorentz's model was that Lorentz *did* have to worry about non-electromagnetic stabilizing forces or he would end up with two different formulae for the longitudinal mass of his electron. How can these two claims by Abraham be reconciled with one another? One's initial reaction might be that Abraham's kinematical argument does not carry over to special relativity because the theory leaves no room for rigid bodies. That in and of itself is certainly true, but it is not the source of the problem. We could run the argument using some appropriate concept of an *approximately* rigid body (and as long as the electron is moving uniformly there is no problem whatsoever on this score). Abraham's argument that kinematic constraints can be used to obviate the need for discussion of the stability of the electron will then go through *as long as we use proper relativistic notions*. From a relativistic point of view, the analysis of Lorentz's model in this section is based on the standard non-covariant definition 12 of four-momentum. If we follow Fermi, Rohrlich and others and use definition 14 (with a fixed unit vector n^μ) instead, the ambiguity in the longitudinal mass of Lorentz's electron simply disappears. After all, under this alternative definition the combination of the energy and momentum of the electron's self-field transforms as a four-vector, even though it is an open system. This, in turn, guarantees—as we saw in eqs. 32–37 at the end of sec. 2—that energy and momentum give the same longitudinal mass. This shows that the ambiguity in the longitudinal mass of Lorentz's electron is not a consequence of the instability of the electron, but an artifact of the definitions of energy and momentum he used. We do not claim great originality for this insight. It is simply a matter of translating Rohrlich's analysis of the "4/3 puzzle" in special relativity (see the discussion following eq. 66) to a pre-relativistic setting.

4.5. Hamiltonian, Lagrangian, and generalized momentum

Poincaré (1906, 524) brought out the inconsistency of Lorentz's model in a slightly different way. He raised the question whether the expressions he found for energy, momentum, and Lagrangian for the field of the moving electron conform to the standard relations between Hamiltonian, Lagrangian, and generalized momentum. For an electron moving in the positive x-direction, these relations are

$$U = \mathbf{P}\cdot\mathbf{v} - L = Pv - L, \quad P = P_x = \frac{dL}{dv}. \tag{78}$$

It turns out that the first relation is satisfied by both the Lorentz and the Bucherer-Langevin model, but that the second is satisfied only by the latter. Using eqs. 62 and 66 for P_{EM} and L_{EM}, respectively, we find

$$P_{\mathrm{EM}}v - L_{\mathrm{EM}} = \left(\frac{4}{3}\gamma l \beta^2 + \frac{l}{\gamma}\right) U'_{\mathrm{EM}} = \gamma l \left(\frac{4}{3}\beta^2 + 1 - \beta^2\right) U'_{\mathrm{EM}}, \tag{79}$$

which does indeed reduce to the expression for U_{EM} found in eq. 57 for any value of l.

We now compute the conjugate momentum,

$$\frac{dL_{\rm EM}}{dv} = -U'_{\rm EM}\frac{d}{dv}\left(\frac{l}{\gamma}\right) = U'_{\rm EM}\left\{l\gamma\frac{v}{c^2} - \frac{1}{\gamma}\frac{dl}{dv}\right\}, \tag{80}$$

where we used eq. 35 for $d\gamma/dv$. For the Lorentz model, with $l = 1$, this reduces to

$$\frac{dL_{\rm EM}}{dv} = \gamma\left(\frac{U'_{\rm EM}}{c^2}\right)v, \tag{81}$$

which differs by the meanwhile familiar factor of 4/3 from the expression for $P_{\rm EM}$ read off from eq. 66 for $l = 1$. For the Bucherer-Langevin model, $l = \gamma^{-1/3}$ and with the help of eq. 75, we find:

$$\frac{dL_{\rm EM}}{dv} = \frac{U'_{\rm EM}}{c^2}\left\{\gamma^{2/3}v + \frac{1}{3}\gamma^{2/3}v\right\} = \frac{4}{3}\gamma^{2/3}\left(\frac{U'_{\rm EM}}{c^2}\right)v. \tag{82}$$

This agrees exactly with eq. 66 for $l = \gamma^{-1/3}$.

The relations 78 are automatically satisfied if $(U/c, \mathbf{P})$ transforms as a four-vector under Lorentz transformations. In that case, we have (see eq. 9):

$$U = \gamma U_0, \;\; P = \gamma\frac{U_0}{c^2}v. \tag{83}$$

Inserting this into $L = Pv - U$, we find

$$L = \gamma U_0\beta^2 - \gamma U_0 = -\gamma U_0(1-\beta^2) = -\frac{U_0}{\gamma}, \tag{84}$$

which, in turn, implies that

$$\frac{dL}{dv} = -U_0\frac{d}{dv}\left(\frac{1}{\gamma}\right) = \frac{U_0}{\gamma^2}\frac{d\gamma}{dv} = \gamma\frac{U_0}{c^2}v, \tag{85}$$

in accordance with eq. 83. This shows once again (cf. the discussion at the end of sec. 4.4) that the inconsistency in Lorentz's model can be taken care of by switching—in relativistic terms—from the standard definition 12 of four-momentum to the Fermi-Rohrlich definition 14 (with fixed n^μ). In that case the energy and momentum of the electron's self-field will satisfy eqs. 83–85 even though it is an open system.

5. POINCARÉ PRESSURE

In this section we give a streamlined version of the argument with which Poincaré (1906, 525–529) introduced what came to be known as "Poincaré pressure" to stabilize Lorentz's purely electromagnetic electron.[43]

The Lagrangian for the electromagnetic field of a moving electron can in all three models (Abraham, Lorentz, Bucherer-Langevin) be written as

(86) $$L_{\mathrm{EM}} = \frac{\varphi(\vartheta/\gamma)}{\gamma^2 r}$$

(Poincaré, 1906, 525), where the argument ϑ/γ of the as yet unknown function φ is the 'ellipticity' (our term) of the "ideal electron" (Poincaré's term for the corresponding state of the moving electron). The ellipticity is the ratio of the radius of the "ideal electron" in the directions perpendicular to the direction of motion ($l\vartheta r$) and its radius ($\gamma l r$) in the direction of motion. This is illustrated in Fig. 2, which is the same as Fig. 1, except that it shows the notation Poincaré used to describe the three electron models.

For the Abraham electron the ellipticity is $1/\gamma$; for both the Lorentz and the Bucherer-Langevin electron it is 1. By examining the Lorentz case, we can determine $\varphi(1)$. Inserting $U_{0_{\mathrm{EM}}} = e^2/8\pi\varepsilon_0\gamma r$, where γr is the radius of the electron at rest in the ether, into eq. 62 for the Lagrangian, we find:

(87) $$L_{\mathrm{EM}_{\mathrm{Lorentz}}} = -\frac{U_{0_{\mathrm{EM}}}}{\gamma} = -\frac{e^2}{8\pi\varepsilon_0\gamma^2 r},$$

Comparison with the general expression for L_{EM} in eq. 86 gives:

(88) $$\varphi(1) = -\frac{e^2}{8\pi\varepsilon_0}.$$

Abraham (1902a, 37) found that the Lagrangian for his electron model has the form

(89) $$L_{\mathrm{EM}_{\mathrm{Abraham}}} = \frac{a}{r}\frac{1-\beta^2}{\beta}\ln\frac{1+\beta}{1-\beta}$$

(Poincaré, 1906, 526). Since $L_{\mathrm{EM}} = \varphi(1)/r$ for $\beta = 0$ (in which case all three electron models coincide), it must be the case that $a = \varphi(1)$. From eqs. 86 and 89 it follows that

(90) $$\varphi(1/\gamma) = \gamma^2 r L_{\mathrm{EM}_{\mathrm{Abraham}}} = \frac{a}{\beta}\ln\frac{1+\beta}{1-\beta}.$$

The Lagrangian for the Lorentz model told us that $\varphi(1) = a = -e^2/8\pi\varepsilon_0$. The Lagrangian for the Abraham model allows us to determine $\varphi'(1)$. We start from eq. 90 and develop both the right-hand side and the argument $1/\gamma$ of φ on the left-hand side to second order in β. This gives (ibid.):

(91) $$\varphi\left(1-\frac{1}{2}\beta^2\right) = a\left(1+\frac{1}{3}\beta^2\right).$$

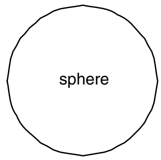
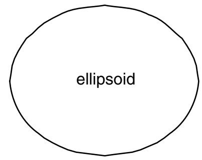
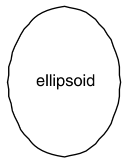
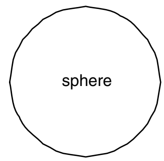
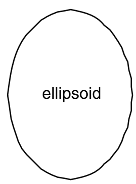
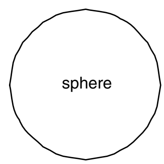
	real electron (in motion) dimensions: $(r, \vartheta r, \vartheta r)$	**ideal electron** (at rest) dimensions: $(\gamma l r, l\vartheta r, l\vartheta r)$
The rigid electron of Abraham $\vartheta = 1$ l arbitrary r constant	 (r, r, r)	 $(\gamma l r, l r, l r)$
The contractile electron of Lorentz and Poincaré $\vartheta = \gamma$ $l = 1$ $\gamma r =$ constant	 $(r, \gamma r, \gamma r)$	 $(\gamma r, \gamma r, \gamma r)$
The contractile electron of constant volume of Bucherer and Langevin $\vartheta = \gamma$ $l = \gamma^{-1/3}$ $\gamma l r = \gamma^{2/3} r =$ constant	 $(r, \gamma r, \gamma r)$	 $(\gamma^{2/3} r, \gamma^{2/3} r, \gamma^{2/3} r)$

FIGURE 2. A moving electron according to the models of Abraham, Lorentz, and Bucherer-Langevin, and the corresponding states at rest in the ether.

Now differentiate both sides:

$$-\beta\varphi'\left(1 - \frac{1}{2}\beta^2\right) = \frac{2}{3}a\beta. \tag{92}$$

It follows that $\varphi'(1) = -(2/3)a$.

As Poincaré notes, all three electron models satisfy a constraint of the form

$$r = b\vartheta^m, \tag{93}$$

where b is a constant and where the exponent m depends on which model we consider. In the Abraham model $\vartheta = 1$ and r is a constant. Hence, $r = b$. In the Lorentz model, $\vartheta = \gamma$ and γr is a constant. It follows that $\vartheta r = b$, or $r = b\vartheta^{-1}$. In the Bucherer-Langevin model, $\vartheta = \gamma$ and $\gamma^{2/3} r$ is a constant. It follows that $\vartheta^{2/3} r = b$, or $r = b\vartheta^{-2/3}$. In other words, the values of m in the three models are

$$\begin{aligned} \text{Abraham}: \ & m = 0, \\ \text{Lorentz}: \ & m = -1, \\ \text{Bucherer} - \text{Langevin}: \ & m = -2/3. \end{aligned} \tag{94}$$

Substituting $r = b\vartheta^m$ into the general expression 86 for the Lagrangian, we find:

$$L_{\text{EM}} = \frac{\varphi(\vartheta/\gamma)}{b\gamma^2\vartheta^m}. \tag{95}$$

Poincaré proceeds to investigate whether this Lagrangian describes a stable physical system. To this end, he checks whether $\partial L_{\text{EM}}/\partial\theta$ vanishes. It turns out that for $m = -2/3$ it does, but that for $m = -1$ it does not. Denote the argument of the function φ with $u \equiv \vartheta/\gamma$.

$$\frac{\partial L_{\text{EM}}}{\partial\vartheta} = \frac{\varphi'(u)}{b\gamma^3\vartheta^m} - \frac{m\varphi(u)}{b\gamma^2\vartheta^{m+1}}. \tag{96}$$

This derivative vanishes if

$$\varphi'(u) = \frac{\gamma m\varphi(u)}{\vartheta} = m\frac{\varphi(u)}{u}. \tag{97}$$

For the Lorentz and Bucherer-Langevin models $u = 1$, and this condition reduces to

$$\varphi'(1) = m\varphi(1). \tag{98}$$

Inserting $\varphi(1) = a$ and $\varphi'(1) = -(2/3)a$, we see that the purely electromagnetic Lagrangian only describes a stable system for $m = -2/3$, which is the value for the Bucherer-Langevin electron. The Lorentz electron calls for an additional term in the Lagrangian.[44] The total Lagrangian is then given by the sum

$$L_{\text{tot}} = L_{\text{EM}} + L_{\text{non-EM}}. \tag{99}$$

Like L_{EM}, $L_{\mathrm{non-EM}}$ is a function of ϑ and r. Treating these variables as independent, we can write the stability conditions for the total Lagrangian as

$$\frac{\partial}{\partial\vartheta}\left(L_{\mathrm{EM}}+L_{\mathrm{non-EM}}\right)=0, \quad \frac{\partial}{\partial r}\left(L_{\mathrm{EM}}+L_{\mathrm{non-EM}}\right)=0. \tag{100}$$

Evaluating the partial derivatives of L_{EM} given by eq. 86,

$$\frac{\partial L_{\mathrm{EM}}}{\partial\vartheta}=\frac{\varphi'(u)}{\gamma^3 r}, \quad \frac{\partial L_{\mathrm{EM}}}{\partial r}=-\frac{\varphi(u)}{\gamma^2 r^2}, \tag{101}$$

and inserting the results into the stability conditions, we find

$$\frac{\partial L_{\mathrm{non-EM}}}{\partial\vartheta}=-\frac{\varphi'(u)}{\gamma^3 r}, \quad \frac{\partial L_{\mathrm{non-EM}}}{\partial r}=\frac{\varphi(u)}{\gamma^2 r^2}. \tag{102}$$

Poincaré (1906, 528–529) continues his analysis without picking a specific model. We shall only do the calculation for the Lorentz model. So we no longer need subscripts such as in eqs. 87 and 89 to distinguish between the models of Abraham and Lorentz. For the Lorentz model $m=-1$, $\gamma=\vartheta$, $r=b/\vartheta$, and $u=1$. Substituting these values into eqs. 102 and using that $\varphi(1)=a$ and $\varphi'(1)=-(2/3)a$, we find:

$$\frac{\partial L_{\mathrm{non-EM}}}{\partial\vartheta}=\frac{2a}{3b\vartheta^2}, \quad \frac{\partial L_{\mathrm{non-EM}}}{\partial r}=\frac{a}{b^2}. \tag{103}$$

These equations are satisfied by a Lagrangian of the form

$$L_{\mathrm{non-EM}}=Ar^3\vartheta^2, \tag{104}$$

where A is a constant. Since $r^3\vartheta^2$ is proportional to the volume V of the moving electron, $L_{\mathrm{non-EM}}$ can be written as

$$L_{\mathrm{non-EM}}=P_{\mathrm{Poincaré}}V, \tag{105}$$

where $P_{\mathrm{Poincaré}}$ is a constant. We chose the letter P because this constant turns out be a (negative) pressure. To determine the constant A, we take the derivative of eq. 104 with respect to ϑ and r, and eliminate r from the results, using $r=b/\vartheta$:

$$\frac{\partial L_{\mathrm{non-EM}}}{\partial\vartheta}=2Ar^3\vartheta=\frac{2Ab^3}{\vartheta^2}, \quad \frac{\partial L_{\mathrm{non-EM}}}{\partial r}=3Ar^2\vartheta^2=3Ab^2. \tag{106}$$

Comparison with eqs. 103 gives:

$$A=\frac{a}{3b^4}. \tag{107}$$

Finally, we write $L_{\mathrm{non-EM}}$ in a form that allows easy comparison with $L_{\mathrm{EM}}=a/\gamma^2 r$ (see eq. 86 with $\varphi(\vartheta/\gamma)=\varphi(1)=a$). Using eq. 107 along with $\vartheta=\gamma$

and $b = \gamma r$, we can rewrite eq. 104 as

(108) $$L_{\text{non-EM}} = \frac{a}{3b^4} r^3 \vartheta^2 = \frac{1}{3}\frac{a}{\gamma^2 r} = \frac{1}{3} L_{\text{EM}} = -\frac{1}{3}\frac{U_{0_{\text{EM}}}}{\gamma},$$

where in the last step we used eq. 62 for $l = 1$. Using that the volume V_0 of Lorentz's electron at rest is equal to γV, we can rewrite this as:

(109) $$L_{\text{non-EM}} = -\frac{1}{3}\frac{U_{0_{\text{EM}}}}{V_0} V.$$

Comparison with expression 105 gives:

(110) $$P_{\text{Poincaré}} = -\frac{1}{3}\frac{U_{0_{\text{EM}}}}{V_0}$$

(Laue, 1911b, 164, eq. 171). Note that this so-called Poincaré pressure is negative. The pressure is present only inside the electron and vanishes outside (Poincaré, 1906, 537).[45] It can be written more explicitly with the help of the ϑ-step-function (defined as: $\vartheta(x) = 0$ for $x < 0$ and $\vartheta(x) = 1$ for $x \geq 0$). For an electron moving through the ether with velocity v in the x-direction, the Poincaré pressure in a co-moving frame (related to a frame at rest in the ether by a Galilean transformation) is:

(111) $$P_{\text{Poincaré}}(\mathbf{x}) = -\frac{1}{3}\frac{U_{0_{\text{EM}}}}{V_0}\vartheta\left(R - \sqrt{\gamma^2 x^2 + y^2 + z^2}\right),$$

where R is the radius of the electron at rest. So there is a sudden drop in pressure at the edge of the electron, which is the only place where forces are exerted.[46] These forces serve two purposes. First, they prevent the electron's surface charge distribution from flying apart under the influence of the Coulomb repulsion between its parts. Second, as the region where $P_{\text{Poincaré}}(\mathbf{x})$ is non-vanishing always coincides with the ellipsoid-shaped region occupied by the moving electron, these forces make the electron contract by a factor γ in the direction of motion.

Relations 78 between Hamiltonian, Lagrangian and generalized momentum, only one of which was satisfied by Lorentz's original purely electromagnetic electron model, are both satisfied once $L_{\text{non-EM}}$ is added to the Lagrangian. Using the total Lagrangian,

(112) $$L_{\text{tot}} = L_{\text{EM}} + L_{\text{non-EM}} = \frac{4}{3} L_{\text{EM}} = -\frac{4}{3}\frac{U_{0_{\text{EM}}}}{\gamma},$$

to compute the total momentum, we find:

(113) $$P_{\text{tot}} = \frac{dL_{\text{tot}}}{dv} = \frac{4}{3}\frac{dL_{\text{EM}}}{dv} = \frac{4}{3}\gamma\frac{U_{0_{\text{EM}}}}{c^2} v,$$

where in the last step we used eq. 81. This is just the electromagnetic momentum $P_{\rm EM}$ found earlier (see eq. 66 for $l=1$). With the help of these expressions for $L_{\rm tot}$ and $P_{\rm tot}$, we can compute the total energy:

$$U_{\rm tot} = P_{\rm tot}v - L_{\rm tot} = \frac{4}{3}\gamma U_{0_{\rm EM}}\beta^2 + \frac{4}{3}\frac{U_{0_{\rm EM}}}{\gamma} = \frac{4}{3}\gamma U_{0_{\rm EM}}. \tag{114}$$

The total energy is the sum of the electromagnetic energy (see eq. 58),

$$U_{\rm EM} = \frac{4}{3}\gamma U_{0_{\rm EM}} - \frac{1}{3}\frac{U_{0_{\rm EM}}}{\gamma}, \tag{115}$$

and the non-electromagnetic energy,

$$U_{\rm non-EM} = \frac{1}{3}\frac{U_{0_{\rm EM}}}{\gamma}, \tag{116}$$

which is minus the product of the Poincaré pressure (see eq. 110) and the volume $V = V_0/\gamma$ of the moving electron. The total energy of the system at rest is

$$U_{0_{\rm tot}} = \frac{4}{3}U_{0_{\rm EM}}, \tag{117}$$

and its rest mass is $m_{0_{\rm tot}} = U_{0_{\rm tot}}/c^2$ accordingly. Eq. 113 can thus be rewritten as

$$P_{\rm tot} = \gamma\left(\frac{U_{0_{\rm tot}}}{c^2}\right)v = \gamma m_{0_{\rm tot}}v. \tag{118}$$

The troublesome factor 4/3 has disappeared.

The total energy and momentum transform as a four-vector under Lorentz transformations. In the system's rest frame its four-momentum is $P^\mu_{0_{\rm tot}} = (U_{0_{\rm tot}}/c, 0, 0, 0)$. In a frame moving with velocity v in the x-direction, it is

$$P^\mu_{\rm tot} = \Lambda^\mu{}_\nu P^\nu_{0_{\rm tot}} = \left(\gamma\frac{U_{0_{\rm tot}}}{c}, \gamma\beta\frac{U_{0_{\rm tot}}}{c}, 0, 0\right), \tag{119}$$

in accordance with eqs. 114, 117, and 118. As we saw at the end of sec. 2, if $(U/c, \mathbf{P})$ transforms as a four-vector, it is guaranteed that energy and momentum lead to the same longitudinal mass. With Poincaré's amendment Lorentz's electron model may no longer be purely electromagnetic—at least it is fully consistent.

As we pointed out earlier, the problem that Abraham found in Lorentz's purely electromagnetic electron model (viz. that momentum and energy lead to different expressions for the longitudinal mass) returns in special relativity as the infamous "4/3 puzzle" of the mass-energy relation of the classical electron. Mathematically, these two problems are identical and the introduction of Poincaré pressure thus takes care of both. In the next section, we shall reintroduce Poincaré pressure à la Max Laue (1911a, 1911b) in his relativistic treatment of Lorentz's electron model.

Before we do so, we need to deal with a serious error committed by Poincaré (1906, 538) in his calculation of the transverse and longitudinal mass of the stabilized Lorentz electron. As a result of this error, Poincaré overestimated what he had accomplished in his paper.[47] The calculations in (Poincaré, 1906) that we have covered so far are all from section 6 of the paper. This section is phrased entirely in terms of energies, momenta, and Lagrangians. The consideration of mass is explicitly postponed (Ibid., 522). In section 4 we showed how Poincaré restated the problem of the ambiguity of the longitudinal mass of Lorentz's electron in terms of the model failing to satisfy one of the standard relations between Hamiltonian, Lagrangian, and generalized momentum (Ibid., 524; cf. sec. 4.5). In this section we traced the steps that Poincaré took in the remainder of section 6 to restore the validity of these relations for Lorentz's model (Ibid., 525–529). This is a completely unobjectionable way to proceed, from a pre-relativistic as well as from a relativistic point of view.[48]

In section 7 of his paper, Poincaré (1906, 531) finally introduces Abraham's definitions 26 of the electromagnetic longitudinal and transverse mass of the electron. And at the end of section 8, at the very end of his discussion of electron models and just before he turns to the problem of gravitation, he computes the mass of the electron in Lorentz's model, limiting himself to what he calls—in scare quotes—the "“experimental mass,” i.e., the mass for small velocities" (Ibid., 538). He writes down the Lagrangian 86 for the special case of the Lorentz electron. Using that $\varphi(\vartheta/\gamma) = \varphi(1) = a$ (where $a = -e^2/8\pi\varepsilon_0$) and $\gamma r = b$ (with b the radius of the electron at rest in the ether), we arrive at the expression given by Poincaré at this point,

$$L_{\text{EM}} = \frac{a}{b}\sqrt{1 - v^2/c^2}, \tag{120}$$

except that Poincaré uses H instead of L_{EM} and sets $c = 1$. For small velocities, eq. 120 reduces to

$$L_{\text{EM}} \approx \frac{a}{b}\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right). \tag{121}$$

Poincaré concludes that for small velocities both the longitudinal and the transverse mass of the electron is given by a/b. Since a is negative, he must have meant $-a/b$. This is just a minor slip. Poincaré's result corresponds to $U_{0_{\text{EM}}}/c^2$,[49] which differs from the result that we found by the infamous factor of 4/3 (see eqs. 117–118). How did Poincaré arrive at his result? It is hard to see how he could have found this in any other way than the following.

Computing the electromagnetic momentum as the generalized momentum corresponding to the Lagrangian 121, one finds

$$P_{\mathrm{EM}} = \frac{dL_{\mathrm{EM}}}{dv} \approx -\frac{a}{b}\frac{v}{c^2}. \tag{122}$$

Inserting this result into definitions 26 for longitudinal and transverse mass, one arrives at:

$$m_{//} = \frac{dP_{\mathrm{EM}}}{dv} \approx -\frac{a}{bc^2}, \quad m_{\perp} = \frac{P_{\mathrm{EM}}}{v} \approx -\frac{a}{bc^2}. \tag{123}$$

This is just the result reported by Poincaré (recall that he set $c = 1$). However, we had no business using eq. 122! As Poincaré himself had pointed out in section 6 of his paper, in the case of the Lorentz model, the electromagnetic momentum P_{EM} is *not* equal to the generalized momentum dL_{EM}/dv. The relation $P = dL/dv$ only holds for the *total* momentum and the *total* Lagrangian. The total Lagrangian is 4/3 times the electromagnetic part. For low velocities it reduces to (cf. eq. 121):

$$L_{\mathrm{tot}} \approx \frac{4}{3}\frac{a}{b}\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right). \tag{124}$$

Replacing L_{EM} by L_{tot} in eqs. 122–123, we find that the low-velocity limit of the electron mass is 4/3 times $-a/b$ or 4/3 times $U_{0_{\mathrm{EM}}}/c^2$, in accordance with what we found above. Unlike the minus sign of $-a/b$ that Poincaré lost in his calculation, the conflation of L_{EM} and L_{tot} has dire consequences. If we use L_{tot} it is immediately obvious that the mass of Lorentz's electron is not of purely electromagnetic origin, whereas if we use L_{EM} we are led to believe that it is. In fact, this is exactly what Poincaré claimed, both at the end of section 8 and in the introduction of his paper. In the introduction, he writes:

> If the inertia of matter is exclusively of electromagnetic origin, as is generally admitted since Kaufmann's experiment, and all forces are of electromagnetic origin (apart from this constant pressure that I just mentioned), the postulate of relativity may be established with perfect rigor. (Poincaré, 1906, 496)

Commenting on this passage, Miller (1973, 248) writes: "However the presence of these stresses [the Poincaré pressure] negates a purely electromagnetic theory of the electron's inertia." We agree. One has to choose between the "postulate of relativity" and mass being "exclusively of electromagnetic origin." Even Poincaré cannot have his cake and eat it too.[50]

6. THE RELATIVISTIC TREATMENT OF THE ELECTRON MODEL OF LORENTZ AS AMENDED BY POINCARÉ

From the point of view of Laue's relativistic continuum mechanics, the problem with Lorentz's fully electromagnetic electron is that it is not a closed system. The four-divergence of the energy-momentum tensor of its electromagnetic field does not vanish. Computing this four-divergence tells us what needs to be added to this energy-momentum tensor to obtain a closed system, i.e., a system with a total energy-momentum tensor such that $\partial_\nu T^{\mu\nu}_{\text{tot}} = 0$. Unsurprisingly, the part that needs to be added is just the energy-momentum tensor for the Poincaré pressure.

The energy-momentum tensor for the electromagnetic field is given by (Jackson, 1975, sec. 12.10)

$$T^{\mu\nu}_{\text{EM}} = \mu_0^{-1}\left(F^{\mu}_{\ \alpha}F^{\alpha\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \tag{125}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor with components (ibid., 550):

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{126}$$

Inserting the components of the field tensor into eq. 125 for the energy-momentum tensor, we recover the familiar expressions for the electromagnetic energy density (cf. eq. 53), (c times) the electromagnetic momentum density (cf. eq. 18), and (minus) the Maxwell stress tensor (cf. eq. 20).

$$\begin{gathered} T^{00}_{\text{EM}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0^{-1}B^2 = u_{\text{EM}}, \\ \left(T^{01}_{\text{EM}}, T^{02}_{\text{EM}}, T^{03}_{\text{EM}}\right) = \left(T^{10}_{\text{EM}}, T^{20}_{\text{EM}}, T^{30}_{\text{EM}}\right) = c\varepsilon_0 \mathbf{E}\times\mathbf{B} = c\mathbf{p}_{\text{EM}}, \\ T^{ij}_{\text{EM}} = -\varepsilon_0\left(E^iE^j - \frac{1}{2}\delta^{ij}E^2\right) - \mu_0^{-1}\left(B^iB^j - \frac{1}{2}\delta^{ij}B^2\right) = -T^{ij}_{\text{Maxwell}}. \end{gathered} \tag{127}$$

We calculate the four-divergence of the energy-momentum tensor for the electromagnetic field of Lorentz's electron in its rest frame. Lorentz invariance guarantees that if the four-divergence of the total energy-momentum tensor vanishes in the rest frame ($\partial_{0_\nu} T^{\mu\nu}_{0_{\text{tot}}} = 0$), it will vanish in all frames ($\partial_\nu T^{\mu\nu}_{\text{tot}} = 0$). In the rest frame, we have

$$T^{\mu\nu}_{0_{\text{EM}}} = \begin{pmatrix} u_{0_{\text{EM}}} & 0 \\ 0 & -T^{ij}_{0_{\text{Maxwell}}} \end{pmatrix}. \tag{128}$$

Consider the four-divergence $\partial_{0_\nu} T^{\mu\nu}_{0_{\rm EM}}$ of this tensor. Since the system is static, only the spatial derivatives, $\partial_{0_j} T^{\mu j}_{0_{\rm EM}}$, give a contribution. Since $T^{0j}_{0_{\rm EM}} = 0$, there will only be contributions for $\mu = i$. Using eq. 127, we can write these contributions as:[51]

$$\partial_{0_j} T^{ij}_{0_{\rm EM}} = -\partial_{0_j} T^{ij}_{0_{\rm Maxwell}} = -\rho_0 E^i_0, \tag{129}$$

The charge density ρ_0 is the surface charge density $\sigma = e/4\pi R^2$ (where e is the charge of the electron and R the radius of the electron in its rest frame):

$$\rho_0 = \sigma\delta(R - r_0), \tag{130}$$

where $r_0 \equiv \sqrt{x_0^2 + y_0^2 + z_0^2}$. Inside the electron there is no electric field (it is a miniature version of Faraday's cage); outside the field is the same as that of a point charge e located at the center of the electron. At $r_0 = R$, right at the surface of the electron, the field has a discontinuity. Its magnitude, E_0, jumps from 0 to $e/4\pi\varepsilon_0 R^2$. At this point we need to use the average of these two values (see, e.g., Griffith, 1999, 102–103). At $r_0 = R$ the field is thus given by

$$E^i_{0_{r_0=R}} = \frac{e}{8\pi\varepsilon_0 R^2}\frac{x^i_0}{R} = \frac{\sigma}{2\varepsilon_0}\frac{x^i_0}{R}. \tag{131}$$

Substituting eqs. 131 and 130 into eq. 129, we find that the divergence of the energy-momentum tensor of the electron's electromagnetic field in its rest frame is:

$$\partial_{0_\nu} T^{\mu\nu}_{0_{\rm EM}} = \begin{cases} \mu = 0: & 0 \\ \mu = i: & -\dfrac{\sigma^2}{2\varepsilon_0}\dfrac{x^i_0}{R}\delta(R - r_0). \end{cases} \tag{132}$$

It vanishes everywhere except at the surface of the electron. To get a total energy-momentum tensor with a four-divergence that vanishes everywhere,

$$\partial_{0_\nu} T^{\mu\nu}_{0_{\rm tot}} = \partial_{0_\nu}\left(T^{\mu\nu}_{0_{\rm EM}} + T^{\mu\nu}_{0_{\rm non-EM}}\right) = 0, \tag{133}$$

we need to add the Poincaré pressure of eq. 111, which in the electron's rest frame is described by the energy-momentum tensor[52]

$$T^{\mu\nu}_{0_{\rm non-EM}} = -\eta^{\mu\nu} P_{\rm Poincar\acute{e}}\vartheta(R - r_0). \tag{134}$$

Using that $\eta^{ij}\partial_j\vartheta(R - r) = \delta(R - r)(x^i/R)$, we find that the four-divergence of this energy-momentum tensor is given by:

$$\partial_{0_\nu} T^{\mu\nu}_{0_{\rm non-EM}} = \begin{cases} \mu = 0: & 0 \\ \mu = i: & -P_{\rm Poincar\acute{e}}\dfrac{x^i_0}{R}\delta(R - r_0). \end{cases} \tag{135}$$

Inserting eq. 110 for the Poincaré pressure, using $U_{0_{\rm EM}} = e^2/8\pi\varepsilon_0 R$, $V_0 = \frac{4}{3}\pi R^3$, and $\sigma = e/4\pi R^2$, we find:

$$P_{\text{Poincaré}} = -\frac{U_{0_{\rm EM}}}{3V_0} = -\frac{e^2}{(8\pi\varepsilon_0 R)(4\pi R^3)} = -\frac{1}{2\varepsilon_0}\left(\frac{e}{4\pi R^2}\right)^2 = -\frac{\sigma^2}{2\varepsilon_0}. \tag{136}$$

This is the expression for the Poincaré pressure given, e.g., in (Lorentz, 1915, 214), (Schwinger, 1983, 376–377, eqs. (24) and (34)), and (Rohrlich, 1997, 1056, eq. (A.4)). Substituting this expression in eq. 135 and comparing the result with eq. 132, we see that the Poincaré pressure indeed ensures that the four-divergence of the electron's total energy-momentum tensor vanishes. The reader is invited to compare this straightforward and physically clearly motivated introduction of Poincaré pressure to (the streamlined version of) Poincaré's own derivation presented in sec. 5.

We now calculate the contributions of $T^{\mu\nu}_{\rm EM}$ and $T^{\mu\nu}_{\rm non-EM}$ to the electron's four-momentum. We begin with the contribution coming from the electron's electromagnetic field:

$$P^{\mu}_{\rm EM} = \frac{1}{c}\int T^{\mu 0}_{\rm EM} d^3x. \tag{137}$$

Using that $T^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma}_0$ and that $d^3x = d^3x_0/\gamma$, we can rewrite this as

$$P^{\mu}_{\rm EM} = \frac{1}{c\gamma}\Lambda^{\mu}{}_{\rho}\Lambda^{0}{}_{\sigma}\int T^{\rho\sigma}_{0_{\rm EM}} d^3x_0. \tag{138}$$

Eq. 128 tells us that there will only be contributions for $\rho\sigma = 00$ and $\rho\sigma = ij$. We denote these contributions as $P^{\mu}_{\rm EM}(00)$ and $P^{\mu}_{\rm EM}(ij)$.

For $P^{\mu}_{\rm EM}(00)$ we have:

$$P^{\mu}_{\rm EM}(00) = \frac{1}{c\gamma}\Lambda^{\mu}{}_{0}\Lambda^{0}{}_{0}\int T^{00}_{0_{\rm EM}} d^3x_0. \tag{139}$$

Since $\Lambda^{\mu}{}_{0} = (\gamma, \gamma\beta, 0, 0)$ (see eq. 7) and the integral over $T^{00}_{0_{\rm EM}}$ gives $U_{0_{\rm EM}}$, this turns into:

$$P^{\mu}_{\rm EM}(00) = \left(\gamma\frac{U_{0_{\rm EM}}}{c}, \gamma\frac{U_{0_{\rm EM}}}{c^2}\mathbf{v}\right). \tag{140}$$

This is just the Lorentz transform of $P^{\mu}_{0_{\rm EM}} = (U_{0_{\rm EM}}/c, 0, 0, 0)$. It is the additional contribution $P^{\mu}_{\rm EM}(ij)$, coming from $T^{ij}_{0_{\rm EM}}$, that is responsible for the fact that the four-momentum of the electron's electromagnetic field does not transform as a four-vector.

For $P^{\mu}_{\rm EM}(ij)$ we have:

$$P^{\mu}_{\rm EM}(ij) = \frac{1}{c\gamma}\Lambda^{\mu}{}_{i}\Lambda^{0}{}_{j}\int T^{ij}_{0_{\rm EM}} d^3x_0. \tag{141}$$

The integrand is minus the Maxwell stress tensor in the electron's rest frame (see eq. 127):

$$T^{ij}_{0_{\rm EM}} = -\varepsilon_0 \begin{pmatrix} E^2_{0_x} - \frac{1}{2}E^2_0 & E_{0_x}E_{0_y} & E_{0_x}E_{0_z} \\ E_{0_y}E_{0_x} & E^2_{0_y} - \frac{1}{2}E^2_0 & E_{0_y}E_{0_z} \\ E_{0_z}E_{0_x} & E_{0_z}E_{0_y} & E^2_{0_z} - \frac{1}{2}E^2_0 \end{pmatrix}. \tag{142}$$

The integrals over the off-diagonal terms are all zero. The integrals over the three diagonal terms are equal to one another and given by

$$\int \varepsilon_0 \left(\frac{1}{2}E^2_0 - \frac{1}{3}E^2_0 \right) d^3x_0 = \frac{1}{3}\int \frac{1}{2}\varepsilon_0 E^2_0 d^3x_0 = \frac{1}{3}U_{0_{\rm EM}}. \tag{143}$$

Since $(1/\gamma)\Lambda^\mu{}_1\Lambda^0{}_1 = (\gamma\beta^2, \gamma\beta, 0, 0)$ and $\Lambda^\mu{}_i\Lambda^0{}_i = 0$ for $i = 2, 3$ (see eq. 7), only the 11-component of eq. 141 is non-zero:

$$P^\mu_{\rm EM}(11) = \left(\frac{1}{3}\gamma\beta^2 \frac{U_{0_{\rm EM}}}{c}, \frac{1}{3}\gamma\frac{U_{0_{\rm EM}}}{c^2}\mathbf{v} \right). \tag{144}$$

Adding eqs. 140 and 144, we find:

$$P^\mu_{\rm EM} = P^\mu_{\rm EM}(00) + P^\mu_{\rm EM}(11) = \left(\gamma\left(1 + \frac{1}{3}\beta^2\right) \frac{U_{0_{\rm EM}}}{c}, \frac{4}{3}\gamma\frac{U_{0_{\rm EM}}}{c^2}\mathbf{v} \right). \tag{145}$$

This, unsurprisingly, is exactly the result we found earlier for the energy and momentum of the electromagnetic field of Lorentz's electron (see eqs. 57 and 66 with $l = 1$ and $U'_{\rm EM} = U_{0_{\rm EM}}$).

The calculation of the contributions to the four-momentum coming from $T^{\mu\nu}_{\rm non-EM}$ is completely analogous to the calculation in eqs. 137–145. We start with:

$$P^\mu_{\rm non-EM} = \frac{1}{c\gamma}\Lambda^\mu{}_\rho\Lambda^0{}_\sigma \int T^{\rho\sigma}_{0_{\rm non-EM}} d^3x_0. \tag{146}$$

Since $T^{\mu\nu}_{0_{\rm non-EM}}$ is diagonal (see eq. 134), there will only be contributions when $\rho = \sigma$. Since $\Lambda^0{}_\mu = (\gamma, \gamma\beta, 0, 0)$, the only contributions will be for $\rho = \sigma = 0$ and $\rho = \sigma = 1$. We denote these by $P^\mu_{\rm non-EM}(00)$ and $P^\mu_{\rm non-EM}(11)$, respectively, and calculate them separately. Since $T^{00}_{0_{\rm non-EM}} = -P_{\rm Poincaré}\vartheta(R - r_0)$ (see eq. 134) and $\int \vartheta(R - r_0) d^3x_0 = V_0$,

$$\int T^{00}_{0_{\rm non-EM}} d^3x_0 = -P_{\rm Poincaré}V_0 = \frac{1}{3}U_{0_{\rm EM}}, \tag{147}$$

where we used eq. 110. Hence,

$$P^\mu_{\rm non-EM}(00) = \left(\gamma\frac{1}{3}\frac{U_{0_{\rm EM}}}{c}, \gamma\frac{1}{3}\frac{U_{0_{\rm EM}}}{c^2}\mathbf{v} \right), \tag{148}$$

which is just the Lorentz transform of $P^{\mu}_{0_{\rm EM}} = \left(\frac{1}{3}U_{0_{\rm EM}}/c, 0, 0, 0\right)$. Similarly, we find:

$$P^{\mu}_{\rm non-EM}(11) = \left(-\frac{1}{3}\gamma\beta^2\frac{U_{0_{\rm EM}}}{c}, -\frac{1}{3}\gamma\frac{U_{0_{\rm EM}}}{c^2}\mathbf{v}\right). \tag{149}$$

Comparing eq. 149 to eq. 144, we see that $P^{\mu}_{\rm non-EM}(11)$ is exactly the opposite of $P^{\mu}_{\rm EM}(11)$:

$$P^{\mu}_{\rm EM}(11) + P^{\mu}_{\rm non-EM}(11) = 0. \tag{150}$$

This is a direct consequence of what is known as *Laue's theorem* (Miller, 1981, 352). This theorem (Laue, 1911a, 539) says that for a "complete [i.e., closed] static system" (*vollständiges statisches System*):

$$\int T^{ij}_{0_{\rm tot}} d^3x_0 = 0. \tag{151}$$

For the electron we have $T^{ij}_{0_{\rm tot}} = T^{ij}_{0_{\rm EM}} + T^{ij}_{0_{\rm non-EM}}$. From eqs. 142–143 we read off that

$$\int T^{ij}_{0_{\rm EM}} d^3x_0 = \begin{cases} i \neq j: & 0 \\ i = j: & \frac{1}{3}U_{0_{\rm EM}}. \end{cases} \tag{152}$$

In analogy with eq. 147, we find:

$$\int T^{ij}_{0_{\rm non-EM}} d^3x_0 = \begin{cases} i \neq j: & 0 \\ i = j: & -\frac{1}{3}U_{0_{\rm EM}}. \end{cases} \tag{153}$$

Laue's theorem thus holds for this system, as it should, and eq. 150 is a direct consequence of this. Using eqs. 138 and 146, we find

$$P^{\mu}_{\rm EM}(ij) + P^{\mu}_{\rm non-EM}(ij) = \frac{1}{c\gamma}\Lambda^{\mu}{}_{i}\Lambda^{0}{}_{j}\int \left(T^{ij}_{0_{\rm EM}} + T^{ij}_{0_{\rm non-EM}}\right) d^3x_0, \tag{154}$$

which by Laue's theorem vanishes, as is confirmed explicitly by eqs. 152–153. Since $P^{\mu}_{\rm EM}(ij) = P^{\mu}_{\rm non-EM}(ij) = 0$ except when $i = j = 1$, the sum of the 11-components considered in eq. 150 is equal to the sum of the ij-components.

Laue's theorem ensures that the four-momentum of a closed static system transforms as a four-vector. The total four-momentum of the electron is the sum of four terms (see eqs. 140, 144, 148, and 149):

$$P^{\mu}_{\rm tot} = P^{\mu}_{\rm EM}(00) + P^{\mu}_{\rm non-EM}(00) + P^{\mu}_{\rm EM}(ij) + P^{\mu}_{\rm non-EM}(ij). \tag{155}$$

The last two terms cancel each other because of Laue's theorem, and all that is left is:

$$P^{\mu}_{\rm tot} = P^{\mu}_{\rm EM}(00) + P^{\mu}_{\rm non-EM}(00). \tag{156}$$

Using eqs. 140 and 148 for these two contributions we recover eq. 119 for the total energy and momentum of the electron:

$$P^{\mu}_{\text{tot}} = \left(\gamma \frac{4}{3}\frac{U_{0_{\text{EM}}}}{c}, \gamma \frac{4}{3}\frac{U_{0_{\text{EM}}}}{c^2}\mathbf{v}\right) = \left(\gamma \frac{U_{0_{\text{tot}}}}{c}, \gamma \frac{U_{0_{\text{tot}}}}{c^2}\mathbf{v}\right). \tag{157}$$

As we pointed out above (see eqs. 133–134 and note 52), we still have a closed system if we set the 00-component of $T^{\mu\nu}_{0_{\text{non-EM}}}$ to zero. This does not affect the result for $P^{\mu}_{\text{non-EM}}(ij)$, which only depends on the ij-components of $T^{\mu\nu}_{0_{\text{non-EM}}}$. $P^{\mu}_{\text{non-EM}}(00)$, however, will be zero if $T^{00}_{0_{\text{non-EM}}} = 0$ (see eq. 146). The total four-momentum will still be a four-vector but compared to eq. 157 the system's rest energy will be smaller by $\frac{1}{3}U_{0_{\text{EM}}}$:

$$P^{\mu}_{\text{tot}} = P^{\mu}_{\text{EM}}(00) = \left(\gamma \frac{U_{0_{\text{EM}}}}{c}, \gamma \frac{U_{0_{\text{EM}}}}{c^2}\mathbf{v}\right). \tag{158}$$

To reiterate: if the stabilizing mechanism for the electron does *not* contribute to the energy in the rest frame but only to the stresses, $T^{00}_{0_{\text{non-EM}}} = 0$ and only the first term in eq. 156 contributes to the four-momentum. In this case, the electron's rest mass is $U_{0_{\text{EM}}}/c^2$ (see eq. 158). If the stabilizing mechanism *does* contribute to the energy in the electron's rest frame, $T^{00}_{0_{\text{non-EM}}} \neq 0$ and both terms in eq. 156 contribute to the four-momentum. If $T^{00}_{0_{\text{non-EM}}} = \frac{1}{3}\left(U_{0_{\text{EM}}}/V_0\right)\vartheta(R - r_0)$, as in Poincaré's specific model (see eqs. 134 and 110), the electron's rest mass is $\frac{4}{3}U_{0_{\text{EM}}}/c^2$ (see eq. 157).[53]

The arbitrariness of the Lorentz-Poincaré electron is much greater than the freedom we have in choosing the 00-component of the energy-momentum tensor for the mechanism stabilizing a spherical surface charge distribution. For starters, we can choose a (surface or volume) charge distribution of any shape we like—a box, a doughnut, a banana, etc. As long as this charge distribution is subject to the Lorentz-FitzGerald contraction, we can turn it into a system with the exact same energy-momentum-mass-velocity relations as the Lorentz-Poincaré electron by adding the appropriate non-electromagnetic stabilizing mechanism.[54] Of course, as the analysis in this section, based on (Laue, 1911a), shows, any closed static system will have the same energy-momentum-mass-velocity relations as the Lorentz-Poincaré electron, no matter whether it consists of charges, electromagnetic fields, and Poincaré pressure or of something else altogether. The only thing that matters is that whatever the electron is made of satisfies Lorentz-invariant laws. The restriction to *static* closed systems, moreover, is completely unnecessary. Any closed system will do.[55] In short, there is nothing we can learn about the nature and structure of the electron from studying its energy-momentum-mass-velocity relations.

Lorentz himself emphasized this in lectures he gave at Caltech in 1922. In a section entitled "Structure of the Electron" in the book based on these lectures and published in 1927, he wrote:

> The formula for momentum was found by a theory in which it was supposed that in the case of the electron the momentum is determined wholly by that of the electromagnetic field [...] This meant that the whole mass of an electron was supposed to be of electromagnetic nature. Then, when the formula for momentum was verified by experiment, it was thought at first that it was thereby proved that electrons have no "material mass." Now we can no longer say this. Indeed, the formula for momentum is a general consequence of the principle of relativity, and a verification of that formula is a verification of the principle and tells us nothing about the nature of mass or of the structure of the electron. (Lorentz, 1927, 125).

By 1922 this point was widely appreciated. In his famous review article on relativity, Pauli (1921, 82–83), for instance, wrote:

> It constituted a definite progress that Lorentz's law of the variability of mass could be derived from the theory of relativity without making any specific assumptions on the electron shape or charge distribution. Also nothing need be assumed about the nature of the mass: [the relativistic formula for the velocity-dependence of mass] is valid for every kind of ponderable mass [...] The old idea that one could distinguish between the "constant" true mass and the "apparent" electromagnetic mass, by means of deflection experiments on cathode rays, can therefore not be maintained.

7. FROM THE ELECTROMAGNETIC VIEW OF NATURE TO RELATIVISTIC CONTINUUM MECHANICS

Experiment was supposed to be the final arbiter in the debate over the electron models of Abraham, Lorentz-Poincaré, and Bucherer-Langevin. Later analysis, however, showed that the results of the experiments of Kaufmann and others were not accurate enough to decide between the different models. They only "indicated a large qualitative increase of mass with velocity" (Zahn and Spees, 1938).[56] All parties involved took these experiments much too seriously, especially when the data favored their own theories. Abraham hyped Kaufmann's results. Lorentz was too eager to believe Bucherer's results, while his earlier concern over Kaufmann's appears to have been somewhat disingenuous. Einstein's cavalier attitude toward Kaufmann's experiments stands in marked contrast to his belief in later results purporting to prove him right.

In Abraham's defense, it should be said that he could also be self-deprecating about his reliance on Kaufmann's data. At the 78th *Versammlung Deutscher Naturforscher und Ärzte* in Stuttgart in 1906, he got quite a few laughs when he joked: "When you look at the numbers you conclude from them that the deviations from the Lorentz theory are at least twice as big as mine, so you may say that the [rigid] sphere theory represents the reflection of β-rays twice as well as the relativity theory [by which Abraham meant Lorentz's electron model in this context]" (quoted in Miller, 1981, sec. 7.4.3, 221).

In 1906 Lorentz gave a series of lectures at Columbia University in New York, which were published in 1909. On the face of it, he seems to have taken Kaufmann's results quite seriously at the time. He wrote: "His [i.e., Kaufmann's] new numbers agree within the limits of experimental errors with the formulae given by Abraham, but [. . .] are decidedly unfavourable to the idea of a contraction such as I attempted to work out" (Lorentz, 1915, 212–213; quoted in Miller, 1981, sec. 12.4.1). Shortly before his departure for New York, he had told Poincaré the same thing: "Unfortunately my hypothesis of the flattening of electrons is in contradiction with Kaufmann's results, and I must abandon it. I am therefore at the end of my rope (*au bout de mon latin*)."[57] These passages strongly suggest that Lorentz took Kaufmann's results much more seriously than Einstein. Miller indeed draws that conclusion. Lorentz expert A. J. Kox, however, has pointed out to one of us (MJ) that Lorentz's reaction was probably more ambivalent (see also Hon, 1995, sec. 6). This is suggested by what Lorentz continues to say after acknowledging the problem with Kaufmann's data in his New York lectures: "Yet, though it seems very likely that we shall have to relinquish this idea altogether, *it is, I think, worth while looking into it somewhat more closely*" (Lorentz, 1915, 213; our italics). Lorentz then proceeds to discuss his idea *at length*.

In response to Kaufmann's alleged refutation of special relativity Einstein wrote in an oft-quoted passage:[58]

> Abraham's and Bucherer's theories of the motion of the electron yield curves that are significantly closer to the observed curve than the curve obtained from the theory of relativity. However, the probability that their theories are correct is rather

> small, in my opinion, because their basic assumptions concerning . . . the moving electron are not suggested by theoretical systems that encompass larger complexes of phenomena (Einstein, 1907b, 439).

This is a fair assessment of Bucherer's theory. Whether it is also a fair assessment of Abraham's electromagnetic program is debatable. This will not concern us here. What we want to point out is that Einstein, like Abraham and Lorentz, took the experimental data much more seriously when they went his way. In early 1917, Friedrich Adler, detained in Vienna awaiting trial for his assassination of the Austrian prime minister Count Stürgkh in November 1916, began sending Einstein letters and manuscripts attacking special relativity.[59] He was still at it in the fall of 1918, when the exchange that is interesting for our purposes took place. Einstein wrote: "for a while Bucherer advocated a theory that comes down to a different choice for l [see eq. 40 and Fig. 1]. But a different choice for l is out of the question now that the laws of motion of the electron have been verified with great precision."[60] From his prison cell in Stein an der Donau Adler replied: "Now, I would be very interested to hear, *which* experiments you see as definitively decisive about the laws of motion of the electron. For as far as my knowledge of the literature goes, I have not found any claim of a final decision."[61] Adler went on to quote remarks from Laue, Lorentz, and the experimentalist Erich Hupka, spanning the years 1910–1915, all saying that this was still an open issue.[62] In his response Einstein cited three recent studies (published between 1914 and 1917), which, he wrote, "have so to speak *conclusively shown* [*sicher bewiesen*] that the relativistic laws of motion of the electron apply (as opposed to, for instance, those of Abraham)" (Einstein's emphasis).[63] Even considering the context in which it was made, this is a remarkably strong statement.

Much more interesting than the agreement between theory and experiment or the lack thereof were the theoretical arguments that Abraham and Lorentz put forward in support of their models. Lorentz was right in thinking that it was no coincidence that his contractile electron exhibited exactly the velocity dependence he needed to account for the absence of ether drift (see the discussion following eq. 72). He could not have known at the time that this particular velocity dependence is a generic feature of relativistic closed systems. As the quotation at the end of sec. 6 shows, he did recognize this later on. Abraham was right that fast electrons call for a new mechanics. His new electromagnetic mechanics is much closer to relativistic mechanics than to Newtonian mechanics. Like Lorentz, he just did not realize that this new mechanics reflected a new kinematics rather than the electromagnetic nature of all matter. Abraham at least came to accept that Minkowski space-time was the natural setting for his electromagnetic program.

Proceeding along similar lines as Abraham in developing his electromagnetic mechanics, we can easily get from Newtonian particle mechanics to relativistic continuum mechanics and back again. The first step is to read $\mathbf{F} = m\mathbf{a}$ as expressing momentum conservation (cf. the discussion following eq. 15 in sec. 2.2). In continuum mechanics, the differential form of the conservation laws is the fundamental law and the integral form is a derived law. In other words, the fundamental conservation laws are expressed in local rather than global terms. This reflects the transition from a particle ontology to a field ontology. Special relativity integrates the laws of momentum and energy conservation. These laws, of course, are Lorentz-invariant rather than Galilean-invariant. We thus arrive at the fundamental law of relativistic continuum mechanics, the Lorentz-invariant differential law of energy-momentum conservation, $\partial_\nu T^{\mu\nu} = 0$. To recap: there are four key elements in the transition from Newtonian particle mechanics based on $\mathbf{F} = m\mathbf{a}$ to relativistic continuum mechanics based on $\partial_\nu T^{\mu\nu} = 0$. They are (in no particular order): the transition from Galilean invariance to Lorentz invariance, the focus on conservation laws rather than force laws, the integration of the laws of energy and momentum conservation, and the transition from a particle ontology to a field ontology.

We now show how, once we have relativistic continuum mechanics, we recover Newtonian particle mechanics. Consider a closed system described by continuous (classical) fields such that the total energy-momentum tensor $T^{\mu\nu}_{\text{tot}}$ of the system can be split into a part describing a localizable particle (e.g., an electron à la Lorentz-Poincaré[64]) and a part describing its environment (e.g., an external electromagnetic field):

$$T^{\mu\nu}_{\text{tot}} = T^{\mu\nu}_{\text{par}} + T^{\mu\nu}_{\text{env}}. \tag{159}$$

Using our fundamental law, $\partial_\nu T^{\mu\nu}_{\text{tot}} = 0$, integrated over space, we find

$$0 = \int \partial_\nu T^{\mu\nu}_{\text{tot}} d^3x = \int \partial_\nu T^{\mu\nu}_{\text{par}} d^3x + \int \partial_\nu T^{\mu\nu}_{\text{env}} d^3x. \tag{160}$$

As long as $T^{\mu\nu}_{\text{par}}$ drops off faster than $1/r^2$ as we go to infinity, Gauss's theorem tells us that

$$\int \partial_i T^{\mu i}_{\text{par}} d^3x = 0. \tag{161}$$

For $\partial_\nu T^{\mu\nu}_{\text{env}}$ we can substitute minus the density f^{μ}_{ext} of the four-force acting on the particle. The spatial components of eq. 160 can thus be written as

(162) $$\frac{d}{dx_0}\int T^{i0}_{\text{par}} d^3x = \int f^{i}_{\text{ext}} d^3x.$$

The right-hand side gives the components of $\mathbf{F}_{\text{external}}$. Since

(163) $$P^{\mu}_{\text{par}} \equiv \frac{1}{c}\int T^{\mu 0}_{\text{par}} d^3x$$

and $x^0 = ct$, the left-hand side is the time derivative of the particle's momentum. Eq. 162 is thus equivalent to

(164) $$\frac{d\mathbf{P}_{\text{par}}}{dt} = \mathbf{F}_{\text{ext}}.$$

This equation has the same form (and the same transformation properties) as Abraham's electromagnetic equation of motion 21. In Abraham's equation, $\mathbf{P}_{\text{par}}$ is the electromagnetic momentum of the electron, and $\mathbf{F}_{\text{ext}}$ is the Lorentz force exerted on the electron by the external fields. Under the appropriate circumstances and with the appropriate identification of the Newtonian mass m, Abraham's electromagnetic equation of motion reduces to Newton's second law, $\mathbf{F} = m\mathbf{a}$ (see eq. 28). The same is true for our more general eq. 164. This equation, however, is not tied to electrodynamics. It is completely agnostic about the nature of both the particle and the external force. The only thing that matters is that it describes systems in Minkowski space-time, which obey relativistic kinematics. $\mathbf{P}_{\text{par}}$ and $\mathbf{F}_{\text{ext}}$, like Abraham's electromagnetic momentum and the Lorentz force, only transform as vectors under Galilean transformations in the limit of low velocities, where Lorentz transformations are indistinguishable from Galilean transformations. They inherit their transformation properties from $\partial_\nu T^{\mu\nu}_{\text{par}}$ and f^{μ}_{ext}, respectively, which transform as four-vectors under Lorentz transformations.

It only makes sense to split the total energy-momentum tensor $T^{\mu\nu}_{\text{tot}}$ into a particle part and an environment part, if the interactions holding the particle together are much stronger than the interactions of the particle with its environment. Typically, therefore, the energy-momentum of the particle taken by itself will very nearly be conserved, i.e.,

(165) $$\partial_\nu T^{\mu\nu}_{\text{par}} \approx 0.$$

This means that the particle's four-momentum will to all intents and purposes transform as a four-vector under Lorentz transformations and satisfy the relations for a strictly closed system (see eqs. 4–13):

(166) $$P^{\mu}_{\text{par}} \equiv \frac{1}{c}\int T^{\mu 0}_{\text{par}} d^3x \approx (\gamma m_0 c, \gamma m_0 \mathbf{v}).$$

Inserting $\mathbf{P}_{\text{par}} = \gamma m_0 \mathbf{v}$ into eq. 164, we can reduce the problem in relativistic continuum mechanics that we started from in eq. 159 to a problem in the relativistic mechanics of point particles. In the limit of small velocities, such problems once again reduce to problems in the Newtonian mechanics of point particles.

To the best of our knowledge, this way of recovering particle mechanics from what might be called 'field mechanics' was first worked out explicitly in the context of general rather than special relativity (Einstein, 1918; Klein, 1918).[65] Relativistic continuum mechanics played a crucial role in the development of general relativity. For one thing, the energy-momentum tensor is the source of the gravitational field in general relativity.[66] Even before the development of general relativity, Einstein recognized the importance of relativistic continuum mechanics. In an unpublished manuscript of 1912, he wrote:

> The general validity of the conservation laws and of the law of the inertia of energy [. . .] suggest that [the symmetric energy-momentum tensor $T^{\mu\nu}$ and the equation $f^{\mu} = -\partial_{\nu} T^{\mu\nu}$] are to be ascribed a general significance, even though they were obtained in a very special case [i.e., electrodynamics]. We owe this generalization, *which is the most important new advance in the theory of relativity*, to the investigations of Minkowski, Abraham, Planck, and Laue (Einstein, 1987–2002, Vol. 4, Doc. 1, [p. 63]; our emphasis).

Einstein went on to give a clear characterization of relativistic continuum mechanics:

> To every kind of material process we want to study, we have to assign a symmetric tensor $(T_{\mu\nu})$ [. . .] Then [$f^{\mu} = -\partial_{\nu} T^{\mu\nu}$] must always be satisfied. The problem to be solved always consists in finding out how $(T_{\mu\nu})$ is to be formed from the variables characterizing the processes under consideration. If several processes can be isolated in the energy-momentum balance that take place in the same region, we have to assign to each individual process its own stress-energy tensor $(T^{(1)}_{\mu\nu})$, etc., and set $(T_{\mu\nu})$ equal to the sum of these individual tensors (ibid.).

As the development of general relativity was demonstrating the importance of continuum mechanics, developments in quantum theory—the Bohr model and Sommerfeld's relativistic corrections to it—rehabilitated particle mechanics, be it of Newtonian or relativistic stripe. As a result, relativistic continuum mechanics proved less important for subsequent developments in areas of physics other than general relativity than Einstein thought in 1912 and than our analysis in this paper suggests. The key factor in this was that it gradually became clear in the 1920s that elementary particles are point-like and not spatially extended like the electron models discussed in this paper. That special relativity precludes the existence of rigid bodies is just one of the problems such models are facing.

In hindsight, Lorentz, the guarded Dutchman, comes out looking much better than Abraham, his impetuous German counterpart. At one point, for instance, Lorentz (1915, 215) cautioned:

> In speculating on the structure of these minute particles we must not forget that there may be many possibilities not dreamt of at present; it may very well be that other internal forces serve to ensure the stability of the system, and perhaps, after all, we are wholly on the wrong track when we apply to the parts of an electron our ordinary notion of force (Lorentz, 1915, 215).

This passage is quoted approvingly by Pais (1972, 83). Even a crude operationalist argument of the young Wolfgang Pauli, which would have made his godfather Ernst Mach proud, can look prescient in hindsight. Criticizing the work of later proponents of the electromagnetic worldview in his review article on relativity, Pauli concluded:

> Finally, a conceptual doubt should be mentioned. The continuum theories make direct use of the ordinary concept of electric field strength, even for the fields in the interior of the electron. This field strength, however, is defined as the force acting on a test particle, and since there are no test particles smaller than an electron or a hydrogen nucleus the field strength at a given point in the interior of such a particle would seem to be unobservable by definition, and thus be fictitious and without physical meaning (Pauli, 1921, 206).

This moved Valentin Bargmann (1960, 189)—who had accompanied Einstein on his quest for a classical unified field theory, a quest very much in the spirit of Abraham's electromagnetic program—to write in the Pauli memorial volume:

> A physicist will feel both pride and humility when he reads Pauli's remarks today. In the light of our present knowledge the attempts which Pauli criticizes may seem hopelessly naïve, although it was certainly sound practice to investigate what the

> profound new ideas of general relativity would contribute to the understanding of the thorny problem of matter (Bargmann, 1960, 189).

Putting such hagiography to one side, we conclude our paper by quoting and commenting on two oft-quoted passages that nicely illustrate some of the key points of our paper. The first is a brief exchange between Planck and Sommerfeld following a lecture by the former at the *Naturforscherversammlung* in Stuttgart on September 19, 1906.[67] Planck talked about "[t]he Kaufmann measurements of the deflectability of β-rays and their relevance for the dynamics of electrons." Abraham, Bucherer,[68] Kaufmann, and Sommerfeld all took part in the discussion afterwards. It was Planck who got to the heart of the matter:

> Abraham is right when he says that the essential advantage of the sphere theory would be that it be a purely electrical theory. If this were feasible, it would be very beautiful indeed, but for the time being it is just a postulate. At the basis of the Lorentz-Einstein theory lies another postulate, namely that no absolute translation can be detected. These two postulates, it seems to me, cannot be combined, and what it comes down to is which postulate one prefers. My sympathies actually lie with the Lorentzian postulate (Planck, 1906b, 761).

Whereupon Sommerfeld, pushing forty, quipped: "I suspect that the gentlemen under forty will prefer the electrodynamical postulate, while those over forty will prefer the mechanical-relativistic postulate" (Ibid.). The reaction of the assembled physicists to Sommerfeld's quick retort has also been preserved in the transcript of this session: "hilarity" (*Heiterkeit*). This exchange between Planck and Sommerfeld is perhaps the clearest statement in the contemporary literature of the dilemma that lies behind the choice between the electron models of Abraham and Lorentz. Physicists had to decide what they thought was more important, full relativity of uniform motion or the reduction of mechanics to electrodynamics. We find it very telling that in 1906 a leader in the field such as Sommerfeld considered the former the conservative and the latter the progressive option. Unlike Abraham, Lorentz, and Planck, however, Sommerfeld did not fully appreciate what was at stake.

First of all, his preference for the "electrodynamical postulate" was mainly because Lorentz's contractile electron was incompatible with superluminal velocities.[69] This can be inferred from a comment on Lorentz's electron model in (Sommerfeld, 1904c). In this paper—translated into Dutch by Peter Debye, Sommerfeld's student at the time (Eckert and Märker, 2000, 148), and communicated to the Amsterdam Academy of Sciences by Lorentz himself—Sommerfeld summarized and simplified his trilogy on electron theory in the proceedings of the Göttingen Academy (Sommerfeld, 1904a, 1904b, 1905a). He wrote:

> As is well-known, Lorentz, for very important reasons, has recently formulated the hypothesis that the shape of the electron is variable, i.e., that for every velocity the electron takes on the shape of a so-called "Heaviside ellipsoid." For velocities greater than that of light this hypothesis cannot be used; one can hardly speak of a "Heaviside hyperboloid" as the shape of the electron (Sommerfeld, 1904c, 433).

Sommerfeld's objections to Lorentz's program were thus not nearly as principled as Abraham's (cf. the passages from Abraham, 1903, quoted in sec. 4.4).

Moreover, from letters he wrote to Wien and Lorentz in November and December of 1906 (letters 102 and 103 in Eckert and Märker, 2000) it appears that Sommerfeld only became familiar with Einstein's work *after* the meeting in Stuttgart. On December 12, 1906, he wrote to Lorentz:

> Meanwhile I have also studied Einstein. It is remarkable to see how he arrives at the exact same results as you do (also with respect to his relative time) despite his very different epistemological point of departure. However, his deformed time, like your deformed electron, does not really sit well with me (Eckert and Märker, 2000, 258).

This passage suggests that Sommerfeld had not read (Einstein, 1905) before the 1906 *Naturforscherversammlung*. So Sommerfeld may not even have realized at the time that there was at least one gentleman well under forty, albeit one not in attendance in Stuttgart, who preferred the "mechanical-relativistic postulate," nor that the mechanics involved need not be Newtonian. By the time of the next *Naturforscherversammlung*, the following year in Dresden, Sommerfeld (1907), still only 39, had jumped ship and had joined the relativity camp (Battimelli, 1981, 150, note 30).[70] In an autobiographical sketch written in 1919, Sommerfeld ruefully looks back on this whole episode. Referring to the trilogy (Sommerfeld, 1904a, 1904b, 1905a), he wrote: "The last of these appeared in the critical year 1905, the birth year of relativity. These difficult and protracted studies, to which I originally attached great value, were therefore condemned to fruitlessness" (Sommerfeld, 1968, Vol. 4, 677).[71]

The second passage that we want to look at comes from Lorentz's important book *The Theory of Electrons*, based on his 1906 lectures in New

York and first published in 1909. Referring to Einstein and special relativity, Lorentz wrote

> His results concerning electromagnetic and optical phenomena (leading to the same contradiction with Kaufmann's results that was pointed out in §179[72]) agree in the main with those which we have obtained in the preceding pages, the chief difference being that Einstein simply postulates what we have deduced, with some difficulty and not altogether satisfactorily, from the fundamental equations of the electromagnetic field. (Lorentz, 1915, 229–230).

The parenthetical reference to "Kaufmann's results" suggests that the famous clause that concludes this sentence—"Einstein simply postulates what we have deduced [...] from the fundamental equations of the electromagnetic field"—refers, at least in part, to Lorentz's own struggles with the velocity dependence of electron mass.[73] The relativistic derivation of these relations is mathematically equivalent to Lorentz's 1899 derivation of them from the requirement, formally identical to the relativity principle, that ether drift can never be detected (see sec. 3, eqs. 45–51). From Lorentz's point of view, the relativistic derivation therefore amounted to nothing more than postulating these relations on the basis of the relativity principle. Lorentz himself had gone to the trouble of producing a concrete model of the electron such that its mass exhibited exactly the desired velocity-dependence (see sec. 4, eqs. 67–73). As we saw at the end of sec. 6, by 1922, if not much earlier, Lorentz had recognized that this had led him on a wild goose chase: "the formula for momentum [of which those for the velocity dependence of mass are a direct consequence] is a general consequence of the principle of relativity [...] and tells us nothing about the nature of mass or of the structure of the electron." This was Lorentz's way of saying what Pais said in the quotation with which we began this paper.

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NOTES

[1]For discussion of and references to the experimental literature, we refer to (Miller, 1981), (Cushing, 1981), and (Hon, 1995). (Pauli, 1921, 83) briefly discusses some of the later experiments. See also (Gerlach, 1933), a review article on electrons first published in the late 1920s.

[2]Moreover, classical electron models have continued to attract attention from (distinguished) physicists (see note 12 below). In addition, the notion of "Poincaré pressure" introduced to stabilize Lorentz's electron (see below) resurfaced in a theory of Einstein (1919), which is enjoying renewed interest (Earman, 2003), as well as in other places (see, e.g., Grøn, 1985, 1988).

[3]See (Sommerfeld, 1904a, 1904b, 1904c, 1905a, 1905b) and (Herglotz, 1903). For a discussion of the development of Sommerfeld's attitude toward the electromagnetic program and special relativity, see (McCormmach, 1970, 490) and (Walter 1999a, 69–73, Forthcoming, sec. 3). On Minkowski and the electromagnetic program, see (Galison, 1979), (Pyenson, 1985, Ch. 4), (Corry, 1997), and (Walter, 1999a, 1999b, Forthcoming).

[4]This particular history of the electron is conspicuously absent, however, from the collection of histories of the electron brought together in (Buchwald and Warwick, 2001). One of us (MJ) bears some responsibility for that and hopes to make amends with this paper.

[5]See also (Darrigol, 2000). We refer to (Janssen, 1995, 2002b) for references to and discussion of earlier literature on this topic.

[6]For more recent commentary, see (Corry, 1999).

[7]As Born explains in introductory comments to the reprint of (Born, 1909b) in a volume with a selection of his papers (Born, 1965, Vol. 1, XIV–XV). For a brief discussion of the debate triggered by Born's work and references to the main contributions to this debate, see the editorial note, "Einstein on length contraction in the theory of relativity," in (Einstein, 1987–2002, Vol. 3, 478–480).

[8]For a brief discussion of this acrimonious exchange, see (Miller, 1981, sec. 1.13.1, especially notes 57 and 58).

[9]For brief discussions, see (Balazs, 1972, 29–30) and (Warwick, 2003, Ch. 8, especially 413–414). We also refer to Warwick's work for British reactions to the predominantly German developments discussed in our paper. See, e.g., (Warwick, 2003, 384) for comments by James Jeans on electromagnetic mass.

[10]For a brief discussion, see (Balazs, 1972, 30)

[11]See also (Cuvaj, 1968). We have benefited from (annotated) translations of Poincaré's paper by Schwartz (1971, 1972) and Kilmister (1970), as well as from the translation of passages from (Poincaré, 1905), the short version of (Poincaré, 1906), by Keswani and Kilmister (1983). A new translation of parts of (Poincaré, 1906) by Scott Walter will appear in (Renn, Forthcoming).

[12]See (Rohrlich, 1960, 1965, 1970, 1997). See also, e.g., (Fermi, 1921, 1922) [cf. note 20 below], (Wilson, 1936), (Dirac, 1938), (Kwal, 1949), (Caldirola, 1956), (Zink, 1966, 1968, 1971), (Pearle, 1982), (Schwinger, 1983) [in a special issue on

the occasion of Dirac's 80th birthday], (Comay, 1991), (Yaghjian, 1992), (Moylan, 1995), and (Hnizdo, 1997).

[13]The letter U rather than E is used for energy to avoid confusion with the electric field. We shall be using SI units throughout. For conversion to other units, see, e.g., (Jackson, 1975, 817–819).

[14]From $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = (c^2 - v^2)dt^2$ it follows that $ds = c\sqrt{1 - v^2/c^2}dt = cdt/\gamma$.

[15]The energy-momentum tensor is typically symmetric. In that case, $T^{i0} = T^{0i}$, which means that the momentum density (T^{i0}/c) equals the energy flow density (cT^{0i}) divided by c^2. As was first noted by Planck, this is one way of expressing the inertia of energy, $E = mc^2$.

[16]Which is why $T^{\mu\nu}$ is also known as the stress-energy tensor or the stress-energy-momentum tensor.

[17]See (Rohrlich, 1965, 89–90, 279–281) or (Janssen, 1995, sec. 2.1.3) for the details of the proof, which is basically an application of the obvious generalization of Gauss's theorem (which says that for any vector field $\mathbf{A}$, $\oint \mathbf{A} \cdot d\mathbf{S} = \int \mathrm{div}\,\mathbf{A}\, d^3x$) from three to four dimensions.

[18]This way of writing P^μ was suggested to us by Serge Rudaz. See (Janssen, 2002b, 440–441, note; 2003, 47) for a more geometrical way of stating the argument below.

[19]As Gordon Fleming (private communication) has emphasized, the rest frame cannot always be uniquely defined. For the systems that will concern us here, this is not a problem. Following Fleming, one can avoid the arbitrary choice of n^μ altogether by accepting that the four-momentum of spatially extended systems is a hyperplane-dependent quantity.

[20]Some of Fermi's earliest papers are on this issue (Miller, 1973, 317). We have not been able to determine what sparked Fermi's interest in this problem. His biographer only devotes one short paragraph to it: "In January 1921, Fermi published his first paper, "On the Dynamics of a Rigid System of Electrical Charges in Translational Motion" [Fermi, 1921]. This subject is of continuing interest; Fermi pursued it for a number of years and even now it occasionally appears in the literature" (Segrè, 1970, 21).

[21](Rohrlich, 1965, 17) notes that Fermi's idea was forgotten and independently rediscovered at least three times, by W. Wilson (1936), by Bernard Kwal (1949), and then by Rohrlich himself (Rohrlich, 1960). This goes to show that John Stachel's meta-theorem—anything worth discovering once in general relativity has been discovered at least twice—also holds for special relativity. In the preface to the second edition of his textbook on special relativity, Aharoni (1965) cites (Rohrlich, 1960) as the motivation for some major revisions of the first edition, published in 1959. In this same preface, Aharoni lists Dirac (1938) and Kwal as rediscoverers. For a concise exposition of Rohrlich's work, see (Aharoni, 1965, sec. 5.5, 160–165).

[22]See (Abraham, 1902a, 25–26; 1903, 110). In both places, he cites (Poincaré, 1900) for the basic idea of ascribing momentum to the electromagnetic field. For

discussion, see (Miller, 1981, sec. 1.10), (Darrigol, 1995; 2000, 361), and (Janssen, 2003, sec. 3)

[23] In fact, another force, a stabilizing force $\mathbf{F}_{\text{stab}}$, needs to be added to keep the charges from flying apart under the influence of their Coulomb repulsion.

[24] See, e.g., (Lorentz, 1904a, sec. 7), (Abraham, 1905, sec. 5), (Jackson, 1975, 238–239), (Janssen, 1995, 56–58), (Griffith, 1999, 351–352). In special relativity, we would write eq. 19 as the integral over the spatial components of the Lorentz four-force density f^{μ}, which is equal to minus $\partial_{\nu} T^{\mu\nu}_{\text{EM}}$, the four-divergence of the energy-momentum tensor for the electron's self-field,

$$F^{i}_{\text{self}} = -\int \partial_{\nu} T^{i\nu}_{\text{EM}}\, d^3x,$$

with $T^{i0}_{\text{EM}} \equiv c\varepsilon_0 (\mathbf{E}\times\mathbf{B})^i$ and $T^{ij}_{\text{EM}} \equiv -T^{ij}_{\text{Maxwell}}$ (cf. eqs. 127 and 129 and note 51).

[25] This assumption may sound innocuous, but under the standard definition 12 of the four-momentum of spatially extended systems, the (ordinary three-)momentum of open systems will in general not be in the direction of motion. Because both Lorentz's and Abraham's electrons are symmetric around an axis in the direction of motion, the momentum of their self-fields is always in the direction of motion, even though these fields by themselves do not constitute closed systems. If a system has momentum that is not in the direction of motion, it will be subject to a turning couple trying to align its momentum with its velocity. Trouton and Noble (1903) tried in vain to detect this effect on a charged capaticor hanging from the ceiling of their laboratory on a torsion wire (cf. Janssen, 2002b, 440–441, note, and Janssen, 1995, especially secs. 1.4.2 and 2.2.5). Ehrenfest (1907) raised the question whether the electron would be subject to a turning couple if it were *not* symmetric around the axis in the direction of motion. Einstein (1907a) countered that the behavior of the electron would be independent of its shape. This exchange between Einstein and Ehrenfest is discussed in (Miller, 1981, sec. 7.4.4.). Laue (1911a) proved Einstein right (see also Pauli, 1921, 186–187). As with the capacitor in the Trouton-Noble experiment, the electromagnetic momentum of the electron is not the only momentum of the system. The non-electromagnetic part of the system also contributes to its momentum. Laue showed that the total momentum of a closed static system is always in the direction of motion. From a modern point of view this is because the four-momentum of a closed system (static or not) transforms as a four-vector under Lorentz transformations. The momenta of open systems, such as the subsystems of a closed static system, need not be in the direction of motion, in which case the system is subject to equal and opposite turning couples. A closed system never experiences a net turning couple. The turning couples on open systems, it turns out, are artifacts of the standard definition 12 of the four-momentum of spatially extended systems. Under the alternative Fermi-Rohrlich definition (see the discussion following eq. 14), there are no turning couples whatsoever (see Butler, 1968; Janssen, 1995; and Teukolsky, 1996).

[26] Substituting the momentum, $p = mv$, of Newtonian mechanics for P_{EM} in eq. 26, we find $m_{//} = m_{\perp} = m$.

[27]But recall that there should be an additional term, $\mathbf{F}_{\text{stab}}$, on the right-hand side of eq. 16 (see note 23).

[28] Substituting the kinetic energy, $U_{\text{kin}} = \frac{1}{2}mv^2$, of Newtonian mechanics for U_{EM} in eq. 31, we find $m_{//} = m$, in accordance with the result found on the basis of eq. 26 and $p = mv$ (see note 26).

[29]The converse is not true. For the electron model of Bucherer and Langevin (see sec. 4) $(U_{\text{EM}}/c, \mathbf{P}_{\text{EM}})$ is not a four-vector, yet U_{EM} and $\mathbf{P}_{\text{EM}}$ give the same longitudinal mass $m_{//}$ (see eqs. 74–77). The same is true for the Newtonian energy $U_{kin} = \frac{1}{2}mv^2$ and the Newtonian momentum $p = mv$ (see notes 26 and 28).

[30]The first relation follows from $\dfrac{d\gamma^{-2}}{dv} = \dfrac{d}{dv}\left(1 - \dfrac{v^2}{c^2}\right)$ or $-2\gamma^{-3}\dfrac{d\gamma}{dv} = -2\dfrac{v}{c^2}$; the second is found with the help of the first:

$$\frac{d(\gamma v)}{dv} = \gamma + v\frac{d\gamma}{dv} = \gamma + \gamma^3\beta^2 = \gamma^3(1 - \beta^2 + \beta^2) = \gamma^3.$$

[31]For more extensive discussion, see (Janssen, 1995, Ch. 3; 2002b; Janssen and Stachel, 2004).

[32]For the magnetic field it is the motion of charges with respect to the ether that matters, not the motion with respect to the lab frame.

[33]For the induced **E** and **B** fields it is the changes in the **B** and **E** fields at fixed points in the ether that matter, not the changes at fixed points in the lab frame.

[34]Lorentz only started using the relativistic transformation formula for non-static charge densities and for current densities in 1915 (Janssen, 1995, secs. 3.5.3 and 3.5.6).

[35]See (Lorentz, 1895, sec. 19–23) for the derivation of this transformation law and (Janssen, 1995, sec. 3.2.5) or (Zahar, 1989, 59–61) for a reconstruction of this derivation in modern notation.

[36]For an elegant and elementary exposition of Planck's derivation, see (Zahar, 1989, sec. 7.1, 227–237). The equations for the relation between $\mathbf{a}'$ and $\mathbf{a}$ can be found on p. 232, eqs. (2)–(4).

[37]Einstein (1905, 919) obtained $m_\perp = \gamma^2 m_0$ instead of $m_\perp = \gamma m_0$, the result obtained by Planck and Lorentz (for $l = 1$). The discrepancy comes from Einstein using $\mathbf{F}' = \mathbf{F}$ instead of $\mathbf{F}' = \text{diag}(1, \gamma, \gamma)\mathbf{F}$, the now standard transformation law for forces used by Lorentz and Planck (Zahar, 1989, 233). Einstein made it clear that he was well aware of the arbitrariness of his definition of force. When (Einstein, 1905) was reprinted in (Blumenthal, 1913), a footnote was added in which Einstein's original definition of force is replaced by the one of Lorentz and Planck. Recently a slip of paper came to light with this footnote in Einstein's own hand. This shows that the footnote was added by Einstein himself and not by Sommerfeld as suggested, e.g., by Miller (1981, 369, 391).

[38]See eq. 126 below for the relation between the (contravariant) electromagnetic field strength tensor $F^{\mu\nu}$ (and its covariant form $F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma}$) and the components of **E** and **B**.

[39]If the Fermi-Rohrlich definition is used, the relation $d^3x = d^3x'/\gamma l^3$ used in going from eq. 54 to eq. 55 no longer holds. Kwal (1949) clearly recognized that this is the source of the problem. In the abstract of his paper he wrote: "The appearance of the factor 1/3 in the expression for the total energy of the moving electron results from the simultaneous use in the calculation of a tensorial quantity (the energy-momentum tensor) and a quantity that is not [a tensor] (the volume element). The difficulty disappears with a tensorial definition of the volume element."

[40]This is an example of what one of us called a "common origin inference" or *COI* in (Janssen, 2002a). The example illustrates how easy it is to overreach with this kind of argument (for other examples see, ibid., 474, 491, 508).

[41]Cf. (Miller, 1981, sec. 1.13.2). Miller cites a letter of January 26, 1905, in which Abraham informed Lorentz of this difficulty. See also (Lorentz, 1915, 213).

[42]Carrying out the differentiation with respect to γ in eq. 76, we find:

$$m_{//} = \frac{c^2}{v}\left(\frac{2}{3}\gamma^{-\frac{1}{3}} + \frac{1}{3}\gamma^{-\frac{7}{3}}\right)\gamma^3 \frac{v}{c^2} m_0,$$

where we used eq. 35 for $d\gamma/dv$. This in turn can be rewritten as

$$m_{//} = \gamma^{\frac{8}{3}}\left(\frac{2}{3} + \frac{1}{3}\gamma^{-2}\right) m_0 = \gamma^{\frac{8}{3}}\left(\frac{2}{3} + \frac{1}{3}\left(1-\beta^2\right)\right) m_0 = \gamma^{\frac{8}{3}}\left(1 - \frac{1}{3}\beta^2\right) m_0.$$

[43]We are grateful to Serge Rudaz for his help in reconstructing this argument.

[44]Referring to (Poincaré, 1885, 1902a, 1902b), Scott Walter (Forthcoming, sec. 1) makes the interesting suggestion that "[s]olving the stability problem of Lorentz's contractile electron was a trivial matter for Poincaré, as it meant transposing to electron theory a special solution to a general problem he had treated earlier at some length: to find the equilibrium form of a rotating fluid mass."

[45]As Miller (1973, 300) points out, in the short announcement of his 1906 paper, Poincaré (1905, 491) mistakenly wrote that the electron "is under the action of constant external pressure" (Keswani and Kilmister, 1983, 352).

[46]For a detailed analysis of the completely analogous case of the forces on a capacitor in the Trouton-Noble experiment, see secs. 2.3.3 and 2.4.2 of (Janssen, 1995).

[47]We are grateful to Scott Walter for reminding us of this problem. We essentially follow the analysis of the problem by Miller (1973, 298–299), although we draw a slightly different conclusion (see note 48). Schwartz (1972, 871) translates the relevant passage from (Poincaré, 1906) but passes over the problem in silence.

[48]One might object, however, that our reading of Poincaré is too charitable. Poincaré certainly does not explicitly say, once he has derived the expression for Poincaré pressure at the end of section 6, that this restores the standard relations between Hamiltonian, Lagrangian, and generalized momentum in Lorentz's model. Yet we take this to be the rationale behind his calculations. Miller (1973, 248) is harder on Poincaré: "contrary to what is sometimes attributed to this paper [Poincaré, 1906], Poincaré never computed the counter term [our eq. 116] necessary to cancel

the second term on the right-hand side of [our eq. 115], nor did he reduce the factor of 4/3 in [the electromagnetic momentum] to unity [compare $\mathbf{P}_{\rm EM}$ in eq. 66 to $\mathbf{P}_{\rm tot}$ in eq. 113]." This is all true. Our rejoinder on behalf of Poincaré is that he did not need to do any of this to remove the inconsistency in Lorentz's model.

[49]Compare eq. 120 to eq. 87, which for small velocities reduces to

$$L_{\rm EM} \approx -U_{0_{\rm EM}}(1 - v^2/2c^2).$$

[50]Shaul Katzir (private communication) has suggested a more charitable interpretation of Poincaré's comments. Poincaré, Katzir suggests, recognized that the electron is not a purely electromagnetic system but believed that its mass is nonetheless given by its electromagnetic momentum through eqs. 26. For the specific model proposed by Poincaré this is not true. The non-electromagnetic piece he added to stabilize Lorentz's electron does contribute to the electron's mass, giving a total mass of $(4/3)U_{0_{\rm EM}}/c^2$. As we shall see in sec. 6, however, it is possible to add a stabilizing piece that does not contribute to the electron's mass (see our remarks following eq. 158).

[51] The derivation of eq. 129 is essentially just the reverse of the derivation of eq. 17 and can be pieced together from the passages we cited for the latter (see note 24). From a relativistic point of view, eq. 129 is immediately obvious since the (four-)gradient of the energy-momemtum tensor gives minus the density of the (four-)force acting on the system (see, e.g., Pauli, 1921, 126, eq. (345)). The right-hand side of eq. 129 is minus the Lorentz force density in the absence of a magnetic field (ibid., 85, eq. (225)).

[52] $T^{00}_{0_{\rm non-EM}}$ can be any function of the spatial coordinates and the system will still be closed. Of course, this component needs to be chosen in such a way that $T^{\mu\nu}_{\rm non-EM}$ continues to transform as a tensor. We ensure this by changing definition 134 to:

$$T^{\mu\nu}_{0_{\rm non-EM}} \equiv -\eta^{\mu\nu} P_{\rm Poincaré}\vartheta(R - r_0) + f(\mathbf{x}_0)\frac{u_0^\mu u_0^\nu}{c^2},$$

where $u^\mu = \gamma(c, \mathbf{v})$ is the electron's four-velocity. The function $f(\mathbf{x}_0)$ can be chosen arbitrarily as long as the energy density is positive definite everywhere. Hence, it must satisfy the condition $f(\mathbf{x}_0) \geq 0$ outside the electron and the condition $f(\mathbf{x}_0) \geq P_{\rm Poincaré}$ inside. If we choose $f(\mathbf{x}_0) = P_{\rm Poincaré}\vartheta(R - r_0)$, the definition above becomes

$$T^{\mu\nu}_{0_{\rm non-EM}} \equiv -\left(\eta^{\mu\nu} - \frac{u_0^\mu u_0^\nu}{c^2}\right) P_{\rm Poincaré}\vartheta(R - r_0),$$

in which case $T^{00}_{0_{\rm non-EM}} = 0$. This definition was proposed by Schwinger (1983, 379, eqs. (42)–(43)).

[53]As Rohrlich (1997, 1056), following (Schwinger, 1983, 374, 379), put it: "The argument over whether $m_{\rm es}$[equal to $U_{0_{\rm EM}}/c^2$ in our notation] or $m_{\rm ed} = 4m_{\rm es}/3$ is the "right" answer is thus resolved: [...] it depends on the model; either value as well as any value in between is possible [as are values greater than $m_{\rm ed}$; cf. note 52 above]. But in all cases, one obtains a four-vector for the stabilized charged sphere". Which

situation obtains cannot be decided experimentally. The rest mass of the electron can be determined, but that value can be represented by U_0/c^2, by $4U_0/3c^2$, or by some other value by adjusting the radius of the electron, for instance, which cannot be determined experimentally.

[54]This stabilizing system will not be as simple as the Poincaré pressure for the Lorentz-Poincaré electron. Without the spherical symmetry of this specific model, eq. 134 for the non-electromagnetic part of the energy-momentum tensor will be more complicated. See (Janssen, 1995, sec. 2.3.3, especially eq. (2.96)) for another simple example, the stabilizing mechanism for the surface charge distribution on a plate capacitor, worked out with the help of Tony Duncan.

[55]See the discussion following eq. 13 and (Janssen, 2003, 46–47).

[56]Quoted in (Miller, 1981, 331). In a review article about electrons originally published in the late 1920s, Walter Gerlach still claimed that the experiments of Bucherer and others decided in favor of the relativistic formula for the velocity dependence of the electron mass. Gerlach concluded: "Today there is therefore no reason to doubt the correctness of the results of the investigations of Bucherer, Wolz, Schaefer, and Neumann that *the experimentally observed velocity-dependence of the electron mass agrees, within the margins to be expected from the sources of error inherent in the method, only with the Lorentz-Einstein theory of the electron*" (Gerlach, 1933, 81). In a footnote, he adds: "Also note in this context the corresponding corroboration on the basis of [De Broglie] "wavelength"-measurements of electrons of different velocity by Ponte [1930]." Inspired by Zahn and Spees, (Rogers et al., 1940) repeated the experiment of the 1910s with sufficient accuracy to distinguish the relativistic prediction from Abraham's. Despite this result, (Faragó and Jánossy, 1957), in a subsequent review of the experimental confirmation of the relativistic formula for the velocity dependence of electron mass, essentially concurred with Zahn and Spees (Battimelli, 1981, 149; note 63 explains the reason for our qualification).

[57]Lorentz to Poincaré, March 8, 1906 (see Miller, 1981, sec. 12.4.1, for the quotation, and pp. 318–319 for a reproduction of the letter in facsimile).

[58]See, e.g., (Holton, 1988, 252–253), (Miller, 1981, sec. 12.4.3), (Hon, 1995, 208), and (Janssen, 2002a, 462, note 9).

[59]See Adler to Einstein, March 9, 1917 (Einstein, 1987–2002, Vol. 8, Doc. 307). In 1909 Adler had supported Einstein's candidacy for a post at the University of Zurich for which both of them had applied (see Einstein to Michele Besso, April 29, 1917 (Einstein, 1987–2002, Vol. 8, Doc. 331)). Einstein reciprocated in 1917 by drafting a petition on behalf of a number of Zurich physicists asking the Austrian authorities for leniency in Adler's case, even as Adler was busying himself with a critique of his benefactor's theories (see the letter to Besso quoted above). A draft of Einstein's petition is reproduced in facsimile in (Renn, 2005, 317). Adler's father, the well-known Austrian social democrat Victor Adler, considered using his son's railings against relativity for an insanity defense. His son, however, was determined to stand by his critique of relativity, even if it meant ending up in front of the firing squad. Adler was in fact sentenced to death but it was clear to all involved that

he would not be executed. The death sentence was commuted to eighteen years in prison on appeal and Adler was pardoned immediately after the war. This bizarre story is related in (Fölsing, 1997, 402–405). For an analysis of the psychology behind Adler's burning martyrdom, see (Ardelt, 1984).

[60]Einstein to Adler, September 29, 1918 (Einstein, 1987–2002, Vol. 8, Doc. 628; translation here and in the following are based on Ann M. Hentschel's).

[61]Adler to Einstein, October 12, 1918 (Einstein, 1987–2002, Vol. 8, Doc. 632; Adler's emphasis).

[62]Cf., however, the quotation from Lorentz in note 72 below.

[63]Einstein to Adler, October 20, 1918 (Einstein, 1987–2002, Vol. 8, Doc. 636). Two of the studies cited by Einstein involved the deflection of fast electrons as in the experiments of Kaufmann, Bucherer, and others; the third—by Karl Glitscher (1917), a student of Sommerfeld—used the fine structure of spectral lines to distinguish between the relativistic and the Abraham prediction for the velocity dependence of the electron mass. The experiment is not mentioned in the review article on electrons by Gerlach (1933), but Faragó and Jánossy (1957, sec. 2) review it very favorably. They write: "Analyzing the available experimental material, we have come to the conclusion that it is the fine-structure splitting in the spectra of atoms of the hydrogen type which give [sic] the only high-precision confirmation of the relativistic law of the variation of electron mass with velocity" (Faragó and Jánossy, 1957, 1417; quoted in Hon, 1995, 197).

[64]In general we need the fields associated with the particle to be sharply peaked around the worldline of the particle, a four-dimensional 'world-tube.'

[65]Einstein and Felix Klein corresponded about this issue in 1918 (Einstein, 1987–2002, Vol. 8, Docs. 554, 556, 561, 566, and 581). See also Hermann Weyl to Einstein, November 16, 1918 (Einstein, 1987–2002, Vol. 8, Doc. 657). A precursor to this approach can be found in (Einstein and Grossmann, 1913, sec. 4), where Einstein pointed out that the geodesic equation, which governs the motion of a test particle in a gravitational field, can be obtained by integrating $T^{\mu\nu}{}_{;\nu} = 0$—the vanishing of the covariant divergence of $T^{\mu\nu}$, the general-relativistic generalization of $\partial_\nu T^{\mu\nu} = 0$—over the 'worldtube' of the corresponding energy-momentum tensor for pressureless dust ("thread of flow" [*Stromfaden*] is the term Einstein used). This argument can also be found in the so-called Zurich Notebook (Einstein, 1987–2002, Vol. 4, Doc. 10, [p. 10] and [p. 58]). For analysis of these passages, see (Norton, 2000, Appendix C) and "A Commentary on the Notes on Gravity in the Zurich Notebook" in (Renn, Forthcoming, sec. 3 and 5.5.10; the relevant pages of the notebook are referred to as '5R' and '43L').

[66]See (Renn and Sauer, Forthcoming) for extensive discussion of the role of the energy-momentum tensor in the research that led to general relativity.

[67]This exchange is also discussed, for instance, in (Miller, 1981, sec. 7.3.4), (McCormmach, 1970, 489–490), and (Jungnickel and McCormmach, 1986, 249–250).

[68]Understandably, Bucherer took exception to the fact that Planck only discussed the electron models of Lorentz and Abraham (Planck, 1906b, 760).

[69]For brief discussions of the debate over superluminal velocities in the years surrounding the advent of special relativity, see (Miller, 1981, 110–111, note 57) and the editorial note, "Einstein on Superluminal Signal Velocities," in (Einstein, 1987–2002, Vol. 5, 56–60).

[70]See (Walter, 1999a, sec. 3.1) for a more charitable assessment of the development of Sommerfeld's views.

[71]We are grateful to Michael Eckert for alerting us to this passage and for providing us with the date of this part of Sommerfeld's autobiographical sketch.

[72] In the second edition, Lorentz added the following footnote at this point: "Later experiments [...] have confirmed [eq. 37] for the transverse electromagnetic mass, so that, in all probability, the only objection that could be raised against the hypothesis of the deformable electron and the principle of relativity has now been removed" (Lorentz, 1915, 339).

[73]For more extensive discussion of this passage, see (Janssen, 1995, sec. 4.3).

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