No. 32

GENERALIZED HYPERGEOMETRIC
SERIES

GENERALIZED
HYPERGEOMETRIC SERIE\$

BY

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PREFACE

Before the year 1923 the literature dealing with generalized hypergeometric series was somewhat scattered, but in that year Professor G. H. Hardy published his paper "A chapter from Ramanujan's note-book" in which he gave an account and proofs of the results then known, most of which had been rediscovered by Ramanujan. Since then numerous papers have been written on the subject, and it seems desirable that the mass of special results, obtained by one method or another, should be collected together. This is the primary object of this tract.

No attempt has been made to give a complete account of the ordinary hypergeometric series. In fact the first chapter simply gives the minimum required for the succeeding chapters. Again, all parts of the subject, such as asymptotic expansions, which definitely belong to function theory, have been deliberately ignored.

Although the main part of the work deals with generalized hypergeometric series, there are also short accounts of Heine's basic hypergeometric series and Appell's hypergeometric functions of two variables.

My thanks are due to Professor G. H. Hardy who made valuable suggestions regarding the general plan of the work, and to Professor L. J. Mordell who suggested the desirability of a tract on this subject.

W. N. B.

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CHAPTER I

THE HYPERGEOMETRIC SERIES

1.1. Introduction. The series*

$$1 + \frac{a \cdot b}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \dots$$

is called the hypergeometric series, and is denoted by $F(a, b; c; z)$.
It is assumed that c is not a negative integer.

The series converges when $|z| < 1$, and also when $z = 1$ provided
that $R(c - a - b) > 0$, and when $z = -1$ provided that

$$R(c - a - b + 1) > 0.$$

For brevity we write

$$(a)_n = a(a+1)(a+2) \dots (a+n-1), \quad (a)_0 = 1,$$

and then

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n.$$

1.2. The differential equation satisfied by $F(a, b; c; z)$.

The differential equation

$$\{\vartheta(\vartheta + c - 1) - z(\vartheta + a)(\vartheta + b)\} y = 0,$$

where ϑ denotes the operator $z\,d/dz$, is evidently satisfied by

$y = \sum\limits_{n=0}^{\infty} A_n z^n$ if

$$(n+1)(c+n) A_{n+1} = (a+n)(b+n) A_n.$$

This is the relation satisfied by consecutive coefficients of the
series $F(a, b; c; z)$, and consequently the equation is satisfied by
the series. The equation can be written as

$$(1) \qquad z(1-z)\frac{d^2y}{dz^2} + \{c - (a+b+1)z\}\frac{dy}{dz} - aby = 0.$$

It is easily seen that the complete solution is

$$y = A\,F(a, b; c; z) + Bz^{1-c}F(a+1-c, b+1-c; 2-c; z),$$

valid for $|z| < 1$.

* Introduced into analysis by Gauss 1.

By changing the dependent variable it can be verified that the function

$$(1-z)^{c-a-b} F (c-a,\ c-b;\ c;\ z)$$

also satisfies (1). This solution can therefore be expressed in terms of the other solutions, and by comparing coefficients we have the relation*

(2) $F (a,\ b;\ c;\ z) = (1-z)^{c-a-b} F (c-a,\ c-b;\ c;\ z).$

By changing z into $1 - \zeta$ in (1) we obtain

$$\zeta (1 - \zeta) \frac{d^2 y}{d\zeta^2} + \{(a+b+1-c) - (a+b+1) \zeta\} \frac{dy}{d\zeta} - aby = 0,$$

which has the solutions

$F (a,\ b;\ a+b+1-c;\ \zeta)$ and $\zeta^{c-a-b} F (c-a,\ c-b;\ 1+c-a-b;\ \zeta).$

There is thus a relation of the form

(3) $F (a,\ b;\ c;\ z) = C\,F (a,\ b;\ a+b+1-c;\ 1-z)$

$\qquad + D (1-z)^{c-a-b} F (c-a,\ c-b;\ 1+c-a-b;\ 1-z),$

where C and D are constants, valid in the region for which $|z| < 1,\ |1-z| < 1$. The constants C and D can be found by putting $z = 0$ and $z = 1$, provided that we know the sum of the hypergeometric series when $z = 1$. This sum will now be obtained.

1.3. Gauss's theorem. We shall prove that, when

$$R (c-a-b) > 0,$$

(1) $$F (a,\ b;\ c;\ 1) = \frac{\Gamma (c) \Gamma (c-a-b)}{\Gamma (c-a) \Gamma (c-b)}.$$

By comparing the coefficients of x^n, it is easily verified that, if $0 \leqslant x < 1$,

$$c\{c - 1 - (2c-a-b-1)x\} F (a,\ b;\ c;\ x)$$

$$+ (c-a)(c-b) xF (a,\ b;\ c+1;\ x)$$

$$= c (c-1)(1-x) F (a,\ b;\ c-1;\ x)$$

$$= c (c-1)\{1 + \sum_{n=1}^{\infty} (u_n - u_{n-1}) x^n\},$$

where u_n is the coefficient of x^n in $F (a, b; c-1; x)$. Now make

* Due to Euler.

$x \to 1$. The right-hand side tends to zero if $u_n \to 0$, which is so when $R (c-a-b) > 0$. Thus

$$F (a,\ b;\ c;\ 1) = \frac{(c-a)(c-b)}{c (c-a-b)} F (a,\ b;\ c+1;\ 1).$$

By repeating this process, we see that

$$F (a,\ b;\ c;\ 1) = \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m} F (a,\ b;\ c+m;\ 1).$$

Now $$\lim_{m \to \infty} \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m} = \frac{\Gamma (c) \Gamma (c-a-b)}{\Gamma (c-a) \Gamma (c-b)}.$$

Also, if $v_n (a, b, c)$ is the coefficient of x^n in $F (a, b; c; x)$, and $m > |c|$, we have

$$| F (a,\ b;\ c+m;\ 1) - 1 | \leqslant \sum_{n=1}^{\infty} |v_n (a, b, c+m)|$$

$$\leqslant \sum_{n=1}^{\infty} v_n (|a|, |b|, m - |c|)$$

$$< \frac{|ab|}{m - |c|} \sum_{n=0}^{\infty} v_n (|a| + 1, |b| + 1, m+1 - |c|).$$

Now the last series converges when $m > |c| + |a| + |b| + 1$, and is a positive decreasing function of m. Hence

$$\lim_{m \to \infty} F (a,\ b;\ c+m;\ 1) = 1,$$

and Gauss's theorem is proved.

When a is a negative integer $-n$, the theorem becomes

$$F (-n,\ b;\ c;\ 1) = \frac{(c-b)_n}{(c)_n},$$

and this is equivalent to Vandermonde's theorem, familiar in connection with one proof of the binomial theorem.

1.4. Connection between hypergeometric functions of z and $1 - z$. We now return to the relation (3) of § 1.2. By putting $z = 0$ and $z = 1$ we have, if $R (c-a-b) > 0$, $R(c) < 1$,

$$\frac{\Gamma (a+b+1-c) \Gamma (1-c)}{\Gamma (a+1-c) \Gamma (b+1-c)} C + \frac{\Gamma (1+c-a-b) \Gamma (1-c)}{\Gamma (1-a) \Gamma (1-b)} D = 1,$$

$$C = \frac{\Gamma (c) \Gamma (c-a-b)}{\Gamma (c-a) \Gamma (c-b)}.$$

We thus find that*

$$(1) \quad F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b+1-c; 1-z)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a, c-b; 1+c-a-b; 1-z).$$

1.5. A definite integral for $F(a, b; c; z)$. Consider the integral

$$I = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,$$

where, for convergence, $R(c) > R(b) > 0$, and $|z| < 1$. It is supposed that the branch of $(1-tz)^{-a}$ is chosen so that $(1-tz)^{-a} \to 1$ as $t \to 0$. Then

$$I = \int_0^1 \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n t^{b+n-1}(1-t)^{c-b-1} dt$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n \frac{\Gamma(b+n)\Gamma(c-b)}{\Gamma(c+n)}$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n,$$

the change in the order of integration and summation being easily justified. We therefore have, under the given conditions,

$$(1) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt.$$

When $z = 1$, the integral on the right reduces to a beta function and we are led again to Gauss's theorem.

Again, if $z = -1$, $a = 1+b-c$, the integral in (1) becomes

$$\int_0^1 t^{b-1}(1-t^2)^{c-b-1} dt,$$

which can be evaluated in terms of gamma functions. This suggests that probably the sum of the series $F(b, 1+b-c; c; -1)$ can be found.

Finally, if $b = 1-a$, $z = \frac{1}{2}$, we are led to the integral

$$\int_0^1 (2t - t^2)^{-a}(1-t)^{c-b-1} dt,$$

* See also Barnes 1 where another method is used to obtain this formula. The method is reproduced in Whittaker and Watson, *Modern Analysis* (ed. 4, 1927), § 14.53.

and, taking $(1-t)^2$ as the new variable, this becomes a beta function. We can thus evaluate $F(a, 1-a; c; \frac{1}{2})$ in terms of gamma functions. The actual formulae will be given in Chapter II.

1.6. Barnes' contour integral for $F(a, b; c; z)$. * Consider the contour integral

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds,$$

where $|\arg(-z)| < \pi$, and the path of integration is curved, if necessary, to separate the poles $s = -a-n$, $s = -b-n$, $(n = 0, 1, 2, \ldots)$ from the poles $s = 0, 1, 2, \ldots$. This contour can always be drawn if a and b are not negative integers, as then none of the decreasing sequences of poles coincides with one of the increasing sequence.

Now,† if $|\arg(s+a)| \leqslant \pi - \delta$, $|\arg s| \leqslant \pi - \delta$, then

$$\log \Gamma(s+a) = (s+a-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log(2\pi) + o(1),$$

when $|s| \to \infty$.

Thus, the integrand, which can be written

$$-\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)} \frac{\pi(-z)^s}{\sin s\pi},$$

is asymptotically equal to

$$-\frac{\pi(-z)^s}{\sin s\pi} \exp[(a+b-c-1)\log s].$$

Putting $s = iv$ on the contour, we see that, for large values of v, the integrand is

$$O[v^{a+b-c-1} \exp\{-v\arg(-z) - \pi|v|\}].$$

Thus the integral is an analytic function of z throughout the domain $|\arg(-z)| \leqslant \pi - \delta$, where δ is any positive number.

Now let C denote the semi-circle of radius $N + \frac{1}{2}$ on the right of the imaginary axis with centre at the origin, N being an integer. As before the integrand is

$$O(N^{a+b-c-1}) \frac{(-z)^s}{\sin s\pi}$$

for large values of N, the implied constant being independent of $\arg s$ when s is on the semi-circle.

* Barnes 1.   † Whittaker and Watson, *Modern Analysis*, § 13.6.

If $s = (N + \frac{1}{2})e^{i\theta}$ and $|z| < 1$, we have

$$\frac{(-z)^s}{\sin s\pi} = O\left[\exp\{(N+\tfrac{1}{2})\cos\theta \log|z| - (N+\tfrac{1}{2})\sin\theta \arg(-z)\right.$$
$$\left. - (N+\tfrac{1}{2})\pi |\sin\theta|\}\right]$$
$$= O\left[\exp\{(N+\tfrac{1}{2})\cos\theta \log|z| - (N+\tfrac{1}{2})\delta |\sin\theta|\}\right],$$

and this is

$$O\left[\exp\{2^{-\frac{1}{2}}(N+\tfrac{1}{2})\log|z|\}\right] \quad \text{if} \quad 0 \leqslant |\theta| \leqslant \tfrac{1}{4}\pi,$$

and $\quad O\left[\exp\{-2^{-\frac{1}{2}}\delta(N+\tfrac{1}{2})\}\right] \quad \text{if} \quad \tfrac{1}{4}\pi \leqslant |\theta| \leqslant \tfrac{1}{2}\pi.$

Hence, if $\log|z|$ is negative, that is if $|z| < 1$, the integrand tends to zero sufficiently rapidly to ensure that $\int_C \to 0$ as $N \to \infty$.

By using Cauchy's theorem for the contour formed by C and the part of the imaginary axis from $i(N+\frac{1}{2})$ to $-i(N+\frac{1}{2})$, and then making $N \to \infty$, we see that, when $|\arg(-z)| \leqslant \pi - \delta$ and $|z| < 1$,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s \, ds$$
$$= \lim_{N \to \infty} \sum_{n=0}^{N} \frac{\Gamma(a+n)\Gamma(b+n)}{n!\,\Gamma(c+n)} z^n,$$

since $\Gamma(-s)$ has a simple pole at $s = n$, $(n = 0, 1, 2, \ldots)$ with residue $(-1)^{n-1}/n!$. Thus the integral represents an analytic function in the region $|\arg(-z)| < \pi$, and when $|z| < 1$ this analytic function may be represented by the series

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; z).$$

The symbol $F(a, b; c; z)$ may therefore be used to denote the more general function defined by the integral when divided by $\Gamma(a)\Gamma(b)/\Gamma(c)$.

1.7. Barnes' lemma.* *If the path of integration is curved so as to separate the increasing and decreasing sequences of poles, then*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)\,ds$$
$$= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}.$$

* Barnes 1.

Write I for the expression on the left. Let C be the semi-circle of radius ρ on the right of the imaginary axis with its centre at the origin, and suppose that $\rho \to \infty$ in such a way that the lower bound of the distance of C from the poles of $\Gamma(\gamma-s)\Gamma(\delta-s)$ is definitely positive. Then

$$\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)$$
$$= \frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(1-\gamma+s)\Gamma(1-\delta+s)} \cdot \frac{\pi^2}{\sin(\gamma-s)\pi \sin(\delta-s)\pi}$$
$$= O\left[s^{\alpha+\beta+\gamma+\delta-2}\exp\{-2\pi|I(s)|\}\right]$$

as $|s| \to \infty$ on the imaginary axis or on C. Thus the original integral converges, and the integral round C tends to zero as $\rho \to \infty$ when $R(\alpha+\beta+\gamma+\delta-1) < 0$. The integral is therefore equal to minus $2\pi i$ times the sum of the residues of the integrand at the poles on the right of the contour. Thus

$$I = \sum_{n=0}^{\infty} \Gamma(\alpha+\gamma+n)\Gamma(\beta+\gamma+n)\Gamma(\delta-\gamma-n)(-1)^n/n!$$
$$+ \sum_{n=0}^{\infty} \Gamma(\alpha+\delta+n)\Gamma(\beta+\delta+n)\Gamma(\gamma-\delta-n)(-1)^n/n!$$
$$= \Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\Gamma(\delta-\gamma)\,F(\alpha+\gamma, \beta+\gamma; 1+\gamma-\delta; 1)$$
$$\qquad + \text{ a similar expression with } \gamma \text{ and } \delta \text{ interchanged.}$$

Using Gauss's theorem we obtain the required result after a little reduction. The formula has been proved only when

$$R(\alpha+\beta+\gamma+\delta-1) < 0,$$

but by the theory of analytic continuation it is true for all values of α, β, γ, δ for which none of the poles of $\Gamma(\alpha+s)\Gamma(\beta+s)$ coincide with any of the poles of $\Gamma(\gamma-s)\Gamma(\delta-s)$.

By writing $s-k$, $\alpha+k$, $\beta+k$, $\gamma-k$, $\delta-k$ for s, α, β, γ, δ, we see that the result is still true when the limits of integration are $k \pm i\infty$, where k is any real constant.

CHAPTER II

GENERALIZED HYPERGEOMETRIC SERIES. FURTHER RESULTS CONCERNING ORDINARY HYPERGEOMETRIC SERIES

2.1. Introductory remarks. In the ordinary hypergeometric series $F(a, b; c; z)$ there are two numerator parameters a, b, and one denominator parameter c. More generally, we can consider the series

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{n! \, (\rho_1)_n \cdots (\rho_q)_n} z^n,$$

which we denote by

$$_pF_q\left[\begin{matrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_p; & z \\ & \rho_1, & \ldots, & \rho_q & \end{matrix}\right].$$

With this notation the series $F(a, b; c; z)$ is denoted by $_2F_1(a, b; c; z)$.

When $p \leqslant q$, the series converges for all values of z. When $p > q+1$, the series converges only for $z = 0$, and is therefore significant only when it terminates.

Usually we shall be concerned with the case when $p = q+1$. Then the series converges when $|z| < 1$, and also when $z = 1$ provided that $R(\Sigma\rho - \Sigma\alpha) > 0$, and when $z = -1$ provided that $R(\Sigma\rho - \Sigma\alpha + 1) > 0$.

The differential equation satisfied by

$$_{p+1}F_p\left[\begin{matrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_{p+1}; & z \\ & \rho_1, & \ldots, & \rho_p & \end{matrix}\right]$$

is, as in § 1.2,

$$\{\vartheta(\vartheta + \rho_1 - 1) \ldots (\vartheta + \rho_p - 1) - z(\vartheta + \alpha_1)(\vartheta + \alpha_2) \ldots (\vartheta + \alpha_{p+1})\}y = 0,$$

where ϑ denotes the operator $z\,d/dz$. The other solutions of this equation are easily found to be

$$z^{1-\rho_1}{}_{p+1}F_p\left[\begin{matrix} 1+\alpha_1-\rho_1, \, 1+\alpha_2-\rho_1, \, 1+\alpha_3-\rho_1, \, \ldots, \, 1+\alpha_{p+1}-\rho_1; \, z \\ 2-\rho_1, \quad 1+\rho_2-\rho_1, \, \ldots, \, 1+\rho_p-\rho_1 \end{matrix}\right]$$

and $p-1$ similar expressions. These solutions are distinct and

valid for $|z| < 1$ if no two of the numbers $1, \rho_1, \rho_2, \ldots, \rho_p$ differ by an integer.

2.2. Saalschütz's theorem. In the formula

$$(1-z)^{a+b-c}{}_2F_1(a, b; c; z) = {}_2F_1(c-a, c-b; c; z),$$

obtained in § 1.2, equate the coefficients of z^n, and we obtain

$$\sum_{r=0}^{n} \frac{(a)_r (b)_r}{r! \, (c)_r} \frac{(c-a-b)_{n-r}}{(n-r)!} = \frac{(c-a)_n (c-b)_n}{n! \, (c)_n}.$$

Hence

$$\sum_{r=0}^{n} \frac{(a)_r (b)_r}{r! \, (c)_r} \frac{(c-a-b)_n (-n)_r}{(1+a+b-c-n)_r \, n!} = \frac{(c-a)_n (c-b)_n}{n! \, (c)_n}.$$

It follows that

$$(1) \qquad {}_3F_2\left[\begin{matrix} a, \, b, \, -n; \, 1 \\ c, \, 1+a+b-c-n \end{matrix}\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n},$$

a result due to Saalschütz.* It sums the series

$$_3F_2\left[\begin{matrix} \alpha_1, & \alpha_2, & \alpha_3; & 1 \\ & \rho_1, & \rho_2 & \end{matrix}\right]$$

when $\rho_1 + \rho_2 = \alpha_1 + \alpha_2 + \alpha_3 + 1$ and one of the numerator parameters is a negative integer. The theorem reduces to Gauss's theorem when $n \to \infty$.†

In future, when the argument z is omitted, it will be assumed that $z = 1$. Saalschütz's theorem can then be written in the form

$$(2) \quad {}_3F_2\left[\begin{matrix} a, \, b, \, c; \\ d, \, e \end{matrix}\right] = \frac{\Gamma(d)\,\Gamma(1+a-e)\,\Gamma(1+b-e)\,\Gamma(1+c-e)}{\Gamma(1-e)\,\Gamma(d-a)\,\Gamma(d-b)\,\Gamma(d-c)},$$

provided that a, b or c is a negative integer and $d+e = a+b+c+1$.

2.3. Kummer's theorem.‡ We shall prove that

$$(1) \qquad {}_2F_1\left[\begin{matrix} a, & b; & -1 \\ & 1+a-b \end{matrix}\right] = \frac{\Gamma(1+a-b)\,\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\,\Gamma(1+\frac{1}{2}a-b)}.$$

As a preliminary lemma we show that

$$(2) \quad {}_2F_1\left[\begin{matrix} a, & b; & z \\ & 1+a-b \end{matrix}\right] = (1-z)^{-a}\,{}_2F_1\left[\begin{matrix} \frac{1}{2}a, \, \frac{1}{2}+\frac{1}{2}a-b; \, -\dfrac{4z}{(1-z)^2} \\ 1+a-b \end{matrix}\right],$$

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* Saalschütz **1, 2.** See also Sheppard **1** and Dougall **1.**
† For the details of the limiting process, see Dougall **1.**
‡ Kummer **1,** p. 53.

a formula* which is valid inside the loop of the curve

$$|4z| = |1 - z|^2$$

which surrounds the origin. The right-hand side of (2) is analytic inside this region, and can therefore be expanded in powers of z when $|z| < 3 - 2\sqrt{2}$. Now the right-hand side of (2) is

$$\sum_{r=0}^{\infty} \frac{(\tfrac{1}{2}a)_r (\tfrac{1}{2} + \tfrac{1}{2}a - b)_r}{r!\,(1 + a - b)_r} (-4z)^r (1 - z)^{-a-2r},$$

and the coefficient of z^n is

$$\sum_{r=0}^{n} \frac{(\tfrac{1}{2}a)_r (\tfrac{1}{2} + \tfrac{1}{2}a - b)_r}{r!\,(1 + a - b)_r} \frac{(-4)^r (a + 2r)_{n-r}}{(n-r)!}$$

$$= \frac{(a)_n}{n!} \sum_{r=0}^{n} \frac{(\tfrac{1}{2} + \tfrac{1}{2}a - b)_r (a + n)_r (-n)_r}{r!\,(1 + a - b)_r (\tfrac{1}{2}a + \tfrac{1}{2})_r}$$

$$= \frac{(a)_n}{n!} \, {}_3F_2 \left[\begin{matrix} \tfrac{1}{2} + \tfrac{1}{2}a - b, \ a + n, \ -n; \\ 1 + a - b, \tfrac{1}{2}a + \tfrac{1}{2} \end{matrix} \right]$$

$$= \frac{(a)_n (b)_n}{n!\,(1 + a - b)_n},$$

by Saalschütz's theorem. The formula (2) is therefore proved when $|z| < 3 - 2\sqrt{2}$, and the complete result follows by analytic continuation.

Now let $z \to -1$, and we find that

$$_2F_1 \left[\begin{matrix} a, \ b; \ -1 \\ 1 + a - b \end{matrix} \right] = 2^{-a} \, {}_2F_1 \left[\begin{matrix} \tfrac{1}{2}a, \ \tfrac{1}{2} + \tfrac{1}{2}a - b; \\ 1 + a - b \end{matrix} \right].$$

The series on the right can be summed by Gauss's theorem, and the required formula is easily obtained.

2.4. Some other sums.
As a preliminary lemma we prove the formula

$$(1) \quad (1 - z)^{-a} \, {}_2F_1 \left[\begin{matrix} a, \ b; \ -z/(1-z) \\ c \end{matrix} \right] = {}_2F_1 \left[\begin{matrix} a, \ c - b; \ z \\ c \end{matrix} \right],$$

valid if $|z| < 1$ and $R(z) < \tfrac{1}{2}$.

* Gauss 1.

The coefficient of z^n on the left is

$$\sum_{r=0}^{n} \frac{(a)_r (b)_r}{r!\,(c)_r} \frac{(-1)^r (a+r)_{n-r}}{(n-r)!} = \frac{(a)_n}{n!} \sum_{r=0}^{n} \frac{(b)_r (-n)_r}{r!\,(c)_r}$$

$$= \frac{(a)_n (c - b)_n}{n!\,(c)_n},$$

by Vandermonde's theorem, and the formula is proved. The argument is valid if $|z| < \tfrac{1}{2}$, and the complete result follows by analytic continuation.

Now let $z \to -1$, and we find that

$$_2F_1 \left[\begin{matrix} a, \ b; \ \tfrac{1}{2} \\ c \end{matrix} \right] = 2^a \, {}_2F_1 \left[\begin{matrix} a, \ c - b; \ -1 \\ c \end{matrix} \right].$$

The series on the right can be summed by Kummer's theorem when either $c = \tfrac{1}{2}(a + b + 1)$ or $a + b = 1$. We thus obtain the formulae

$$(2) \quad {}_2F_1 \left[\begin{matrix} a, \ b; \ \tfrac{1}{2} \\ \tfrac{1}{2}(a + b + 1) \end{matrix} \right] = \frac{\Gamma(\tfrac{1}{2}) \Gamma(\tfrac{1}{2} + \tfrac{1}{2}a + \tfrac{1}{2}b)}{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}a) \Gamma(\tfrac{1}{2} + \tfrac{1}{2}b)},$$

$$(3) \quad {}_2F_1 \left[\begin{matrix} a, \ 1 - a; \ \tfrac{1}{2} \\ c \end{matrix} \right] = \frac{\Gamma(\tfrac{1}{2}c) \Gamma(\tfrac{1}{2}c + \tfrac{1}{2})}{\Gamma(\tfrac{1}{2}c + \tfrac{1}{2}a) \Gamma(\tfrac{1}{2} + \tfrac{1}{2}c - \tfrac{1}{2}a)}.$$

The formula (2) is due to Gauss.*

2.5. Standard types of generalized hypergeometric series.†
When the parameters of the series

$$_{p+1}F_p \left[\begin{matrix} \alpha_1, \ \alpha_2, \ \ldots, \ \alpha_{p+1}; \ z \\ \rho_1, \ \ldots, \ \rho_p \end{matrix} \right]$$

are such that $\Sigma\rho = \Sigma\alpha + 1$ (which is satisfied by the series $_3F_2$ in Saalschütz's theorem) the series will be said to be *Saalschützian*.

If the parameters satisfy the relations

$$1 + \alpha_1 = \rho_1 + \alpha_2 = \ldots = \rho_p + \alpha_{p+1},$$

the series will be said to be *well-poised*.

The series will be called *nearly-poised* if all but one of the pairs of parameters have the same sum. If

$$\rho_1 + \alpha_2 = \rho_2 + \alpha_3 = \ldots = \rho_p + \alpha_{p+1},$$

* Gauss 1.

† The names 'Saalschützian', 'well-poised' and 'nearly-poised' are due to Whipple 3, 2 and 5. Whipple applied the term 'Saalschützian' to terminating series only.

so that the breakdown in the equality of sums of pairs of para-
meters occurs with the first pair (regarding 1 as the first denomin-
ator parameter) we shall call the series a *nearly-poised series of
the first kind*. If, however, the breakdown occurs with the last
pair, so that

$$1 + \alpha_1 = \rho_1 + \alpha_2 = \ldots = \rho_{p-1} + \alpha_p,$$

we shall call the series a *nearly-poised series of the second kind*.

It will be noticed that the series in Kummer's theorem is well-
poised, while the series in Gauss's theorem is nearly-poised.

CHAPTER III

SERIES OF THE TYPE $_3F_2$ WITH UNIT ARGUMENT

3.1. Dixon's theorem. The theorem of Saalschütz gives one
case in which the series $_3F_2$ with unit argument can be summed.
In this paragragh we show that any convergent well-poised
series of this type can be evaluated in terms of gamma functions.
The formula is*

$$(1) \quad {}_3F_2\left[\begin{matrix} a, & b & c; \\ & 1+a-b, & 1+a-c \end{matrix}\right]$$

$$= \frac{\Gamma(1+\tfrac{1}{2}a)\,\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+\tfrac{1}{2}a-b-c)}{\Gamma(1+a)\,\Gamma(1+\tfrac{1}{2}a-b)\,\Gamma(1+\tfrac{1}{2}a-c)\,\Gamma(1+a-b-c)},$$

and includes as a special case the sum of the cubes of the co-
efficients in the binomial expansion.† It reduces to Kummer's
theorem when $c \to -\infty$.

Now, by Gauss's theorem,

$$\frac{\Gamma(1+a+2n)\,\Gamma(1+a-b-c)}{\Gamma(1+a-b+n)\,\Gamma(1+a-c+n)} = {}_2F_1\left[\begin{matrix} b+n, & c+n; \\ & 1+a+2n \end{matrix}\right].$$

Thus

$$\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(1+a-b)\,\Gamma(1+a-c)}\,{}_3F_2\left[\begin{matrix} a, & b, & c; \\ & 1+a-b, & 1+a-c \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(1+a-b+n)\,\Gamma(1+a-c+n)}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(1+a+2n)\,\Gamma(1+a-b-c)}\,{}_2F_1\left[\begin{matrix} b+n, & c+n; \\ & 1+a+2n \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n+m)\,\Gamma(c+n+m)}{n!\,m!\,\Gamma(1+a+2n+m)\,\Gamma(1+a-b-c)}$$

$$= \sum_{p=0}^{\infty}\sum_{n=0}^{p} \frac{\Gamma(a+n)\,\Gamma(b+p)\,\Gamma(c+p)}{n!\,(p-n)!\,\Gamma(1+a+n+p)\,\Gamma(1+a-b-c)}$$

$$= \sum_{p=0}^{\infty} \frac{\Gamma(a)\,\Gamma(b+p)\,\Gamma(c+p)}{p!\,\Gamma(1+a-b-c)\,\Gamma(1+a+p)}\,{}_2F_1\left[\begin{matrix} a, & -p; & -1 \\ & 1+a+p \end{matrix}\right]$$

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* Dixon 2. The proof given here is due to Watson 3.
† Morley 1. See also Dixon 1, Richmond 1, MacMahon 1.

$$= \sum_{p=0}^{\infty} \frac{\Gamma(a)\,\Gamma(b+p)\,\Gamma(c+p)\,\Gamma(1+\tfrac{1}{2}a)}{p!\,\Gamma(1+a-b-c)\,\Gamma(1+a)\,\Gamma(1+\tfrac{1}{2}a+p)}$$

$$= \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}\,{}_2F_1\!\left[\begin{array}{c} b,\,c; \\ 1+\tfrac{1}{2}a \end{array}\right]$$

$$= \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(1+\tfrac{1}{2}a)\,\Gamma(1+\tfrac{1}{2}a-b-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)\,\Gamma(1+\tfrac{1}{2}a-b)\,\Gamma(1+\tfrac{1}{2}a-c)}$$

and the formula is proved. In the analysis* we have used the theorems of Kummer and Gauss.

3.2. Some transformations of series $_3F_2$.

In this paragraph we prove two fundamental relations, one involving two series $_3F_2$, and the other involving three series of this type. The first formula is†

$$(1) \quad {}_3F_2\!\left[\begin{array}{c} a,\,b,\,c; \\ e,\,f \end{array}\right] = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\,{}_3F_2\!\left[\begin{array}{c} e-a,\,f-a,\,s; \\ s+b,\,s+c \end{array}\right],$$

where $s = e+f-a-b-c$.

The proof proceeds on similar lines to the proof of Dixon's theorem, Gauss's theorem being used in the analysis,

$$\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(e)\,\Gamma(f)}\,{}_3F_2\!\left[\begin{array}{c} a,\,b,\,c; \\ e,\,f \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(e+n)\,\Gamma(f+n)}$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(b+n)\,\Gamma(c+n)}{n!\,\Gamma(e+f-a+n)}\,{}_2F_1\!\left[\begin{array}{c} e-a,\,f-a; \\ e+f-a+n \end{array}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(b+n)\,\Gamma(c+n)\,\Gamma(e-a+m)\,\Gamma(f-a+m)}{n!\,m!\,\Gamma(e+f-a+n+m)\,\Gamma(e-a)\,\Gamma(f-a)}$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(e-a+m)\,\Gamma(f-a+m)\,\Gamma(b)\,\Gamma(c)}{m!\,\Gamma(e+f-a+m)\,\Gamma(e-a)\,\Gamma(f-a)}\,{}_2F_1\!\left[\begin{array}{c} b,\,c; \\ e+f-a+m \end{array}\right]$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(e-a+m)\,\Gamma(f-a+m)\,\Gamma(b)\,\Gamma(c)\,\Gamma(e+f-a-b-c+m)}{m!\,\Gamma(e+f-a-b+m)\,\Gamma(e+f-a-c+m)\,\Gamma(e-a)\,\Gamma(f-a)}$$

$$= \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(e+f-a-b-c)}{\Gamma(e+f-a-b)\,\Gamma(e+f-a-c)}\,{}_3F_2\!\left[\begin{array}{c} e-a,\,f-a,\,e+f-a-b-c; \\ e+f-a-b,\,e+f-a-c \end{array}\right]$$

* The justification of the interchange in the order of summation is similar to that of the next paragraph.   † Thomae 1, equation 11.

and this is the formula stated. The argument requires that the double series should be absolutely convergent. This is certainly so if the real part of $e+f-a$ is sufficiently large. Suppose, for example, that $e+f-a > 2r+1$, where r is an integer. Then*

$$\Gamma(e+f-a+m+n) > \Gamma(n+r+1)\,\Gamma(m+r+1)$$

and the double series may be compared with the product of two absolutely convergent simple series.

The second fundamental relation is†

$$(2) \quad {}_3F_2\!\left[\begin{array}{c} a,\,b,\,c; \\ e,\,f \end{array}\right] = \frac{\Gamma(1-a)\,\Gamma(e)\,\Gamma(f)\,\Gamma(c-b)}{\Gamma(e-b)\,\Gamma(f-b)\,\Gamma(1+b-a)\,\Gamma(c)}$$

$$\times\,{}_3F_2\!\left[\begin{array}{c} b,\,b-e+1,\,b-f+1; \\ 1+b-c,\,1+b-a \end{array}\right]$$

+ a similar expression with b and c interchanged.

To prove this consider the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\pm i\pi s}\,\frac{\Gamma(-s)\,\Gamma(a+s)\,\Gamma(b+s)\,\Gamma(c+s)}{\Gamma(e+s)\,\Gamma(f+s)}\,ds,$$

where the contour is curved, as usual, to separate the increasing and decreasing sequences of poles. As in the proof of Barnes' lemma this integral is equal to minus $2\pi i$ times the sum of the residues at poles of the integrand on the right of the contour. Similarly by taking a large semi-circle on the left of the contour we can prove that the integral is equal to $2\pi i$ times the sum of the residues at poles on the left of the contour. Equating these two results we have

$$\frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(e)\,\Gamma(f)}\,{}_3F_2\!\left[\begin{array}{c} a,\,b,\,c; \\ e,\,f \end{array}\right]$$

$$= \sum_{a,\,b,\,c} e^{\pm i\pi a}\,\frac{\Gamma(a)\,\Gamma(b-a)\,\Gamma(c-a)}{\Gamma(e-a)\,\Gamma(f-a)}\,{}_3F_2\!\left[\begin{array}{c} a,\,1+a-e,\,1+a-f; \\ 1+a-b,\,1+a-c \end{array}\right].$$

Now multiply these two relations by $e^{\mp i\pi a}$ and subtract, and we obtain (2).

* Cf. Hardy 2, p. 500.
† Thomae 1, p. 72. See also Hardy 2, p. 501.

3.3. Watson's theorem on the sum of a $_3F_2$. Using the transformation (1) of § 3.2 we have

$$_3F_2\begin{bmatrix} a, & b, & c; \\ \tfrac{1}{2}(a+b+1), & 2c \end{bmatrix} = \frac{\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2})\,\Gamma(2c)\,\Gamma(c+\tfrac{1}{2}-\tfrac{1}{2}a-\tfrac{1}{2}b)}{\Gamma(a)\,\Gamma(c+\tfrac{1}{2}-\tfrac{1}{2}a+\tfrac{1}{2}b)\,\Gamma(2c+\tfrac{1}{2}-\tfrac{1}{2}a-\tfrac{1}{2}b)}$$

$$\times\,_3F_2\begin{bmatrix} 2c-a, & \tfrac{1}{2}(1+b-a), & c+\tfrac{1}{2}(1-a-b); \\ 2c+\tfrac{1}{2}(1-a-b), & c+\tfrac{1}{2}(1-a+b) \end{bmatrix}.$$

The series on the right can be summed by Dixon's theorem, and we find that

$$(1) \quad _3F_2\begin{bmatrix} a, & b, & c; \\ \tfrac{1}{2}(a+b+1), & 2c \end{bmatrix}$$

$$= \frac{\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}+c)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}a+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}a-\tfrac{1}{2}b+c)}{\Gamma(\tfrac{1}{2}+\tfrac{1}{2}a)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}b)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}a+c)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}b+c)}.$$

This formula was given by Watson* in the case when a is a negative integer, and the more general case was given by Whipple.† When $c\to\infty$, the theorem reduces to § 2.4 (2).

3.4. Whipple's theorem on the sum of a $_3F_2$. We now prove that, when $a+b=1$ and $e+f=2c+1$,

$$(1) \quad _3F_2\begin{bmatrix} a, b, c; \\ e, f \end{bmatrix}$$

$$= \frac{\pi\,\Gamma(e)\,\Gamma(f)}{2^{2c-1}\,\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}e)\,\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}f)\,\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}e)\,\Gamma(\tfrac{1}{2}b+\tfrac{1}{2}f)},$$

a result given by Whipple.‡

Using the transformation (1) of § 3.2, we see that, under the given conditions,

$$_3F_2\begin{bmatrix} a, b, c; \\ e, f \end{bmatrix} = \frac{\Gamma(e)\,\Gamma(f)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b+c)\,\Gamma(2c)}\,_3F_2\begin{bmatrix} e-a, f-a, c; \\ b+c, 2c \end{bmatrix}.$$

The series on the right can be summed by Watson's theorem, and the result follows. When we substitute for b and e and let $c\to\infty$, the theorem reduces to § 2.4 (3).

3.5. The functions Fp and Fn. The fundamental relations of § 3.2 are only two of many relations obtained in 1879 by Thomae§ who approached the subject through the calculus of

* Watson 5. See also Hardy 2. † Whipple 1.
‡ Whipple 1. § Thomae 1.

finite differences. In 1923 Whipple* introduced a notation which provided a clue to the numerous formulae obtained by Thomae. Let $r_0, r_1, r_2, r_3, r_4, r_5$ be six parameters such that

$$\sum_{i=0}^{5} r_i = 0.$$

With these parameters associate numbers α and β such that

$$\alpha_{lmn} = \tfrac{1}{2} + r_l + r_m + r_n,$$

$$\beta_{mn} = 1 + r_m - r_n.$$

Define functions Fp and Fn by the equations

$$Fp\,(0;\,4,\,5) = \frac{1}{\Gamma(\alpha_{123})\,\Gamma(\beta_{40})\,\Gamma(\beta_{50})}\,_3F_2\begin{bmatrix} \alpha_{145}, & \alpha_{245}, & \alpha_{345}; \\ \beta_{40}, & \beta_{50} \end{bmatrix},$$

$$Fn\,(0;\,4,\,5) = \frac{1}{\Gamma(\alpha_{045})\,\Gamma(\beta_{04})\,\Gamma(\beta_{05})}\,_3F_2\begin{bmatrix} \alpha_{023}, & \alpha_{013}, & \alpha_{012}; \\ \beta_{04}, & \beta_{05} \end{bmatrix}.$$

The Fn function is derived from the corresponding Fp function by changing the signs of all the r's.

By permutation of the suffixes $60\,Fp$'s and $60\,Fn$'s can be found.

If $\quad \alpha_{145}=a, \quad \alpha_{245}=b, \quad \alpha_{345}=c, \quad \beta_{40}=e, \quad \beta_{50}=f,$

$$\alpha_{123} \equiv e+f-a-b-c = s,$$

so that the hypergeometric function occurring in the definition of $Fp\,(0;4,5)$ is $_3F_2[a,b,c;e,f]$, then all the α's and β's can be expressed in terms of a, b, c, e and f. They are set out in Table I.

Table I.

Expressions for α's and β's in terms of a, b, c, e, f
$(s = e+f-a-b-c).$

$2=1-c$	$\alpha_{123}=s$	$\beta_{01}=2-s-a$	$\beta_{20}=s+b$	$\beta_{40}=e$
$3=1-b$	$\alpha_{124}=e-c$	$\beta_{02}=2-s-b$	$\beta_{21}=1-a+b$	$\beta_{41}=1+b+c-f$
$=1-f+a$	$\alpha_{125}=f-c$	$\beta_{03}=2-s-c$	$\beta_{23}=1+b-c$	$\beta_{42}=1+a+c-f$
$=1-e+a$	$\alpha_{134}=e-b$	$\beta_{04}=2-e$	$\beta_{24}=1-a-c+f$	$\beta_{43}=1+a+b-f$
$=1-a$	$\alpha_{135}=f-b$	$\beta_{05}=2-f$	$\beta_{25}=1-a-c+e$	$\beta_{45}=1+e-f$
$=1-f+b$	$\alpha_{145}=a$	$\beta_{10}=s+a$	$\beta_{30}=s+c$	$\beta_{50}=f$
$=1-e+b$	$\alpha_{234}=e-a$	$\beta_{12}=1+a-b$	$\beta_{31}=1+c-a$	$\beta_{51}=1+b+c-e$
$=1-f+c$	$\alpha_{235}=f-a$	$\beta_{13}=1+a-c$	$\beta_{32}=1-b+c$	$\beta_{52}=1+a+c-e$
$=1-e+c$	$\alpha_{245}=b$	$\beta_{14}=1-b-c+f$	$\beta_{34}=1-a-b+f$	$\beta_{53}=1+a+b-e$
$=1-s$	$\alpha_{345}=c$	$\beta_{15}=1-b-c+e$	$\beta_{35}=1-a-b+e$	$\beta_{54}=1-e+f$

* Whipple 1. Cf. Barnes 2.

In Tables II A and II B the parameters of the Fp's and Fn's are given in terms of a, b, c, e, f.

Table II A.

$$Fp\,(u;\,v,\,w) = \frac{1}{\Gamma\,(\alpha_{xyz})\,\Gamma\,(\beta_{vu})\,\Gamma\,(\beta_{wu})}\,{}_3F_2\left[\begin{matrix}\alpha_{vwx},\ \alpha_{vwy},\ \alpha_{vwz};\\ \beta_{vu},\,\beta_{wu}\end{matrix}\right].$$

	v, w	Numerator parameters			Denominator parameters	
Fp (0)	4, 5	a,	b,	c	e,	f
	2, 3	s,	$e-a$,	$f-a$	$s+b$,	$s+c$
	1, 4	a,	$e-b$,	$e-c$	e,	$s+a$
Fp (1) [Fp (2) and Fp (3) are of this type]	0, 2	$1-e+b$,	$1-f+b$,	$1-a$	$1+b-a$,	$2-s-a$
	0, 4	$1-s$,	$1-f+b$,	$1-f+c$	$1+b+c-f$,	$2-s-a$
	2, 3	$e-a$,	$f-a$,	$1-a$	$1-a+b$,	$1-a+c$
	2, 4	b,	$e-a$,	$1-f+b$	$1+b+c-f$,	$1+b-a$
	4, 5	$1-s$,	b,	c	$1+b+c-f$,	$1+b+c-e$
Fp (4) [Fp (5) is of this type]	0, 1	$1-e+a$,	$1-b$,	$1-c$	$2-e$,	$1+f-b-c$
	0, 5	$1-e+a$,	$1-e+b$,	$1-e+c$	$2-e$,	$1-e+f$
	1, 2	$f-c$,	$1-c$,	s	$1+f-a-c$,	$1+f-b-c$
	1, 5	$1-e+a$,	$f-c$,	$f-b$	$1+f-e$,	$1+f-b-c$

Table II B.

$$Fn\,(u;\,v,\,w) = \frac{1}{\Gamma\,(\alpha_{uvw})\,\Gamma\,(\beta_{uv})\,\Gamma\,(\beta_{uw})}\,{}_3F_2\left[\begin{matrix}\alpha_{uyz},\ \alpha_{uzx},\ \alpha_{uxy};\\ \beta_{uv},\,\beta_{uw}\end{matrix}\right].$$

	v, w	Numerator parameters			Denominator parameters	
Fn (0)	4, 5	$1-a$,	$1-b$,	$1-c$	$2-e$,	$2-f$
	2, 3	$1-s$,	$1-e+a$,	$1-f+a$	$2-s-b$,	$2-s-c$
	1, 4	$1-a$,	$1-e+b$,	$1-e+c$	$2-e$,	$2-s-a$
Fn (1) [Fn (2) and Fn (3) are of this type]	0, 2	$e-b$,	$f-b$,	a	$1+a-b$,	$s+a$
	0, 4	s,	$f-b$,	$f-c$	$1-b-c+f$,	$s+a$
	2, 3	$1-e+a$,	$1-f+a$,	a	$1+a-b$,	$1+a-c$
	2, 4	$1-b$,	$1-e+a$,	$f-b$	$1-b-c+f$,	$1+a-b$
	4, 5	s,	$1-b$,	$1-c$	$1-b-c+f$,	$1-b-c+e$
Fn (4) [Fn (5) is of this type]	0, 1	$e-a$,	b,	c	e,	$1-f+b+c$
	0, 5	$e-a$,	$e-b$,	$e-c$	e,	$1+e-f$
	1, 2	$1-f+c$,	c,	$1-s$	$1-f+a+c$,	$1-f+b+c$
	1, 5	$e-a$,	$1-f+c$,	$1-f+b$	$1+e-f$,	$1-f+b+c$

In these tables only representative forms are given. The permutation of the indices 1, 2, 3 corresponds with the permutation of a, b, c, whilst the permutation of 4 and 5 corresponds with the

permutation of e, f. Thus Fp (2) and Fp (3) are of the same type as Fp (1) in the sense that they can be derived from it by the interchange of b or c with a. For example, by comparison with Fp (1; 0, 2) it is seen that Fp (2; 0, 1) has the parameters

$$1-e+a,\ \ 1-f+a,\ \ 1-b;\ \ 1+a-b,\ \ 2-s-b.$$

The condition that the series Fp $(u;\,v,\,w)$ may be convergent is $R\,(\alpha_{xyz}) > 0$. It will be noticed that $\Gamma\,(\alpha_{xyz})$ occurs in the denominator in the definition of the Fp function. The condition for $Fn\,(u;\,v,\,w)$ to be convergent is $R\,(\alpha_{uvw}) > 0$.

3.6. Transformations of series $_3F_2$ with unit argument.

We now turn to the fundamental relations of § 3.2. The formula (1) of that paragraph can be written, in our new notation,

(1) $$Fp\,(0;\,4,\,5) = Fp\,(0;\,2,\,3).$$

By interchanging r_4 and r_1 we find that

$$Fp\,(0;\,1,\,5) = Fp\,(0;\,2,\,3),$$

and thus

(2) $$Fp\,(0;\,4,\,5) = Fp\,(0;\,1,\,5).$$

Accordingly all the permutations of the indices 1 to 5 are legitimate, and we see that all the ten expressions Fp $(0;\,v,\,w)$ are equal* and may be conveniently denoted by Fp (0). Similar results are true for the other Fp's and the Fn's. Thus the 60 series Fp may be divided into six groups of 10, the members of any one group being all equal. A similar remark applies to the 60 series Fn.

3.7. Three-term relations.

Now turn to the relation (2) of § 3.2. In our present notation this can be written

(1) $$\frac{\sin\pi\beta_{23}}{\pi\Gamma\,(\alpha_{023})}\,Fp\,(0) = \frac{Fn\,(2)}{\Gamma\,(\alpha_{134})\,\Gamma\,(\alpha_{135})\,\Gamma\,(\alpha_{345})}$$
$$-\frac{Fn\,(3)}{\Gamma\,(\alpha_{124})\,\Gamma\,(\alpha_{125})\,\Gamma\,(\alpha_{245})}.$$

Changing the signs of the r's, we obtain

(2) $$\frac{\sin\pi\beta_{32}}{\pi\Gamma\,(\alpha_{145})}\,Fn\,(0) = \frac{Fp\,(2)}{\Gamma\,(\alpha_{025})\,\Gamma\,(\alpha_{024})\,\Gamma\,(\alpha_{012})}$$
$$-\frac{Fp\,(3)}{\Gamma\,(\alpha_{035})\,\Gamma\,(\alpha_{034})\,\Gamma\,(\alpha_{013})}.$$

* Barnes 2, Hardy 2, Whipple 1.

By combining three equations like (1) it is found* that

(3) $$\frac{\sin \pi \beta_{45}}{\Gamma(\alpha_{012})\,\Gamma(\alpha_{013})\,\Gamma(\alpha_{023})}\,Fp\,(0)$$
$$+\frac{\sin \pi \beta_{50}}{\Gamma(\alpha_{124})\,\Gamma(\alpha_{134})\,\Gamma(\alpha_{234})}\,Fp\,(4)$$
$$+\frac{\sin \pi \beta_{04}}{\Gamma(\alpha_{125})\,\Gamma(\alpha_{135})\,\Gamma(\alpha_{235})}\,Fp\,(5)=0,$$

and, changing the signs of the r's,

(4) $$\frac{\sin \pi \beta_{54}}{\Gamma(\alpha_{345})\,\Gamma(\alpha_{245})\,\Gamma(\alpha_{145})}\,Fn\,(0)$$
$$+\frac{\sin \pi \beta_{05}}{\Gamma(\alpha_{035})\,\Gamma(\alpha_{025})\,\Gamma(\alpha_{015})}\,Fn\,(4)$$
$$+\frac{\sin \pi \beta_{40}}{\Gamma(\alpha_{034})\,\Gamma(\alpha_{024})\,\Gamma(\alpha_{014})}\,Fn\,(5)=0.$$

Now eliminate $Fn\,(2)$ from the relation of the type (1) connecting $Fp\,(5)$, $Fn\,(0)$, $Fn\,(2)$ and the relation of the type (2) connecting $Fn\,(2)$, $Fp\,(0)$, $Fp\,(5)$. It follows that†

(5) $$\frac{Fp\,(0)}{\Gamma(\alpha_{120})\,\Gamma(\alpha_{130})\,\Gamma(\alpha_{230})\,\Gamma(\alpha_{240})\,\Gamma(\alpha_{140})\,\Gamma(\alpha_{340})}$$
$$+\frac{\sin \pi \beta_{05}\,Fn\,(0)}{\pi \Gamma(\alpha_{123})\,\Gamma(\alpha_{124})\,\Gamma(\alpha_{134})\,\Gamma(\alpha_{234})}=K_0\,Fp\,(5),$$

where

$$\pi^3 K_0 = \sin \pi \alpha_{145} \sin \pi \alpha_{245} \sin \pi \alpha_{345} + \sin \pi \alpha_{123} \sin \pi \beta_{40} \sin \pi \beta_{50},$$

or, in terms of the r's,

$$4\pi^3 K_0 = \sum_{n=1}^{5} \cos \pi\,(r_0 + 2r_n) - \cos 3\pi r_0.$$

The analogue of (5) is

(6) $$\frac{Fn\,(0)}{\Gamma(\alpha_{345})\,\Gamma(\alpha_{245})\,\Gamma(\alpha_{145})\,\Gamma(\alpha_{135})\,\Gamma(\alpha_{235})\,\Gamma(\alpha_{125})}$$
$$+\frac{\sin \pi \beta_{50}\,Fp\,(0)}{\pi \Gamma(\alpha_{045})\,\Gamma(\alpha_{035})\,\Gamma(\alpha_{025})\,\Gamma(\alpha_{015})}=K_0\,Fn\,(5).$$

All the three-term relations between the 120 hypergeometric series are typified by the equations (1) to (6).

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* Cf. Thomae 1, equation 46, Hardy 2, equation 7.1.
† Cf. Thomae 1, equation 53. The discussion given here is due to Whipple 1.

3.8. An example. As an example of the use of the tables consider the formula*

(1) $$_3F_2\left[\begin{array}{c}a,\ b,\ c;\\ e,\ f\end{array}\right]=\frac{\Gamma(e)\,\Gamma(e-a-b)}{\Gamma(e-a)\,\Gamma(e-b)}\,_3F_2\left[\begin{array}{c}a,\ b,\ f-c;\\ a+b-e+1,\ f\end{array}\right]$$
$$+\frac{\Gamma(e)\,\Gamma(f)\,\Gamma(a+b-e)\,\Gamma(e+f-a-b-c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(f-c)\,\Gamma(e+f-a-b)}$$
$$\times\,_3F_2\left[\begin{array}{c}e-a,\ e-b,\ e+f-a-b-c;\\ e-a-b+1,\ e+f-a-b\end{array}\right].$$

By the use of Tables IIA and IIB we identify the series as multiples of $Fp\,(0;4,5)$, $Fn\,(5;0,3)$ and $Fn\,(3;0,5)$. Hence, after division by $\Gamma(s)\,\Gamma(e)\,\Gamma(f)$, the formula may be written (with the help of Table I) in the form

$$Fp\,(0;\,4,\,5)=\frac{\pi}{\sin \pi \beta_{53}}\frac{\Gamma(\alpha_{035})}{\Gamma(\alpha_{234})\,\Gamma(\alpha_{134})\,\Gamma(\alpha_{123})}\,Fn\,(5;0,3)$$
$$+\frac{\pi}{\sin \pi \beta_{35}}\frac{\Gamma(\alpha_{035})}{\Gamma(\alpha_{145})\,\Gamma(\alpha_{245})\,\Gamma(\alpha_{125})}\,Fn\,(3;0,5).$$

This is equivalent to §3.7 (1) with the indices 2 and 5 interchanged.

The formula (1) has an interesting connection with Saalschütz's theorem. If $e+f=a+b+c+1$, the first series on the right reduces to a $_2F_1$ which can be summed by Gauss's theorem, and we obtain†

(2) $$_3F_2\left[\begin{array}{c}a,\ b,\ e+f-a-b-1;\\ e,\ \ \ \ f\end{array}\right]$$
$$=\frac{\Gamma(e)\,\Gamma(f)\,\Gamma(e-a-b)\,\Gamma(f-a-b)}{\Gamma(e-a)\,\Gamma(e-b)\,\Gamma(f-a)\,\Gamma(f-b)}$$
$$+\frac{1}{a+b-e}\frac{\Gamma(e)\,\Gamma(f)}{\Gamma(a)\,\Gamma(b)\,\Gamma(e+f-a-b)}$$
$$\times\,_3F_2\left[\begin{array}{c}e-a,\ e-b,\ 1;\\ e-a-b+1,\ e+f-a-b\end{array}\right].$$

If a or b is a negative integer, the second term on the right vanishes, and we obtain Saalschütz's theorem. Thus (2) gives the form which Saalschütz's theorem assumes when we remove the

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* Hardy 2, equation 5.2.
† Saalschütz 2. See also Hardy 2.

restriction that one of the numerator parameters must be a negative integer.

3.9. Terminating series. When the parameter c is a negative integer, say $-m$, the series $_3F_2[a,b,c;e,f]$ terminates. In this case the series can be written in the reverse order, and we find that, when $\alpha_{345} \equiv c = -m$,

(1) $\Gamma(\alpha_{124})\,\Gamma(\alpha_{125})\,Fp\,(0) = (-1)^m\,\Gamma(\alpha_{023})\,\Gamma(\alpha_{013})\,Fn\,(3).$

This equation is a degenerate form of § 3.7 (1).

There are 18 terminating series altogether. Three of these are forms of $Fp\,(0)$, namely $Fp\,(0;4,5)$, $Fp\,(0;3,5)$ and $Fp\,(0;3,4)$. When reversed these three give $Fn\,(3,1,2)$, $Fn\,(4;1,2)$ and $Fn\,(5;1,2)$. The relations between the 18 series are shown by the equations*

$\Gamma(\alpha_{123})\,\Gamma(\alpha_{124})\,\Gamma(\alpha_{125})\,Fp\,(0)$

$= \Gamma(\alpha_{023})\,\Gamma(\alpha_{024})\,\Gamma(\alpha_{025})\,Fp\,(1)$

$= \Gamma(\alpha_{013})\,\Gamma(\alpha_{014})\,\Gamma(\alpha_{015})\,Fp\,(2)$

$= (-1)^m\,\Gamma(\alpha_{123})\,\Gamma(\alpha_{023})\,\Gamma(\alpha_{013})\,Fn\,(3)$

$= (-1)^m\,\Gamma(\alpha_{124})\,\Gamma(\alpha_{024})\,\Gamma(\alpha_{014})\,Fn\,(4)$

$= (-1)^m\,\Gamma(\alpha_{125})\,\Gamma(\alpha_{025})\,\Gamma(\alpha_{015})\,Fn\,(5).$

The other series such as $Fn\,(0)$ do not give any specially simple relations.

CHAPTER IV

METHODS OF OBTAINING TRANSFORMATIONS OF HYPERGEOMETRIC SERIES; (1) BY SUMMING SERIES OF LOWER ORDER

4.1. Introductory remarks. We now consider various methods by which transformations of generalized hypergeometric series have been obtained. The argument z will usually be equal to unity and will therefore be omitted, but later we shall derive some transformations of series for which $z = -1$.

In § 3.1 we have shown how Dixon's theorem can be obtained from Kummer's theorem by using Gauss's theorem in the analysis. Whipple* extended this method to obtain transformations of both well-poised and nearly-poised series. Instead of following Whipple's method we use a method† which gives results for series of higher order when the series terminate. In this chapter we shall be concerned only with terminating series except in the case of some deductions in § 4.4 and § 4.6.

4.2. A method of obtaining transformations. Suppose we have the formula

(1) $\displaystyle\sum_{r=0}^{n} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-n)_r (\kappa_1+n)_{r(\mu_1-1)} (\kappa_2+n)_{r(\mu_2-1)} (l_1)_{\nu_1 r} (-1)^r c^r}{r!\,(\beta_1)_r (\beta_2)_r (l_1+n)_{r(\nu_1-1)} (\kappa_1)_{\mu_1 r} (\kappa_2)_{\mu_2 r}}$

$\qquad\qquad = \dfrac{(\rho_1)_n (\rho_2)_n (\rho_3)_n}{(\sigma_1)_n (\sigma_2)_n} \dfrac{(l_1)_n b^n}{(\kappa_1)_n (\kappa_2)_n},$

giving the sum of a certain hypergeometric series. Then

$F\begin{bmatrix} \rho_1,\ \rho_2,\ \rho_3,\ a_1,\ a_2,\ -m;\ b \\ \sigma_1,\ \sigma_2,\ p_1,\ p_2,\quad p_3 \end{bmatrix} = \displaystyle\sum_{n=0}^{m} \frac{(a_1)_n (a_2)_n (-m)_n (\kappa_1)_n (\kappa_2)_n}{n!\,(p_1)_n (p_2)_n (p_3)_n (l_1)_n}$

$\times \displaystyle\sum_{r=0}^{n} \frac{(\alpha_1)_r (\alpha_2)_r (\alpha_3)_r (-n)_r (\kappa_1+n)_{r(\mu_1-1)} (\kappa_2+n)_{r(\mu_2-1)} (l_1)_{\nu_1 r} (-1)^r c^r}{r!\,(\beta_1)_r (\beta_2)_r (l_1+n)_{r(\nu_1-1)} (\kappa_1)_{\mu_1 r} (\kappa_2)_{\mu_2 r}}$

$= \displaystyle\sum_{r=0}^{m}\sum_{n=r}^{m} \frac{(a_1)_n (a_2)_n (-m)_n (\kappa_1+\mu_1 r)_{n-r} (\kappa_2+\mu_2 r)_{n-r} (\alpha_1)_r (\alpha_2)_r (\alpha_3)_r c^r}{(n-r)!\,(p_1)_n (p_2)_n (p_3)_n (l_1+\nu_1 r)_{n-r}\, r!\,(\beta_1)_r (\beta_2)_r}.$

Putting $n = r + t$, we get

$$\sum_{r=0}^{m} \sum_{t=0}^{m-r} \frac{(a_1)_{r+t}(a_2)_{r+t}(-m)_{r+t}(\kappa_1 + \mu_1 r)_t(\kappa_2 + \mu_2 r)_t(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r c^r}{r!\,t!\,(p_1)_{r+t}(p_2)_{r+t}(p_3)_{r+t}(l_1 + \nu_1 r)_t(\beta_1)_r(\beta_2)_r}.$$

Thus, if (1) is true, we have

$$(2) \quad F\left[\begin{array}{c} \rho_1,\ \rho_2,\ \rho_3,\ a_1,\ a_2,\ -m;\ b \\ \sigma_1,\ \sigma_2,\ p_1,\ p_2,\ \ p_3 \end{array}\right]$$

$$= \sum_{r=0}^{m} \frac{(a_1)_r(a_2)_r(-m)_r(\alpha_1)_r(\alpha_2)_r(\alpha_3)_r c^r}{r!\,(p_1)_r(p_2)_r(p_3)_r(\beta_1)_r(\beta_2)_r}$$

$$\times F\left[\begin{array}{c} \kappa_1 + \mu_1 r,\ \kappa_2 + \mu_2 r,\ a_1 + r,\ a_2 + r,\ -m + r; \\ l_1 + \nu_1 r,\ p_1 + r,\ p_2 + r,\ \ p_3 + r \end{array}\right].$$

In these formulae there may be any number of the quantities $\alpha, \beta, \rho, \sigma, \kappa, l, a, p$. The numbers μ and ν may be positive or negative integers provided that $(a)_{-n}$ is replaced by $(-1)^n/(1-a)_n$.

The series on the left of (1) can be expressed as a generalized hypergeometric series. If we can sum the series on the right of (2) in terms of gamma functions, the right-hand side of (2) can also be expressed as a generalized hypergeometric series. Thus, from a known sum of a generalized hypergeometric series we have a method of deducing transformations of such series.*

4.3. Transformations of well-poised series. In our first application of the method we use Saalschütz's theorem in the form

$$_3F_2\left[\begin{array}{c} 1+a-b-c,\ a+n,\ -n; \\ 1+a-b,\ 1+a-c \end{array}\right] = \frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n}.$$

We thus obtain

$$(1) \quad _{s+4}F_{s+3}\left[\begin{array}{c} a,\ \ b,\ \ \ c,\ \ a_1,\ a_2,\ \ldots,\ a_s,\ -m; \\ 1+a-b,\ 1+a-c,\ p_1,\ p_2,\ \ldots,\ p_s,\ p_{s+1} \end{array}\right]$$

$$= \sum_{r=0}^{m} \frac{(-4)^r(1+a-b-c)_r(\tfrac{1}{2}a)_r(\tfrac{1}{2}a+\tfrac{1}{2})_r(a_1)_r(a_2)_r \ldots (a_s)_r(-m)_r}{r!\,(1+a-b)_r(1+a-c)_r(p_1)_r(p_2)_r \ldots (p_{s+1})_r}$$

$$\times\ _{s+2}F_{s+1}\left[\begin{array}{c} a+2r,\ a_1+r,\ a_2+r,\ \ldots,\ a_s+r,\ -m+r; \\ p_1+r,\ p_2+r,\ \ldots,\ p_s+r,\ p_{s+1}+r \end{array}\right].$$

* The method given by Bailey 3 is slightly more general than that given here. The number λ of that paper is here taken equal to unity, the case which gives the most interesting formulae.

Taking $s = 1$ and choosing the parameters so that Dixon's theorem sums the series $_3F_2$ on the right, we find that

$$(2) \quad _5F_4\left[\begin{array}{c} a,\ \ \ b,\ \ \ \ c,\ \ \ \ d,\ \ \ -m; \\ 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m(1+\tfrac{1}{2}a-d)_m}{(1+\tfrac{1}{2}a)_m(1+a-d)_m}\ _4F_3\left[\begin{array}{c} 1+a-b-c,\ \tfrac{1}{2}a,\ d,\ -m; \\ 1+a-b,\ 1+a-c,\ d-\tfrac{1}{2}a-m \end{array}\right].$$

The series on the right of (2) is Saalschützian and so can be summed when it reduces to a $_3F_2$. Taking $b = 1 + \tfrac{1}{2}a$ we obtain the formula*

$$(3) \quad _5F_4\left[\begin{array}{c} a,\ 1+\tfrac{1}{2}a,\ \ \ c,\ \ \ \ \ d,\ \ \ \ -m; \\ \tfrac{1}{2}a,\ \ 1+a-c,\ 1+a-d,\ 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m(1+a-c-d)_m}{(1+a-c)_m(1+a-d)_m}.$$

Now choose the parameters in (1) so that the series on the right can be summed by (3), and we obtain the transformation

$$(4) \quad _7F_6\left[\begin{array}{c} a,1+\tfrac{1}{2}a,\ \ \ b,\ \ \ \ \ c,\ \ \ \ \ d,\ \ \ \ \ e,\ \ \ -m; \\ \tfrac{1}{2}a,\ \ 1+a-b,1+a-c,1+a-d,1+a-e,1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\ _4F_3\left[\begin{array}{c} 1+a-b-c,\ d,\ e,\ -m; \\ 1+a-b,\ 1+a-c,\ d+e-a-m \end{array}\right],$$

a result due to Whipple.† It transforms a terminating well-poised $_7F_6$ into a Saalschützian $_4F_3$, and conversely transforms any terminating Saalschützian $_4F_3$ into a well-poised $_7F_6$. When $e = \tfrac{1}{2}a$ it reduces to (2).

In the particular case when

$$1 + 2a = b + c + d + e - m,$$

the series on the right of (4) reduces to a $_3F_2$ which can be summed by Saalschütz's theorem and we obtain Dougall's theorem,‡

* Dougall 1, equation 9; Hardy 2, equation 3.1; Whipple 2, equation 5.2. The formula is true, with slight modifications when m is not a positive integer. For the more general case see § 4.4 (1).

† Whipple 2. See also Whipple 4. The formula is also true with slight modifications when $1+a-b-c$ is a negative integer and m is not a positive integer. See § 4.4.

‡ Dougall 1, equation 6. See also Hardy 2. The formula is fundamental in Hardy's paper.

namely

$$(5) \quad {}_7F_6\left[\begin{array}{cccccc} a, 1+\tfrac{1}{2}a, & b, & c, & d, & e, & -m; \\ \tfrac{1}{2}a, & 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m (1+a-b-c)_m (1+a-b-d)_m (1+a-c-d)_m}{(1+a-b)_m (1+a-c)_m (1+a-d)_m (1+a-b-c-d)_m},$$

provided that $1+2a=b+c+d+e-m.$

This formula sums a terminating well-poised $_7F_6$, with the special form of the second parameter, when the sum of the denominator parameters exceeds the sum of the numerator parameters by two.

Dougall's theorem cannot be used on the right of (1) for general values of r. We therefore apply the method of § 4.2 to Dougall's theorem itself, which can be written in the form

$$_7F_6\left[\begin{array}{cccccc} k, 1+\tfrac{1}{2}k, k+b-a, k+c-a, k+d-a, & a+n, & -n; \\ \tfrac{1}{2}k, & 1+a-b, 1+a-c, 1+a-d, 1+k-a-n, 1+k+n \end{array}\right]$$

$$= \frac{(1+k)_n (b)_n (c)_n (d)_n}{(a-k)_n (1+a-b)_n (1+a-c)_n (1+a-d)_n},$$

where $k = 1+2a-b-c-d$. This is equivalent to

$$\sum_{r=0}^{n} \frac{(-1)^r (k)_r (1+\tfrac{1}{2}k)_r (k+b-a)_r (k+c-a)_r (k+d-a)_r}{r! (\tfrac{1}{2}k)_r (1+a-b)_r (1+a-c)_r (1+a-d)_r (1+k+n)_r}$$

$$\times (a+n)_r (a-k+n)_{-r} (-n)_r$$

$$= \frac{(1+k)_n (b)_n (c)_n (d)_n}{(a-k)_n (1+a-b)_n (1+a-c)_n (1+a-d)_n},$$

and so, by the process of § 4.2, we deduce that

$$(6) \quad {}_{s+5}F_{s+4}\left[\begin{array}{cccccccc} a, & b, & c, & d, & a_1, a_2, ..., a_s, -m; \\ & 1+a-b, 1+a-c, 1+a-d, p_1, p_2, ..., p_s, p_{s+1} \end{array}\right]$$

$$= \sum_{r=0}^{m} \frac{(k)_r (k+b-a)_r (k+c-a)_r (k+d-a)_r (\tfrac{1}{2}a)_r (\tfrac{1}{2}+\tfrac{1}{2}a)_r}{r! (\tfrac{1}{2}k)_r (\tfrac{1}{2}+\tfrac{1}{2}k)_r (1+a-b)_r (1+a-c)_r (1+a-d)_r}$$

$$\times \frac{(a_1)_r ... (a_s)_r (-m)_r}{(p_1)_r ... (p_{s+1})_r}$$

$$\times {}_{s+3}F_{s+2}\left[\begin{array}{cccccc} a+2r, & a-k, & a_1+r, ..., a_s+r, & -m+r; \\ & 1+k+2r, p_1+r, ..., p_s+r, p_{s+1}+r \end{array}\right].$$

In this formula we can choose the parameters so that Dougall's theorem sums the series on the right, and we obtain the transformation*

$$(7) \quad {}_9F_8\left[\begin{array}{ccccc} a, 1+\tfrac{1}{2}a, & b, & c, & d, & e, \\ \tfrac{1}{2}a, & 1+a-b, 1+a-c, 1+a-d, 1+a-e, \end{array}\right.$$
$$\left.\begin{array}{ccc} f, & g, & -m; \\ 1+a-f, 1+a-g, 1+a+m \end{array}\right]$$

$$= \frac{(1+a)_m (1+k-e)_m (1+k-f)_m (1+k-g)_m}{(1+k)_m (1+a-e)_m (1+a-f)_m (1+a-g)_m}$$

$$\times {}_9F_8\left[\begin{array}{ccccc} k, 1+\tfrac{1}{2}k, k+b-a, k+c-a, k+d-a, & e, \\ \tfrac{1}{2}k, & 1+a-b, 1+a-c, 1+a-d, 1+k-e, \end{array}\right.$$
$$\left.\begin{array}{ccc} f, & g, & -m; \\ 1+k-f, 1+k-g, 1+k+m \end{array}\right],$$

where $k = 1+2a-b-c-d$, and the parameters are subject to the restriction that

$$b+c+d+e+f+g-m = 2+3a.$$

This formula, which is one of the most general known transformations of terminating well-poised series, connects two well-poised series, either of which is of general type except for the second parameter, and the restriction that the sum of the denominator parameters exceeds the sum of the numerator parameters by two.

4.4. Some deductions from the formulae of § 4.3. Dougall's theorem includes as limiting cases some of the previous results. For example, if we replace d, e by $1+a-d, 1+a-e$, and then let $a \to \infty$, we obtain Saalschütz's theorem. If we substitute for b and let $m \to \infty$ we obtain†

$$(1) \quad {}_5F_4\left[\begin{array}{ccccc} a, 1+\tfrac{1}{2}a, & c, & d, & e; \\ \tfrac{1}{2}a, & 1+a-c, 1+a-d, 1+a-e \end{array}\right]$$

$$= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)},$$

* Bailey 3.

† The justification of the passage to the limit is not particularly difficult. The details are given by Dougall 1, § 8.

which generalizes § 4.3 (3). The last formula reduces to Dixon's theorem when $e = \frac{1}{2}a$.

Similarly, if we let $m \to \infty$ in Whipple's transformation § 4.3 (4), we derive the formula*

$$(2) \quad {}_6F_5\left[\begin{array}{cccccc} a, & 1+\frac{1}{2}a, & b, & c, & d, & e; & -1 \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{array}\right]$$

$$= \frac{\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)}\, {}_3F_2\left[\begin{array}{ccc} 1+a-b-c, & d, & e; \\ 1+a-b, & 1+a-c \end{array}\right],$$

which expresses a well-poised ${}_6F_5$ with argument -1 in terms of a ${}_3F_2$, or conversely expresses any ${}_3F_2$ with unit argument in terms of a well-poised ${}_6F_5$.

From (2), taking $b + c = 1 + a$, we obtain

$$(3) \quad {}_4F_3\left[\begin{array}{cccc} a, & 1+\frac{1}{2}a, & d, & e; & -1 \\ & \frac{1}{2}a, & 1+a-d, & 1+a-e \end{array}\right]$$

$$= \frac{\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)},$$

a result which can also be derived from (1) by making $c \to -\infty$.

Dougall's theorem is itself an obvious particular case of § 4.3 (7), obtained by taking $k = a - b$, when the series on the right reduces to unity.

Now let $m \to \infty$ in § 4.3 (7), after replacing k and b by their values in terms of the other parameters. In doing this some care is necessary since, when m is large, the terms near both ends of the series on the right are important while the terms in the middle become negligible. We therefore suppose for convenience that m is odd, divide the series on the right into two parts of $\frac{1}{2}(m+1)$ terms each, and reverse the terms of the second half. There is then no great difficulty in justifying the limiting process, and we obtain the formula†

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* Whipple 2, equation 6.3.

† See Bailey 5 where the details are given. Another method of obtaining (4) is given in Chapter VI.

$$(4) \quad {}_7F_6\left[\begin{array}{ccccccc} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{array}\right]$$

$$= \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1+a-e-f-g)}{\Gamma(1+a)\,\Gamma(1+a-f-g)\,\Gamma(1+a-g-e)\,\Gamma(1+a-e-f)}$$

$$\times {}_4F_3\left[\begin{array}{cc} 1+a-c-d, & e, f, g; \\ 1+a-c, & 1+a-d, & e+f+g-a \end{array}\right]$$

$$+ \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)}{\Gamma(1+a)\,\Gamma(1+a-c-d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(g)}$$

$$\times \frac{\Gamma(e+f+g-1-a)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-e-f-g)\,\Gamma(2+2a-d-e-f-g)}$$

$$\times {}_4F_3\left[\begin{array}{c} 2+2a-c-d-e-f-g,\, 1+a-f-g,\, 1+a-g-e,\, 1+a-e-f; \\ 2+a-e-f-g,\ 2+2a-c-e-f-g,\ 2+2a-d-e-f-g \end{array}\right].$$

This relation is true if

$$R(2+2a-c-d-e-f-g) > 0.$$

It is a generalization of Whipple's transformation § 4.3 (4). It will be noticed that, when one of $e, f, g, 1+a-c-d$ is a negative integer $-n$, the second term on the right vanishes owing to the presence of $\Gamma(-n)$ in the denominator. We thus see that

$$(5) \quad {}_7F_6\left[\begin{array}{ccccccc} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g; \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{array}\right]$$

$$= \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1+a-e-f-g)}{\Gamma(1+a)\,\Gamma(1+a-f-g)\,\Gamma(1+a-g-e)\,\Gamma(1+a-e-f)}$$

$$\times {}_4F_3\left[\begin{array}{cc} 1+a-c-d, & e, f, g; \\ 1+a-c, & 1+a-d, & e+f+g-a \end{array}\right],$$

provided only that the series on the right terminates and the series on the left converges. This is the form in which Whipple* stated his theorem.

Again, if in § 4.3 (7) we substitute for g and let $f \to \infty$, or substitute for d and let $c \to \infty$, we obtain relations between well-poised ${}_7F_6$. Such relations will be discussed in Chapter VII.

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* Whipple 2, equation 7.7. See also Whipple 4.

4.5. Transformations of nearly-poised series of the second kind. In §4.3(1) take $s = 0$ and sum the series on the right by Vandermonde's theorem. We thus obtain the formula

$$(1) \quad {}_4F_3\left[\begin{array}{cccc} a, & b, & c, & -m; \\ & 1+a-b, & 1+a-c, & w \end{array}\right]$$

$$= \frac{(w-a)_m}{(w)_m}$$

$$\times {}_5F_4\left[\begin{array}{ccccc} 1+a-w, & \tfrac{1}{2}a, & \tfrac{1}{2}(1+a), & 1+a-b-c, & -m; \\ 1+a-b, & 1+a-c, & \tfrac{1}{2}(1+a-w-m), & 1+\tfrac{1}{2}(a-w-m) \end{array}\right],$$

which is due to Whipple,[*] and transforms a nearly-poised ${}_4F_3$ into a Saalschützian ${}_5F_4$.

If the series on the right of (1) reduces to a ${}_3F_2$ it can be summed by Saalschütz's theorem. We thus derive the following special cases:

$$(1.1) \quad {}_3F_2\left[\begin{array}{ccc} a, & 1+\tfrac{1}{2}a, & -m; \\ & \tfrac{1}{2}a, & w \end{array}\right] = \frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m},$$

$$(1.2) \quad {}_3F_2\left[\begin{array}{ccc} a, & b, & -m; \\ & 1+a-b, & 1+2b-m \end{array}\right] = \frac{(a-2b)_m(1+\tfrac{1}{2}a-b)_m(-b)_m}{(1+a-b)_m(\tfrac{1}{2}a-b)_m(-2b)_m},$$

$$(1.3) \quad {}_4F_3\left[\begin{array}{cccc} a, & 1+\tfrac{1}{2}a, & b, & -m; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+2b-m \end{array}\right] = \frac{(a-2b)_m(-b)_m}{(1+a-b)_m(-2b)_m},$$

$$(1.4) \quad {}_4F_3\left[\begin{array}{cccc} a, & 1+\tfrac{1}{2}a, & b, & -m; \\ & \tfrac{1}{2}a, & 1+a-b, & 2+2b-m \end{array}\right]$$

$$= \frac{(a-2b-1)_m(\tfrac{1}{2}+\tfrac{1}{2}a-b)_m(-b-1)_m}{(1+a-b)_m(\tfrac{1}{2}a-b-\tfrac{1}{2})_m(-2b-1)_m}.$$

Of these formulae (1.1) is the only one that we can use on the right of §4.3 (1), and in this case we derive the formula[†]

$$(2) \quad {}_5F_4\left[\begin{array}{ccccc} a, & 1+\tfrac{1}{2}a, & b, & c, & -m; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & w \end{array}\right]$$

$$= \frac{(w-a-1-m)(w-a)_{m-1}}{(w)_m}$$

$$\times {}_5F_4\left[\begin{array}{ccccc} 1+\tfrac{1}{2}a, & \tfrac{1}{2}+\tfrac{1}{2}a, & 1+a-b-c, & 1+a-w, & -m; \\ \tfrac{1}{2}(3+a-w-m), & 1+\tfrac{1}{2}(a-w-m), & 1+a-b, & 1+a-c \end{array}\right],$$

* Whipple 5.   † Bailey 3, equation 8.5.

which expresses a nearly-poised ${}_5F_4$ in terms of a Saalschützian ${}_5F_4$.

We now turn to §4.3 (6) and choose the parameters so that the series on the right can be summed by Saalschütz's theorem, (1.3), (1.2) or (1.4). We thus derive the four following transformations[*]

$$(3) \quad {}_5F_4\left[\begin{array}{ccccc} a, & b, & c, & d, & -m; \\ & 1+a-b, & 1+a-c, & 1+a-d, & w \end{array}\right]$$

$$= \frac{(1+2k-a)_m(1+k-a)_m}{(1+k)_m(1+2k-2a)_m}$$

$$\times {}_9F_8\left[\begin{array}{ccccccc} k, & 1+\tfrac{1}{2}k, & \tfrac{1}{2}a, & \tfrac{1}{2}+\tfrac{1}{2}a, & k+b-a, & k+c-a, & k+d-a, \\ & \tfrac{1}{2}k, & 1+k-\tfrac{1}{2}a, & \tfrac{1}{2}+k-\tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, \end{array}\right.$$

$$\left.\begin{array}{cc} 1+a-w, & -m; \\ k-a+w, & 1+k+m \end{array}\right]$$

where $\qquad k = 1+2a-b-c-d, \quad w = 2a-2k-m;$

$$(4) \quad {}_6F_5\left[\begin{array}{cccccc} a, & 1+\tfrac{1}{2}a, & b, & c, & d, & -m; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & w \end{array}\right]$$

$$= \frac{(2k-a)_m(k-a)_m}{(1+k)_m(2k-2a)_m}$$

$$\times {}_9F_8\left[\begin{array}{ccccccc} k, & 1+\tfrac{1}{2}k, & \tfrac{1}{2}+\tfrac{1}{2}a, & 1+\tfrac{1}{2}a, & k+b-a, & k+c-a, & k+d-a, \\ & \tfrac{1}{2}k, & \tfrac{1}{2}+k-\tfrac{1}{2}a, & k-\tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, \end{array}\right.$$

$$\left.\begin{array}{cc} 1+a-w, & -m; \\ k-a+w, & 1+k+m \end{array}\right]$$

where $w = 1+2a-2k-m$, and k is the same as before;

$$(5) \quad {}_5F_4\left[\begin{array}{ccccc} a, & b, & c, & d, & -m; \\ & 1+a-b, & 1+a-c, & 1+a-d, & w \end{array}\right]$$

$$= \frac{(k-a)_m(1+2k-a)_{m-1}(2k-a+2m)}{(1+k)_m(2k-2a)_m}$$

$$\times {}_9F_8\left[\begin{array}{ccccccc} k, & 1+\tfrac{1}{2}k, & \tfrac{1}{2}+\tfrac{1}{2}a, & \tfrac{1}{2}a, & k+b-a, & k+c-a, & k+d-a, \\ & \tfrac{1}{2}k, & \tfrac{1}{2}+k-\tfrac{1}{2}a, & 1+k-\tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, \end{array}\right.$$

$$\left.\begin{array}{cc} 1+a-w, & -m; \\ k-a+w, & 1+k+m \end{array}\right],$$

where $w = 1+2a-2k-m;$

* Bailey 3, equations 8.1–8.4.

$$(6) \quad {}_6F_5\left[\begin{array}{cccccc} a, & 1+\tfrac12 a, & b, & c, & d, & -m; \\ & \tfrac12 a, & 1+a-b, & 1+a-c, & 1+a-d, & w \end{array}\right]$$

$$= \frac{(k-a-1)_m (2k-a)_{m-1}(2k-a+2m-1)}{(1+k)_m (2k-2a-1)_m}$$

$$\times {}_9F_8\left[\begin{array}{cccccc} k, & 1+\tfrac12 k, & \tfrac12+\tfrac12 a, & 1+\tfrac12 a, & k+b-a, & k+c-a, & k+d-a, \\ & \tfrac12 k, & \tfrac12+k-\tfrac12 a, & k-\tfrac12 a, & 1+a-b, & 1+a-c, & 1+a-d, \end{array}\right.$$

$$\left.\begin{array}{cc} 1+a-w, & -m; \\ k-a+w, & 1+k+m \end{array}\right],$$

where $w = 2 + 2a - 2k - m$.

The last four formulae all express nearly-poised series in terms of well-poised series. In (3) and (4) the nearly-poised series are Saalschützian, and in (5) and (6) they are such that the sum of the denominator parameters exceeds that of the numerator parameters by two.

4.6. Transformations of nearly-poised series of the first kind. A terminating series can evidently have its terms written in the reverse order. A nearly-poised series of the second kind then becomes a similar series of the first kind,* while a Saalschützian series or a well-poised series remains of the same type. Thus from the formulae of the last paragraph we can derive transformations of nearly-poised series of the first kind. The results obtained from § 4.5 (1) and (2) are†

$$(1) \quad {}_4F_3\left[\begin{array}{cccc} a, & b, & c, & -m; \\ & \kappa-b, & \kappa-c, & \kappa+m; \end{array}\right] = \frac{(\kappa)_m (\kappa-b-c)_m}{(\kappa-b)_m (\kappa-c)_m}$$

$$\times {}_5F_4\left[\begin{array}{ccccc} \tfrac12\kappa-\tfrac12 a, & \tfrac12+\tfrac12\kappa-\tfrac12 a, & b, & c, & -m; \\ \kappa-a, & \tfrac12\kappa, & \tfrac12+\tfrac12\kappa, & b+c-\kappa+1-m \end{array}\right],$$

$$(2) \quad {}_5F_4\left[\begin{array}{ccccc} a, & 1+\tfrac12\kappa, & b, & c, & -m; \\ & \tfrac12\kappa, & 1+\kappa-b, & 1+\kappa-c, & 1+\kappa+m \end{array}\right]$$

$$= \frac{(1+\kappa)_m (1+\kappa-b-c)_m}{(1+\kappa-b)_m (1+\kappa-c)_m}$$

$$\times {}_5F_4\left[\begin{array}{ccccc} \tfrac12\kappa-\tfrac12 a, & \tfrac12+\tfrac12\kappa-\tfrac12 a, & b, & c, & -m; \\ 1+\kappa-a, & \tfrac12\kappa, & \tfrac12+\tfrac12\kappa, & b+c-\kappa-m \end{array}\right].$$

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* Whipple 5, § 6.

† (1) is due to Whipple 5, and (2) to Bailey 3. For the results derived from § 4.5, (3)-(6), see Bailey 3.

It should be noticed that if, in (1), we let m tend to infinity through integral values, we obtain the formula*

$$(3) \quad {}_3F_2\left[\begin{array}{ccc} a, & b, & c; & -1 \\ & \kappa-b, & \kappa-c \end{array}\right]$$

$$= \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)}{\Gamma(\kappa)\,\Gamma(\kappa-b-c)}\,{}_4F_3\left[\begin{array}{cccc} b, & c, & \tfrac12(\kappa-a), & \tfrac12(1+\kappa-a); \\ & \kappa-a, & \tfrac12\kappa, & \tfrac12(\kappa+1) \end{array}\right],$$

which transforms a non-terminating nearly-poised ${}_3F_2$ with argument -1. The series on the right is not Saalschützian.

When $c = \tfrac12\kappa$ in (3), the relation expresses any ${}_2F_1$ with argument -1 in terms of a ${}_3F_2$. This ${}_3F_2$ can be transformed by the formulae of Chapter III, and so a large number of series can be found allied to a given ${}_2F_1$ with argument -1. Such series have been considered in detail by Whipple.†

4.7. The case when a is a negative integer. In certain applications of nearly-poised series of the second kind the parameter a is a negative integer. Returning to § 4.3 it is evident that formula (1) of that paragraph is still true if a is a negative integer $-n$, and $-m$ is not necessarily a negative integer. Thus, taking $s = 0$ and using Vandermonde's theorem, we derive the formula‡

$$(1) \quad {}_4F_3\left[\begin{array}{cccc} -n, & b, & c, & d; \\ & 1-n-b, & 1-n-c, & w \end{array}\right] = \frac{(w-d)_n}{(w)_n}$$

$$\times {}_5F_4\left[\begin{array}{ccccc} d, & 1-n-b-c, & -\tfrac12 n, & \tfrac12-\tfrac12 n, & 1-n-w; \\ 1-n-b, & 1-n-c, & \tfrac12(1+d-w-n), & 1+\tfrac12(d-w-n) \end{array}\right].$$

This appears to be the most interesting formula of its type.

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* Whipple 5.

† Whipple 8.

‡ Whipple 5, equation 6.6.

CHAPTER V

METHODS OF OBTAINING TRANSFORMATIONS OF HYPERGEOMETRIC SERIES; (2) BY DOUGALL'S METHOD AND CARLSON'S THEOREM

5.1. An elementary proof of Dougall's theorem. When transformations of terminating series have been discovered, they are usually capable of being proved in a very simple way. We begin by proving Dougall's theorem in a manner substantially equivalent to his original proof.*

Writing f in place of $-m$, the theorem becomes

$$_7F_6\left[\begin{array}{ccccccc} a, & 1+\tfrac{1}{2}a, & b, & c, & d, & e, & f; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{array}\right]$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

$$\times \frac{\Gamma(1+a-b-c-d)\Gamma(1+a-b-c-f)\Gamma(1+a-b-d-f)}{\Gamma(1+a-b-f)\Gamma(1+a-c-f)\Gamma(1+a-d-f)} \frac{\Gamma(1+a-c-d-f)}{\Gamma(1+a-b-c-d-f)}$$

provided that $1+2a = b+c+d+e+f$, and f is a negative integer.

Suppose the theorem is true when $-f = 0, 1, 2, \ldots, m-1$. We shall prove it true when $f = -m$, and then the result will follow by induction.

Now, by symmetry, the result is true if $-c$ or $-d$ has one of the values $0, 1, 2, \ldots, m-1$, that is if $-c$ or $b+c+e+f-1-2a$ has one of these values. It is therefore true in particular when $f = -m$ and c has one of $2m$ values. But, when $f = -m$, we can multiply by $(1+a-c)_m(1+a-b-c-d)_m$ and the formula states the equality of two polynomials of degree $2m$ in c. Thus, if we can prove the equality for one more value of c, the result will be proved. We choose the value $c = a+m$ which is a pole of the last term only of the series, and the result is easily verified.

* Dougall 1. See also Hardy 2.

5.2. An elementary proof of a transformation of well-poised series $_9F_8$. Similarly we can prove the more general relation §4.3(7) connecting two well-poised $_9F_8$. This relation can be written

$$_9F_8\left[\begin{array}{cccccc} a, & 1+\tfrac{1}{2}a, & b, & c, & d, & e, \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, \end{array}\right.$$
$$\left.\begin{array}{ccc} f, & g, & h; \\ 1+a-f, & 1+a-g, & 1+a-h \end{array}\right]$$

$$= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-h)}{\Gamma(1+a)\Gamma(1+a-e-f)\Gamma(1+a-e-g)\Gamma(1+a-e-h)}$$

$$\times \frac{\Gamma(1+a-f-g-h)\Gamma(1+a-e-g-h)\Gamma(1+a-e-f-h)}{\Gamma(1+a-f-g)\Gamma(1+a-f-h)\Gamma(1+a-g-h)} \frac{\Gamma(1+a-e-f-g)}{\Gamma(1+a-e-f-g-h)}$$

$$\times {}_9F_8\left[\begin{array}{cccccc} k, & 1+\tfrac{1}{2}k, & k+b-a, & k+c-a, & k+d-a, & e, \\ & \tfrac{1}{2}k, & 1+a-b, & 1+a-c, & 1+a-d, & 1+k-e, \end{array}\right.$$
$$\left.\begin{array}{ccc} f, & g, & h; \\ 1+k-f, & 1+k-g, & 1+k-h \end{array}\right]$$

where

$$k = 1+2a-b-c-d,$$

$$b+c+d+e+f+g+h = 2+3a,$$

and h is a negative integer.

Suppose this result is true when $-h$ has any one of the values $0, 1, 2, \ldots, m-1$. Then, by symmetry, it is true when $-e$, $-f$ or $-g$ has one of these values. We proceed to prove that the result is true when $h = -m$, and then the theorem will follow by induction.

If $h = -m$, the formula can be written

$$(1) \qquad (1+k)_m(1+a-e)_m(1+a-f)_m(1+k-e-f)_m$$

$$\times {}_9F_8\left[\begin{array}{cccccc} a, & 1+\tfrac{1}{2}a, & b, & c, & d, & e, \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, \end{array}\right.$$
$$\left.\begin{array}{ccc} f, & 1+a+k-e-f+m, & -m; \\ 1+a-f, & e+f-k-m, & 1+a+m \end{array}\right]$$

$$= (1+a)_m (1+k-e)_m (1+k-f)_m (1+a-e-f)_m$$

$$\times {}_9F_8 \begin{bmatrix} k, & 1+\tfrac{1}{2}k, & k+b-a, & k+c-a, & k+d-a, & e, \\ & \tfrac{1}{2}k, & 1+a-b, & 1+a-c, & 1+a-d, & 1+k-e, \\[1em] & & f, & 1+a+k-e-f+m, & -m; \\ & & 1+k-f, & e+f-a-m, & 1+k+m \end{bmatrix}$$

where $k = 1 + 2a - b - c - d$.

By hypothesis this is true when f or $1+a+k-e-f+m (=g)$ has one of the values $0, -1, -2, \ldots, -m+1$, that is for $2m$ values of f.

But the expressions on each side of (1) are polynomials in f of degree $2m$. It will therefore be sufficient if these expressions are equal for one other value of f to establish the fact that (1) is an identity. We choose the value $f = k + m$, and then we require

$$_7F_6 \begin{bmatrix} a, & 1+\tfrac{1}{2}a, & b, & c, & d, & f, & -m; \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-f, & 1+a+m \end{bmatrix}$$

$$= \frac{(1+a)_m (k+b-a)_m (k+c-a)_m (k+d-a)_m}{(k-a)_m (1+a-b)_m (1+a-c)_m (1+a-d)_m}.$$

This is true by Dougall's theorem, and so the proof is complete.

5.3. Carlson's theorem.

When a transformation is known to be true for terminating series, it can sometimes be shown to be true also (with slight modifications) for non-terminating series, by using a theorem due to Carlson.* To prove this theorem we require some preliminary lemmas.

Lemma 1. *If $f(z)$ is regular in a region D and on its boundary C (a simple closed curve), and if $|f(z)| \leqslant M$ on C, then $|f(z)| \leqslant M$ at all interior points of D.*

We first notice that if $\phi(x)$ is continuous, $\phi(x) \leqslant k$, and

$$\frac{1}{b-a} \int_a^b \phi(x)\,dx \geqslant k,$$

* Carlson 1. See also Wigert 1, Riesz 1 and Hardy 1. The proof given here follows that given by Titchmarsh, *Theory of functions* (1932), Chapter V.

then $\phi(x) = k$. For if $\phi(\xi) < k$, there is an interval $(\xi - \delta, \xi + \delta)$ in which $\phi(x) \leqslant k - \epsilon$ (say), and

$$\int_a^b \phi(x)\,dx \leqslant 2\delta(k-\epsilon) + (b-a-2\delta)k = (b-a)k - 2\delta\epsilon,$$

which contradicts the hypotheses.

Now suppose that, at an interior point z_0 of D, $|f(z)|$ has a value at least equal to its value anywhere else. Let Γ be a circle centre z_0 lying entirely in D. Then

$$(1) \qquad f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\,dz.$$

Putting $z - z_0 = re^{i\theta}$, $f(z)/f(z_0) = \rho e^{i\phi}$, so that ρ and ϕ are functions of θ, we may write (1) as

$$(2) \qquad 1 = \frac{1}{2\pi} \int_0^{2\pi} \rho e^{i\phi}\,d\theta.$$

Hence

$$1 \leqslant \frac{1}{2\pi} \int_0^{2\pi} \rho\,d\theta.$$

But by hypothesis $\rho \leqslant 1$, and so $\rho = 1$ for all values of θ.

Taking the real part of (2), we now obtain

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \cos\phi\,d\theta,$$

and so $\cos\phi = 1$. Hence $f(z) = f(z_0)$ on Γ, and so everywhere; that is $f(z)$ is a constant.

Thus $|f(z)| < M$ at all interior points of D unless $f(z)$ is a constant.

Lemma 2. *Let $f(z)$ be an analytic function of $z (= re^{i\theta})$, regular in the region D between two straight lines making an angle π/α at the origin, and on the lines themselves. Suppose that*

$$(3) \qquad |f(z)| \leqslant M$$

on the lines, and that, as $r \to \infty$,

$$(4) \qquad f(z) = O(e^{r\beta}),$$

where $\beta < \alpha$, uniformly in the angle. Then the inequality (3) holds throughout the region D.

We may suppose, without loss of generality, that the two lines are $\theta = \pm \frac{1}{2}\pi/\alpha$.

Let
$$F(z) = e^{-\epsilon z^\gamma} f(z),$$

where $\beta < \gamma < \alpha$ and $\epsilon > 0$. Then

(5)
$$|F(z)| = e^{-\epsilon r^\gamma \cos\gamma\theta} |f(z)|.$$

On the lines $\theta = \pm \frac{1}{2}\pi/\alpha$, $\cos\gamma\theta > 0$ since $\gamma < \alpha$. Hence on these lines
$$|F(z)| \leqslant |f(z)| \leqslant M.$$

Also on the arc $|\theta| \leqslant \frac{1}{2}\pi/\alpha$ of the circle $|z| = R$,
$$|F(z)| \leqslant e^{-\epsilon R^\gamma \cos\frac{1}{2}\gamma\pi/\alpha} |f(z)| < A e^{R^\beta - \epsilon R^\gamma \cos\frac{1}{2}\gamma\pi/\alpha}$$

and the right-hand side $\to 0$ as $R \to \infty$. Hence, if R is sufficiently large, $|F(z)| \leqslant M$ on this arc also. Thus, by lemma 1, $|F(z)| \leqslant M$ throughout the interior of the region $|\theta| \leqslant \frac{1}{2}\pi/\alpha$, $r \leqslant R$; that is, since R is arbitrarily large, throughout the region D. Hence by (5)
$$|f(z)| \leqslant M e^{\epsilon r^\gamma}$$

in D, and, making $\epsilon \to 0$, the result stated follows.

Lemma 3. *Suppose that $f(z)$ is regular and of the form $O(e^{k|z|})$ for $\theta_1 \leqslant \theta \leqslant \theta_2$, where $\theta_2 - \theta_1 < \pi$. Suppose also that $f(z) = O(e^{h_1|z|})$ when $\theta = \theta_1$, and $f(z) = O(e^{h_2|z|})$ when $\theta = \theta_2$. Let $H(\theta)$ be the function of the form $a\cos\theta + b\sin\theta$ which takes the values h_1, h_2 at θ_1, θ_2. Then*
$$f(z) = O(e^{H(\theta)r})$$

uniformly in the angle $\theta_1 \leqslant \theta \leqslant \theta_2$.

The value of $H(\theta)$ is easily seen to be
$$H(\theta) = \frac{h_1 \sin(\theta_2 - \theta) + h_2 \sin(\theta - \theta_1)}{\sin(\theta_2 - \theta_1)}.$$

Let
$$F(z) = f(z) e^{-(a - ib)z}.$$
Then

(6)
$$|F(z)| = |f(z)| e^{-H(\theta)r},$$

and so, if r is large enough,
$$|F(re^{i\theta_1})| = O(e^{(h_1 - H(\theta_1))r}) = O(1).$$

A similar result holds for $F(re^{i\theta_2})$. Hence by lemma 2, $F(z)$ is bounded in the angle (θ_1, θ_2) and the result stated follows from (6).

Lemma 4. *If $f(z)$ is regular and of the form $O(e^{k|z|})$ for $R(z) \geqslant 0$, and $f(z) = O(e^{-a|z|})$, where $a > 0$, on the imaginary axis, then $f(z)$ is identically zero.*

We apply lemma 3 to $f(z)$ with $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$, $h_1 = k$, $h_2 = -a$. Then

(7)
$$f(z) = O\{e^{(k\cos\theta - a|\sin\theta|)r}\}$$

for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

Similarly, by taking $\theta_1 = -\frac{1}{2}\pi$, $\theta_2 = 0$, $h_1 = -a$, $h_2 = k$ in lemma 3, we find that (7) also holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$.

Let
$$F(z) = e^{\omega z} f(z),$$

where ω is a (large) positive number. Then by (7) there is a constant M, independent of ω, such that

(8)
$$|F(z)| \leqslant M e^{\{(k+\omega)\cos\theta - a|\sin\theta|\}r}$$

for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. In particular we have

(9)
$$|F(z)| \leqslant M$$

for $\theta = \pm\frac{1}{2}\pi$ and $\theta = \pm\alpha$, where $\alpha = \arctan\{(k + \omega)/a\}$.

We can now apply lemma 2 to each of the three angles $(-\frac{1}{2}\pi, -\alpha)$, $(-\alpha, \alpha)$ and $(\alpha, \frac{1}{2}\pi)$. It follows that (9) actually holds for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Hence
$$|f(z)| \leqslant M e^{-\omega r \cos\theta}$$

and, making $\omega \to \infty$, it follows that $|f(z)| = 0$. This proves the lemma.*

Carlson's theorem. *If $f(z)$ is regular and of the form $O(e^{k|z|})$, where $k < \pi$, for $R(z) \geqslant 0$, and if $f(z) = 0$ for $z = 0, 1, 2, \ldots$, then $f(z)$ is identically zero.*

Consider the function
$$F(z) = f(z) \operatorname{cosec} \pi z.$$

On the circles $|z| = n + \frac{1}{2}$, where n is a positive integer, $\operatorname{cosec} \pi z$ is bounded. Hence $F(z) = O(e^{k|z|})$ on the circles and also on the imaginary axis. Since $F(z)$ is regular, it follows that, if
$$n - \tfrac{1}{2} < |z| < n + \tfrac{1}{2},$$
$$F(z) = O(e^{k(n+\frac{1}{2})}) = O(e^{k|z|}),$$

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* Also due to Carlson 1.

and so $F(z)$ is of this form throughout $R(z) \geqslant 0$. Also

$$F(z) = O\left(e^{(k-\eta)|z|}\right)$$

on the imaginary axis. The result therefore follows from lemma 4.

5.4. An example on Carlson's theorem.

We shall illustrate the use of Carlson's theorem by applying it (with $k = 0$) to Whipple's transformation of a well-poised $_7F_6$, §4.4(5). We shall assume that the result is known to be true when g is a negative integer. This has been proved in §4.3, but it can also be proved by Dougall's method.* Writing $-k-z$ for g, where k is a positive integer, the theorem to be proved is

$$(1) \quad {}_7F_6 \begin{bmatrix} a, & 1+\tfrac{1}{2}a, & c, & d, & e, & f, \\ & \tfrac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, \\ & & & & & -k-z; \\ & & & & & 1+a+k+z \end{bmatrix}$$

$$= \frac{\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(1+a+k+z)\,\Gamma(1+a-e-f+k+z)}{\Gamma(1+a)\,\Gamma(1+a-e-f)\,\Gamma(1+a-f+k+z)\,\Gamma(1+a-e+k+z)}$$

$$\times {}_4F_3 \begin{bmatrix} 1+a-c-d, & e, & f, & -k-z; \\ 1+a-c, & 1+a-d, & e+f-a-k-z \end{bmatrix}$$

and it is known to be true when $z = 0, 1, 2, \ldots$.

We proceed to prove that (1) is true when $1+a-c-d$ is a negative integer and the series on the left is convergent, so that

$$(2) \qquad R\{4 + 4a - 2(c+d+e+f) + 2k + 2z\} > 0.$$

The series $_6F_5$ derived from the left-hand side of (1) by suppressing the parameters involving z is absolutely and uniformly convergent for real values of a, c, d, e, f such that all the denominator parameters are positive and

$$(3) \qquad 3 + 3a - 2(c+d+e+f) > 0.$$

Also, if $R(z) \geqslant 0$,

$$\left| \frac{-k-z+r}{1+a+k+z+r} \right| < 1,$$

* See Whipple 4 where Carlson's theorem is also used to prove the result when the series on the left does not terminate. Carlson's theorem had previously been used by Hardy 2 to prove § 4.4 (1).

for $r = 0, 1, 2, \ldots$ since $1 + a > 0$. Hence the function $_7F_6$ is absolutely and uniformly convergent for $R(z) \geqslant 0$, and is regular and bounded in this region.

Now, if also

$$(4) \qquad 1 + a + k - e - f > 0,$$

the function

$$\frac{\Gamma(1+a+k+z)\,\Gamma(1+a-e-f+k+z)}{\Gamma(1+a-f+k+z)\,\Gamma(1+a-e+k+z)}$$

is regular for $R(z) \geqslant 0$. Since it tends to unity when $|z| \to \infty$, it is bounded in the half-plane. The series $_4F_3$ consists of a finite number of terms, and each term is bounded for $R(z) \geqslant 0$ if

$$(5) \qquad 1 + 2a + k - c - d - e - f > 0,$$

in which case $e + f - a - k - z$ cannot be a negative integer $\geqslant 1 + a - c - d$. Accordingly, if $f(z)$ is the difference between the two sides of (1), the conditions of Carlson's theorem are satisfied by $f(z)$. It follows that (1) holds so long as the parameters are subject to the conditions (3), (4) and (5) and the restrictions that the denominator parameters not involving z are real and positive. These conditions may be removed by analytic continuation, and (1) is true provided that $1 + a - c - d$ is a negative integer and the series on the left is convergent.

It will be noticed that the formula § 4.3 (7) connecting two terminating series $_9F_8$ cannot possibly be generalized in this way owing to the presence of a denominator parameter of the form $A - m$ on both sides of the formula.

CHAPTER VI

METHODS OF OBTAINING TRANSFORMATIONS OF HYPERGEOMETRIC SERIES; (3) BY BARNES' CONTOUR INTEGRALS

6.1. Introductory remarks. In Chapter V we saw how transformations of non-terminating series can sometimes be derived by a use of Carlson's theorem, and in § 4.4 some transformations of such series were obtained by a limiting process from transformations connecting terminating series of higher orders. In this chapter a direct method* is given in which free use is made of contour integrals of Barnes' type.

6.2. Barnes' second lemma. We now prove the formula†

$$(1) \quad \frac{1}{2\pi i} \int \frac{\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(\alpha_3+s)\,\Gamma(1-\beta_1-s)\,\Gamma(-s)\,ds}{\Gamma(\beta_2+s)}$$

$$= \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(1-\beta_1+\alpha_1)\,\Gamma(1-\beta_1+\alpha_2)\,\Gamma(1-\beta_1+\alpha_3)}{\Gamma(\beta_2-\alpha_1)\,\Gamma(\beta_2-\alpha_2)\,\Gamma(\beta_2-\alpha_3)}$$

provided that $\beta_1+\beta_2=\alpha_1+\alpha_2+\alpha_3+1$. The path of integration is a line parallel to the imaginary axis except that it is curved, if necessary, so that the decreasing sequences of poles lie to the left, and the increasing sequences of poles to the right of the contour.‡ All the integrals in this chapter are of this type.

By Barnes' lemma (§ 1.7) we have

$$\frac{1}{2\pi i} \int \Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(n-s)\,\Gamma(\beta_1-\alpha_1-\alpha_2-s)\,ds$$

$$= \frac{\Gamma(\alpha_1+n)\,\Gamma(\alpha_2+n)\,\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)}{\Gamma(\beta_1+n)}.$$

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* Bailey 8.
† Barnes 2.
‡ The integral is taken from $c-i\infty$ to $c+i\infty$. In some papers the integrals are taken in the opposite direction, and so variations in sign occur.

Thus

$${}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2\end{matrix}\right]=\frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}$$

$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\frac{(\alpha_3)_n}{(\beta_2)_n}\frac{1}{2\pi i}\int \Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(n-s)\,\Gamma(\beta_1-\alpha_1-\alpha_2-s)\,ds$$

$$= \frac{\Gamma(\beta_1)}{\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}$$

$$\times \frac{1}{2\pi i}\int \Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(\beta_1-\alpha_1-\alpha_2-s)\,\Gamma(-s)\,{}_2F_1\left[\begin{matrix}\alpha_3,-s;\\ \beta_2\end{matrix}\right]ds$$

and so

$$(2) \quad {}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2\end{matrix}\right]=\frac{\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\beta_1-\alpha_1)\,\Gamma(\beta_1-\alpha_2)\,\Gamma(\beta_2-\alpha_3)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(\beta_1-\alpha_1-\alpha_2-s)\,\Gamma(\beta_2-\alpha_3+s)\,\Gamma(-s)\,ds}{\Gamma(\beta_2+s)}.$$

The interchange in the order of summation and integration can easily be justified if $R(\beta_2-\alpha_3+s)>0$. Now take $\beta_1=\alpha_3$; the series on the left can be summed by Gauss's theorem, and the lemma is proved.

If the integral in (1) is evaluated in terms of hypergeometric series by considering the residues at poles on the right of the contour, we obtain a relation which reduces to Saalschütz's theorem when one of the parameters α_1, α_2, α_3 is a negative integer.*

6.3. Integrals representing well-poised series. From Barnes' second lemma it is easily verified that

$$\frac{\Gamma(\alpha_1+n)\,\Gamma(\alpha_2+n)\,\Gamma(\alpha_3+n)}{\Gamma(\kappa-\alpha_1+n)\,\Gamma(\kappa-\alpha_2+n)\,\Gamma(\kappa-\alpha_3+n)}$$

$$= \frac{1}{\Gamma(\kappa-\alpha_2-\alpha_3)\,\Gamma(\kappa-\alpha_3-\alpha_1)\,\Gamma(\kappa-\alpha_1-\alpha_2)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(\alpha_3+s)\,\Gamma(\kappa-\alpha_1-\alpha_2-\alpha_3-s)\,\Gamma(n-s)\,ds}{\Gamma(\kappa+n+s)}.$$

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* The relation similarly obtained from (2) is equivalent to § 3.8 (1).

It follows, by expansion and the interchange of the order of summation and integration, that

$$(1) \quad F\begin{bmatrix} a, & \alpha_1, & \alpha_2, & \alpha_3, & \rho_1, \rho_2, \cdots \rho_r; \\ & \kappa - \alpha_1, \kappa - \alpha_2, \kappa - \alpha_3, \sigma_1, \sigma_2, \cdots \sigma_r \end{bmatrix}$$

$$= \frac{\Gamma(\kappa - \alpha_1)\,\Gamma(\kappa - \alpha_2)\,\Gamma(\kappa - \alpha_3)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)\,\Gamma(\kappa - \alpha_2 - \alpha_3)\,\Gamma(\kappa - \alpha_1 - \alpha_3)\,\Gamma(\kappa - \alpha_1 - \alpha_2)}$$

$$\times \frac{1}{2\pi i} \int \frac{\Gamma(\alpha_1 + s)\,\Gamma(\alpha_2 + s)\,\Gamma(\alpha_3 + s)\,\Gamma(\kappa - \alpha_1 - \alpha_2 - \alpha_3 - s)\,\Gamma(-s)}{\Gamma(\kappa + s)}$$

$$\times F\begin{bmatrix} a, & \rho_1, & \rho_2, & \ldots, & \rho_r, & -s; \\ & \sigma_1, & \sigma_2, & \ldots, \sigma_r, & \kappa + s \end{bmatrix} ds.$$

Thus, if we can sum the series on the right of (1) in terms of gamma functions, we can find an integral of Barnes' type representing the series on the left. If we use Dixon's theorem on the right, we obtain an integral representing the well-poised series

$$_5F_4\begin{bmatrix} a, & b, & c, & d, & e; \\ & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e \end{bmatrix},$$

and when $b = 1 + \tfrac{1}{2}a$ the integral can be evaluated by Barnes' second lemma, giving the formula § 4.4 (1). We therefore adjust the parameters in (1) so that the series on the right can be summed by that formula, and we obtain

$$(2) \quad _7F_6\begin{bmatrix} a, 1 + \tfrac{1}{2}a, & b, & c, & d, & e, & f; \\ \tfrac{1}{2}a, & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e, & 1 + a - f \end{bmatrix}$$

$$= \frac{\Gamma(1 + a - b)\,\Gamma(1 + a - c)\,\Gamma(1 + a - d)\,\Gamma(1 + a - e)\,\Gamma(1 + a - f)}{\Gamma(1 + a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(1 + a - c - d)\,\Gamma(1 + a - b - d)}$$
$$\frac{}{\Gamma(1 + a - b - c)\,\Gamma(1 + a - e - f)}$$

$$\times \frac{1}{2\pi i} \int \frac{\Gamma(b + s)\,\Gamma(c + s)\,\Gamma(d + s)\,\Gamma(1 + a - e - f + s)}{\Gamma(1 + a - b - c - d - s)\,\Gamma(-s)\,ds} \frac{\Gamma(1 + a - b - c - d - s)\,\Gamma(-s)\,ds}{\Gamma(1 + a - e + s)\,\Gamma(1 + a - f + s)}.$$

When $f = -n$, a negative integer, and $1 + 2a = b + c + d + e - n$, the integral on the right can be evaluated by Barnes' second lemma, and we obtain Dougall's theorem. We cannot use this theorem on the right of (1) and so the process comes to an end for well-poised series.

If we evaluate the integral on the right of (2) by considering the residues at poles on the right of the contour, we obtain the transformation § 4.4 (4) of a well-poised $_7F_6$ in terms of two Saalschützian $_4F_3$.

6.4. Integrals representing nearly-poised series of the first kind.
Now in § 6.3 (1) take $r = 0$ and sum the $_2F_1$ on the right by Gauss's theorem, and we find that

$$(1) \quad _4F_3\begin{bmatrix} a, & b, & c, & d; \\ & \kappa - b, & \kappa - c, & \kappa - d \end{bmatrix}$$

$$= \frac{\Gamma(\kappa - b)\,\Gamma(\kappa - c)\,\Gamma(\kappa - d)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(\kappa - c - d)\,\Gamma(\kappa - b - d)\,\Gamma(\kappa - b - c)}$$

$$\times \frac{1}{2\pi i} \int \frac{\Gamma(b + s)\,\Gamma(c + s)\,\Gamma(d + s)\,\Gamma(\kappa - a + 2s)}{\Gamma(\kappa - b - c - d - s)\,\Gamma(-s)\,ds} \frac{\Gamma(\kappa - b - c - d - s)\,\Gamma(-s)\,ds}{\Gamma(\kappa - a + s)\,\Gamma(\kappa + 2s)}$$

If $d = \tfrac{1}{2} + \tfrac{1}{2}\kappa$, $c = \tfrac{1}{2}\kappa$, we can evaluate the integral on the right by Barnes' second lemma, and (changing κ into $1 + \kappa$) we find that

$$(2) \quad _3F_2\begin{bmatrix} a, 1 + \tfrac{1}{2}\kappa, & b; \\ \tfrac{1}{2}\kappa, & 1 + \kappa - b \end{bmatrix}$$

$$= \frac{\Gamma(\tfrac{1}{2}\kappa)\,\Gamma(1 + \tfrac{1}{2}\kappa - \tfrac{1}{2}a)\,\Gamma(1 + \kappa - b)\,\Gamma(\kappa - a - 2b)}{\Gamma(1 + \tfrac{1}{2}\kappa)\,\Gamma(\tfrac{1}{2}\kappa - \tfrac{1}{2}a)\,\Gamma(1 + \kappa - a - b)\,\Gamma(\kappa - 2b)}.$$

Now use this result on the right of § 6.3 (1) and we obtain

$$(3) \quad _5F_4\begin{bmatrix} a, 1 + \tfrac{1}{2}\kappa, & b, & c, & d; \\ \tfrac{1}{2}\kappa, & 1 + \kappa - b, & 1 + \kappa - c, & 1 + \kappa - d \end{bmatrix}$$

$$= \frac{\kappa - a}{\kappa} \frac{\Gamma(1 + \kappa - b)\,\Gamma(1 + \kappa - c)\,\Gamma(1 + \kappa - d)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(1 + \kappa - c - d)\,\Gamma(1 + \kappa - b - d)}$$
$$\frac{}{\Gamma(1 + \kappa - b - c)}$$

$$\times \frac{1}{2\pi i} \int \frac{\Gamma(b + s)\,\Gamma(c + s)\,\Gamma(d + s)\,\Gamma(\kappa - a + 2s)}{\Gamma(1 + \kappa - b - c - d - s)\,\Gamma(-s)\,ds} \frac{\Gamma(1 + \kappa - b - c - d - s)\,\Gamma(-s)\,ds}{\Gamma(1 + \kappa - a + s)\,\Gamma(\kappa + 2s)}.$$

6.5. Transformations of nearly-poised series of the first kind.
We now evaluate the integral on the right of § 6.4 (1) by

considering the residues at poles on the right of the contour, and so obtain the formula

$$(1) \quad {}_4F_3\left[\begin{matrix} a, & b, & c, & d; \\ & \kappa-b, & \kappa-c, & \kappa-d \end{matrix}\right]$$

$$= \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa-d)\,\Gamma(\kappa-b-c-d)}{\Gamma(\kappa-c-d)\,\Gamma(\kappa-b-d)\,\Gamma(\kappa-b-c)\,\Gamma(\kappa)}$$

$$\times {}_5F_4\left[\begin{matrix} b, & c, & d, & \tfrac{1}{2}(\kappa-a), & \tfrac{1}{2}(1+\kappa-a); \\ \kappa-a, & \tfrac{1}{2}\kappa, & \tfrac{1}{2}(\kappa+1), & 1-\kappa+b+c+d \end{matrix}\right]$$

$$+ \frac{\Gamma(\kappa-b)\,\Gamma(\kappa-c)\,\Gamma(\kappa-d)\,\Gamma(b+c+d-\kappa)\,\Gamma(3\kappa-a-2b-2c-2d)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(2\kappa-a-b-c-d)\,\Gamma(3\kappa-2b-2c-2d)}$$

$$\times {}_5F_4\left[\begin{matrix} \kappa-c-d, & \kappa-b-d, & \kappa-b-c, & \tfrac{3}{2}\kappa-\tfrac{1}{2}a-b-c-d, \\ & & & \tfrac{1}{2}+\tfrac{3}{2}\kappa-\tfrac{1}{2}a-b-c-d; \\ 1+\kappa-b-c-d, & 2\kappa-a-b-c-d, & & \\ & & \tfrac{3}{2}\kappa-b-c-d, & \tfrac{1}{2}+\tfrac{3}{2}\kappa-b-c-d \end{matrix}\right].$$

This is a generalization of § 4.6 (1), and expresses a nearly-poised ${}_4F_3$ in terms of two Saalschützian ${}_5F_4$.

Similarly from § 6.4 (3) we obtain a transformation of a nearly-poised ${}_5F_4$ into two Saalschützian ${}_5F_4$, and this is a generalization of § 4.6 (2).

6.6. The integral analogue of Dougall's theorem.

The second lemma of Barnes may be regarded as the integral analogue of Saalschütz's theorem of which it gives a generalization. We may similarly enquire whether there is an integral analogous to Dougall's theorem.

First consider the integral

$$\frac{1}{2\pi i}\int \frac{\begin{matrix}\Gamma(a+s)\,\Gamma(1+\tfrac{1}{2}a+s)\,\Gamma(b+s)\,\Gamma(c+s)\,\Gamma(d+s)\\ \Gamma(b-a-s)\,\Gamma(-s)\,ds\end{matrix}}{\Gamma(\tfrac{1}{2}a+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)}$$

which is analogous to a well-poised series ${}_5F_4$. By considering the residues at poles on the right of the contour, the integral can be expressed in terms of two well-poised ${}_5F_4$ which can be summed by § 4.4 (1). We can thus evaluate the integral in terms of gamma functions, and we find after some reduction that

$$(1) \quad \frac{1}{2\pi i}\int \frac{\begin{matrix}\Gamma(a+s)\,\Gamma(1+\tfrac{1}{2}a+s)\,\Gamma(b+s)\,\Gamma(c+s)\,\Gamma(d+s)\\ \Gamma(b-a-s)\,\Gamma(-s)\,ds\end{matrix}}{\Gamma(\tfrac{1}{2}a+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)}$$

$$= \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(b+c-a)\,\Gamma(b+d-a)}{2\Gamma(1+a-c-d)\,\Gamma(b+c+d-a)}.$$

Similarly the more general integral

$$\frac{1}{2\pi i}\int \frac{\begin{matrix}\Gamma(a+s)\,\Gamma(1+\tfrac{1}{2}a+s)\,\Gamma(b+s)\,\Gamma(c+s)\,\Gamma(d+s)\\ \Gamma(e+s)\,\Gamma(f+s)\,\Gamma(b-a-s)\,\Gamma(-s)\,ds\end{matrix}}{\begin{matrix}\Gamma(\tfrac{1}{2}a+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)\\ \Gamma(1+a-e+s)\,\Gamma(1+a-f+s)\end{matrix}}$$

can be expressed in terms of two well-poised series ${}_7F_6$, but these can only be evaluated by Dougall's theorem when they terminate, and then the contour cannot be drawn to separate the increasing and decreasing sequences of poles. We can, however, evaluate the integral in another way.

From Barnes' second lemma we have

$$\frac{\Gamma(d+s)\,\Gamma(e+s)\,\Gamma(f+s)}{\Gamma(1+a-d+s)\,\Gamma(1+a-e+s)\,\Gamma(1+a-f+s)}$$

$$= \frac{1}{\Gamma(1+a-e-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-e)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(1+a-d-e-f-t)\,\Gamma(s-t)\,dt}{\Gamma(1+a+s+t)}.$$

Thus our integral is equal to*

$$\frac{1}{2\pi i}\int \frac{\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(1+a-d-e-f-t)\,dt}{\Gamma(1+a-e-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-e)}$$

$$\times \frac{1}{2\pi i}\int \frac{\begin{matrix}\Gamma(a+s)\,\Gamma(1+\tfrac{1}{2}a+s)\,\Gamma(b+s)\,\Gamma(c+s)\,\Gamma(s-t)\\ \Gamma(b-a-s)\,\Gamma(-s)\,ds\end{matrix}}{\Gamma(\tfrac{1}{2}a+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a+s+t)}.$$

* For the justification of the interchange in the order of integration cf. Whittaker and Watson, *Modern Analysis*, § 14.53. The lower bound of the distance between the s and t contours is supposed to be definitely positive.

The integration with respect to s can be performed by means of (1), and we obtain

$$\frac{\Gamma(b)\,\Gamma(c)\,\Gamma(b+c-a)}{2\Gamma(1+a-e-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-e)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(b-a-t)\,\Gamma(1+a-d-e-f-t)\,\Gamma(-t)\,dt}{\Gamma(1+a-c+t)\,\Gamma(b+c-a-t)}.$$

This integral can be evaluated by Barnes' second lemma when

$$1+2a = b+c+d+e+f.$$

With this restriction we thus find that

$$(2)\quad \frac{1}{2\pi i}\int \frac{\Gamma(a+s)\,\Gamma(1+\tfrac12 a+s)\,\Gamma(b+s)\,\Gamma(c+s)\,\Gamma(d+s)}{\Gamma(\tfrac12 a+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)}$$

$$\times \frac{\Gamma(e+s)\,\Gamma(f+s)\,\Gamma(b-a-s)\,\Gamma(-s)\,ds}{\Gamma(1+a-e+s)\,\Gamma(1+a-f+s)}$$

$$= \frac{\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(b+c-a)\,\Gamma(b+d-a)}{2\Gamma(1+a-d-e)\,\Gamma(1+a-c-e)\,\Gamma(1+a-c-d)}$$

$$\times \frac{\Gamma(b+e-a)\,\Gamma(b+f-a)}{\Gamma(1+a-c-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-e-f)}.$$

This is the integral analogue of Dougall's theorem. By considering the residues at poles on the right of the contour, we obtain the formula

$$(3)\quad {}_7F_6\left[\begin{matrix} a,\ 1+\tfrac12 a, & b, & c, & d, & e, & f; \\ \tfrac12 a, & 1+a-b,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(b-a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-c-e)\,\Gamma(1+a-c-d)}$$

$$\times \frac{\Gamma(b+c-a)\,\Gamma(b+d-a)\,\Gamma(b+e-a)\,\Gamma(b+f-a)}{\Gamma(1+a-c-f)\,\Gamma(1+a-d-f)\,\Gamma(1+a-e-f)}$$

$$- \frac{\Gamma(1+2b-a)\,\Gamma(b+c-a)\,\Gamma(b+d-a)\,\Gamma(b+e-a)\,\Gamma(b+f-a)}{\Gamma(1+b-c)\,\Gamma(1+b-d)\,\Gamma(1+b-e)\,\Gamma(1+b-f)}$$

$$\times \frac{\Gamma(a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(b-a)\,\Gamma(1+a)\,\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)}$$

$$\times {}_7F_6\left[\begin{matrix} 2b-a,\ 1+b-\tfrac12 a, & b, & b+c-a, \\ b-\tfrac12 a, & 1+b-a,\ 1+b-c, \end{matrix}\right.$$

$$\left.\begin{matrix} b+d-a,\ b+e-a,\ b+f-a; \\ 1+b-d,\ 1+b-e,\ 1+b-f \end{matrix}\right],$$

where $1+2a = b+c+d+e+f$.

This is the form assumed by Dougall's theorem when we remove the restriction that one of the parameters must be a negative integer.

6.7. A method of obtaining transformations of integrals of Barnes' type. In the formula §6.6 (2) write $k = 1+2a-c-d-e$, and replace a,c,d,e,k,b by $k, k+c-a, k+d-a, k+e-a, a, a+t$, and we find that

$$\frac{\Gamma(a+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(-t)}{\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)}$$

$$= \frac{2\Gamma(c)\,\Gamma(d)\,\Gamma(e)}{\Gamma(a-k)\,\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(k+s)\,\Gamma(1+\tfrac12 k+s)\,\Gamma(k+c-a+s)\,\Gamma(k+d-a+s)\,\Gamma(k+e-a+s)}{\Gamma(\tfrac12 k+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)\,\Gamma(1+a-e+s)}$$

$$\times \frac{\Gamma(-t+s)\,\Gamma(a+t+s)\,\Gamma(a-k+t-s)\,\Gamma(-s)\,ds}{\Gamma(1+k+t+s)}.$$

Now multiply by

$$\frac{\Gamma(\rho_1+t)\,\Gamma(\rho_2+t)\,\Gamma(b-a-t)}{\Gamma(\sigma_1+t)},$$

integrate with respect to t, and then put $t+s$ for t on the right. We thus obtain

$$(1)\quad \frac{1}{2\pi i}\int \frac{\Gamma(a+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(\rho_1+t)\,\Gamma(\rho_2+t)\,\Gamma(b-a-t)\,\Gamma(-t)\,dt}{\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)\,\Gamma(\sigma_1+t)}$$

$$= \frac{2\Gamma(c)\,\Gamma(d)\,\Gamma(e)}{\Gamma(a-k)\,\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)}$$

$$\times \frac{1}{2\pi i} \int \frac{\begin{matrix}\Gamma(k+s)\,\Gamma(1+\tfrac{1}{2}k+s)\,\Gamma(k+c-a+s)\\ \Gamma(k+d-a+s)\,\Gamma(k+e-a+s)\,\Gamma(-s)\,ds\end{matrix}}{\Gamma(\tfrac{1}{2}k+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)\,\Gamma(1+a-e+s)}$$

$$\times \frac{1}{2\pi i} \int \frac{\begin{matrix}\Gamma(a+2s+t)\,\Gamma(a-k+t)\,\Gamma(\rho_1+s+t)\\ \Gamma(\rho_2+s+t)\,\Gamma(b-a-s-t)\,\Gamma(-t)\,dt\end{matrix}}{\Gamma(1+k+2s+t)\,\Gamma(\sigma_1+s+t)},$$

where $k=1+2a-c-d-e$.

In this formula there may be any number of the quantities ρ and σ. If we can integrate with respect to t on the right, we can obtain a relation between two integrals of Barnes' type.

6.8. An integral related to well-poised series. In the formula § 6.7 (1) choose the parameters ρ, σ so that we can evaluate the t integral by § 6.6 (2). We thus find that

(1) $$\frac{1}{2\pi i}\int \frac{\Gamma(a+t)\,\Gamma(1+\tfrac{1}{2}a+t)\,\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)}{\Gamma(\tfrac{1}{2}a+t)\,\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)}$$

$$\times \frac{\Gamma(f+t)\,\Gamma(g+t)\,\Gamma(h+t)\,\Gamma(b-a-t)\,\Gamma(-t)\,dt}{\Gamma(1+a-f+t)\,\Gamma(1+a-g+t)\,\Gamma(1+a-h+t)}$$

$$= \frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f+b-a)\,\Gamma(g+b-a)\,\Gamma(h+b-a)}{\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)\,\Gamma(1+a-g-h)}$$
$$\frac{}{\Gamma(1+a-f-h)\,\Gamma(1+a-f-g)}$$

$$\times \frac{1}{2\pi i}\left\{ \frac{\Gamma(k+s)\,\Gamma(1+\tfrac{1}{2}k+s)\,\Gamma(b+s)}{\Gamma(\tfrac{1}{2}k+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)\,\Gamma(1+a-e+s)} \right.$$
$$\frac{\Gamma(k+c-a+s)\,\Gamma(k+d-a+s)}{}$$

$$\times \frac{\Gamma(k+e-a+s)\,\Gamma(f+s)\,\Gamma(g+s)\,\Gamma(h+s)\,\Gamma(b-k-s)\,\Gamma(-s)\,ds}{\Gamma(1+k-f+s)\,\Gamma(1+k-g+s)\,\Gamma(1+k-h+s)},$$

where $k=1+2a-c-d-e$, and the parameters are connected by the relation

(2) $$2+3a=b+c+d+e+f+g+h.$$

From this formula we can, in the usual way, obtain a relation connecting four well-poised series of the type $_9F_8$. If we write

$$V(a;b,c,d,e,f,g,h)$$
$$= \frac{\Gamma(1+a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(g)\,\Gamma(h)}{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)}$$
$$\frac{}{\Gamma(1+a-f)\,\Gamma(1+a-g)\,\Gamma(1+a-h)}$$

$$\times \, _9F_8\left[\begin{matrix} a, & 1+\tfrac{1}{2}a, & b, & c, & d, \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, \end{matrix}\right.$$
$$\left.\begin{matrix} e, & f, & g, & h; \\ 1+a-e, & 1+a-f, & 1+a-g, & 1+a-h \end{matrix}\right]$$

the formula can be written

(3) $\operatorname{cosec}(b-a)\pi$

$$\times \{V(a;\,b,\,c,\,d,\,e,\,f,\,g,\,h)-V(2b-a;\,b,\,b+c-a,\,b+d-a,$$
$$b+e-a,\,b+f-a,\,b+g-a,\,b+h-a)\}$$

$$= \frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f+b-a)\,\Gamma(g+b-a)}{\Gamma(1+a-d-e)\,\Gamma(1+a-c-e)\,\Gamma(1+a-c-d)}$$
$$\frac{\Gamma(h+b-a)\operatorname{cosec}(1+a-f-g-h)\pi}{\Gamma(1+a-g-h)\,\Gamma(1+a-f-h)\,\Gamma(1+a-f-g)}$$

$$\times \{V(1+2a-c-d-e;\,b,\,1+a-d-e,$$
$$1+a-c-e,\,1+a-c-d,\,f,\,g,\,h)$$

$$-V(2b-2a-1+c+d+e;\,b,\,b-a+c,\,b-a+d,$$
$$b-a+e,\,1+a-g-h,\,1+a-f-h,\,1+a-f-g)\},$$

provided that (2) is satisfied. Any one of the series $_9F_8$ is of general type except for the second parameter and the restriction that the sum of the denominator parameters exceeds the sum of the numerator parameters by two. When f, g or h is a negative integer, the formula reduces to § 4.3 (7).

The parameters of the second series are obtained from those of the first series by the addition of $b-a$ to each, the parameter so obtained from b becoming the first parameter of the second series. We shall say that the two series are 'complementary with respect to the parameter b.' It will be noticed that the two series on the right of (3) are also complementary with respect to the parameter b.

6.9. Integrals related to Saalschützian nearly-poised series. Now in the formula § 6.7 (1) take $\rho_1=f$, $b=1+2k-a-f$, so that we can integrate on the right by means of Barnes' second lemma. We are thus led to the formula

$$(1)\quad \frac{1}{2\pi i}\int \frac{\Gamma(a+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(f+t)\,\Gamma(1+2k-2a-f-t)\,\Gamma(-t)\,dt}{\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)\,\Gamma(1+a-e+t)}$$

$$=\frac{2\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(1+2k-2a)}{\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)\,\Gamma(1+k-a)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(k+s)\,\Gamma(1+\tfrac12 k+s)\,\Gamma(a+2s)\,\Gamma(f+s)\,\Gamma(k+c-a+s)}{\Gamma(\tfrac12 k+s)\,\Gamma(1+2k-a+2s)\,\Gamma(1+k-f+s)}$$

$$\times \frac{\Gamma(k+d-a+s)\,\Gamma(k+e-a+s)\,\Gamma(1+2k-a-f+s)\,\Gamma(1+k-a-f-s)\,\Gamma(-s)\,ds}{\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)\,\Gamma(1+a-e+s)},$$

where $k=1+2a-c-d-e$.

From this formula we find, in the usual way, a relation in which there are two nearly-poised series $_5F_4$ on the left and two well-poised series $_9F_8$ on the right. The nearly-poised series are

$$_5F_4\left[\begin{array}{ccccc} a, & c, & d, & e, & f; \\ & 1+a-c, & 1+a-d, & 1+a-e, & 2a+f-2k \end{array}\right]$$

and

$$_5F_4\left[\begin{array}{c} 1+2k-2a,\ \ 1+2k-2a-f+c,\ \ 1+2k-2a-f+d, \\ 1+2k-2a-f+e,\ \ 1+2k-a-f; \\ 2+2k-a-f-c,\ \ 2+2k-a-f-d, \\ 2+2k-a-f-e,\ \ 2+2k-2a-f \end{array}\right],$$

while one series $_9F_8$ has the numerator parameters $k,\ 1+\tfrac12 k,\ \tfrac12 a,$ $\tfrac12+\tfrac12 a, f,\ k+c-a,\ k+d-a,\ k+e-a,\ 1+2k-a-f,$ and the other $_9F_8$ is complementary to this with respect to the parameter $1+2k-a-f$. The nearly-poised series are Saalschützian in type. One is of the first kind and the other of the second kind. When f is a negative integer the relation reduces to §4.5(3), while if $1+2k-2a+e-f$ is a negative integer the relation reduces to a transformation of a Saalschützian nearly-poised series $_5F_4$ of the first kind into a well-poised $_9F_8$. This is equivalent to the transformation obtained from §4.5(3) by the reversal of series, referred to in §4.6.

As a particular case of (1) take $d=1+\tfrac12 a,\ e=\tfrac12+\tfrac12 a$, so that

$k=a-c-\tfrac12$, and the integral on the right can be evaluated by §6.6(1). Replacing f by d, we obtain the formula

$$(2)\quad \frac{1}{2\pi i}\int \frac{\Gamma(a+t)\,\Gamma(1+\tfrac12 a+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(-2c-d-t)\,\Gamma(-t)\,dt}{\Gamma(\tfrac12 a+t)\,\Gamma(1+a-c+t)}$$

$$=\frac{\Gamma(1+a)\,\Gamma(c)\,\Gamma(d)\,\Gamma(-2c)\,\Gamma(a-2c-d)\,\Gamma(-c-d)}{2\Gamma(a-2c)\,\Gamma(1+a-c-d)\,\Gamma(-c)}.$$

Now in §6.7(1) take $\rho_1=1+\tfrac12 a$, $\rho_2=f$, $\sigma_1=\tfrac12 a$. The integration on the right with respect to t can be performed by (2) provided that $b=2k-a-f$, and we are led to the formula

$$(3)\quad \frac{1}{2\pi i}\int \frac{\Gamma(a+t)\,\Gamma(1+\tfrac12 a+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(e+t)\,\Gamma(f+t)}{\Gamma(\tfrac12 a+t)\,\Gamma(1+a-c+t)\,\Gamma(1+a-d+t)}$$

$$\times \frac{\Gamma(2k-2a-f-t)\,\Gamma(-t)\,dt}{\Gamma(1+a-e+t)}$$

$$=\frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(2k-2a)}{\Gamma(k+c-a)\,\Gamma(k+d-a)\,\Gamma(k+e-a)\,\Gamma(k-a)}$$

$$\times \frac{1}{2\pi i}\int \frac{\Gamma(k+s)\,\Gamma(1+\tfrac12 k+s)\,\Gamma(k+c-a+s)\,\Gamma(k+d-a+s)\,\Gamma(k+e-a+s)}{\Gamma(\tfrac12 k+s)\,\Gamma(1+a-c+s)\,\Gamma(1+a-d+s)\,\Gamma(1+a-e+s)}$$

$$\times \frac{\Gamma(1+a+2s)\,\Gamma(f+s)\,\Gamma(2k-a-f+s)\,\Gamma(k-a-f-s)\,\Gamma(-s)\,ds}{\Gamma(2k-a+2s)\,\Gamma(1+k-f+s)},$$

where $k=1+2a-c-d-e$.

This formula leads to a relation involving two Saalschützian nearly-poised series $_6F_5$ and two well-poised $_9F_8$ which are complementary with respect to the parameter $2k-a-f$. One of the nearly-poised series is of the first kind and one of the second kind. When f is a negative integer the relation reduces to §4.5(4), and when $2k-2a+e-f$ is a negative integer the relation reduces to the corresponding transformation of a nearly-poised series of the first kind.

Generalizations of §4.5(5) and (6) can be found in a manner

entirely analogous to that of §4.5. In each case we obtain a relation involving two nearly-poised series, one of each kind, and two well-poised $_9F_8$.

It will be noticed that no direct generalizations of §4.5(1) and (2) have been given. The formulae obtained in these cases involve five series instead of three or four as previously obtained. In each case two of the series are nearly-poised and of the second kind, one is nearly-poised and of the first kind, and the other two are Saalschützian in type.

CHAPTER VII

FURTHER TRANSFORMATIONS OF WELL-POISED SERIES

7.1. Introductory remarks. The formula §4.4(5) transforms a well-poised series $_7F_6$ into a Saalschützian $_4F_3$ provided that the latter series terminates. There are thus two distinct cases, one in which the $_7F_6$ terminates and another in which the $_7F_6$ does not terminate although the $_4F_3$ does.

We suppose that

$$u + v + w = x + y + z - n + 1,$$

and then the formula in the two cases can be written

$$(1) \quad {}_4F_3 \begin{bmatrix} x, \ y, \ z, \ -n; \\ u, \ v, \ w \end{bmatrix}$$

$$= \frac{\Gamma(v+w-x)\,\Gamma(1+y-u)\,\Gamma(1+z-u)\,\Gamma(1-n-u)}{\Gamma(1+y-n-u)\,\Gamma(1+z-n-u)\,\Gamma(1+y+z-u)\,\Gamma(1-u)}$$

$$\times {}_7F_6 \begin{bmatrix} a, \ 1+\tfrac{1}{2}a, \ w-x, \ v-x, \quad y, \qquad\qquad z, \\ \tfrac{1}{2}a, \quad v, \quad w, \quad 1+z-n-u, \ 1+y-n-u, \\ \qquad\qquad\qquad\qquad\qquad\qquad -n; \\ \qquad\qquad\qquad\qquad\qquad 1+y+z-u \end{bmatrix},$$

where $a = y+z-n-u = w+v-x-1$; and

$$(2) \quad {}_4F_3 \begin{bmatrix} x, \ y, \ z, \ -n; \\ u, \ v, \ w \end{bmatrix}$$

$$= \frac{\Gamma(v+w+n)\,\Gamma(1+x-u)\,\Gamma(1+y-u)\,\Gamma(1+z-u)}{\Gamma(1+y+z-u)\,\Gamma(1+z+x-u)\,\Gamma(1+x+y-u)\,\Gamma(1-u)}$$

$$\times {}_7F_6 \begin{bmatrix} a', \ 1+\tfrac{1}{2}a', \ w+n, \ v+n, \quad x, \qquad\qquad y, \\ \tfrac{1}{2}a', \quad v, \quad w, \quad 1+y+z-u, \ 1+z+x-u, \\ \qquad\qquad\qquad\qquad\qquad\qquad z; \\ \qquad\qquad\qquad\qquad\qquad 1+x+y-u \end{bmatrix},$$

where $a' = x+y+z-u = v+w+n-1.$

It is evident that §4.4(5) and these equivalent formulae can be used to find additional relations connecting two well-poised series or connecting a well-poised series and a Saalschützian series. These relations have been fully worked out by Whipple.*

* Whipple 3.

7.2. A relation connecting terminating Saalschützian $_4F_3$.

When the $_7F_6$ in § 7.1 (1) is transformed by § 4.4 (5), with $g = -n$, into a Saalschützian $_4F_3$, we obtain the formula

$$(1) \quad {}_4F_3\left[\begin{matrix} x, \, y, \, z, \, -n; \\ u, \, v, \, w \end{matrix}\right]$$

$$= \frac{(v-z)_n \, (w-z)_n}{(v)_n \, (w)_n} \, {}_4F_3\left[\begin{matrix} u-x, \, u-y, \, z, \, -n; \\ 1-v+z-n, \, 1-w+z-n, \, u \end{matrix}\right],$$

where, of course, the parameters are subject to the condition

$$u+v+w = x+y+z-n+1.$$

The formula (1) can be obtained immediately by equating the coefficients of ζ^n on the two sides of the relation

$$\begin{aligned} & {}_2F_1\left[\begin{matrix} x, \, y; \, \zeta \\ u \end{matrix}\right] {}_2F_1\left[\begin{matrix} 1-n-v, \, 1-n-w; \, \zeta \\ 1-n-z \end{matrix}\right] \\ & = {}_2F_1\left[\begin{matrix} u-x, \, u-y; \, \zeta \\ u \end{matrix}\right] {}_2F_1\left[\begin{matrix} v-z, \, w-z; \, \zeta \\ 1-n-z \end{matrix}\right], \end{aligned}$$

which is an immediate consequence of § 1.2 (2).

The series occurring in (1) can both be reversed, and so we obtain two more Saalschützian series related to the given $_4F_3$. By interchanging the parameters x, y, z or u, v, w in (1) we obtain 9 distinct $_4F_3$ related to the given $_4F_3$, apart from the 10 equivalent series obtained by merely reversing the order of the terms.

7.3. Notation for terminating well-poised $_7F_6$.

The method of obtaining transformations of a given terminating well-poised $_7F_6$ is simple. Starting from § 4.4 (5), with $g = -n$, we obtain a Saalschützian $_4F_3$ from which other related $_4F_3$ can be derived by § 7.2 (1). These $_4F_3$ can now be transformed by § 7.1 (1) and (2) into well-poised series, either terminating or non-terminating. The number of series involved is, however, fairly large, and so it is convenient to use a notation* analogous to that of § 3.5.

Let $r_1, r_2, \ldots, r_6$ be six parameters such that

$$\Sigma r = 0,$$

and write ϕ for the fraction $\frac{1}{3}(n-1)$.

* Whipple 3.

We write

$$\epsilon_{ij} = r_i + r_j - \phi,$$

$$\delta_{ij} = r_i - r_j - n.$$

Then evidently

$$\epsilon_{12} + \epsilon_{34} + \epsilon_{56} = 1 - n$$

and

$$\delta_{12} = \epsilon_{13} - \epsilon_{23} - n.$$

Now let

$$(1) \quad S(1, 2, 3) = \frac{(-1)^n \, \Gamma(1-\epsilon_{56}) \, \Gamma(1-\epsilon_{46}) \, \Gamma(1-\epsilon_{45})}{\Gamma(1-n-\epsilon_{56}) \, \Gamma(1-n-\epsilon_{46}) \, \Gamma(1-n-\epsilon_{45})}$$

$$\times {}_4F_3\left[\begin{matrix} \epsilon_{23}, \, \epsilon_{13}, \, \epsilon_{12}, \, -n; \\ 1-n-\epsilon_{56}, \, 1-n-\epsilon_{46}, \, 1-n-\epsilon_{45} \end{matrix}\right],$$

$$(2) \quad W(1; 4) = \frac{\Gamma(1+\delta_{14}) \, \Gamma(1-\epsilon_{24}) \, \Gamma(1-\epsilon_{34}) \, \Gamma(1-\epsilon_{45}) \, \Gamma(1-\epsilon_{46})}{\Gamma(1+n+\delta_{14}) \, \Gamma(1-n-\epsilon_{24}) \, \Gamma(1-n-\epsilon_{34}) \, \Gamma(1-n-\epsilon_{45}) \, \Gamma(1-n-\epsilon_{46})}$$

$$\times {}_7F_6\left[\begin{matrix} \delta_{14}, \, 1+\frac{1}{2}\delta_{14}, \, \epsilon_{16}, \, \epsilon_{15}, \, \epsilon_{13}, \\ \frac{1}{2}\delta_{14}, \, 1-n-\epsilon_{46}, \, 1-n-\epsilon_{45}, \, 1-n-\epsilon_{34}, \\ \epsilon_{12}, \, -n; \\ 1-n-\epsilon_{24}, \, 1+n+\delta_{14} \end{matrix}\right],$$

all permutations of the numbers $1, 2, \ldots, 6$ being allowed. These definitions may also be written in the forms

$$(3) \quad S(1, 2, 3) = \sum_{p=0}^{n} {}_nC_p (\epsilon_{23})_p (\epsilon_{13})_p (\epsilon_{12})_p (\epsilon_{56})_{n-p} (\epsilon_{46})_{n-p} (\epsilon_{45})_{n-p},$$

$$(4) \quad W(1; 4) = (-1)^n \sum_{p=0}^{n} {}_nC_p \frac{r_1 - r_4 + 2p - n}{r_1 - r_4}$$

$$\times \frac{(\epsilon_{12})_p (\epsilon_{13})_p (\epsilon_{15})_p (\epsilon_{16})_p (\epsilon_{42})_{n-p} (\epsilon_{43})_{n-p} (\epsilon_{45})_{n-p} (\epsilon_{46})_{n-p}}{(1-r_1+r_4)_{n-p} (1+r_1-r_4)_p}.$$

The equation § 7.1 (1) can now be written

$$(5) \quad S(1, 2, 3) = W(1; 4).$$

As the series involved in $S(1, 2, 3)$ and $W(1; 4)$ terminate, they may each be written in the reverse order, and (3) and (4) show that

$$(6) \quad S(1, 2, 3) \equiv S(4, 5, 6),$$

$$(7) \quad W(1; 4) \equiv W(4; 1).$$

The equation (5) may now be used repeatedly. Thus

$$S(1, 2, 3) = W(1; 4) = S(1, 2, 5) = W(2; 6) = S(2, 3, 5) = \ldots.$$

Formally there are 20 S's and 30 W's, but each series is counted twice if we make no distinction between a given series and that obtained by reversal of the terms.

There are therefore 10 distinct S's and 15 W's. The 25 series are all equal. The identities involved in addition to (5), (6) and (7) are

(8) $S(1,2,3) = S(1,2,4) \equiv S(3,5,6)$,

(9) $W(1;4) = W(2;6) \equiv W(6;2)$,

(10) $W(1;4) = W(1;5) \equiv W(5;1)$,

(11) $S(1,2,3) = W(2;1) \equiv W(1;2)$.

7.4. Notation for non-terminating well-poised $_7F_6$.

In §7.1 (2) the first of the parameters of the series on the right is

$$a' = v + w + n - 1 = -\phi - 2r_4 - r_5 - r_6.$$

We write $\lambda_{4;56} = -\phi - 2r_4 - r_5 - r_6$

and

(1) $W(4; 5, 6)$

$$= \frac{\Gamma(1+\lambda_{4;56})\,\Gamma(1-\epsilon_{14})\,\Gamma(1-\epsilon_{24})\,\Gamma(1-\epsilon_{34})\,\Gamma(1-\epsilon_{45})\,\Gamma(1-\epsilon_{46})}{\Gamma(1+n+\delta_{14})\,\Gamma(1+n+\delta_{24})\,\Gamma(1+n+\delta_{34})\,\Gamma(1-n-\epsilon_{45})\,\Gamma(1-n-\epsilon_{46})\,\Gamma(\epsilon_{56})}$$

$$\times {}_7F_6 \left[\begin{array}{c} \lambda_{4;56}, \; 1+\tfrac{1}{2}\lambda_{4;56}, \quad 1-\epsilon_{45}, \quad 1-\epsilon_{46}, \quad \epsilon_{23}, \\ \tfrac{1}{2}\lambda_{4;56}, \quad 1-n-\epsilon_{46}, \; 1-n-\epsilon_{45}, \; 1+n+\delta_{14}, \\ \epsilon_{13}, \qquad \epsilon_{12}; \\ 1+n+\delta_{24}, \; 1+n+\delta_{34} \end{array} \right],$$

and then §7.1 (2) can be written in the form

(2) $S(1, 2, 3) = W(4; 5, 6)$.

The series in $W(4; 5, 6)$ is convergent when $R(\epsilon_{56}) > 0$.

Since $S(4,5,6)$ is the same as $S(1,2,3)$ written in the reverse order, there are six non-terminating well-poised series corresponding with each Saalschützian series. It should be noticed, however, that not all the six are convergent since the ϵ's have a negative sum. The total number of non-terminating $_7F_6$ derived from the 10 equal Saalschützian series is 60.

By combining (2) with the relations between the S's, we find the following formulae:

(3) $W(4; 5, 6) = S(1, 2, 3) \equiv S(4, 5, 6)$,

(4) $W(4; 5, 6) = S(1, 2, 4) \equiv S(3, 5, 6)$,

(5) $W(4; 5, 6) = S(1, 2, 5) \equiv S(3, 4, 6)$,

(6) $W(4; 5, 6) = W(1; 4) \equiv W(4; 1)$,

(7) $W(4; 5, 6) = W(1; 5) \equiv W(5; 1)$,

(8) $W(4; 5, 6) = W(1; 2) \equiv W(2; 1)$,

(9) $W(4; 5, 6) = W(4; 5) \equiv W(5; 4)$,

(10) $W(4; 5, 6) = W(5; 6) \equiv W(6; 5)$,

(11) $W(4; 5, 6) = W(5; 4, 6)$,

(12) $W(4; 5, 6) = W(5; 1, 6)$,

(13) $W(4; 5, 6) = W(5; 1, 2)$,

(14) $W(4; 5, 6) = W(5; 1, 4)$,

(15) $W(4; 5, 6) = W(4; 1, 5)$,

(16) $W(4; 5, 6) = W(4; 1, 2)$,

(17) $W(4; 5, 6) = W(1; 2, 3)$,

(18) $W(4; 5, 6) = W(1; 2, 4)$,

(19) $W(4; 5, 6) = W(1; 2, 5)$,

(20) $W(4; 5, 6) = W(1; 5, 6)$,

(21) $W(4; 5, 6) = W(1; 4, 5)$.

Thus associated with a given non-terminating well-poised series there are 3 distinct Saalschützian series, 5 terminating well-poised series, and 11 non-terminating well-poised series. Also, associated with a given terminating well-poised $_7F_6$ there are 2 distinct Saalschützian series, 3 terminating well-poised series, and 8 non-terminating well-poised series.

As in §3.5 we can work out the parameters of the various series associated with a given well-poised series, either terminating or non-terminating. Tables of these parameters have been given by Whipple* and are set out below. The second parameters of the

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* Whipple 3. Whipple also gives the parameters of the well-poised series associated with a given Saalschützian series.

well-poised series are omitted, and only numerator parameters of these series are given.

Table I. *Parameters of associated series. Master series well-poised and terminating*

$$s = c + d + e + f - 2a - n - 1$$

$W(1; 2)$ $a;\; c, d, e, f,\; -n$

$W(2; 1)$ $-a-2n;\; c-a-n,\; d-a-n,\; e-a-n,\; f-a-n,\; -n$

$W(1; 3)$ $s+a-c;\; s, d, e, f,\; -n$

$W(3; 1)$ $c-a-s-2n;\; c-a-n,\; 1+a-e-f,\; 1+a-d-f,\; 1+a-d-e,\; -n$

$W(2; 3)$ $s-c-n;\; s,\; d-a-n,\; e-a-n,\; f-a-n,\; -n$

$W(3; 2)$ $c-s-n;\; c,\; 1+a-e-f,\; 1+a-d-f,\; 1+a-d-e,\; -n$

$W(3; 4)$ $c-d-n;\; c,\; c-a-n,\; 1+a-d-f,\; 1+a-d-e,\; -n$

$S(1, 2, 3)$ $s,\; c,\; c-a-n,\; -n;\; c+f-a-n,\; c+e-a-n,\; c+d-a-n$

$S(4, 5, 6)$ $1+a-c-d,\; 1+a-c-f,\; 1+a-c-e,\; -n;\; 1-n-s,\; 1-n-c,\; 1+a-c$

$S(1, 3, 4)$ $c,\; d,\; 1+a-e-f,\; -n;\; 1+a-e,\; 1+a-f,\; c+d-a-n$

$S(2, 5, 6)$ $e-a-n,\; f-a-n,\; 1+a-c-d,\; -n;\; 1-n-c,\; 1-n-d,\; e+f-a-n$

$W(1; 2, 3)$ $1-s-n-c;\; 1-s,\; 1-c;\; 1+a-c-d,\; 1+a-c-e,\; 1+a-c-f$

$W(2; 1, 3)$ $1-s+a-c;\; 1-s,\; 1+n+a-c;\; 1+a-c-d,\; 1+a-c-e,\; 1+a-c-f$

$W(3; 1, 2)$ $1+a-2c;\; 1-c,\; 1+n+a-c;\; 1+a-c-d,\; 1+a-c-e,\; 1+a-c-f$

$W(1; 3, 4)$ $1-c-d-n;\; 1-c,\; 1-d;\; e-a-n,\; f-a-n,\; 1+a-c-d$

$W(2; 3, 4)$ $e+f-s;\; 1-c+a+n,\; 1-d+a+n;\; e, f,\; 1+a-c-d$

$W(3; 1, 4)$ $e+f-c-a-n;\; 1-c,\; e+f-a;\; e-a-n,\; f-a-n,\; 1+a-c-d$

$W(3; 2, 4)$ $e+f-c;\; 1-c+a+n,\; e+f-a;\; e, f,\; 1+a-c-d$

$W(3; 4, 5)$ $s-c;\; e+f-a,\; d+f-a;\; s, f,\; f-a-n$

Table II. *Parameters of associated series. Master series well-poised and non-terminating*

$$t = c + d - e - f - g - 2n$$

$W(4; 5, 6)$ $c+d-n-1;\; c, d;\; e, f, g$

$W(5; 4, 6)$ $c-t-n;\; c,\; 1-t;\; e, f, g$

$W(5; 1, 6)$ $1+e-t-d;\; 1-d+e+n,\; 1-t;\; e,\; 1-n-f-t,\; 1-n-g-t$

$W(5; 1, 2)$ $1+e+f-2d+n;\; 1-d+e+n,\; 1-d+f+n;\; 1-n-g-t,\; c-g-n,\; 1-$

$W(5; 1, 4)$ $c-d+e;\; 1-d+e+n,\; c;\; e,\; c-f-n,\; c-g-n$

$W(4; 1, 5)$ $c+e+t-1;\; e+t+n,\; c;\; e,\; c-f-n,\; c-g-n$

$W(4; 1, 2)$ $e+f+2t+n-1;\; e+t+n,\; f+t+n;\; d-g-n,\; c-g-n,\; t$

$W(1; 2, 3)$ $1-f-g-n;\; 1-f,\; 1-g;\; 1-c,\; 1-d,\; t$

$W(1; 2, 4)$ $e-g+t;\; 1-g,\; e+t+n;\; d-g-n,\; c-g-n,\; t$

$W(1; 2, 5)$ $1-d+e-g;\; 1-g,\; 1-d+e+n;\; 1-g-t-n,\; c-g-n,\; 1-d$

$W(1; 5, 6)$ $1-c-d+2e+n;\; 1-d+e+n,\; 1-c+e+n;\; e,\; 1-n-f-t,\; 1-n-g-$

$W(1; 4, 5)$ $c+e-f-g-n;\; e+t+n,\; 1-d+e+n;\; e,\; c-f-n,\; c-g-n$

Table II (*cont.*)

$S(1, 2, 3)$ $e, f, g,\; -n;\; c-n,\; d-n,\; 1-t-n$

$S(4, 5, 6)$ $1-c,\; 1-d, t,\; -n;\; 1-e-n,\; 1-f-n,\; 1-g-n$

$S(1, 2, 4)$ $1-n-e-t,\; 1-n-f-t,\; g,\; -n;\; 1-d+g,\; 1-c+g,\; 1-t-n$

$S(3, 5, 6)$ $d-g-n,\; c-g-n,\; t,\; -n;\; e+t, f+t,\; 1-g-n$

$S(1, 2, 5)$ $d-e-n,\; d-f-n,\; g,\; -n;\; g+t,\; 1-c+g,\; d-n$

$S(3, 4, 6)$ $1-n-g-t,\; c-g-n,\; 1-d,\; -n;\; 1-d+e,\; 1-d+f,\; 1-g-n$

$W(1; 4)$ $c+d-e-2n-1;\; f, g,\; d-e-n,\; c-e-n,\; -n$

$W(4; 1)$ $1-c-d+e;\; 1-n-f-t,\; 1-n-g-t,\; 1-c,\; 1-d,\; -n$

$W(1; 5)$ $f+g-d;\; f, g,\; 1-n-e-t,\; c-e-n,\; -n$

$W(5; 1)$ $d-f-g-2n;\; d-f-n,\; d-g-n,\; 1-c, t,\; -n$

$W(1; 2)$ $f-e-n;\; f,\; 1-n-c-t,\; d-e-n,\; c-e-n,\; -n$

$W(4; 5)$ $1-d-t-n;\; 1-n-e-t,\; 1-n-f-t,\; 1-n-g-t,\; 1-d,\; -n$

$W(5; 4)$ $d+t-n-1;\; d-e-n,\; d-f-n,\; d-g-n, t,\; -n$

$W(5; 6)$ $d-c-n;\; d-e-n,\; d-f-n,\; d-g-n,\; 1-c,\; -n$

The parameters tabulated for $S(1, 2, 3)$ are $\epsilon_{23}, \epsilon_{13}, \epsilon_{12}, -n$; $1-n-\epsilon_{56}, 1-n-\epsilon_{46}, 1-n-\epsilon_{45}$. Those tabulated for $W(1; 4)$ are $\delta_{14}; \epsilon_{16}, \epsilon_{15}, \epsilon_{13}, \epsilon_{12}, -n$, and the parameters shown for $W(4; 5, 6)$ are $\lambda_{4;56}; 1-\epsilon_{45}, 1-\epsilon_{46}; \epsilon_{23}, \epsilon_{13}, \epsilon_{12}$.

7.5. Transformations of unrestricted well-poised $_7F_6$.

In the transformations given so far in this chapter, a parameter or a linear combination of parameters has been restricted to be a negative integer. We now consider transformations when there is no such restriction.*

The formula § 4.4 (4) is a formula of this type. It is convenient to write

$$W(a;\; c, d, e, f, g)$$

$$= {}_7F_6\left[\begin{array}{ccccccc} a, & 1+\tfrac{1}{2}a, & c, & d, & e, & f, & g; \\ & \tfrac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{array}\right].$$

Now transform the series

$$W(1+2a-e-f-g;\; c, d,\; 1+a-f-g,\; 1+a-e-g,\; 1+a-e-f)$$

by § 4.4 (4) and we obtain the same series $_4F_3$ as occur in that formula. We thus find that

* Bailey 12.

(1) $W(a; c, d, e, f, g)$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(2+2a-e-f-g)}{\Gamma(1+a)\,\Gamma(1+a-c-d)} \cdot \frac{\dfrac{\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-e-f-g)}}{\Gamma(2+2a-d-e-f-g)}$$

$$\times W(1+2a-e-f-g;\ c, d,\ 1+a-f-g,\ 1+a-e-g,\ 1+a-e-f).$$

On duplicating this formula we obtain

(2) $W(a; c, d, e, f, g)$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(g)\,\Gamma(2+2a-d-e-f-g)\,\Gamma(2+2a-c-e-f-g)}$$

$$\times \frac{\Gamma(3+3a-c-d-e-f-2g)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-d-f-g)\,\Gamma(2+2a-c-d-e-g)}$$

$$\times W(2+3a-c-d-e-f-2g;\ 1+a-c-g,\ 1+a-d-g,$$
$$1+a-e-g,\ 1+a-f-g,\ 2+2a-c-d-e-f-g).$$

If g or $1+a-e-f$ is a negative integer, (1) gives a relation between a terminating and a non-terminating well-poised series, whereas if c is a negative integer the formula reduces to one connecting two terminating series. Similarly (2) gives, in certain circumstances, a relation between two series, only one of which terminates.

The series on the right of (2) can now be transformed by § 4.4 (4), and we thus derive the formula

(3) $\displaystyle {}_7F_6\!\left[\begin{array}{ccccccc} a, & 1+\tfrac12 a, & c, & d, & e, & f, & g; \\ & \tfrac12 a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{array}\right]$

$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(g)\,\Gamma(1+a-c-f)\,\Gamma(1+a-d-f)}$$

$$\times \frac{\Gamma(g-f)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(1+a-e-f)\,\Gamma(2+2a-c-d-e-g)}$$

$$\times {}_4F_3\!\left[\begin{array}{c} 1+a-c-g,\ 1+a-d-g,\ 1+a-e-g,\ f; \\ 2+2a-c-d-e-g,\ 1+a-g,\ 1+f-g \end{array}\right]$$

$$+ \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-g)}{\Gamma(1+a)\,\Gamma(f)\,\Gamma(1+a-c-g)\,\Gamma(1+a-d-g)}$$

$$\times \frac{\Gamma(f-g)\,\Gamma(2+2a-c-d-e-f-g)}{\Gamma(1+a-e-g)\,\Gamma(2+2a-c-d-e-f)}$$

$$\times {}_4F_3\!\left[\begin{array}{c} 1+a-c-f,\ 1+a-d-f,\ 1+a-e-f,\ g; \\ 2+2a-c-d-e-f,\ 1+a-f,\ 1+g-f \end{array}\right].$$

This formula and § 4.4 (4) appear to be the only formulae of their type. They generalize all Whipple's formulae expressing a well-poised ${}_7F_6$ in terms of a Saalschützian ${}_4F_3$. The formulae (1) and (2) do not, however, generalize all Whipple's formulae expressing a well-poised series in terms of another well-poised series. Apparently the generalizations of the other formulae are not of such a simple type.

7.6. Transformations of well-poised ${}_9F_8$. The formula § 4.3 (7) transforms a terminating well-poised ${}_9F_8$ into another series of the same type. It would seem at first sight as if we could take the parameters in different sets of three and so obtain several formulae connecting well-poised ${}_9F_8$. It appears however that, apart from the mere reversal of series, there is only one other formula of this type, obtained by duplicating § 4.3 (7). This formula is*

(1) $\displaystyle {}_9F_8\!\left[\begin{array}{cccccc} a, & 1+\tfrac12 a, & b, & c, & d, & e, \\ & \tfrac12 a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, \end{array}\right.$

$$\left.\begin{array}{ccc} f, & g, & -n; \\ 1+a-f, & 1+a-g, & 1+a+n \end{array}\right]$$

$$= \frac{(1+a)_n\,(1+a-b-c)_n\,(1+a-b-d)_n\,(1+a-b-e)_n}{(1+a-b)_n\,(1+a-c)_n\,(1+a-d)_n\,(1+a-e)_n} \cdot \frac{(1+a-b-f)_n\,(g)_n}{(1+a-f)_n\,(g-b)_n}$$

$$\times {}_9F_8\!\left[\begin{array}{cccc} b-g-n, & 1+\tfrac12(b-g-n), & b, & 1+a-c-g, \\ & \tfrac12(b-g-n), & 1-g-n, & b+c-a-n, \end{array}\right.$$

$$\left.\begin{array}{cccc} 1+a-d-g, & 1+a-e-g, & 1+a-f-g, & b-a-n, & -n; \\ b+d-a-n, & b+e-a-n, & b+f-a-n, & 1+a-g, & 1+b-g \end{array}\right],$$

where $2+3a = b+c+d+e+f+g-n$.

Similarly, by duplicating § 6.8 (3), we obtain, with the notation of that paragraph,

* Bailey 12. See also Whipple 10 where the formula is proved by Dougall's method. We can derive § 4.3 (7) from this formula by duplication.

(2) $\operatorname{cosec}(b-a)\pi.\{V(a; b, c, d, e, f, g, h)$

$\qquad - V(2b-a; b, b+c-a, b+d-a, b+e-a, b+f-a, b+g-a,$

$\qquad\qquad\qquad\qquad\qquad b+h-a)\}$

$$= \frac{\Gamma(c)\,\Gamma(d)\,\Gamma(e)\,\Gamma(f)\,\Gamma(c+b-a)\,\Gamma(d+b-a)}{\Gamma(1+a-c-g)\,\Gamma(1+a-d-g)\,\Gamma(1+a-e-g)\,\Gamma(1+a-f-g)}$$

$$\times \frac{\Gamma(e+b-a)\,\Gamma(f+b-a)\operatorname{cosec}(g-h)\pi}{\Gamma(1+a-c-h)\,\Gamma(1+a-d-h)\,\Gamma(1+a-e-h)\,\Gamma(1+a-f-h)}$$

$$\times \{V(b-g+h; b, 1+a-c-g, 1+a-d-g, 1+a-e-g,$$

$$\qquad\qquad\qquad\qquad 1+a-f-g, h, h+b-a)$$

$$- V(b+g-h; b, 1+a-c-h, 1+a-d-h, 1+a-e-h,$$

$$\qquad\qquad\qquad\qquad 1+a-f-h, g, g+b-a)\}$$

provided that
$$2+3a = b+c+d+e+f+g+h.$$

The transformation §7.5(3) can be deduced from (1) by a limiting process analogous to that by which §4.4(4) was deduced from the previous relation connecting two well-poised series $_9F_8$.

CHAPTER VIII

BASIC HYPERGEOMETRIC SERIES

8.1. Introductory remarks. The hypergeometric series was generalized in a different way by Heine* who considered the series

$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}z + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}z^2 + \cdots,$$

which reduces to $_2F_1(\alpha, \beta; \gamma; z)$ when $q \to 1$.

We shall follow Heine in writing α, β, γ instead of q^α, q^β, q^γ, and define the basic series by

$$_2\Phi_1(\alpha, \beta; \gamma; z)$$

$$= 1 + \frac{(1-\alpha)(1-\beta)}{(1-q)(1-\gamma)}z + \frac{(1-\alpha)(1-\alpha q)(1-\beta)(1-\beta q)}{(1-q)(1-q^2)(1-\gamma)(1-\gamma q)}z^2 + \cdots,$$

where $|q| < 1$, $|z| < 1$.

For brevity we write

$$(a)_{q,n} = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \quad (a)_{q,0} = 1,$$

and then
$$_2\Phi_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^\infty \frac{(\alpha)_{q,n}(\beta)_{q,n}}{(q)_{q,n}(\gamma)_{q,n}}z^n,$$

and, more generally,

$$_r\Phi_s\begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_r; z \\ \rho_1, \ldots, \rho_s \end{bmatrix} = \sum_{n=0}^\infty \frac{(\alpha_1)_{q,n}(\alpha_2)_{q,n}\cdots(\alpha_r)_{q,n}}{(q)_{q,n}(\rho_1)_{q,n}\cdots(\rho_s)_{q,n}}z^n.$$

8.2. Some elementary results. As particular cases of the series $_2\Phi_1$ we have

(1) $\quad \dfrac{z}{1-q}{}_2\Phi_1(q, q; q^2; z) = \dfrac{z}{1-q} + \dfrac{z^2}{1-q^2} + \dfrac{z^3}{1-q^3} + \cdots,$

(2) $\quad \dfrac{z}{1-q^{\frac12}}{}_2\Phi_1(q, q^{\frac12}; q^{\frac32}; z) = \dfrac{z}{1-q^{\frac12}} + \dfrac{z^2}{1-q^{\frac32}} + \dfrac{z^3}{1-q^{\frac52}} + \cdots,$

(3) $\quad {}_2\Phi_1(q, -1; -q; z) = 1 + \dfrac{2z}{1+q} + \dfrac{2z^2}{1+q^2} + \dfrac{2z^3}{1+q^3} + \cdots.$

* E. Heine, *Theorie der Kugelfunctionen*, 1 (1878), pp. 97–125.

If we divide (2) by $z^{\frac{1}{2}}$, and replace q, z by q^2, qe^{2ix}, where x is real, the imaginary part of the series becomes

$$\frac{q^{\frac{1}{2}}\sin x}{1-q} + \frac{q^{\frac{3}{2}}\sin 3x}{1-q^3} + \frac{q^{\frac{5}{2}}\sin 5x}{1-q^5} + \dots\,,$$

which is the series for $\dfrac{Kk}{2\pi}\,\mathrm{sn}\,\dfrac{2Kx}{\pi}$.

Similarly from (3) we can derive a series connected with $\mathrm{dn}\,(2Kx/\pi)$.

We now show that

(4) $${}_1\Phi_0(a;\,z) = \prod_{n=0}^{\infty}\left(\frac{1-aq^n z}{1-q^n z}\right).$$

By subtracting series term by term, it is easily shown that

$${}_1\Phi_0(a;z) - {}_1\Phi_0(a;qz) = (1-a)\,z\,{}_1\Phi_0(aq;z),$$

$${}_1\Phi_0(a;z) - a\,{}_1\Phi_0(a;qz) = (1-a)\,{}_1\Phi_0(aq;z).$$

Eliminating the series which occurs on the right, we have

$${}_1\Phi_0(a;\,z) = \frac{1-az}{1-z}\,{}_1\Phi_0(a;\,qz),$$

and thus

$${}_1\Phi_0(a;\,z) = \frac{(1-az)(1-aqz)\dots(1-aq^{n-1}z)}{(1-z)(1-qz)\dots(1-q^{n-1}z)}\,{}_1\Phi_0(a;\,q^n z).$$

Now make $n\to\infty$ and (4) follows at once.

As particular cases of this result we note that, if $a = 0$,

(5) $$1 + \frac{z}{1-q} + \frac{z^2}{(1-q)(1-q^2)} + \dots = \frac{1}{(1-z)(1-qz)(1-q^2z)\dots}\,,$$

and, if z is replaced by z/a and then $a\to\infty$,

(6) $$1 - \frac{z}{1-q} + \frac{qz^2}{(1-q)(1-q^2)} - \dots + \frac{(-1)^n q^{\frac{1}{2}n(n-1)}z^n}{(1-q)(1-q^2)\dots(1-q^n)} + \dots$$
$$= (1-z)(1-qz)(1-q^2z)\dots\,.$$

Another consequence* of (4) is that

(7) $${}_1\Phi_0(a;z)\,{}_1\Phi_0(b;az) = {}_1\Phi_0(ab;z).$$

8.3. The analogue of Dougall's theorem.
The method of Chapter IV for obtaining transformations of generalized hyper-

* For the results of this paragraph see Heine, *loc. cit.*

geometric series does not appear to be capable of generalization so as to apply to basic series. The transformations of well-poised series, however, have their analogues for basic series, with the possible exception of the transformations of non-terminating well-poised ${}_9F_8$. We first prove the analogue of Dougall's theorem, namely*

(1) $${}_8\Phi_7\left[\begin{array}{ccccccc} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-N};\ q \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{N+1} \end{array}\right]$$

$$= \frac{(aq)_{q,N}\,(aq/cd)_{q,N}\,(aq/bd)_{q,N}\,(aq/bc)_{q,N}}{(aq/b)_{q,N}\,(aq/c)_{q,N}\,(aq/d)_{q,N}\,(aq/bcd)_{q,N}},$$

provided that $bcde = a^2 q^{N+1}$, and N is a positive integer.

It will be noticed that the effect of the presence of the four elements $q\sqrt{a}$, $-q\sqrt{a}$, $\sqrt{a}$, $-\sqrt{a}$ in the function on the left is merely the insertion of the factor $(1-aq^{2n})/(1-a)$ in the general term of the series.

The proof follows the same general lines as the proof of Dougall's theorem given in §5.1. Writing f in place of q^{-N}, the theorem becomes

$${}_8\Phi_7\left[\begin{array}{cccccccc} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f;\ q \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{array}\right]$$

$$= \prod_{n=1}^{\infty}\left[\frac{(1-aq^n)(1-aq^n/cd)(1-aq^n/bd)(1-aq^n/bc)}{(1-aq^n/b)(1-aq^n/c)(1-aq^n/d)(1-aq^n/f)}\right.$$

$$\times \left.\frac{(1-aq^n/bf)(1-aq^n/cf)(1-aq^n/df)(1-aq^n/bcdf)}{(1-aq^n/bcd)(1-aq^n/bcf)(1-aq^n/bdf)(1-aq^n/cdf)}\right],$$

provided that $a^2 q = bcdef$, and f is of the form q^{-N} where N is a positive integer.

Suppose the theorem is true when $f = 1, q^{-1}, q^{-2}, \dots, q^{-(N-1)}$. We shall prove it true when $f = q^{-N}$, and then the result will follow by induction. Now by symmetry the result is true if c or d has one of the values $1, q^{-1}, \dots, q^{-(N-1)}$, that is if c or $a^2 q/bcef$ has one of these values. It is therefore true in particular when $f = q^{-N}$ and c has one of $2N$ values. But when $f = q^{-N}$ we can multiply by $(aq/c)_{q,N}\,(aq/bcd)_{q,N}$ and the formula states the equality of two polynomials of degree $2N$ in c. Thus, if we can prove the

* Jackson 1.

equality for one more value of c, the result will be established. We choose the value $c = aq^N$, which is a pole of the last term only of the series, and the result is easily verified.

8.4. The analogue of Saalschütz's theorem.

In the formula just proved substitute for e, replace d by aq/d, and let $a \to 0$. We thus obtain the formula*

$$(1) \quad {}_3\Phi_2 \begin{bmatrix} b, c, q^{-N}; q \\ d, bcq^{1-N}/d \end{bmatrix} = \frac{(d/b)_{q,N} (d/c)_{q,N}}{(d)_{q,N} (d/bc)_{q,N}},$$

which is the analogue of Saalschütz's theorem.

By comparing the coefficients of powers of z and using (1), it is easy to prove the formula†

$$(2) \quad {}_2\Phi_1 \begin{bmatrix} \alpha, \beta; z \\ \gamma \end{bmatrix} = {}_1\Phi_0 \left[\alpha\beta/\gamma; z \right] {}_2\Phi_1 \begin{bmatrix} \gamma/\alpha, \gamma/\beta; \alpha\beta z/\gamma \\ \gamma \end{bmatrix}.$$

This is the analogue of § 1.2 (2).

Finally, if we let $N \to \infty$ in (1), we obtain the analogue of Gauss's theorem, namely

$$(3) \quad {}_2\Phi_1 \begin{bmatrix} b, c; d/bc \\ d \end{bmatrix} = \prod_{n=0}^{\infty} \left[\frac{(1 - dq^n/b)(1 - dq^n/c)}{(1 - dq^n)(1 - dq^n/bc)} \right].$$

8.5. Transformations of well-poised basic series.

The argument of § 5.2 can be used, with trivial alterations in the wording, to prove the transformation‡

$$(1) \quad {}_{10}\Phi_9 \begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, g, h, j; q \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h, aq/j \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1 - aq^n)(1 - aq^n/fg)(1 - aq^n/fh)(1 - aq^n/fj)}{(1 - aq^n/f)(1 - aq^n/g)(1 - aq^n/h)(1 - aq^n/j)} \right.$$

$$\left. \times \frac{(1 - aq^n/gh)(1 - aq^n/gj)(1 - aq^n/hj)(1 - aq^n/fghj)}{(1 - aq^n/ghj)(1 - aq^n/hjf)(1 - aq^n/jfg)(1 - aq^n/fgh)} \right]$$

$$\times {}_{10}\Phi_9 \begin{bmatrix} k, q\sqrt{k}, -q\sqrt{k}, kc/a, kd/a, ke/a, f, g, h, j; q \\ \sqrt{k}, -\sqrt{k}, aq/c, aq/d, aq/e, kq/f, kq/g, kq/h, kq/j \end{bmatrix},$$

where $k = a^2q/cde$ and $cdefghj = a^3q^2$, and f, g, h or j is of the form q^{-N} where N is a positive integer or zero.

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* Watson 6. Watson derives (1) from (2). (1) had been given previously by F. H. Jackson.

† Heine, *loc. cit.* p. 115, formula 13.   ‡ Bailey 4.

This is the analogue of the transformation § 4.3 (7) connecting two terminating well-poised series ${}_9F_8$. It includes the analogue of Dougall's theorem as an obvious particular case (when $cd = aq$).

If we substitute for k and j, and make $e \to \infty$, the formula becomes*

$$(2) \quad {}_8\Phi_7 \begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, g; a^2q^2/cdefg \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq/g \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1 - aq^n)(1 - aq^n/fg)(1 - aq^n/ge)(1 - aq^n/ef)}{(1 - aq^n/e)(1 - aq^n/f)(1 - aq^n/g)(1 - aq^n/efg)} \right]$$

$$\times {}_4\Phi_3 \begin{bmatrix} aq/cd, e, f, g; q \\ efg/a, aq/c, aq/d \end{bmatrix},$$

where e, f or g is of the form q^{-N}. This is the analogue of Whipple's formula § 4.3 (4) transforming a well-poised ${}_7F_6$ into a Saalschützian ${}_4F_3$. When $d = 1$ the series on the left of (2) reduces to unity, and we again obtain the analogue of Saalschütz's theorem. The formula (2) is due to Watson who used it to prove the Rogers-Ramanujan identities.†

Now in (1) replace c and k by their values in terms of the other parameters, putting $j = q^{-N}$, and then let $N \to \infty$, and in the same way as we obtained § 4.4 (4) we derive the formula

$$(3) \quad {}_8\Phi_7 \begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, f, g, h; a^2q^2/defgh \\ \sqrt{a}, -\sqrt{a}, aq/d, aq/e, aq/f, aq/g, aq/h \end{bmatrix}$$

$$= \prod_{n=1}^{\infty} \left[\frac{(1 - aq^n)(1 - aq^n/fg)(1 - aq^n/fh)(1 - aq^n/gh)}{(1 - aq^n/f)(1 - aq^n/g)(1 - aq^n/h)(1 - aq^n/fgh)} \right]$$

$$\times {}_4\Phi_3 \begin{bmatrix} aq/de, f, g, h; q \\ aq/d, aq/e, fgh/a \end{bmatrix}$$

$$+ \prod_{n=1}^{\infty} \left[\frac{(1 - aq^n)(1 - aq^n/de)(1 - fq^{n-1})(1 - gq^{n-1})(1 - hq^{n-1})}{(1 - aq^n/d)(1 - aq^n/e)(1 - aq^n/f)(1 - aq^n/g)(1 - aq^n/h)} \right.$$

$$\left. \times \frac{(1 - a^2q^{n+1}/dfgh)(1 - a^2q^{n+1}/efgh)}{(1 - a^2q^{n+1}/defgh)(1 - q^{n-2}fgh/a)} \right]$$

$$\times {}_4\Phi_3 \begin{bmatrix} aq/gh, aq/fh, aq/fg, a^2q^2/defgh; q \\ aq^2/fgh, a^2q^2/dfgh, a^2q^2/efgh \end{bmatrix}.$$

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* Watson 6.   † Given in the next paragraph.

This is the analogue of § 4.4 (4) and generalizes (2). It also shows that (2) is true provided only that the series on the right terminates and the series on the left converges, a fact which Watson stated was probable.

It is evident that other transformations of well-poised basic series could be worked out in a way entirely analogous to that of Chapter VII.

8.6. Some limiting cases. Now in the formula § 8.5 (2) make c, d, e, f and g tend to infinity. The process of making c, d, e, and f tend to infinity presents no theoretical difficulty when $g = q^{-N}$, since we are dealing with terminating series. To justify the process of subsequently making $N \to \infty$ through integral values, an appeal must be made to Tannery's theorem.

Now
$$\lim_{c \to \infty} \frac{(c)_{q,n}}{(aq/c)_{q,n}} \, c^{-n} = (-1)^n q^{\frac{1}{2}n(n-1)},$$

with similar formulae in d, e, f and g. Hence

$$\lim_{(c,d,e,f,g \to \infty)} {}_8\Phi_7 \left[\begin{array}{ccccccc} a, & q\sqrt{a}, & -q\sqrt{a}, & c, & d, & e, & f, & g; & \dfrac{a^2q^2}{cdefg} \\[2mm] & \sqrt{a}, & -\sqrt{a}, & aq/c, & aq/d, & aq/e, & aq/f, & aq/g \end{array} \right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(a)_{q,n}}{(q)_{q,n}} \frac{1 - aq^{2n}}{1-a} (-1)^{5n} q^{\frac{5}{2}n(n-1)} a^{2n} q^{2n}.$$

Also

$$\lim_{(e,f,g \to \infty)} \prod_{n=1}^{\infty} \left[\frac{(1 - aq^n)(1 - aq^n/fg)(1 - aq^n/ge)(1 - aq^n/ef)}{(1 - aq^n/e)(1 - aq^n/f)(1 - aq^n/g)(1 - aq^n/efg)} \right]$$

$$= \prod_{n=1}^{x} (1 - aq^n),$$

and

$$\lim_{(c,d,e,f,g \to \infty)} {}_4\Phi_3 \left[\begin{array}{c} aq/cd, \ e, \ f, \ g; \ q \\ efg/a, \ aq/c, \ aq/d \end{array} \right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\{(-1)^n q^{\frac{1}{2}n(n-1)}\}^3}{(-1)^n q^{\frac{1}{2}n(n-1)}/a^n} \frac{q^n}{(1-q)(1-q^2)\ldots(1-q^n)}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{a^n q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}.$$

We thus have the formula

(1) $$1 + \sum_{n=1}^{\infty} (-1)^n a^{2n} q^{\frac{1}{2}n(5n-1)} (1 - aq^{2n})$$

$$\times \frac{(1-aq)(1-aq^2)\ldots(1-aq^{n-1})}{(1-q)(1-q^2)\ldots(1-q^n)}$$

$$= \prod_{n=1}^{\infty} (1 - aq^n) \cdot \left[1 + \sum_{n=1}^{\infty} \frac{a^n q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)} \right].$$

By putting $a = 1$ and $a = q$ in this result, and using Jacobi's well-known formula

$$\prod_{n=1}^{\infty} [(1 - q^{2n-1}z)(1 - q^{2n-1}/z)(1 - q^{2n})] = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n$$

to express the series on the left of (1) as one of two products, we obtain the Rogers-Ramanujan identities[*]

(2) $$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots$$

$$= \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})\ldots},$$

(3) $$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \cdots$$

$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\ldots}.$$

In these formulae the indices of the powers of q in the numerators on the left are n^2 and $n(n+1)$, while, in the products on the right, the indices of the powers of q form two arithmetic progressions with difference 5.

Now in the formula § 8.5 (2) let $cd = aq$ and let e, f, $g \to \infty$, and we obtain

(4) $$1 + \sum_{n=1}^{\infty} (-1)^n a^n q^{\frac{1}{2}n(3n-1)} (1 - aq^{2n})$$

$$\times \frac{(1-aq)(1-aq^2)\ldots(1-aq^{n-1})}{(1-q)(1-q^2)\ldots(1-q^n)}$$

$$= \prod_{n=1}^{\infty} (1 - aq^n).$$

[*] For the history of these formulae see Rogers and Ramanujan 1, or Ramanujan, *Collected Papers* (1927), p. 344. See also Rogers 1 and 2, Schur 1. The proof given here is due to Watson 6.

For $a = 1$ this gives

$$(5) \qquad 1 + \sum_{n=1}^{\infty} (-1)^n \{q^{\frac{1}{2}n(3n-1)} + q^{\frac{1}{2}n(3n+1)}\} = \prod_{n=1}^{\infty} (1 - q^n),$$

a classical result (due to Euler) in the theory of partitions.

Again, if we put $cd = aq$, $e = \sqrt{(aq)}$, and let $f, g \to \infty$, we obtain

$$(6) \qquad 1 + \sum_{n=1}^{\infty} a^{\frac{1}{2}n} q^{n(n-\frac{1}{2})} (1 - aq^{2n}) \frac{(1-aq)(1-aq^2)\dots(1-aq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)}$$

$$= \prod_{n=1}^{\infty} \left(\frac{1-aq^n}{1-a^{\frac{1}{2}}q^{n-\frac{1}{2}}} \right).$$

For example, if $a = 1$, we have, replacing q by q^2,

$$(7) \qquad 1 + \sum_{n=1}^{\infty} q^{\frac{1}{2}n(n+1)} = \prod_{n=1}^{\infty} \left(\frac{1-q^{2n}}{1-q^{2n-1}} \right),$$

a result due to Gauss.

Lastly, if we put $c = \sqrt{a}$, $d = -\sqrt{a}$, and let $e, f, g \to \infty$, we obtain the relation

$$(8) \qquad 1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\dots(1-aq^{n-1})}{(1-q)(1-q^2)\dots(1-q^n)} a^n q^{\frac{1}{2}n(3n+1)}$$

$$= \prod_{n=1}^{\infty} (1 - aq^n) \cdot \left[1 + \sum_{n=1}^{\infty} \frac{a^n q^{n^2} (1+q)(1+q^2)\dots(1+q^n)}{(1-q)(1-q^2)\dots(1-q^n) \times (1-aq^2)(1-aq^4)\dots(1-aq^{2n})} \right]$$

and, in particular, when $a = 1$,

$$(9) \qquad 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\{(1-q)(1-q^2)\dots(1-q^n)\}^2} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right).$$

When $a = q$ we find from (8), after using (7), that

$$(10) \qquad 1 + \sum_{n=1}^{\infty} \frac{(1+q)(1+q^2)\dots(1+q^n) q^{n(n+1)}}{(1-q)(1-q^2)\dots(1-q^n)(1-q^3)(1-q^5)\dots(1-q^{2n+1})}$$

$$= \prod_{n=1}^{\infty} \left[\frac{1-q^{6n}}{(1-q^{6n-3})(1-q^{n+1})} \right].$$

Evidently a large number of such formulae could be found, but sufficient have been given to indicate the possibilities of the transformations given in this chapter.

CHAPTER IX
APPELL'S HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES

9.1. Definitions of Appell's functions. We have seen how the hypergeometric series can be generalized by simply increasing the number of parameters. Some other generalizations have been studied by Appell* in which the number of variables is increased.

Consider the two hypergeometric series

$$F(\alpha, \beta; \gamma; x), \qquad F(\alpha', \beta'; \gamma'; y).$$

If we form their product we obtain a double series, depending on the two variables x and y, in which the general term is

$$\frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{m! \, n! \, (\gamma)_m (\gamma')_n} x^m y^n.$$

Now replace one, two or three of the products $(\alpha)_m(\alpha')_n$, $(\beta)_m(\beta')_n$, $(\gamma)_m(\gamma')_n$ by the corresponding expressions

$$(\alpha)_{m+n}, \quad (\beta)_{m+n}, \quad (\gamma)_{m+n}.$$

There are five possibilities, one of which gives the series

$$\sum\sum \frac{(\alpha)_{m+n} (\beta)_{m+n}}{m! \, n! \, (\gamma)_{m+n}} x^m y^n,$$

which is simply the expansion of the function

$$F(\alpha, \beta; \gamma; x+y).$$

The four remaining possibilities lead to the definitions of Appell's hypergeometric functions of two variables, namely

$$(1) \quad F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum\sum \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! \, n! \, (\gamma)_{m+n}} x^m y^n,$$

$$(2) \quad F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y) = \sum\sum \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! \, n! \, (\gamma)_m (\gamma')_n} x^m y^n,$$

$$(3) \quad F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y) = \sum\sum \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{m! \, n! \, (\gamma)_{m+n}} x^m y^n,$$

$$(4) \quad F_4(\alpha, \beta; \gamma, \gamma'; x, y) = \sum\sum \frac{(\alpha)_{m+n} (\beta)_{m+n}}{m! \, n! \, (\gamma)_m (\gamma')_n} x^m y^n.$$

* See Appell and Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques* (1926).

The double series are absolutely convergent for

(1a) $\qquad |x| < 1, \quad |y| < 1;$

(2a) $\qquad |x| + |y| < 1;$

(3a) $\qquad |x| < 1, \quad |y| < 1;$

(4a) $\qquad |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1.$

To prove these statements we know that the general term of F_1 is

$$A_{m,n} x^m y^n = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')} \times \frac{\Gamma(\alpha+m+n)\,\Gamma(\beta+m)\,\Gamma(\beta'+n)}{\Gamma(\gamma+m+n)\,\Gamma(m+1)\,\Gamma(n+1)} x^m y^n,$$

and, using Stirling's formula, we see that for large values of m and n,

$$A_{m,n} \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta')} m^{\beta-1} n^{\beta'-1} (m+n)^{\alpha-\gamma}.$$

Write $\qquad R(\alpha) = \alpha_1, \quad R(\beta) = \beta_1, \quad R(\beta') = \beta_1', \quad R(\gamma) = \gamma_1,$

and let N be a number greater than the modulus of

$$\Gamma(\gamma)/\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\beta').$$

Write also a and b for $|x|$ and $|y|$. Then, for all large enough values of m and n,

$$|A_{m,n} x^m y^n| < \frac{N}{(m+n)^{\gamma_1-\alpha_1}\, m^{1-\beta_1}\, n^{1-\beta_1'}} a^m b^n$$

from which we conclude that F_1 is convergent when a and b are less than unity.

It can be shown in the same way that the series is divergent if a or b is greater than unity.

Similarly, if $A_{m,n} x^m y^n$ denotes the general term in F_2, we find that

$$|A_{m,n} x^m y^n| < N \frac{(m+n)!}{m!\,n!} (m+n)^{\alpha_1-1} m^{\beta_1-\gamma_1} n^{\beta_1'-\gamma_1'} a^m b^n.$$

Let k be a positive number greater than both $\beta_1 - \gamma_1$ and $\beta_1' - \gamma_1'$. Then

$$m^{\beta_1-\gamma_1} n^{\beta_1'-\gamma_1'} < m^k n^k \leqslant \frac{(m+n)^{2k}}{4^k}.$$

The series of moduli, therefore, has its terms less than those of the series

$$\frac{N}{4^k} \Sigma\Sigma \frac{(m+n)!}{m!\,n!} (m+n)^{2k+\alpha_1-1} a^m b^n,$$

which is $\qquad \dfrac{N}{4^k} \sum_{r=0}^{\infty} r^{2k+\alpha_1-1} (a+b)^r$

and is convergent when $a+b < 1$.

For the series F_3 we find, for the general term $A_{m,n} x^m y^n$, that

$$|A_{m,n} x^m y^n| < N m^{\alpha_1+\beta_1-2} n^{\alpha_1'+\beta_1'-2} (m+n)^{1-\gamma_1}$$

$$\times \frac{m!\,n!}{(m+n)!} a^m b^n$$

$$< N m^{\alpha_1+\beta_1-2} n^{\alpha_1'+\beta_1'-2} (m+n)^{1-\gamma_1} a^m b^n,$$

and so the series F_3 is convergent when a and b are less than unity.

Finally, for F_4 we find that

$$|A_{m,n} x^m y^n| < N (m+n)^{\alpha_1+\beta_1-2} m^{1-\gamma_1} n^{1-\gamma_1'} \left\{ \frac{(m+n)!}{m!\,n!} \right\}^2 a^m b^n.$$

Let k be a positive number greater than both $1-\gamma_1$ and $1-\gamma_1'$, and then

$$|A_{m,n} x^m y^n| < \frac{N}{4^k} (m+n)^{2k+\alpha_1+\beta_1-2} \left\{ \frac{(m+n)!}{m!\,n!} \right\}^2 a^m b^n.$$

Grouping together those terms of the series

$$\Sigma\Sigma (m+n)^{2k+\alpha_1+\beta_1-2} \left\{ \frac{(m+n)!}{m!\,n!} \right\}^2 a^m b^n$$

for which $m+n=r$, we obtain

$$\sum_{r=0}^{\infty} r^{2k+\alpha_1+\beta_1-2} \left\{ a^r + \binom{r}{1}^2 a^{r-1} b + \binom{r}{2}^2 a^{r-2} b^2 + \ldots + b^r \right\}$$

which is less than

$$\sum_{r=0}^{\infty} r^{2k+\alpha_1+\beta_1-2} (\sqrt{a}+\sqrt{b})^{2r},$$

and this series is convergent if $\sqrt{a}+\sqrt{b} < 1$.

It will be noticed that the functions all reduce to the ordinary hypergeometric series $F(\alpha, \beta; \gamma; x)$ when y is zero. The first three functions also reduce to $F(\alpha, \beta; \gamma; x)$ when β' is zero.

9.2. The partial differential equations satisfied by the functions.

Consider the function F_1, and let

$$z = F_1(\alpha; \beta, \beta'; \gamma; x, y) = \Sigma\Sigma A_{m,n} x^m y^n.$$

Then

$$A_{m+1,n} = \frac{(\alpha+m+n)(\beta+m)}{(m+1)(\gamma+m+n)} A_{m,n}.$$

Write $\vartheta = x\dfrac{\partial}{\partial x}$, $\phi = y\dfrac{\partial}{\partial y}$. Then, as in § 1.2, we see that F_1 satisfies the differential equation

$$\left\{(\vartheta + \phi + \alpha)(\vartheta + \beta) - \frac{1}{x}\vartheta(\vartheta + \phi + \gamma - 1)\right\} z = 0,$$

and a similar result is obtained by considering the relation between $A_{m,n}$ and $A_{m,n+1}$.

Now write p, q, r, s, t for the first and second order partial derivatives, and we find that the function F_1 satisfies the equations

$$F_1 \begin{cases} x(1-x)r + y(1-x)s + \{\gamma - (\alpha + \beta + 1)x\}p - \beta yq - \alpha\beta z = 0, \\ y(1-y)t + x(1-y)s + \{\gamma - (\alpha + \beta' + 1)y\}q - \beta'xp - \alpha\beta'z = 0 \end{cases}.$$

Similarly we find that the other functions satisfy the equations

$$F_2 \begin{cases} x(1-x)r - xys + \{\gamma - (\alpha + \beta + 1)x\}p - \beta yq - \alpha\beta z = 0, \\ y(1-y)t - xys + \{\gamma' - (\alpha + \beta' + 1)y\}q - \beta'xp - \alpha\beta'z = 0 \end{cases},$$

$$F_3 \begin{cases} x(1-x)r + ys + \{\gamma - (\alpha + \beta + 1)x\}p - \alpha\beta z = 0, \\ y(1-y)t + xs + \{\gamma - (\alpha' + \beta' + 1)y\}q - \alpha'\beta'z = 0 \end{cases},$$

$$F_4 \begin{cases} x(1-x)r - y^2 t - 2xys + \{\gamma - (\alpha + \beta + 1)x\}p \\ \qquad\qquad - (\alpha + \beta + 1)yq - \alpha\beta z = 0, \\ y(1-y)t - x^2 r - 2xys + \{\gamma' - (\alpha + \beta + 1)y\}q \\ \qquad\qquad - (\alpha + \beta + 1)xp - \alpha\beta z = 0 \end{cases}.$$

9.3. Expression of the functions F_1, F_2, F_3 in terms of definite integrals.

The functions F_1, F_2, F_3 can be expressed in terms of double integrals. The formulae are

$$(1) \quad \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta - \beta')}{\Gamma(\gamma)} F_1(\alpha; \beta, \beta'; \gamma; x, y)$$

$$= \iint u^{\beta-1} v^{\beta'-1}(1 - u - v)^{\gamma-\beta-\beta'-1}(1 - ux - vy)^{-\alpha} du\, dv$$

taken over the triangle $u \geqslant 0$, $v \geqslant 0$, $u + v \leqslant 1$;

$$(2) \quad \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta)\Gamma(\gamma' - \beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y)$$

$$= \int_0^1\!\!\int_0^1 u^{\beta-1} v^{\beta'-1}(1 - u)^{\gamma-\beta-1}(1 - v)^{\gamma'-\beta'-1}(1 - ux - vy)^{-\alpha} du\, dv;$$

$$(3) \quad \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma - \beta - \beta')}{\Gamma(\gamma)} F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y)$$

$$= \iint u^{\beta-1} v^{\beta'-1}(1 - u - v)^{\gamma-\beta-\beta'-1}(1 - ux)^{-\alpha}(1 - vy)^{-\alpha'} du\, dv,$$

taken over the triangle $u \geqslant 0$, $v \geqslant 0$, $u + v \leqslant 1$. The parameters are, of course, supposed to be such that the double integrals are convergent. The formulae are readily proved by expanding the integrand in powers of x and y and integrating term by term. There appears to be no simple integral representation of this type for the function F_4.

The function F_1 can also be expressed by a simple integral, the formula being

$$(4) \quad \frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} F_1(\alpha; \beta, \beta'; \gamma; x, y)$$

$$= \int_0^1 u^{\alpha-1}(1 - u)^{\gamma-\alpha-1}(1 - ux)^{-\beta}(1 - uy)^{-\beta'} du.$$

The four functions can also be expressed as double contour integrals taken along contours of Barnes' type.

9.4. Transformations of the functions F_1 and F_2.

Consider the integral

$$\int_0^1 u^{\alpha-1}(1 - u)^{\gamma-\alpha-1}(1 - ux)^{-\beta}(1 - uy)^{-\beta'} du,$$

which occurs in § 9.3 (4). There exist five changes of variable which leave unaltered the form of the integral, namely

$$u = 1 - v, \quad u = \frac{v}{1 - x + vx}, \quad u = \frac{v}{1 - y + vy},$$

$$u = \frac{1 - v}{1 - vx}, \quad u = \frac{1 - v}{1 - vy}.$$

On making these changes in the variable, we are immediately led to the following transformations:

(1) $F_1(\alpha; \beta, \beta'; \gamma; x, y)$
$$= (1-x)^{-\beta}(1-y)^{-\beta'} F_1\left(\gamma - \alpha; \beta, \beta'; \gamma; -\frac{x}{1-x}, -\frac{y}{1-y}\right),$$

(2) $= (1-x)^{-\alpha} F_1\left(\alpha; \gamma - \beta - \beta', \beta'; \gamma; -\frac{x}{1-x}, \frac{y-x}{1-x}\right),$

(3) $= (1-y)^{-\alpha} F_1\left(\alpha; \beta, \gamma - \beta - \beta'; \gamma; \frac{x-y}{1-y}, -\frac{y}{1-y}\right),$

(4) $= (1-x)^{\gamma-\alpha-\beta}(1-y)^{-\beta'} F_1\left(\gamma - \alpha; \gamma - \beta - \beta', \beta'; \gamma; x, \frac{x-y}{1-y}\right),$

(5) $= (1-x)^{-\beta}(1-y)^{\gamma-\alpha-\beta'} F_1\left(\gamma - \alpha; \beta, \gamma - \beta - \beta'; \gamma; \frac{y-x}{1-x}, y\right).$

When $\beta' = 0$, (1) and (2) reduce to § 2.4 (1) and (4) reduces to § 1.2 (2).

The above formulae show that there are at least six solutions of the differential equations satisfied by F_1. It has been shown that there are 60 integrals of these equations,* each integral involving a function F_1, and that there is a linear relation connecting any four of these integrals. The 60 solutions correspond to Kummer's 24 solutions of the hypergeometric equation.

Similarly by considering the double integral

$$\int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha} du\, dv,$$

which occurs in § 9.3 (2), and making the substitutions

(a) $u = 1 - u', \quad v = v',$

(b) $u = u', \quad v = 1 - v',$

(c) $u = 1 - u', \quad v = 1 - v',$

we deduce the formulae

(6) $F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y)$
$$= (1-x)^{-\alpha} F_2\left(\alpha; \gamma - \beta, \beta'; \gamma, \gamma'; -\frac{x}{1-x}, \frac{y}{1-x}\right),$$

* A table of these 60 solutions is reproduced in Appell and Kampé de Fériet, *loc. cit.* pp. 62–64.

(7) $= (1-y)^{-\alpha} F_2\left(\alpha; \beta, \gamma' - \beta'; \gamma, \gamma'; \frac{x}{1-y}, -\frac{y}{1-y}\right),$

(8) $= (1-x-y)^{-\alpha} F_2\left(\alpha; \gamma - \beta, \gamma' - \beta'; \gamma, \gamma'; -\frac{x}{1-x-y}, -\frac{y}{1-x-y}\right).$

There do not appear to be similar transformations for the functions F_3 and F_4.

9.5. Cases of reducibility of F_1, F_2, F_3. As particular cases of the formulae (4), (3) and (6) of § 9.4, we have

(1) $F_1(\alpha; \beta, \beta'; \gamma; x, x)$
$$= (1-x)^{\gamma-\alpha-\beta-\beta'} F(\gamma - \alpha, \gamma - \beta - \beta'; \gamma; x)$$
$$= F(\alpha, \beta + \beta'; \gamma; x),$$

(2) $F_1(\alpha; \beta, \beta'; \beta + \beta'; x, y) = (1-y)^{-\alpha} F\left(\alpha, \beta; \beta + \beta'; \frac{x-y}{1-y}\right),$

(3) $F_2(\alpha; \beta, \beta'; \beta, \gamma'; x, y) = (1-x)^{-\alpha} F\left(\alpha, \beta'; \gamma'; \frac{y}{1-x}\right).$

Of these formulae the second shows that the function F_1 reduces to an ordinary hypergeometric function when $\gamma = \beta + \beta'$, and the third shows that F_2 similarly reduces when $\gamma = \beta$ (or, by symmetry, when $\gamma' = \beta'$).

Now, from the definition of F_1, we have

$$F_1(\alpha; \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} F(\alpha + m, \beta'; \gamma + m; y) x^m$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} (1-y)^{-\beta'} F\left(\gamma - \alpha, \beta'; \gamma + m; -\frac{y}{1-y}\right) x^m$$

$$= (1-y)^{-\beta'} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_m (\gamma - \alpha)_n (\beta')_n}{m! n! (\gamma)_{m+n}} x^m \left(-\frac{y}{1-y}\right)^n,$$

and so

(4) $F_1(\alpha; \beta, \beta'; \gamma; x, y)$
$$= (1-y)^{-\beta'} F_3\left(\alpha, \gamma - \alpha; \beta, \beta'; \gamma; x, -\frac{y}{1-y}\right).$$

Thus the function F_1 can always be expressed in terms of F_3.

Conversely, the function $F_3(\alpha, \alpha'; \beta, \beta'; \gamma; x, y)$ reduces to the function F_1 when $\gamma = \alpha + \alpha'$.

Now F_1 reduces to an ordinary hypergeometric series when $\gamma = \beta + \beta'$. Thus the function F_3 similarly reduces when

$$\gamma = \alpha + \alpha' = \beta + \beta',$$

the formula being

(5) $F_3(\alpha, \gamma - \alpha; \beta, \gamma - \beta; \gamma; x, y)$
$$= (1-y)^{\alpha + \beta - \gamma} F(\alpha, \beta; \gamma; x + y - xy).$$

We now show that the function F_1 can always be expressed in terms of F_2. For

$$(1-y)^{-\beta'} F_2\left(\alpha; \beta, \beta'; \gamma, \alpha; x, -\frac{y}{1-y}\right)$$

$$= (1-y)^{-\beta'} \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} x^m F\left(\alpha + m, \beta'; \alpha; -\frac{y}{1-y}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} x^m F(\beta', -m; \alpha; y)$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} x^m \frac{(\alpha - \beta')_m}{(\alpha)_m} F(\beta', -m; 1 + \beta' - \alpha - m; 1 - y),$$

the last step following from § 1.4 (1), though it can easily be verified by simple algebra since the series terminate. We thus have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(\beta)_m (\alpha - \beta')_m (\beta')_n (-m)_n}{m! \, n! \, (\gamma)_m (1 + \beta' - \alpha - m)_n} x^m (1-y)^n$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\beta)_{n+s} (\alpha - \beta')_{n+s} (\beta')_n (-1)^n x^{n+s} (1-y)^n}{s! \, n! \, (\gamma)_{n+s} (1 + \beta' - \alpha - n - s)_n}$$

$$= \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\beta)_{n+s} (\alpha - \beta')_s (\beta')_n}{s! \, n! \, (\gamma)_{n+s}} x^{n+s} (1-y)^n,$$

and so

(6) $(1-y)^{-\beta'} F_2\left(\alpha; \beta, \beta'; \gamma, \alpha; x, -\frac{y}{1-y}\right)$

$$= F_1[\beta; \alpha - \beta', \beta'; \gamma; x, x(1-y)],$$

which proves the result.

This formula also shows that the function F_2 reduces to F_1

when $\gamma' = \alpha$. Now the function F_1 on the right of (6) reduces to an ordinary hypergeometric function when $\gamma = \alpha$. Thus

$$F_2(\alpha; \beta, \beta'; \gamma, \gamma'; x, y)$$

similarly reduces when $\gamma = \gamma' = \alpha$. The formula can be written in the form

(7) $F_2(\alpha; \beta, \beta'; \alpha, \alpha; x, y)$

$$= (1-x)^{-\beta} (1-y)^{-\beta'} F\left(\beta, \beta'; \alpha; \frac{xy}{(1-x)(1-y)}\right),$$

and this can be proved very easily by expanding the right-hand side in powers of x and y.

9.6. A case of reducibility of F_4. The cases of reducibility given in § 9.5 have all been known for a considerable time, and are all given in the treatise by Appell and Kampé de Fériet. In this paragraph a formula will be given which has only been discovered quite recently.* This formula is

(1) $F_4[\alpha, \beta; \gamma, \alpha + \beta - \gamma + 1; z(1-Z), Z(1-z)]$
$$= F(\alpha, \beta; \gamma; z) F(\alpha, \beta; \alpha + \beta - \gamma + 1; Z),$$

and is valid inside simply-connected regions surrounding $z = 0$, $Z = 0$ for which

$$|z(1-Z)|^{\frac{1}{2}} + |Z(1-z)|^{\frac{1}{2}} < 1.$$

If we change z, Z into $1 - Z, 1 - z$, we see that

(2) $F_4[\alpha, \beta; \gamma, \alpha + \beta - \gamma + 1; z(1-Z), Z(1-z)]$
$$= F(\alpha, \beta; \gamma; 1 - Z) F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z),$$

inside simply-connected regions surrounding $z = 1$, $Z = 1$ which satisfy the same inequality as before.

The formulae (1) and (2) give the complete expression of F_4 in terms of ordinary hypergeometric functions when $\gamma + \gamma' = \alpha + \beta + 1$.

To prove (1) we first consider the function

$$(1-x)^{-\alpha} (1-y)^{-\beta} F_4\left(\alpha, \beta; \gamma, \gamma'; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right),$$

which is an analytic function of x and y when $|x|$ and $|y|$ are sufficiently small, and can therefore be expanded in a double

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* Bailey 11 and 13.

series of powers of x and y. The coefficient of $x^m y^n$ in this expansion is

$$\sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(\alpha)_{r+s}(\beta)_{r+s}}{r!\,s!\,(\gamma)_r(\gamma')_s} \frac{(-1)^{r+s}(\alpha+r+s)_{m-r}(\beta+r+s)_{n-s}}{(m-r)!\,(n-s)!}$$

$$= \frac{(\alpha)_m(\beta)_n}{m!\,n!} \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(\alpha+m)_s(\beta+n)_r(-m)_r(-n)_s}{r!\,s!\,(\gamma)_r(\gamma')_s}$$

$$= \frac{(\alpha)_m(\beta)_n}{m!\,n!} F(\alpha+m, -n; \gamma') F(\beta+n, -m; \gamma)$$

$$= \frac{(\alpha)_m(\beta)_n(\gamma'-\alpha-m)_n(\gamma-\beta-n)_m}{m!\,n!\,(\gamma)_m(\gamma')_n}$$

$$= \frac{(\alpha)_m(\beta)_n(1+\alpha-\gamma')_m(1+\beta-\gamma)_n(\gamma-\beta)_{m-n}}{m!\,n!\,(\gamma)_m(\gamma')_n(1+\alpha-\gamma')_{m-n}}.$$

Now, if $\gamma+\gamma'=\alpha+\beta+1$, the factors*

$$(\gamma-\beta)_{m-n} \quad \text{and} \quad (1+\alpha-\gamma')_{m-n}$$

cancel, and so we obtain

$$(1-x)^{-\alpha}(1-y)^{-\beta}$$

$$\times F_4\left(\alpha, \beta; \gamma, \gamma'; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)}\right)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m(\beta)_n(\gamma-\beta)_m(\gamma'-\alpha)_n}{m!\,n!\,(\gamma)_m(\gamma')_n} x^m y^n$$

$$= F(\alpha, \gamma-\beta; \gamma; x) F(\beta, \gamma'-\alpha; \gamma'; y)$$

$$= (1-x)^{-\alpha}(1-y)^{-\beta} F\left(\alpha, \beta; \gamma; -\frac{x}{1-x}\right) F\left(\alpha, \beta; \gamma'; -\frac{y}{1-y}\right),$$

and this is equivalent to the formula stated. It has been proved for small enough values of $|x|$ and $|y|$, and therefore of $|z|$ and $|Z|$, and the complete result follows by analytic continuation.

If α is a negative integer, (1) can be written

$$F_4[-n, \beta+n; \gamma, \beta-\gamma+1; zZ, (1-z)(1-Z)]$$

$$= F(-n, \beta+n; \gamma; z) F(-n, \beta+n; \beta-\gamma+1; 1-Z).$$

* If $m < n$, an expression such as $(a)_{m-n}$ must be replaced by $(-1)^{n-m}/(1-a)_{n-m}$.

The second series on the right can be rearranged in powers of Z, and we obtain

$$(3) \quad F_4[-n, \beta+n; \gamma, \beta-\gamma+1; zZ, (1-z)(1-Z)]$$

$$= \frac{(-1)^n(\gamma)_n}{(\beta-\gamma+1)_n} F(-n, \beta+n; \gamma; z) F(-n, \beta+n; \gamma; Z),$$

a formula due to Watson.*

There are other cases in which Appell's functions of two variables reduce to ordinary hypergeometric functions, but the cases given above include those in which the number of conditions satisfied by the parameters is as small as possible.

* Watson 2. See also Watson, *Theory of Bessel Functions* (1922), § 11.6, where the formula is used to prove Bateman's expansion.

CHAPTER X

SOME MISCELLANEOUS RESULTS

10.1. The theorems of Cayley and Orr. In 1858 Cayley* published, without proof, the theorem that, if

$$(1-z)^{\alpha+\beta-\gamma} F(2\alpha, 2\beta; 2\gamma; z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$F(\alpha, \beta; \gamma+\tfrac{1}{2}; z) F(\gamma-\alpha, \gamma-\beta; \gamma+\tfrac{1}{2}; z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\gamma+\tfrac{1}{2})_n} a_n z^n.$$

Cayley stated that he had discovered the result in discussing certain relations in planetary theory. It was not until forty years later that a proof was published by Orr,† who discussed the differential equation satisfied by the product of two hypergeometric series, and obtained several additional results. The main results given by Orr may be stated as follows:

If $(1-z)^{\alpha+\beta-\gamma-\frac{1}{2}} F(2\alpha, 2\beta; 2\gamma; z) = \sum a_n z^n,$

then

$$F(\alpha, \beta; \gamma; z) F(\gamma-\alpha+\tfrac{1}{2}, \gamma-\beta+\tfrac{1}{2}; \gamma+1; z) = \sum \frac{(\gamma+\tfrac{1}{2})_n}{(\gamma+1)_n} a_n z^n;$$

and if $(1-z)^{\alpha+\beta-\gamma-\frac{1}{2}} F(2\alpha-1, 2\beta; 2\gamma-1; z) = \sum a_n z^n,$

then $F(\alpha, \beta; \gamma; z) F(\gamma-\alpha+\tfrac{1}{2}, \gamma-\beta-\tfrac{1}{2}; \gamma; z) = \sum \frac{(\gamma-\tfrac{1}{2})_n}{(\gamma)_n} a_n z^n.$

Several proofs of these results have been given,‡ but it is only during the last few years that a proof with any real claim to simplicity has been discovered.§

If we compare coefficients of z^n in the products, the identities to be proved are seen to be

$$(1) \quad {}_3F_2\left[\begin{matrix} 2\alpha, 2\beta, -n; \\ 2\gamma, 1+\alpha+\beta-\gamma-n \end{matrix}\right] = \frac{(\gamma-\alpha)_n (\gamma-\beta)_n}{(\gamma)_n (\gamma-\alpha-\beta)_n}$$

$$\times {}_4F_3\left[\begin{matrix} \alpha, \beta, \tfrac{1}{2}-\gamma-n, -n; \\ \gamma+\tfrac{1}{2}, 1+\alpha-\gamma-n, 1+\beta-\gamma-n \end{matrix}\right],$$

* Cayley 1.   † Orr 1.
‡ Edwardes 1, Watson 4, Whipple 6 and 7.
§ For the proof given here see Whipple 7.

$$(2) \quad {}_3F_2\left[\begin{matrix} 2\alpha, 2\beta, -n; \\ 2\gamma, \tfrac{1}{2}+\alpha+\beta-\gamma-n \end{matrix}\right] = \frac{(\gamma-\alpha+\tfrac{1}{2})_n (\gamma-\beta+\tfrac{1}{2})_n}{(\gamma+\tfrac{1}{2})_n (\gamma+\tfrac{1}{2}-\alpha-\beta)_n}$$

$$\times {}_4F_3\left[\begin{matrix} \alpha, \beta, -\gamma-n, -n; \\ \gamma, \tfrac{1}{2}+\alpha-\gamma-n, \tfrac{1}{2}+\beta-\gamma-n \end{matrix}\right],$$

$$(3) \quad {}_3F_2\left[\begin{matrix} 2\alpha-1, 2\beta, -n; \\ 2\gamma-1, \tfrac{1}{2}+\alpha+\beta-\gamma-n \end{matrix}\right] = \frac{(\gamma-\alpha+\tfrac{1}{2})_n (\gamma-\beta-\tfrac{1}{2})_n}{(\gamma-\tfrac{1}{2})_n (\gamma+\tfrac{1}{2}-\alpha-\beta)_n}$$

$$\times {}_4F_3\left[\begin{matrix} \alpha, \beta, 1-\gamma-n, -n; \\ \gamma, \tfrac{1}{2}+\alpha-\gamma-n, \tfrac{3}{2}+\beta-\gamma-n \end{matrix}\right].$$

It is convenient first to prove (2) and (3). The series on the right of (2) and (3) are Saalschützian. If we change α, γ into $\alpha+\tfrac{1}{2}, \gamma+\tfrac{1}{2}$ in (3), the right-hand side of (3) can then be transformed into the right-hand side of (2) by § 7.2 (1), the transformation of terminating Saalschützian ${}_4F_3$. Thus (3) follows from (2).

To prove (2) we use the transformation of Saalschützian ${}_4F_3$ already referred to, the transformation of a nearly-poised ${}_3F_2$ into a Saalschützian ${}_4F_3$ (§ 4.5 (1) with $c = \tfrac{1}{2}+\tfrac{1}{2}a$), and also the formula*

$${}_3F_2\left[\begin{matrix} a, b, -n; \\ e, f \end{matrix}\right] = \frac{(e-a)_n (f-a)_n}{(e)_n (f)_n} {}_3F_2\left[\begin{matrix} 1-s, a, -n; \\ 1+a-f-n, 1+a-e-n \end{matrix}\right],$$

where $s = e+f-a-b+n.$

We thus find that

$${}_3F_2\left[\begin{matrix} 2\alpha, 2\beta, -n; \\ 2\gamma, \tfrac{1}{2}+\alpha+\beta-\gamma-n \end{matrix}\right]$$

$$= \frac{(\tfrac{1}{2}+\alpha-\beta+\gamma)_n (2\gamma-2\alpha)_n}{(2\gamma)_n (\tfrac{1}{2}-\alpha-\beta+\gamma)_n} {}_3F_2\left[\begin{matrix} 2\alpha, \tfrac{1}{2}+\alpha+\beta-\gamma, -n; \\ \tfrac{1}{2}+\alpha-\beta+\gamma, 1+2\alpha-2\gamma-n \end{matrix}\right]$$

$$= \frac{(\tfrac{1}{2}+\alpha-\beta+\gamma)_n}{(\tfrac{1}{2}-\alpha-\beta+\gamma)_n} {}_4F_3\left[\begin{matrix} \alpha, \gamma-\beta, 2\gamma+n, -n; \\ \gamma, \gamma+\tfrac{1}{2}, \tfrac{1}{2}+\alpha-\beta+\gamma \end{matrix}\right]$$

$$= \frac{(\tfrac{1}{2}-\alpha+\gamma)_n (\tfrac{1}{2}-\beta+\gamma)_n}{(\tfrac{1}{2}-\alpha-\beta+\gamma)_n (\gamma+\tfrac{1}{2})_n} {}_4F_3\left[\begin{matrix} \alpha, \beta, -\gamma-n, -n; \\ \gamma, \tfrac{1}{2}+\alpha-\gamma-n, \tfrac{1}{2}+\beta-\gamma-n \end{matrix}\right],$$

and (2) is proved.

* In the notation of Chapter III this is the relation between $Fp(0; 4, 5)$ and $Fp(2; 4, 5)$. It can also be deduced from § 7.2 (1) by substituting for u in terms of the other parameters and making x tend to infinity.

To prove (1), multiply (2) by γ, change α, γ into $\alpha + 1$, $\gamma + 1$ in (3) and multiply by α, and subtract term by term. We then find that

$$
{}_3F_2\left[\begin{matrix} 2\alpha, \ 2\beta, \ -n; \\ 2\gamma + 1, \ \tfrac{1}{2} + \alpha + \beta - \gamma - n \end{matrix}\right]
$$

$$
= \frac{(\gamma - \alpha + \tfrac{1}{2})_n \, (\gamma - \beta + \tfrac{1}{2})_n}{(\gamma + \tfrac{1}{2})_n \, (\gamma + \tfrac{1}{2} - \alpha - \beta)_n} \, {}_4F_3\left[\begin{matrix} \alpha, \ \beta, \ -\gamma - n, \ -n; \\ \gamma + 1, \ \tfrac{1}{2} + \alpha - \gamma - n, \ \tfrac{1}{2} + \beta - \gamma - n \end{matrix}\right].
$$

This is (1) with $\gamma + \tfrac{1}{2}$ instead of γ, and so Cayley's theorem is proved.

When $\gamma = \alpha + \beta$, Cayley's theorem becomes

$$
(4) \qquad \left\{ F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right] \right\}^2 = {}_3F_2\left[\begin{matrix} 2\alpha, \ 2\beta, \ \alpha + \beta; \ z \\ 2\alpha + 2\beta, \ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right],
$$

a result due to Clausen.[*]

Similarly from Orr's theorems we obtain

$$
(5) \quad F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \alpha + \beta - \tfrac{1}{2} \end{matrix}\right] F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right] = {}_3F_2\left[\begin{matrix} 2\alpha, \ 2\beta, \ \alpha + \beta; \ z \\ 2\alpha + 2\beta - 1, \ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right],
$$

$$
(6) \quad F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \alpha + \beta - \tfrac{1}{2} \end{matrix}\right] F\left[\begin{matrix} \alpha, \ \beta - 1; \ z \\ \alpha + \beta - \tfrac{1}{2} \end{matrix}\right] = {}_3F_2\left[\begin{matrix} 2\alpha, 2\beta - 1, \alpha + \beta - 1; z \\ 2\alpha + 2\beta - 2, \ \alpha + \beta - \tfrac{1}{2} \end{matrix}\right],
$$

both of which were given by Orr.

10.2. Some similar results. Further results of a similar nature[†] can be obtained by using the transformation §4.5 (1) of a nearly-poised series into a Saalschützian ${}_5F_4$. When the nearly-poised series is also Saalschützian, the ${}_5F_4$ reduces to a ${}_4F_3$. We can thus prove for example that, if

$$
(1 - z)^{\alpha + \beta - \gamma - \tfrac{1}{2}} \, {}_3F_2\left[\begin{matrix} 2\alpha, \ 2\beta, \ \gamma; \ z \\ 2\gamma, \ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right] = \Sigma \, a_n z^n,
$$

then

$$
F\left[\begin{matrix} \alpha, \beta; z \\ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right] F\left[\begin{matrix} \tfrac{1}{2} + \gamma - \alpha, \ \tfrac{1}{2} + \gamma - \beta; z \\ 2\gamma - \alpha - \beta + \tfrac{1}{2} \end{matrix}\right] = \Sigma \, \frac{(\gamma + \tfrac{1}{2})_n}{(2\gamma - \alpha - \beta + \tfrac{1}{2})_n} \, a_n z^n.
$$

The identity implied is

$$
(1) \quad {}_4F_3\left[\begin{matrix} 2\alpha, \ 2\beta, \ \gamma, \ -n; \\ 2\gamma, \ \alpha + \beta + \tfrac{1}{2}, \ \tfrac{1}{2} - n + \alpha + \beta - \gamma \end{matrix}\right]
$$

$$
= \frac{(\tfrac{1}{2} + \gamma - \alpha)_n (\tfrac{1}{2} + \gamma - \beta)_n}{(\gamma + \tfrac{1}{2})_n (\tfrac{1}{2} - \alpha - \beta + \gamma)_n}
$$

$$
\times {}_4F_3\left[\begin{matrix} \alpha, \ \beta, \ \tfrac{1}{2} + \alpha + \beta - 2\gamma - n, \ -n; \\ \tfrac{1}{2} + \alpha - \gamma - n, \ \tfrac{1}{2} + \beta - \gamma - n, \ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right].
$$

[*] Clausen 1.   [†] Bailey 14.

To prove this identity, we have

$$
{}_4F_3\left[\begin{matrix} 2\alpha, \ 2\beta, \ \gamma, \ -n; \\ 2\gamma, \ \alpha + \beta + \tfrac{1}{2}, \ \tfrac{1}{2} - n + \alpha + \beta - \gamma \end{matrix}\right]
$$

$$
= \frac{(2\gamma - 2\alpha)_n (\tfrac{1}{2} + \alpha - \beta + \gamma)_n}{(2\gamma)_n (\tfrac{1}{2} - \alpha - \beta + \gamma)_n}
$$

$$
\times {}_4F_3\left[\begin{matrix} 2\alpha, \ \tfrac{1}{2} + \alpha + \beta - \gamma, \ \tfrac{1}{2} + \alpha - \beta, \ -n; \\ \tfrac{1}{2} + \alpha - \beta + \gamma, \ \tfrac{1}{2} + \alpha + \beta, \ 1 - n + 2\alpha - 2\gamma \end{matrix}\right]
$$

$$
= \frac{(\tfrac{1}{2} + \alpha - \beta + \gamma)_n}{(\tfrac{1}{2} - \alpha - \beta + \gamma)_n} \, {}_4F_3\left[\begin{matrix} \alpha, \ \alpha + \tfrac{1}{2}, \ 2\gamma + n, \ -n; \\ \tfrac{1}{2} + \alpha - \beta + \gamma, \ \gamma + \tfrac{1}{2}, \ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right]
$$

$$
= \frac{(\tfrac{1}{2} + \gamma - \alpha)_n (\tfrac{1}{2} + \gamma - \beta)_n}{(\gamma + \tfrac{1}{2})_n (\tfrac{1}{2} - \alpha - \beta + \gamma)_n}
$$

$$
\times {}_4F_3\left[\begin{matrix} \alpha, \ \beta, \ \tfrac{1}{2} + \alpha + \beta - 2\gamma - n, \ -n; \\ \tfrac{1}{2} + \beta - \gamma - n, \ \tfrac{1}{2} + \alpha - \gamma - n, \ \alpha + \beta + \tfrac{1}{2} \end{matrix}\right].
$$

Here we have used §7.2 (1), then §4.5 (1), and finally §7.2 (1) again.

Further results containing more parameters can be obtained from §4.7 (1). Replacing d in that formula by $1 - n - d$, it is easily shown that, if

$$
F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \gamma \end{matrix}\right] F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \delta \end{matrix}\right] = \Sigma \, a_n z^n,
$$

then

$$
{}_4F_3\left[\begin{matrix} \alpha, \ \beta, \ \tfrac{1}{2}(\gamma + \delta), \ \tfrac{1}{2}(\gamma + \delta - 1); \ 4z(1 - z) \\ \alpha + \beta, \ \gamma, \ \delta \end{matrix}\right] = \Sigma \, \frac{(\gamma + \delta - 1)_n}{(\alpha + \beta)_n} \, a_n z^n.
$$

Similarly, by changing b, w into $1 - n - b$, $1 - n - w$ in the same formula, we see that, if

$$
F\left[\begin{matrix} \alpha, \ \beta; \ z \\ \gamma \end{matrix}\right] F\left[\begin{matrix} \alpha, \ \delta; \ z \\ \gamma \end{matrix}\right] = \Sigma \, a_n z^n,
$$

then

$$
(1 - z)^{-\alpha} \, {}_4F_3\left[\begin{matrix} \alpha, \ \beta, \ \delta, \ \gamma - \alpha; \ -z^2/4 \, (1 - z) \\ \gamma, \ \tfrac{1}{2}(\beta + \delta), \ \tfrac{1}{2}(\beta + \delta + 1) \end{matrix}\right] = \Sigma \, \frac{(\gamma)_n}{(\beta + \delta)_n} \, a_n z^n.
$$

As a particular case of the first of these results, take $\gamma = \delta = \beta$, and we obtain the quadratic transformation of Gauss

$$(2) \qquad F\begin{bmatrix} \alpha, \ \beta; \ 4z(1-z) \\ \alpha+\beta+\tfrac{1}{2} \end{bmatrix} = F\begin{bmatrix} 2\alpha, \ 2\beta; \ z \\ \alpha+\beta+\tfrac{1}{2} \end{bmatrix}.$$

Again, taking $\delta = \alpha + \beta - \gamma + 1$, we have

$$(3) \qquad F\begin{bmatrix} \alpha, \ \beta; \ z \\ \gamma \end{bmatrix} F\begin{bmatrix} \alpha, \ \beta; \ z \\ \alpha+\beta-\gamma+1 \end{bmatrix}$$

$$= {}_4F_3\begin{bmatrix} \alpha, \ \beta, \ \tfrac{1}{2}(\alpha+\beta+1), \ \tfrac{1}{2}(\alpha+\beta); \ 4z(1-z) \\ \alpha+\beta, \ \gamma, \ \alpha+\beta-\gamma+1 \end{bmatrix},$$

which can also be deduced from § 9.6 (1).

10.3. Darling's theorems on products. From § 1.2 (2) it is evident that

$$(1) \qquad {}_2F_1(\alpha, \ \beta; \ \gamma; \ z) {}_2F_1(1-\alpha, \ 1-\beta; \ 2-\gamma; \ z)$$

$$= {}_2F_1(\alpha+1-\gamma, \ \beta+1-\gamma; \ 2-\gamma; \ z) {}_2F_1(\gamma-\alpha, \ \gamma-\beta; \ \gamma; \ z).$$

This relation and § 1.2 (2) have been generalized by Darling,* and the generalizations apply to series of any order. For series of the type ${}_3F_2$ the formulae are

$$(2) \qquad {}_3F_2\begin{bmatrix} \alpha, \ \beta, \ \gamma; \ z \\ \delta, \ \epsilon \end{bmatrix} {}_3F_2\begin{bmatrix} 1-\alpha, \ 1-\beta, \ 1-\gamma; \ z \\ 2-\delta, \ 2-\epsilon \end{bmatrix}$$

$$= \frac{\epsilon-1}{\epsilon-\delta} {}_3F_2\begin{bmatrix} \alpha+1-\delta, \ \beta+1-\delta, \ \gamma+1-\delta; \ z \\ 2-\delta, \ \epsilon+1-\delta \end{bmatrix}$$

$$\times {}_3F_2\begin{bmatrix} \delta-\alpha, \ \delta-\beta, \ \delta-\gamma; \ z \\ \delta, \ \delta+1-\epsilon \end{bmatrix}$$

$$+ \frac{\delta-1}{\delta-\epsilon} {}_3F_2\begin{bmatrix} \alpha+1-\epsilon, \ \beta+1-\epsilon, \ \gamma+1-\epsilon; \ z \\ 2-\epsilon, \ \delta+1-\epsilon \end{bmatrix}$$

$$\times {}_3F_2\begin{bmatrix} \epsilon-\alpha, \ \epsilon-\beta, \ \epsilon-\gamma; \ z \\ \epsilon, \ \epsilon+1-\delta \end{bmatrix},$$

---

* Darling 2. For the proofs given here see Bailey 9 and Burchnall 1.

and

$$(3) \qquad (1-z)^{\alpha+\beta+\gamma-\delta-\epsilon} {}_3F_2\begin{bmatrix} \alpha, \ \beta, \ \gamma; \ z \\ \delta, \ \epsilon \end{bmatrix}$$

$$= \frac{\epsilon-1}{\epsilon-\delta} {}_3F_2\begin{bmatrix} \delta-\alpha, \ \delta-\beta, \ \delta-\gamma; \ z \\ \delta, \delta+1-\epsilon \end{bmatrix} {}_3F_2\begin{bmatrix} \epsilon-\alpha, \ \epsilon-\beta, \ \epsilon-\gamma; \ z \\ \epsilon-1, \ \epsilon+1-\delta \end{bmatrix}$$

$$+ \frac{\delta-1}{\delta-\epsilon} {}_3F_2\begin{bmatrix} \epsilon-\alpha, \ \epsilon-\beta, \ \epsilon-\gamma; \ z \\ \epsilon, \ \epsilon+1-\delta \end{bmatrix} {}_3F_2\begin{bmatrix} \delta-\alpha, \ \delta-\beta, \ \delta-\gamma; \ z \\ \delta-1, \ \delta+1-\epsilon \end{bmatrix},$$

which reduce to (1) and § 1.2 (2) when $\gamma = \epsilon \to \infty$.

In order to give a proof which is applicable to series of any order, it is convenient to write

$$A = {}_3F_2\begin{bmatrix} \alpha, \ \beta, \ \gamma; \ z \\ \delta, \ \epsilon \end{bmatrix},$$

$$B = z^{1-\delta} {}_3F_2\begin{bmatrix} \alpha+1-\delta, \ \beta+1-\delta, \ \gamma+1-\delta; \ z \\ 2-\delta, \ \epsilon+1-\delta \end{bmatrix},$$

$$C = z^{1-\epsilon} {}_3F_2\begin{bmatrix} \alpha+1-\epsilon, \ \beta+1-\epsilon, \ \gamma+1-\epsilon; \ z \\ 2-\epsilon, \ \delta+1-\epsilon \end{bmatrix},$$

$$A' = {}_3F_2\begin{bmatrix} 1-\alpha, \ 1-\beta, \ 1-\gamma; \ z \\ 2-\delta, \ 2-\epsilon \end{bmatrix},$$

$$B' = z^{\delta-1} {}_3F_2\begin{bmatrix} \delta-\alpha, \ \delta-\beta, \ \delta-\gamma; \ z \\ \delta, \ \delta+1-\epsilon \end{bmatrix},$$

$$C' = z^{\epsilon-1} {}_3F_2\begin{bmatrix} \epsilon-\alpha, \ \epsilon-\beta, \ \epsilon-\gamma; \ z \\ \epsilon, \ \epsilon+1-\delta \end{bmatrix},$$

and

$$\Delta = \begin{vmatrix} \dfrac{dB}{dz}, & \dfrac{dC}{dz} \\ B, & C \end{vmatrix}.$$

Then (2) and (3) can be written as*

$$(4) \qquad AA' = \frac{\epsilon-1}{\epsilon-\delta} BB' + \frac{\delta-1}{\delta-\epsilon} CC',$$

$$(5) \qquad \Delta = (\epsilon-\delta) z^{1-\delta-\epsilon}(1-z)^{\delta+\epsilon-\alpha-\beta-\gamma-1} A'.$$

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* (5) expresses A' in terms of B, C and their differential coefficients, whereas (3) expresses A in terms of B', C'. The two formulae are, however, equivalent.

By comparing the coefficients of z^n in (2), we see that the formula to be proved is

$$\sum_{r=0}^{n} \frac{(-1)^r (\alpha - r)_n (\beta - r)_n (\gamma - r)_n}{r!(n-r)!(\delta - 1 - r)_{n+1}(\epsilon - 1 - r)_{n+1}}$$

$$+ \sum_{r=0}^{n} \frac{(-1)^r (\alpha + 1 - \delta - r)_n (\beta + 1 - \delta - r)_n (\gamma + 1 - \delta - r)_n}{r!(n-r)!(1 - \delta - r)_{n+1}(\epsilon - \delta - r)_{n+1}}$$

$$+ \sum_{r=0}^{n} \frac{(-1)^r (\alpha + 1 - \epsilon - r)_n (\beta + 1 - \epsilon - r)_n (\gamma + 1 - \epsilon - r)_n}{r!(n-r)!(1 - \epsilon - r)_{n+1}(\delta - \epsilon - r)_{n+1}} = 0.$$

Now consider the integral

$$\int \frac{(\alpha - s)_n (\beta - s)_n (\gamma - s)_n \, ds}{(-s)_{n+1}(\delta - 1 - s)_{n+1}(\epsilon - 1 - s)_{n+1}},$$

taken round a large circle $|s| = R$. This integral evidently tends to zero as $R \to \infty$, and, equating to zero the sum of the residues at the poles, we obtain the required identity.

To prove (5) we consider the integral

$$\int \frac{(\alpha - s)_n (\beta - s)_n (\gamma - s)_n \, ds}{(-s)_{n-m+1}(\delta - 1 - s)_{n+1}(\epsilon - 1 - s)_{n+1}},$$

where $m = 0$, 1, 2. The case when $m = 0$ has already been considered and leads to (4). When $m = 1$ and 2 we find similarly that

$$(6) \qquad A' \frac{dA}{dz} = \frac{\epsilon - 1}{\epsilon - \delta} B' \frac{dB}{dz} + \frac{\delta - 1}{\delta - \epsilon} C' \frac{dC}{dz},$$

$$(7) \qquad A' \frac{d^2A}{dz^2} = \frac{\epsilon - 1}{\epsilon - \delta} B' \frac{d^2B}{dz^2} + \frac{\delta - 1}{\delta - \epsilon} C' \frac{d^2C}{dz^2} + \frac{a}{z^2(1-z)},$$

where a is a constant. In the case when $m = 2$ the integrand is $-\frac{1}{s} + O\left(\frac{1}{s^2}\right)$ and so the integral $\to -2\pi i$ as $R \to \infty$. The term $a/z^2(1-z)$ arises from the series $\sum_{n=0}^{\infty} z^{n-2}$. When $n \leqslant m - 1$, $(-s)_{n-m+1}$ must be replaced by $(-1)^{m-n-1}/(1+s)_{m-n-1}$, and the identity to be proved connects two series instead of three. The corresponding powers of z in (6) and (7) are then negative.

From (4), (6) and (7) we find that

$$A' \Delta_1 = \frac{a\Delta}{z^2(1-z)},$$

where

$$\Delta_1 = \begin{vmatrix} \dfrac{d^2A}{dz^2}, & \dfrac{d^2B}{dz^2}, & \dfrac{d^2C}{dz^2} \\[2mm] \dfrac{dA}{dz}, & \dfrac{dB}{dz}, & \dfrac{dC}{dz} \\[2mm] A, & B, & C \end{vmatrix}.$$

Evidently

$$\frac{d\Delta_1}{dz} = \begin{vmatrix} \dfrac{d^3A}{dz^3}, & \dfrac{d^3B}{dz^3}, & \dfrac{d^3C}{dz^3} \\[2mm] \dfrac{dA}{dz}, & \dfrac{dB}{dz}, & \dfrac{dC}{dz} \\[2mm] A, & B, & C \end{vmatrix}.$$

But, by § 2.1, A, B, C satisfy a differential equation of the form

$$\frac{d^3y}{dz^3} + P \frac{d^2y}{dz^2} + Q \frac{dy}{dz} + Ry = 0,$$

where

$$P = \frac{\delta + \epsilon + 1 - (\alpha + \beta + \gamma + 3)z}{z(1-z)}.$$

It follows that

$$\frac{d\Delta_1}{dz} = -P\Delta_1,$$

so that

$$\Delta_1 = be^{-\int P \, dz} = bz^{-(\delta + \epsilon + 1)}(1-z)^{\delta + \epsilon - \alpha - \beta - \gamma - 2},$$

where b is a constant. Thus

$$\Delta = M A' z^{1 - \delta - \epsilon}(1-z)^{\delta + \epsilon - \alpha - \beta - \gamma - 1},$$

where M is a constant. Equating coefficients of $z^{1-\delta-\epsilon}$ we find that $M = \epsilon - \delta$, and so (5) is proved.

Exactly similar results are true for series of any order, and they can be proved in the same way.

Similar results are also true for basic series. Thus, by considering the integral

$$\int \frac{(\alpha s)_{q,n} (\beta s)_{q,n} (\gamma s)_{q,n} \, ds}{(s)_{q,n+1}(\delta s/q)_{q,n+1}(\epsilon s/q)_{q,n+1}}$$

round a large circle, we can prove that

$$(8) \quad {}_3\Phi_2\begin{bmatrix} \alpha, \ \beta, \ \gamma; \ z \\ \delta, \ \epsilon \end{bmatrix} {}_3\Phi_2\begin{bmatrix} q/\alpha, \ q/\beta, \ q/\gamma; \ \alpha\beta\gamma qz/\delta\epsilon \\ q^2/\delta, \ q^2/\epsilon \end{bmatrix}$$

$$= \frac{\delta\,(q-\epsilon)}{q\,(\delta-\epsilon)}{}_3\Phi_2\begin{bmatrix} q\alpha/\delta, \ q\beta/\delta, \ q\gamma/\delta; \ z \\ q^2/\delta, \ q\epsilon/\delta \end{bmatrix} {}_3\Phi_2\begin{bmatrix} \delta/\alpha, \ \delta/\beta, \ \delta/\gamma; \ \alpha\beta\gamma qz/\delta\epsilon \\ \delta, \ q\delta/\epsilon \end{bmatrix}$$

$$+ \frac{\epsilon\,(q-\delta)}{q\,(\epsilon-\delta)}{}_3\Phi_2\begin{bmatrix} q\alpha/\epsilon, \ q\beta/\epsilon, \ q\gamma/\epsilon; \ z \\ q^2/\epsilon, \ q\delta/\epsilon \end{bmatrix} {}_3\Phi_2\begin{bmatrix} \epsilon/\alpha, \ \epsilon/\beta, \ \epsilon/\gamma; \ \alpha\beta\gamma qz/\delta\epsilon \\ \epsilon, \ q\epsilon/\delta \end{bmatrix}.$$

This is the analogue of (2). The analogue of (3) is

$$(9) \quad {}_3\Phi_2\begin{bmatrix} \alpha, \ \beta, \ \gamma; \ \delta\epsilon z/\alpha\beta\gamma \\ \delta, \ \epsilon \end{bmatrix} {}_1\Phi_0\begin{bmatrix} \delta\epsilon \\ \overline{\alpha\beta\gamma}; \ z \end{bmatrix}$$

$$= \frac{\delta\,(q-\epsilon)}{q\,(\delta-\epsilon)}{}_3\Phi_2\begin{bmatrix} \delta/\alpha, \ \delta/\beta, \ \delta/\gamma; \ qz \\ \delta, \ q\delta/\epsilon \end{bmatrix} {}_3\Phi_2\begin{bmatrix} \epsilon/\alpha, \ \epsilon/\beta, \ \epsilon/\gamma; \ z \\ \epsilon/q, \ q\epsilon/\delta \end{bmatrix}$$

$$+ \frac{\epsilon\,(q-\delta)}{q\,(\epsilon-\delta)}{}_3\Phi_2\begin{bmatrix} \epsilon/\alpha, \ \epsilon/\beta, \ \epsilon/\gamma; \ qz \\ \epsilon, \ q\epsilon/\delta \end{bmatrix} {}_3\Phi_2\begin{bmatrix} \delta/\alpha, \ \delta/\beta, \ \delta/\gamma; \ z \\ \delta/q, \ q\delta/\epsilon \end{bmatrix}.$$

If we put $\gamma = \epsilon$ and then let $\epsilon \to 0$, this reduces to Heine's formula § 8.4 (2).

10.4. Partial sums of hypergeometric series.

A number of theorems have recently been given which express the sum of n terms of an ordinary hypergeometric series with unit argument in terms of an infinite series of the type ${}_3F_2$. The subject had previously received some attention from Hill and Whipple,* but new interest was aroused by a theorem due to Ramanujan, who stated that

$$(1) \quad \frac{1}{n}+\left(\frac{1}{2}\right)^2\frac{1}{n+1}+\left(\frac{1.3}{2.4}\right)^2\frac{1}{n+2}+\ldots$$

$$= \left\{\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})}\right\}^2\left\{1+\left(\frac{1}{2}\right)^2+\left(\frac{1.3}{2.4}\right)^2+\ldots \text{ to } n \text{ terms}\right\}.$$

This was proved by Watson and Darling, and generalized by Whipple, Hodgkinson and Bailey.† The method used by Watson

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* Hill and Whipple 1, and Hill 1 and 2.

† Watson 7, Darling 1, Whipple 9, Hodgkinson 1, Bailey 6 and 7.

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is particularly simple. In § 3.8 (1) put $c = f + n - 1$, where n is a positive integer. Then

$${}_3F_2\begin{bmatrix} a, \ b, \ f+n-1; \\ e, \ f \end{bmatrix} = \frac{\Gamma(e)\,\Gamma(e-a-b)}{\Gamma(e-a)\,\Gamma(e-b)}{}_3F_2\begin{bmatrix} a, \ b, \ 1-n; \\ a+b-e+1, \ f \end{bmatrix}.$$

Now let $e \to a + b + n$, and we get*

$$(2) \quad {}_2F_1\begin{bmatrix} a, \ b; \\ f \end{bmatrix} \text{ to } n \text{ terms}$$

$$= \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(n)\,\Gamma(a+b+n)}{}_3F_2\begin{bmatrix} a, \ b, \ f+n-1; \\ f, \ a+b+n \end{bmatrix}.$$

Ramanujan's result is the particular case of (2) when $a = b = \frac{1}{2}$, $f = 1$. The method of proof applies when $f \geqslant a + b$.

A more general result is†

$$(3) \quad \frac{\Gamma(x+m)\,\Gamma(y+m)}{\Gamma(m)\,\Gamma(x+y+m)}{}_3F_2\begin{bmatrix} x, \ y, \ v+m-1; \\ v, \ x+y+m \end{bmatrix} \text{ to } n \text{ terms}$$

$$= \frac{\Gamma(x+n)\,\Gamma(y+n)}{\Gamma(n)\,\Gamma(x+y+n)}{}_3F_2\begin{bmatrix} x, \ y, \ v+n-1; \\ v, \ x+y+n \end{bmatrix} \text{ to } m \text{ terms}.$$

We can evidently suppose that $n > m$. Then, in the terms of the series on the left, the factors $v + r$ in the denominator cancel with factors in the numerator when $r \geqslant m - 1$. Thus if we multiply by $(v)_{m-1}$ we obtain two polynomials in v of degree $m - 1$. If therefore we can prove that these polynomials are equal for m values of v, we have established the result.

Now for each of the m values $v = -n + 1, \ -n, \ \ldots, \ -n - m + 2$, the partial series become complete hypergeometric series which can be summed by Saalschütz's theorem and the verification is immediate. When $m \to \infty$ the theorem gives (2), and the proof is valid for all values of the parameters.

The series ${}_3F_2$ in (2) can be transformed in many ways by the transformations of Chapter III. In particular, by using the

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* Bailey 6.

† Bailey 7. Three proofs are given there.

relation between $Fp\,(0;4,5)$, $Fp\,(4;0,1)$ and $Fp\,(1;0,4)$, we find that

(4) $_2F_1[\alpha,\beta;\gamma]$ to n terms

$$=\frac{\Gamma(1+\alpha-\gamma)\,\Gamma(1+\beta-\gamma)}{\Gamma(1-\gamma)\,\Gamma(1-\gamma+\alpha+\beta)}\left\{1-\frac{(\alpha)_n\,(\beta)_n}{n!\,(\gamma-1)_n}\,_3F_2\left[\begin{array}{c}1-\alpha,\,1-\beta,\,n;\\2-\gamma,\,n+1\end{array}\right]\right\},$$

a result which is due to Whipple.*

Again from (2), using the relation $Fp\,(0;4,5)=Fp\,(0;1,4)$, we find that

(5) $_2F_1[\alpha,\beta;\gamma]$ to n terms

$$=\frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)}{\Gamma(n)\,\Gamma(\alpha+\beta-\gamma+1)\,\Gamma(\gamma+n)}\,_3F_2\left[\begin{array}{c}\gamma-\alpha,\,\gamma-\beta,\,\gamma-1+n;\\\gamma,\,\gamma+n\end{array}\right],$$

which was given by Hodgkinson.† In both (4) and (5) it is assumed that $R\,(\alpha+\beta+1-\gamma)>0$.

Now Whipple's transformation § 4.4 (5) can be written

$$_4F_3\left[\begin{array}{c}t,\,x,\,y,\,z;\\u,\,v,\,w\end{array}\right]$$

$$=\frac{\Gamma(v+w-t)\,\Gamma(1+x-u)\,\Gamma(1+y-u)\,\Gamma(1+z-u)}{\Gamma(1+y+z-u)\,\Gamma(1+z+x-u)\,\Gamma(1+x+y-u)\,\Gamma(1-u)}$$

$$\times\,_7F_6\left[\begin{array}{c}a,\,1+\tfrac12 a,\,w-t,\,v-t,\,x,\,y,\,z;\\\tfrac12 a,\,v,\,w,\,1+y+z-u,\,1+z+x-u,\,1+x+y-u\end{array}\right],$$

where $a=x+y+z-u,\quad u+v+w-t-x-y-z=1,$

and one of t, x, y, z is a negative integer.

Put $t=1-n$ and let $u\to1-n$. Then we find that‡

(6) $_3F_2\left[\begin{array}{c}x,\,y,\,z;\\v,\,w\end{array}\right]$ to n terms

$$=\frac{\Gamma(v+w+n-1)\,\Gamma(x+n)\,\Gamma(y+n)\,\Gamma(z+n)}{\Gamma(n)\,\Gamma(y+z+n)\,\Gamma(z+x+n)\,\Gamma(x+y+n)}$$

$$\times\,_7F_6\left[\begin{array}{c}a,\,1+\tfrac12 a,\,w+n-1,\,v+n-1,\,x,\,y,\,z;\\\tfrac12 a,\,v,\,w,\,y+z+n,\,z+x+n,\,x+y+n\end{array}\right],$$

° Whipple 9. See also Hodgkinson 1; his first formula is equivalent to (4).
† Hodgkinson 1.   ‡ Bailey 6.

where $a=x+y+z+n-1$ and the series on the left is restricted to be Saalschützian. This reduces to (2) when we substitute for w and let $z\to\infty$.

The well-poised $_7F_6$ in (6) can be transformed in various ways into two Saalschützian $_4F_3$ by § 4.4 (4) and § 7.5 (3). In particular, using the latter formula, we find that

(7) $_3F_2\left[\begin{array}{c}x,\,y,\,z;\\v,\,w\end{array}\right]$ to n terms

$$=\frac{\Gamma(x+n)\,\Gamma(y+n)\,\Gamma(z+n)}{\Gamma(n)\,(v+n-1)\,(w+n-1)}$$

$$\times\left\{\frac{\Gamma(w)\,\Gamma(w-v)}{\Gamma(v+n-1)\,\Gamma(w-x)\,\Gamma(w-y)\,\Gamma(w-z)}\right.$$

$$\times\,_4F_3\left[\begin{array}{c}v-x,\,v-y,\,v-z,\,v+n-1;\\v,\,v+1-w,\,v+n\end{array}\right]$$

$$+\frac{\Gamma(v)\,\Gamma(v-w)}{\Gamma(w+n-1)\,\Gamma(v-x)\,\Gamma(v-y)\,\Gamma(v-z)}$$

$$\times\,_4F_3\left[\begin{array}{c}w-x,\,w-y,\,w-z,\,w+n-1;\\w,\,w+1-v,\,w+n\end{array}\right]\right\},$$

where $v+w=x+y+z+1$. This result was given by Darling.*

Similarly from § 6.8 (3) and § 7.6 (2), the relations connecting four well-poised $_9F_8$, by putting $c=1-n$ where n is a positive integer, and then letting $b\to a+n$, we obtain two formulae each of which gives the sum of n terms of the series

$$_7F_6\left[\begin{array}{cccccc}a,\,1+\tfrac12 a,&d,&e,&f,&g,&h;\\\tfrac12 a,&1+a-d,&1+a-e,&1+a-f,&1+a-g,&1+a-h\end{array}\right],$$

where $1+2a=d+e+f+g+h$, in terms of two infinite well-poised series. The formulae are very complicated and their only interest lies in the fact of their existence.

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* Darling 2, p. 335.

EXAMPLES

1. Prove the following results, subject to convergence conditions:

(i) $s+(s+2)\left(\dfrac{s}{1}\right)^3+(s+4)\left\{\dfrac{s(s+1)}{1.2}\right\}^3+\ldots$

$$=\frac{\sin s\pi}{\pi}\frac{\Gamma\{\tfrac{1}{2}(s+1)\}\Gamma\{\tfrac{1}{2}(1-3s)\}}{[\Gamma\{\tfrac{1}{2}(1-s)\}]^2},$$

(ii) $s-(s+2)\left(\dfrac{s}{1}\right)^3+(s+4)\left\{\dfrac{s(s+1)}{1.2}\right\}^3-\ldots=\dfrac{\sin s\pi}{\pi}$,

(iii) $1+3\dfrac{x-1}{x+1}+5\dfrac{(x-1)(x-2)}{(x+1)(x+2)}+\ldots=x$,

(iv) $1-3\dfrac{x-1}{x+1}+5\dfrac{(x-1)(x-2)}{(x+1)(x+2)}-\ldots=0$,

(v) $1-3\left(\dfrac{x-1}{x+1}\right)^3+5\left\{\dfrac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^3-\ldots=\dfrac{\{\Gamma(x+1)\}^3\Gamma(3x-1)}{\{\Gamma(2x)\}^3}$,

(vi) $1-\dfrac{1}{3}\dfrac{x-1}{x+1}+\dfrac{1}{5}\dfrac{(x-1)(x-2)}{(x+1)(x+2)}-\ldots=\dfrac{2^{4x}\{\Gamma(x+1)\}^4}{4x\{\Gamma(2x+1)\}^2}$,

(vii) $1+\left(\dfrac{x-1}{x+1}\right)^2+\left\{\dfrac{(x-1)(x-2)}{(x+1)(x+2)}\right\}^2+\ldots=\dfrac{2x}{4x-1}\dfrac{\{\Gamma(x+1)\}^4\Gamma(4x+1)}{\{\Gamma(2x+1)\}^4}$,

(viii) $1-\dfrac{2xyz}{(x+1)(y+1)(z+1)}$

$+\dfrac{2x(x-1)y(y-1)z(z-1)}{(x+1)(x+2)(y+1)(y+2)(z+1)(z+2)}-\ldots$

$$=\frac{\Gamma(x+1)\Gamma(y+1)\Gamma(z+1)\Gamma(x+y+z+1)}{\Gamma(y+z+1)\Gamma(z+x+1)\Gamma(x+y+1)}.$$

[Dougall and Ramanujan. See Hardy 2.]

2. Prove the results:

(i) $1-5\left(\dfrac{1}{2}\right)^3+9\left(\dfrac{1.3}{2.4}\right)^3-13\left(\dfrac{1.3.5}{2.4.6}\right)^3+\ldots=\dfrac{2}{\pi}$,

(ii) $1+9\left(\dfrac{1}{4}\right)^4+17\left(\dfrac{1.5}{4.8}\right)^4+25\left(\dfrac{1.5.9}{4.8.12}\right)^4+\ldots=\dfrac{2\sqrt{2}}{\sqrt{\pi}\{\Gamma(\tfrac{3}{4})\}^2}$,

(iii) $1-5\left(\dfrac{1}{2}\right)^5+9\left(\dfrac{1.3}{2.4}\right)^5-13\left(\dfrac{1.3.5}{2.4.6}\right)^5+\ldots=\dfrac{2}{\{\Gamma(\tfrac{3}{4})\}^4}$,

(iv) $1-\left(\dfrac{1}{2}\right)^3+\left(\dfrac{1.3}{2.4}\right)^3-\left(\dfrac{1.3.5}{2.4.6}\right)^3+\ldots=\left\{\dfrac{\Gamma(\tfrac{9}{8})}{\Gamma(\tfrac{5}{4})\Gamma(\tfrac{7}{8})}\right\}^2$.

[Ramanujan. See Hardy 3 and Whipple 4.]

3. Prove that

(i) $\ _3F_2\left[\begin{array}{c}\tfrac{1}{2},\ \tfrac{1}{2}+x,\ \tfrac{1}{2}-x;\ -1\\ 1-x,\ 1+x\end{array}\right]$

$$=\frac{\pi^2 x}{\sqrt{2}.\sin\pi x\,\Gamma(\tfrac{1}{2}x+\tfrac{5}{8})\Gamma(\tfrac{1}{2}x+\tfrac{7}{8})\Gamma(\tfrac{5}{8}-\tfrac{1}{2}x)\Gamma(\tfrac{7}{8}-\tfrac{1}{2}x)},$$

(ii) $\ _4F_3\left[\begin{array}{c}4x+4y,\quad 6x,\quad 6y,\quad \tfrac{1}{2}+x+y;\ -1\\ 1-2x+4y,\ 1+4x-2y,\ \tfrac{1}{2}+3x+3y\end{array}\right]$

$$=\frac{\Gamma(\tfrac{1}{2})\,\Gamma(1+4x-2y)\,\Gamma(1-2x+4y)\,\Gamma(\tfrac{1}{2}+3x+3y)\,\Gamma(1+x+y)}{\Gamma(\tfrac{1}{2}+3x)\,\Gamma(\tfrac{1}{2}+3y)\,\Gamma(1+x-2y)\,\Gamma(1-2x+y)\,\Gamma(1+4x+4y)},$$

(iii) $\ _6F_5\left[\begin{array}{c}a,\ 1+\tfrac{1}{2}a,\quad \tfrac{1}{2}+x,\quad \tfrac{1}{2}-x,\quad \tfrac{1}{2}+y,\quad \tfrac{1}{2}-y;\ -1\\ \tfrac{1}{2}a,\ \tfrac{1}{2}+a-x,\ \tfrac{1}{2}+a+x,\ \tfrac{1}{2}+a-y,\ \tfrac{1}{2}+a+y\end{array}\right]$

$$=\frac{\pi\,\Gamma(\tfrac{1}{2}+a+x)\,\Gamma(\tfrac{1}{2}+a-x)\,\Gamma(\tfrac{1}{2}+a+y)\,\Gamma(\tfrac{1}{2}+a-y)}{2^{2a-1}\Gamma(a)\,\Gamma(1+a)\,\Gamma\{\tfrac{1}{2}(1+a+x+y)\}\,\Gamma\{\tfrac{1}{2}(1+a+x-y)\}}$$
$$\times\frac{1}{\Gamma\{\tfrac{1}{2}(1+a-x+y)\}\,\Gamma\{\tfrac{1}{2}(1+a-x-y)\}}.$$

[See Whipple 2, where further results of this kind are given.]

4. By comparing coefficients of powers of x, prove the formulae:

(i) $\ _1F_1(\alpha;\rho;x)\,_1F_1(\alpha;\rho;-x)=\,_2F_3(\alpha,\rho-\alpha;\rho,\tfrac{1}{2}\rho,\tfrac{1}{2}\rho+\tfrac{1}{2};\tfrac{1}{4}x^2)$,

(ii) $\left\{1+\dfrac{x}{(1!)^3}+\dfrac{x^2}{(2!)^3}+\ldots\right\}\left\{1-\dfrac{x}{(1!)^3}+\dfrac{x^2}{(2!)^3}-\ldots\right\}$

$$=1-\frac{3!x^2}{(1!2!)^3}+\frac{6!x^4}{(2!4!)^3}-\frac{9!x^6}{(3!6!)^3}+\ldots,$$

(iii) $\ _2F_1\{\alpha,\beta;\alpha+\beta+\tfrac{1}{2};4x(1-x)\}=\,_2F_1(2\alpha,2\beta;\alpha+\beta+\tfrac{1}{2};x)$,

(iv) $(1-x)^{-a}\,_3F_2\left[\begin{array}{c}\tfrac{1}{2}a,\ \tfrac{1}{2}+\tfrac{1}{2}a,\ 1+a-b-c;\ \dfrac{-4x}{(1-x)^2}\\ 1+a-b,\quad 1+a-c\end{array}\right]$

$$=\,_3F_2\left[\begin{array}{c}a,\quad b,\quad c;\ x\\ 1+a-b,\ 1+a-c\end{array}\right],$$

each of the formulae (iii) and (iv) being valid inside a certain region surrounding the origin. For example, (iii) is valid inside the loop of the lemniscate $|4x(1-x)|=1$ which surrounds the origin. Show that inside the other loop of this lemniscate

(v) $\ _2F_1\{\alpha,\beta;\alpha+\beta+\tfrac{1}{2};4x(1-x)\}=\,_2F_1(2\alpha,2\beta;\alpha+\beta+\tfrac{1}{2};1-x)$.

[See Bailey 1, where further formulae of this type are given. (i) and (ii) are due to Ramanujan, (iii) is Gauss's quadratic transformation, and (iv) is due to Whipple.]

5. From Ex. 4 (iv) deduce §4.5(1) by multiplying by $(1-x)^{w+m-1}$ and equating coefficients of x^m.

[Bailey 2.]

6. By comparing coefficients of powers of x, prove that

$(1+x)(1-x)^{-a-1}\,_3F_2\left[\begin{array}{c}\tfrac{1}{2}+\tfrac{1}{2}a,\ 1+\tfrac{1}{2}a,\ 1+a-b-c;\ \dfrac{-4x}{(1-x)^2}\\ 1+a-b,\quad 1+a-c\end{array}\right]$

$$=\,_4F_3\left[\begin{array}{c}a,\ 1+\tfrac{1}{2}a,\quad b,\quad c;\ x\\ \tfrac{1}{2}a,\ 1+a-b,\ 1+a-c\end{array}\right].$$

By multiplying by $(1-x)^{w+m-1}$ and equating coefficients of x^m, deduce § 4.5 (2).

[Bailey 3.]

7. Obtain the formula § 3.8 (1) by multiplying § 1.4 (1) by $z^{d-1}(1-z)^{f-d-1}$ and integrating from 0 to 1.

Similarly from § 1.2 (2) deduce the formula

$$_3F_2\left[\begin{array}{c}\alpha_1,\ \alpha_2,\ \alpha_3;\\ \beta_1,\ \beta_2\end{array}\right]=\frac{\Gamma(\beta_2)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}$$

$$\times\,_3F_2\left[\begin{array}{c}\beta_1-\alpha_1,\ \beta_1-\alpha_2,\ \alpha_3;\\ \beta_1,\ \beta_1+\beta_2-\alpha_1-\alpha_2\end{array}\right].$$

[Hardy 2.]

8. Prove the identity

$$_7F_6\left[\begin{array}{c}a,\ 1+\tfrac12 a,\ \ \tfrac12 d,\ \ \ \tfrac12+\tfrac12 d,\ \ a-d,\ 1+2a-d+m,\ \ -m;\\ \tfrac12 a,\ \ 1+a-\tfrac12 d,\ \tfrac12+a-\tfrac12 d,\ 1+d,\ \ \ d-a-m,\ \ \ \ 1+a+m\end{array}\right]$$

$$=\frac{(1+a)_m(1+2a-2d)_m}{(1+a-d)_m(1+2a-d)_m},$$

where m is a positive integer.

[Bailey 3.]

9. Prove that

$$_3F_2\left[\begin{array}{c}\nu+\mu+1,\ \nu+\tfrac12,\ \nu-\mu+1;\\ 2\nu+2,\ \nu+\tfrac32\end{array}\right]=\frac{\Gamma(2\nu+2)}{2\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)}$$

$$\times\left\{\psi\left(\frac{\nu+\mu+2}{2}\right)+\psi\left(\frac{\nu-\mu+2}{2}\right)-\psi\left(\frac{\nu+\mu+1}{2}\right)-\psi\left(\frac{\nu-\mu+1}{2}\right)\right\},$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$.

[Watson 1. See also Hardy 2.]

10. Evaluate the integral

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(-s)\,\Gamma(1-\delta-s)\,\Gamma(1-\epsilon-s)\,ds}{\Gamma(1-\alpha-s)\,\Gamma(1-\beta-s)\,\Gamma(1-\gamma-s)}$$

by considering the poles on the two sides of the contour, and hence show that

$$_3F_2\left[\begin{array}{c}\alpha,\ \beta,\ \gamma;\\ \delta,\ \epsilon\end{array}\right]=\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(1-\gamma)\,\Gamma(\delta)\,\Gamma(\delta-\epsilon)}{\Gamma(2-\delta)\,\Gamma(1-\epsilon)\,\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta)\,\Gamma(\delta-\gamma)}$$

$$\times\,_3F_2\left[\begin{array}{c}\alpha+1-\delta,\ \beta+1-\delta,\ \gamma+1-\delta;\\ 2-\delta,\ \epsilon+1-\delta\end{array}\right]$$

$$+\frac{\Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(1-\gamma)\,\Gamma(\epsilon)\,\Gamma(\epsilon-\delta)}{\Gamma(2-\epsilon)\,\Gamma(1-\delta)\,\Gamma(\epsilon-\alpha)\,\Gamma(\epsilon-\beta)\,\Gamma(\epsilon-\gamma)}$$

$$\times\,_3F_2\left[\begin{array}{c}\alpha+1-\epsilon,\ \beta+1-\epsilon,\ \gamma+1-\epsilon;\\ 2-\epsilon,\ \delta+1-\epsilon\end{array}\right],$$

provided that $R(\delta+\epsilon-\alpha-\beta-\gamma)>0.$

Also deduce this result from § 10.3 (2). Obtain similar results for series of any order.

[Darling 2. The second method is used there.]

11. If

$$I(x,y,z,t;v,w)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(x+s)\,\Gamma(y+s)\,\Gamma(z+s)\,\Gamma(t+s)}{\Gamma(v+s)\,\Gamma(w+s)}\Gamma(v+w-x-y-z-t-s)\,\Gamma(-s)\,ds,$$

prove that

$$I(x,y,z,t;v,w)$$

$$=\frac{\Gamma(z)\,\Gamma(t)\,\Gamma(v+w-x-z-t)\,\Gamma(v+w-y-z-t)}{\Gamma(v-x)\,\Gamma(v-y)\,\Gamma(w-z)\,\Gamma(w-t)}$$

$$\times\,I(x,\ y,\ w-z,\ w-t;\ v+w-z-t,\ w)$$

$$=\frac{\Gamma(x)\,\Gamma(y)\,\Gamma(v+w-x-z-t)\,\Gamma(v+w-y-z-t)}{\Gamma(v-z)\,\Gamma(v-t)\,\Gamma(w-z)\,\Gamma(w-t)}$$

$$\times\,I(v-t,\ w-t,\ z,\ v+w-x-y-t;\ v+w-x-t,\ v+w-y-t)$$

$$=\frac{\Gamma(x)\,\Gamma(y)\,\Gamma(z)\,\Gamma(t)\,\Gamma(v+w-x-z-t)\,\Gamma(v+w-y-z-t)}{\Gamma(v-x)\,\Gamma(v-y)\,\Gamma(v-z)\,\Gamma(v-t)}$$

$$\times\frac{\Gamma(v+w-x-y-z)\,\Gamma(v+w-x-y-t)}{\Gamma(w-x)\,\Gamma(w-y)\,\Gamma(w-z)\,\Gamma(w-t)}$$

$$\times\,I(w-x,\ w-y,\ w-z,\ w-t;\ w,\ v+2w-x-y-z-t).$$

Hence obtain three formulae each connecting four Saalschützian series of the type $_4F_3$. Deduce expressions for the sum of n terms of a Saal-schützian $_3F_2$ in terms of two non-terminating Saalschützian $_4F_3$.

[Use § 6.3 (2).]

12. If

$$a_{rs}=\frac{(f)_{r+s}}{(f-\tfrac12 a)_r\,(f-\tfrac12 a)_s},$$

prove that the equations

$$\sum_{r=0}^{\nu}a_{rs}x_{r,n}=1,\quad(s=0,\ 1,\ 2,\ ...,\ n)$$

are satisfied by

$$x_{r,n}=\frac{a}{a+2r}\frac{(1+\tfrac12 a)_n}{n!}\frac{(-n)_r}{r!}\frac{(f-\tfrac12 a)_r}{(f)_r}.$$

Prove also that

$$\sum_{r=0}^{n}x_{r,n}=\,_2F_1\left[\begin{array}{c}\tfrac12 a,\ \tfrac12 a;\\ f\end{array}\right]\text{ to }n+1\text{ terms,}$$

and hence that

$$\lim_{n\to\infty}\sum_{r=0}^{n}x_{r,n}=\frac{\Gamma(f)\,\Gamma(f-a)}{\{\Gamma(f-\tfrac12 a)\}^2}.$$

[The particular case when $f=\tfrac12(a+5)$ was given by Chapman 1.]

13. Prove that

$$F_4(\alpha,\ \alpha+\tfrac12-\beta;\ \gamma,\ \beta+\tfrac12;\ x,\ y^2)$$

$$=(1-y)^{-2\alpha}F_2\left[\alpha;\ \alpha+\tfrac12-\beta,\ \beta;\ \gamma,\ 2\beta;\ \frac{x}{(1-y)^2},\ \frac{-4y}{(1-y)^2}\right].$$

[Appell and Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques*, p. 27.]

14. Prove that

$$F(\alpha, \beta; \gamma; z) F(\alpha, \beta; \gamma; Z)$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F_4[\alpha, \beta; \gamma, \alpha+\beta-\gamma+1; zZ, (1-z)(1-Z)]$$

$$+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\{(1-z)(1-Z)\}^{\gamma-\alpha-\beta}$$

$$\times F_4[\gamma-\beta, \gamma-\alpha; \gamma, \gamma-\alpha-\beta+1; zZ, (1-z)(1-Z)].$$

Prove also that

$$F(\alpha, \beta; \gamma; z) F(\alpha, \beta; \gamma; Z)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\gamma-\alpha)_m(\gamma-\beta)_m}{m!\, n!\, (\gamma)_m (\gamma)_{2m+n}} (zZ)^m (z+Z-zZ)^n.$$

[Watson 2. The first formula can be derived from § 9.6 (1) and § 1.4 (1).]

15. If

$$H\begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix} = \frac{1}{(\gamma-1)(\delta-1)} + \frac{\alpha\beta}{(\gamma-1)\gamma(\delta-1)\delta}$$

$$+ \frac{\alpha(\alpha+1)\beta(\beta+1)}{(\gamma-1)\gamma(\gamma+1)(\delta-1)\delta(\delta+1)} + \dots,$$

prove that
$$H\begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix} = H\begin{pmatrix} \gamma-\alpha, & \gamma-\beta \\ \gamma, & \gamma+\delta-\alpha-\beta \end{pmatrix}.$$

Prove also that, if $H_r\begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix}$ denotes the sum of the first r terms of $H\begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix}$, then

$$H_r\begin{pmatrix} \alpha, \beta \\ \gamma, \delta \end{pmatrix} = H\begin{pmatrix} \gamma-\alpha, & \gamma-\beta \\ \gamma, & \gamma+\delta-\alpha-\beta \end{pmatrix}$$

$$- \frac{(\alpha)_r(\beta)_r}{(\gamma-1)_r(\delta-1)_r} H\begin{pmatrix} \gamma-\alpha, & \gamma-\beta \\ \gamma+r, & \gamma+\delta-\alpha-\beta \end{pmatrix}.$$

Deduce from the first result that

$$\frac{1}{100^2} + \frac{1}{101^2} + \frac{1}{102^2} + \frac{1}{103^2} + \dots = 0.0100502\dots,$$

and show that a million terms of the given series give only the same order of accuracy as three terms of the transformed series.

[Hill and Whipple 1.]

16. Show that, if

$$(1-z)^{\alpha+\beta-\gamma} {}_3F_2\begin{bmatrix} 2\alpha, 2\beta, \gamma+\frac{1}{2}; z \\ 2\gamma, \alpha+\beta+\frac{1}{2} \end{bmatrix} = \Sigma a_n z^n,$$

then $\quad F\begin{bmatrix} \alpha, \beta; z \\ \alpha+\beta+\frac{1}{2} \end{bmatrix} F\begin{bmatrix} \gamma-\alpha, \gamma-\beta; z \\ 2\gamma-\alpha-\beta+\frac{1}{2} \end{bmatrix} = \Sigma \frac{(\gamma)_n}{(2\gamma-\alpha-\beta+\frac{1}{2})_n} a_n z^n.$

Deduce that

$$F\begin{bmatrix} \alpha, \beta; z \\ \alpha+\beta+\frac{1}{2} \end{bmatrix} F\begin{bmatrix} \frac{1}{2}-\alpha, \frac{1}{2}-\beta; z \\ \frac{3}{2}-\alpha-\beta \end{bmatrix} = {}_3F_2\begin{bmatrix} \alpha-\beta+\frac{1}{2}, \beta-\alpha+\frac{1}{2}, \frac{1}{2}; z \\ \alpha+\beta+\frac{1}{2}, \frac{3}{2}-\alpha-\beta \end{bmatrix}.$$

[For the first result see Bailey 14. The second result was given by Orr 1.]

17. If

$$C = z^{1-\zeta} {}_4F_3\begin{bmatrix} \alpha+1-\zeta, \beta+1-\zeta, \gamma+1-\zeta, \delta+1-\zeta; z \\ 2-\zeta, \epsilon+1-\zeta, \theta+1-\zeta \end{bmatrix},$$

$$D = z^{1-\theta} {}_4F_3\begin{bmatrix} \alpha+1-\theta, \beta+1-\theta, \gamma+1-\theta, \delta+1-\theta; z \\ 2-\theta, \epsilon+1-\theta, \zeta+1-\theta \end{bmatrix},$$

$$A' = {}_4F_3\begin{bmatrix} 1-\alpha, 1-\beta, 1-\gamma, 1-\delta; z \\ 2-\epsilon, 2-\zeta, 2-\theta \end{bmatrix},$$

$$B' = z^{\epsilon-1} {}_4F_3\begin{bmatrix} \epsilon-\alpha, \epsilon-\beta, \epsilon-\gamma, \epsilon-\delta; z \\ \epsilon, \epsilon+1-\zeta, \epsilon+1-\theta \end{bmatrix},$$

prove that

$$D\frac{dC}{dz} - C\frac{dD}{dz} = \frac{\theta-\zeta}{\epsilon-1} z^{3-\epsilon-\zeta-\theta}(1-z)^{\epsilon+\zeta+\theta-1-\alpha-\beta-\gamma-\delta}\left(A'\frac{dB'}{dz} - B'\frac{dA'}{dz}\right).$$

Show that, by taking

$$\epsilon+\zeta+\theta-1-\alpha-\beta-\gamma-\delta = 0$$

and equating coefficients of z^n, the formula § 7.6 (1) is obtained.
[For the first result see Darling 2.]

18. Assuming Murphy's formula

$$P_n(\mu) = F(n+1, -n; 1; \tfrac{1}{2}-\tfrac{1}{2}\mu)$$

for the Legendre polynomial of order n, prove that

$$P_n(\mu) = \frac{(2n)!}{2^{2n}(n!)^2} z^n F(\tfrac{1}{2}, -n; \tfrac{1}{2}-n; z^{-2}),$$

where $z = \mu + \sqrt{(\mu^2-1)}$.

Hence prove the formula of Neumann and Adams

$$P_p(\mu) P_q(\mu) = \sum_{r=0}^{q} \frac{A_{p-r}A_r A_{q-r}}{A_{p+q-r}}\left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) P_{p+q-2r}(\mu)$$

where
$$A_r = \frac{1.3.5\dots(2r-1)}{r!} \text{ and } p > q.$$

More generally, if

$$P_n^m(\mu) = \frac{(n+m)!}{2^m m!(n-m)!}(\mu^2-1)^{\frac{1}{2}m} F(m-n, m+n+1; m+1; \tfrac{1}{2}-\tfrac{1}{2}\mu),$$

where m and n are positive integers and $m \leqslant n$, prove that

$$P_n^m(\mu) = \frac{2^{m-2n}(2n)!}{n!(n-m)!} z^{n-m}(\mu^2-1)^{\frac{1}{2}m} F(\tfrac{1}{2}+m, m-n; \tfrac{1}{2}-n; z^{-2})$$

$$= \frac{2^{-m-2n}(2n)!}{n!(n-m)!} z^{n+m}(\mu^2-1)^{-\frac{1}{2}m} F(\tfrac{1}{2}-m, -m-n; \tfrac{1}{2}-n; z^{-2}),$$

and

$$(\mu^2-1)^{\frac{1}{2}m} P_p^m(\mu) P_q^m(\mu) = \frac{(p+m)!(q+m)!}{2^m(p-m)!(q-m)!}$$

$$\times \sum_{r=0}^{q-m} \frac{A_{r,m}A_{p-r}^m A_{q-r}^m}{A_{p+q+m-r}^{-m}}\left(\frac{2p+2q+2m-4r+1}{2p+2q+2m-2r+1}\right) P_{p+q+m-2r}^m(\mu),$$

where
$$A_s^m = \frac{(\tfrac{1}{2})_s}{(s+m)!}, \quad A_{r,m} = \frac{(\tfrac{1}{2}-m)_r}{r!},$$

and p, q, m are positive integers such that $p \geqslant q+2m$, $q \geqslant m$.

[See Bailey 10, where other references are given.]

19. If

$$P_n^{(\alpha,\beta)}(x) = (-1)^n \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\, F(-n,\, n+\alpha+\beta+1;\, \alpha+1;\, \tfrac{1}{2}-\tfrac{1}{2}x),$$

prove that

$$\sum_{n=0}^{\infty} \frac{n!\,(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}\, t^n P_n^{(\alpha,\beta)}(\cos 2\phi)\, P_n^{(\alpha,\beta)}(\cos 2\Phi)$$

$$= (1+t)^{-\alpha-\beta-1}\, F_4[\tfrac{1}{2}(\alpha+\beta+1),\ \tfrac{1}{2}(\alpha+\beta+2);\ \alpha+1,\ \beta+1;\ a^2/k^2,\ b^2/k^2]$$

and

$$\sum_{n=0}^{\infty} \frac{n!\,(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n}\, (2n+\alpha+\beta+1)\, t^n P_n^{(\alpha,\beta)}(\cos 2\phi)\, P_n^{(\alpha,\beta)}(\cos 2\Phi)$$

$$= \frac{(\alpha+\beta+1)(1-t)}{(1+t)^{\alpha+\beta+2}}\, F_4[\tfrac{1}{2}(\alpha+\beta+2),\ \tfrac{1}{2}(\alpha+\beta+3);\ \alpha+1,\ \beta+1;\ a^2/k^2,\ b^2/k^2],$$

where $a = \sin\phi\sin\Phi$, $b = \cos\phi\cos\Phi$, $k = \tfrac{1}{2}(t^{-\frac{1}{2}}+t^{\frac{1}{2}})$. Deduce in particular that

$$\sum_{n=0}^{\infty} \frac{(-1)^n(\alpha+\beta+1)_n}{(\beta+1)_n}\, t^n P_n^{(\alpha,\beta)}(x)$$

$$= \frac{1}{(1+t)^{\alpha+\beta-1}}\, F\left[\begin{array}{cc} \tfrac{1}{2}(\alpha+\beta+1), & \tfrac{1}{2}(\alpha+\beta+2); \\ & \beta+1 \end{array} \ \frac{2t(1+x)}{(1+t)^2} \right].$$

[Cf. Watson **8**, where the sum of the second series is expressed as an integral of an elementary function.]

20. Prove the formulae

(i) $F_4[\alpha,\,\beta;\,\gamma,\,1+\alpha-\gamma;\,z(1-Z),\,Z(1-z)]$

$$= F_2[\alpha;\,\beta,\,\beta;\,\gamma,\,1+\alpha-\gamma;\,z,\,Z],$$

(ii) $F_4\left[\alpha,\,\beta;\,\gamma,\,\beta;\,-\dfrac{x}{(1-x)(1-y)},\,-\dfrac{y}{(1-x)(1-y)}\right]$

$$= (1-x)^\alpha(1-y)^\alpha\, F_1[\alpha;\,\gamma-\beta,\,1+\alpha-\gamma;\,\gamma;\,x,\,xy].$$

From (ii) deduce the formulae

(iii) $F_4\left[\alpha,\,\beta;\,\alpha,\,\beta;\,-\dfrac{x}{(1-x)(1-y)},\,-\dfrac{y}{(1-x)(1-y)}\right]$

$$= (1-xy)^{-1}(1-x)^\beta(1-y)^\alpha,$$

(iv) $F_4\left[\alpha,\,\beta;\,\beta,\,\beta;\,-\dfrac{x}{(1-x)(1-y)},\,-\dfrac{y}{(1-x)(1-y)}\right]$

$$= (1-x)^\alpha(1-y)^\alpha\, F[\alpha,\,1+\alpha-\beta;\,\beta;\,xy],$$

(v) $F_4\left[\alpha,\,\beta;\,1+\alpha-\beta,\,\beta;\,-\dfrac{x}{(1-x)(1-y)},\,-\dfrac{y}{(1-x)(1-y)}\right]$

$$= (1-y)^\alpha\, F\left[\alpha,\,\beta;\,1+\alpha-\beta;\,-\frac{x(1-y)}{1-x}\right].$$

[Bailey **15**.]

BIBLIOGRAPHY

[This list is largely additional to the bibliographies given in Klein's *Vorlesungen über die hypergeometrische Funktion* (revised by Haupt, 1933) and Appell and Kampé de Fériet's *Fonctions hypergéométriques et hypersphériques* (1926).]

W. N. Bailey

1. Products of generalized hypergeometric series, *Proc. London Math. Soc.* (2), 28 (1928), 242–254.

2. Transformations of generalized hypergeometric series, *Proc. London Math. Soc.* (2), 29 (1929), 495–502.

3. Some identities involving generalized hypergeometric series, *Proc. London Math. Soc.* (2), 29 (1929), 503–516.

4. An identity involving Heine's basic hypergeometric series, *Journal London Math. Soc.*, 4 (1929), 254–257.

5. An extension of Whipple's theorem on well-poised hypergeometric series, *Proc. London Math. Soc.* (2), 31 (1930), 505–511.

6. The partial sum of the coefficients of the hypergeometric series, *Journal London Math. Soc.*, 6 (1931), 40–41.

7. On one of Ramanujan's theorems, *Journal London Math. Soc.*, 7 (1932), 34–36.

8. Some transformations of generalized hypergeometric series, and contour integrals of Barnes' type, *Quart. J. of Math.* (Oxford), 3 (1932), 168–182.

9. On certain relations between hypergeometric series of higher order, *Journal London Math. Soc.*, 8 (1933), 100–107.

10. On the product of two Legendre polynomials, *Proc. Camb. Phil. Soc.*, 29 (1933), 173–177.

11. A reducible case of the fourth type of Appell's hypergeometric functions of two variables, *Quart. J. of Math.* (Oxford), 4 (1933), 305–308.

12. Transformations of well-poised hypergeometric series, *Proc. London Math. Soc.* (2), 36 (1934), 235–240.

13. On the reducibility of Appell's function F_4, *Quart. J. of Math.* (Oxford), 5 (1934), 291–292.

14. Some theorems concerning products of hypergeometric series, *Proc. London Math. Soc.* (2), 38 (1935), 377–384.

15. Some infinite integrals involving Bessel functions, *Proc. London Math. Soc.* [In course of publication.]

E. W. Barnes

1. A new development of the theory of the hypergeometric functions, *Proc. London Math. Soc.* (2), 6 (1908), 141–177.

2. A transformation of generalized hypergeometric series, *Quart. J. of Math.*, 41 (1910), 136–140.

J. L. Burchnall

1. A relation between hypergeometric series, *Quart. J. of Math.* (Oxford), 3 (1932), 318–320.

F. Carlson

1. Sur une classe de séries de Taylor, dissertation, Upsala (1914).

A. Cayley

1. On a theorem relating to hypergeometric series, *Phil. Mag.* (4), 16 (1858), 356–357. [*Collected papers*, 3, 268–269.]

S. Chapman

1. Some ratios of infinite determinants occurring in the kinetic theory of gases, *Journal London Math. Soc.*, 8 (1933), 266–272.

T. Clausen

1. Ueber die Fälle wenn die Reihe $y = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \ldots$ ein quadrat von der Form $z = 1 + \frac{\alpha' \beta' \gamma'}{1 \cdot \delta' \epsilon'} x + \ldots$ hat, *Journal für Math.*, 3 (1828), 89–95.

H. B. C. Darling

1. On a proof of one of Ramanujan's theorems, *Journal London Math. Soc.*, 5 (1930), 8–9.

2. On certain relations between hypergeometric series of higher orders, *Proc. London Math. Soc.* (2), 34 (1932), 323–339.

A. C. Dixon

1. On the sum of the cubes of the coefficients in a certain expansion by the binomial theorem, *Messenger of Math.*, 20 (1891), 79–80.

2. Summation of a certain series, *Proc. London Math. Soc.* (1), 35 (1903), 285–289.

3. On a certain double integral, *Proc. London Math. Soc.* (2), 2 (1905), 8–15.

J. Dougall

1. On Vandermonde's theorem and some more general expansions, *Proc. Edinburgh Math. Soc.*, 25 (1907), 114–132.

D. Edwardes

1. An expansion in factorials similar to Vandermonde's theorem, and applications, *Messenger of Math.*, 52 (1923), 129–136.

A. R. Forsyth

1. On linear differential equations, *Quart. J. of Math.*, 19 (1883), 292–337.

C. Fox

1. The expression of hypergeometric series in terms of similar series, *Proc. London Math. Soc.* (2), 26 (1927), 201–210.

C. F. Gauss

1. Disquisitiones generales circa seriem infinitam, *Ges. Werke*, 3 (1866), 123–163 and 207–229.

G. H. Hardy

1. On two theorems of F. Carlson and S. Wigert, *Acta Math.* 42 (1920), 327–339.

2. A chapter from Ramanujan's note-book, *Proc. Camb. Phil. Soc.*, 21 (1923), 492–503.

3. Some formulae of Ramanujan, *Proc. London Math. Soc.* (2), 22 (1923), xii–xiii (Records for 14 Dec., 1922).

M. J. M. Hill

1. On a formula for the sum of a finite number of terms of the hypergeometric series when the fourth element is equal to unity, *Proc. London Math. Soc.* (2), 5 (1907), 335–341.

2. On a formula for the sum of a finite number of terms of the hypergeometric series when the fourth element is unity. (Second communication.) *Proc. London Math. Soc.* (2), 6 (1908), 339–348.

M. J. M. Hill and F. J. W. Whipple

1. A reciprocal relation between generalized hypergeometric series, *Quart. J. of Math.*, 41 (1910), 128–135.

J. Hodgkinson

1. Note on one of Ramanujan's theorems, *Journal London Math. Soc.*, 6 (1931), 42–43.

F. H. Jackson

1. Summation of q-hypergeometric series, *Messenger of Math.*, 50 (1921), 101–112.

E. E. Kummer

1. Ueber die hypergeometrische Reihe, *Journal für Math.*, 15 (1836), 39–83.

P. A. MacMahon

1. The sums of the powers of the binomial coefficients, *Quart. J. of Math.*, 33 (1902), 274–288.

F. Morley

1. On the series $1 + \left(\frac{p}{1}\right)^3 + \left\{\frac{p\,(p+1)}{1.2}\right\}^3 + \ldots$, *Proc. London Math. Soc.* (1), 34 (1902), 397–402.

W. McF. Orr

1. Theorems relating to the product of two hypergeometric series, *Trans. Camb. Phil. Soc.*, 17 (1899), 1–15.

C. T. Preece

1. Dougall's theorem on hypergeometric functions, *Proc. Camb. Phil. Soc.*, 21 (1923), 595–598.

2. The product of two generalized hypergeometric functions, *Proc. London Math. Soc.* (2), 22 (1924), 370–380.

H. W. Richmond

1. The sum of the cubes of the coefficients in $(1-x)^{2n}$, *Messenger of Math.*, 21 (1892), 77–78.

M. Riesz

1. Sur le principe de Phragmén-Lindelöf, *Proc. Camb. Phil. Soc.*, 20 (1920), 205–207; and correction, *ibid.* 21 (1921), 6.

L. J. Rogers

1. Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* (1), 25 (1894), 318–343.

2. On two theorems of combinatory analysis and some allied identities, *Proc. London Math. Soc.* (2), 16 (1917), 315–336.

L. J. Rogers and S. Ramanujan

1. Proof of certain identities in combinatory analysis (with a prefatory note by G. H. Hardy), *Proc. Camb. Phil. Soc.*, 19 (1919), 211–216.

L. Saalschütz

1. Eine Summationsformel, *Zeitschrift für Math. u. Phys.*, 35 (1890), 186–188.

2. Über einen Spezialfall der hypergeometrischen Reihe dritter Ordnung, *Zeitschrift für Math. u. Phys.*, 36 (1891), 278–295 and 321–327.

I. Schur

1. Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *Berliner Sitzungsberichte*, 1917, No. 23, 301–321.

J. H. C. Searle

1. The summation of certain series, *Messenger of Math.*, 38 (1909), 138–144.

W. F. Sheppard

1. Summation of the coefficients of some terminating hypergeometric series, *Proc. London Math. Soc.* (2), 10 (1912), 469–478.

J. Thomae

1. Ueber die Funktionen welche durch Reihen von der Form dargestellt werden: $1 + \frac{pp'p''}{1q'q''} + \ldots$, *Journal für Math.*, 87 (1879), 26–73.

G. N. Watson

1. The integral formula for generalized Legendre functions, *Proc. London Math. Soc.* (2), 17 (1918), 241–246.

2. The product of two hypergeometric functions, *Proc. London Math. Soc.* (2), 20 (1922), 189–195.

3. Dixon's theorem on generalized hypergeometric functions, *Proc. London Math. Soc.* (2), 22 (1924), xxxii–xxxiii (Records for 17 May, 1923).

4. The theorems of Clausen and Cayley on products of hypergeometric functions, *Proc. London Math. Soc.* (2), 22 (1924), 163–170.

5. A note on generalized hypergeometric series, *Proc. London Math. Soc.* (2), 23 (1925), xiii–xv (Records for 8 Nov., 1923).

6. A new proof of the Rogers-Ramanujan identities, *Journal London Math. Soc.*, 4 (1929), 4–9.

7. Theorems stated by Ramanujan (VIII): Theorems on divergent series, *Journal London Math. Soc.*, 4 (1929), 82–86.

8. Notes on generating functions of polynomials: (4) Jacobi polynomials, *Journal London Math. Soc.*, 9 (1934), 22–28.

F. J. W. Whipple

1. A group of generalized hypergeometric series: relations between 120 allied series of the type $F\,[a, b, c; d, e]$, *Proc. London Math. Soc.* (2), 23 (1925), 104–114.

2. On well-poised series, generalized hypergeometric series having parameters in pairs, each pair with the same sum, *Proc. London Math. Soc.* (2), 24 (1926), 247–263.

3. Well-poised series and other generalized hypergeometric series, *Proc. London Math. Soc.* (2), 25 (1926), 525–544.

4. A fundamental relation between generalized hypergeometric series, *Journal London Math. Soc.*, 1 (1926), 138–145.

5. Some transformations of generalized hypergeometric series, *Proc. London Math. Soc.* (2), 26 (1927), 257–272.

F. J. W. Whipple (*cont.*)

6. Algebraic proofs of the theorems of Cayley and Orr concerning the products of certain hypergeometric series, *Journal London Math. Soc.*, 2 (1927), 85–90.

7. On a formula implied in Orr's theorems concerning the products of hypergeometric series, *Journal London Math. Soc.*, 4 (1929), 48–50.

8. On series allied to the hypergeometric series with argument − 1, *Proc. London Math. Soc.* (2), 30 (1930), 81–94.

9. The sum of the coefficients of a hypergeometric series, *Journal London Math. Soc.*, 5 (1930), 192.

10. On transformations of terminating well-poised hypergeometric series of type $_9F_8$, *Journal London Math. Soc.*, 9 (1934), 137–140.

S. Wigert

1. Sur un théorème concernant les fonctions entières, *Arkiv för Mat. Ast. o. Fys.*, 11 (1916), No. 22.