THE BARGAINING PROBLEM[1]

By John F. Nash, Jr.

A new treatment is presented of a classical economic problem, one which occurs in many forms, as bargaining, bilateral monopoly, etc. It may also be regarded as a nonzero-sum two-person game. In this treatment a few general assumptions are made concerning the behavior of a single individual and of a group of two individuals in certain economic environments. From these, the solution (in the sense of this paper) of the classical problem may be obtained. In the terms of game theory, values are found for the game.

INTRODUCTION

A two-person bargaining situation involves two individuals who have the opportunity to collaborate for mutual benefit in more than one way. In the simpler case, which is the one considered in this paper, no action taken by one of the individuals without the consent of the other can affect the well-being of the other one.

The economic situations of monopoly versus monopsony, of state trading between two nations, and of negotiation between employer and labor union may be regarded as bargaining problems. It is the purpose of this paper to give a theoretical discussion of this problem and to obtain a definite "solution"—making, of course, certain idealizations in order to do so. A "solution" here means a determination of the amount of satisfaction each individual should expect to get from the situation, or, rather, a determination of how much it should be worth to each of these individuals to have this opportunity to bargain.

This is the classical problem of exchange and, more specifically, of bilateral monopoly as treated by Cournot, Bowley, Tintner, Fellner, and others. A different approach is suggested by von Neumann and Morgenstern in *Theory of Games and Economic Behavior*[2] which permits the identification of this typical exchange situation with a nonzero sum two-person game.

In general terms, we idealize the bargaining problem by assuming that the two individuals are highly rational, that each can accurately compare his desires for various things, that they are equal in bargaining skill, and that each has full knowledge of the tastes and preferences of the other.

[1] The author wishes to acknowledge the assistance of Professors von Neumann and Morgenstern who read the original form of the paper and gave helpful advice as to the presentation.

[2] John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 1944 (Second Edition, 1947), pp. 15–31.

In order to give a theoretical treatment of bargaining situations we abstract from the situation to form a mathematical model in terms of which to develop the theory.

In making our treatment of bargaining we employ a numerical utility, of the type developed in *Theory of Games*, to express the preferences, or tastes, of each individual engaged in bargaining. By this means we bring into the mathematical model the desire of each individual to maximize his gain in bargaining. We shall briefly review this theory in the terminology used in this paper.

UTILITY THEORY OF THE INDIVIDUAL

The concept of an "anticipation" is important in this theory. This concept will be explained partly by illustration. Suppose Mr. Smith knows he will be given a new Buick tomorrow. We may say that he has a Buick anticipation. Similarly, he might have a Cadillac anticipation. If he knew that tomorrow a coin would be tossed to decide whether he would get a Buick or a Cadillac, we should say that he had a $\frac{1}{2}$ Buick, $\frac{1}{2}$ Cadillac anticipation. Thus an anticipation of an individual is a state of expectation which may involve the certainty of some contingencies and various probabilities of other contingencies. As another example, Mr. Smith might know that he will get a Buick tomorrow and think that he has half a chance of getting a Cadillac too. The $\frac{1}{2}$ Buick, $\frac{1}{2}$ Cadillac anticipation mentioned above illustrates the following important property of anticipations: if $0 \leqslant p \leqslant 1$ and A and B represent two anticipations, there is an anticipation, which we represent by $pA + (1 - p)\,B$, which is a probability combination of the two anticipations where there is a probability p of A and $1 - p$ of B.

By making the following assumptions we are enabled to develop the utility theory of a single individual:

1. An individual offered two possible anticipations can decide which is preferable or that they are equally desirable.

2. The ordering thus produced is transitive; if A is better than B and B is better than C then A is better than C.

3. Any probability combination of equally desirable states is just as desirable as either.

4. If A, B, and C are as in assumption (2), then there is a probability combination of A and C which is just as desirable as C. This amounts to an assumption of continuity.

5. If $0 \leqslant p \leqslant 1$ and A and B are equally desirable, then $pA + (1 - p)\,C$ and $pB + (1 - p)\,C$ are equally desirable. Also, if A and B are equally desirable, A may be substituted for B in any desirability ordering relationship satisfied by B.

These assumptions suffice to show the existence of a satisfactory utility function, assigning a real number to each anticipation of an individual. This utility function is not unique, that is, if u is such a function then so also is $au + b$, provided $a > 0$. Letting capital letters represent anticipations and small ones real numbers, such a utility function will satisfy the following properties:

(a) $u(A) > u(B)$ is equivalent to A is more desirable than B, etc.

(b) If $0 \leqslant p \leqslant 1$ then $u[pA + (1 - p) B] = pu(A) + (1 - p) u(B)$. This is the important linearity property of a utility function.

TWO PERSON THEORY

In *Theory of Games and Economic Behavior* a theory of n-person games is developed which includes as a special case the two-person bargaining problem. But the theory there developed makes no attempt to find a value for a given n-person game, that is, to determine what it is worth to each player to have the opportunity to engage in the game. This determination is accomplished only in the case of the two-person zero sum game.

It is our viewpoint that these n-person games should have values; that is, there should be a set of numbers which depend continuously upon the set of quantities comprising the mathematical description of the game and which express the utility to each player of the opportunity to engage in the game.

We may define a two-person anticipation as a combination of two one-person anticipations. Thus we have two individuals, each with a certain expectation of his future environment. We may regard the one-person utility functions as applicable to the two-person anticipations, each giving the result it would give if applied to the corresponding one-person anticipation which is a component of the two-person anticipation. A probability combination of two two-person anticipations is defined by making the corresponding combinations for their components. Thus if $[A, B]$ is a two-person anticipation and $0 \leqslant p \leqslant 1$, then

$$p[A, B] + (1 - p)[C, D]$$

will be defined as

$$[pA + (1 - p)C, pB + (1 - p)D].$$

Clearly the one-person utility functions will have the same linearity property here as in the one-person case. From this point onwards when the term anticipation is used it shall mean two-person anticipation.

In a bargaining situation one anticipation is especially distinguished; this is the anticipation of no cooperation between the bargainers. It is natural, therefore, to use utility functions for the two individuals which assign the number zero to this anticipation. This still leaves each individual's utility function determined only up to multiplication by a positive real number. Henceforth any utility functions used shall be understood to be so chosen.

We may produce a graphical representation of the situation facing the two by choosing utility functions for them and plotting the utilities of all available anticipations in a plane graph.

It is necessary to introduce assumptions about the nature of the set of points thus obtained. We wish to assume that this set of points is compact and convex, in the mathematical senses. It should be convex since an anticipation which will graph into any point on a straight line segment between two points of the set can always be obtained by the appropriate probability combination of two anticipations which graph into the two points. The condition of compactness implies, for one thing, that the set of points must be bounded, that is, that they can all be inclosed in a sufficiently large square in the plane. It also implies that any continuous function of the utilities assumes a maximum value for the set at some point of the set.

We shall regard two anticipations which have the same utility for any utility function corresponding to either individual as equivalent so that the graph becomes a complete representation of the essential features of the situation. Of course, the graph is only determined up to changes of scale since the utility functions are not completely determined.

Now since our solution should consist of *rational* expectations of gain by the two bargainers, these expectations should be realizable by an appropriate agreement between the two. Hence, there should be an available anticipation which gives each the amount of satisfaction he should expect to get. It is reasonable to assume that the two, being rational, would simply agree to that anticipation, or to an equivalent one. Hence, we may think of one point in the set of the graph as representing the solution, and also representing all anticipations that the two might agree upon as fair bargains. We shall develop the theory by giving conditions which should hold for the relationship between this solution point and the set, and from these deduce a simple condition determining the solution point. We shall consider only those cases in which there is a possibility that both individuals could gain from the situation. (This does not exclude cases where, in the end, only one individual could have benefited because the "fair bargain" might consist of an agreement to use a probability method to decide who is to gain in the end. Any probability combination of available anticipations is an available anticipation.)

Let u_1 and u_2 be utility functions for the two individuals. Let $c(S)$ represent the solution point in a set S which is compact and convex and includes the origin. We assume:

6. If α is a point in S such that there exists another point β in S with the property $u_1(\beta) > u_1(\alpha)$ and $u_2(\beta) > u_2(\alpha)$, then $\alpha \neq c(S)$.

7. If the set T contains the set S and $c(T)$ is in S, then $c(T) = c(S)$.

We say that a set S is symmetric if there exist utility operators u_1 and u_2 such that when (a, b) is contained in S, (b, a) is also contained in S; that is, such that the graph becomes symmetrical with respect to the line $u_1 = u_2$.

8. If S is symmetric and u_1 and u_2 display this, then $c(S)$ is a point of the form (a, a), that is, a point on the line $u_1 = u_2$.

The first assumption above expresses the idea that each individual wishes to maximize the utility to himself of the ultimate bargain. The third expresses equality of bargaining skill. The second is more complicated. The following interpretation may help to show the naturalness of this assumption: If two rational individuals would agree that $c(T)$ would be a fair bargain if T were the set of possible bargains, then they should be willing to make an agreement, of lesser restrictiveness, not to attempt to arrive at any bargains represented by points outside of the set S if S contained $c(T)$. If S were contained in T this would reduce their situation to one with S as the set of possibilities. Hence $c(S)$ should equal $c(T)$.

We now show that these conditions require that the solution be the point of the set in the first quadrant where $u_1 u_2$ is maximized. We know some such point exists from the compactness. Convexity makes it unique.

Let us now choose the utility functions so that the above-mentioned point is transformed into the point (1, 1). Since this involves the multiplication of the utilities by constants, (1, 1) will now be the point of maximum $u_1 u_2$. For no points of the set will $u_1 + u_2 > 2$, now, since if there were a point of the set with $u_1 + u_2 > 2$ at some point on the line segment between (1, 1) and that point, there would be a value of $u_1 u_2$ greater than one (see Figure 1).

We may now construct a square in the region $u_1 + u_2 \leqslant 2$ which is symmetrical in the line $u_1 = u_2$, which has one side on the line $u_1 + u_2 = 2$, and which completely encloses the set of alternatives. Considering the square region formed as the set of alternatives, instead of the older set, it is clear that (1, 1) is the only point satisfying assumptions (6) and (8). Now using assumption (7) we may conclude that (1, 1) must also be the solution point when our original (transformed) set is the set of alternatives. This establishes the assertion.

We shall now give a few examples of the application of this theory.

EXAMPLES

Let us suppose that two intelligent individuals, Bill and Jack, are in a position where they may barter goods but have no money with which

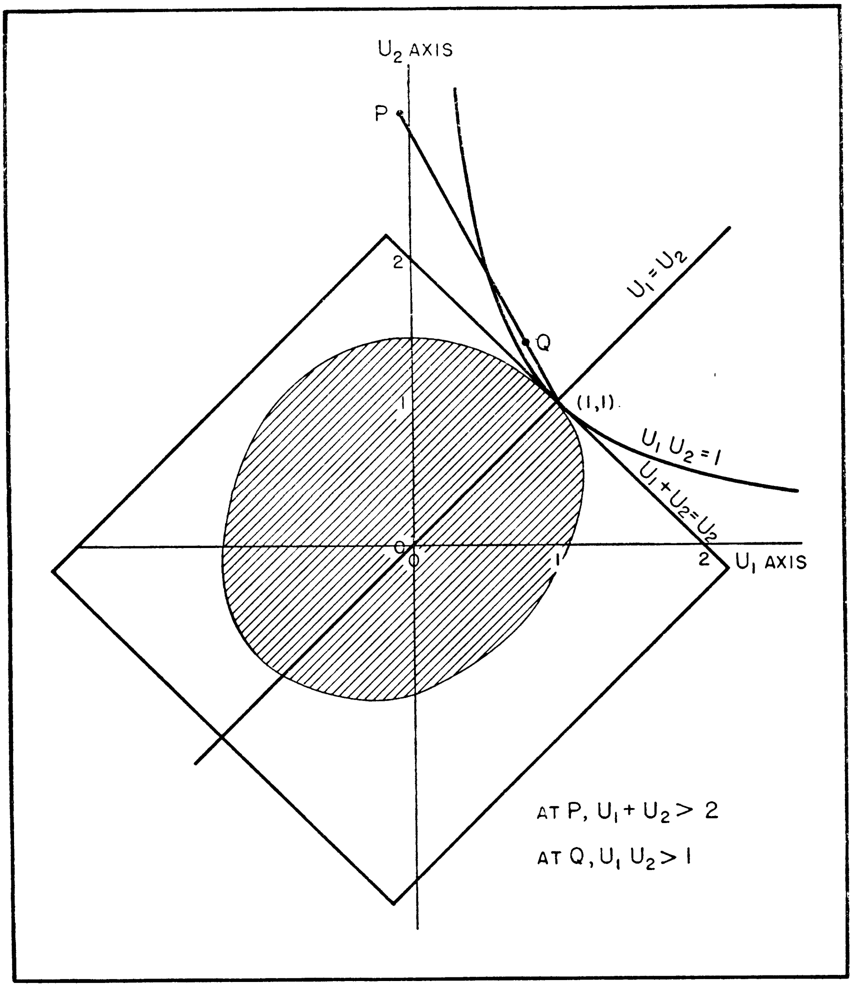

FIGURE 1

to facilitate exchange. Further, let us assume for simplicity that the utility to either individual of a portion of the total number of goods involved is the sum of the utilities to him of the individual goods in that portion. We give below a table of goods possessed by each individual with the utility of each to each individual. The utility functions used for the two individuals are, of course, to be regarded as arbitrary.

Bill's goods	*Utility to Bill*	*Utility to Jack*
book	2	4
whip	2	2
ball	2	1
bat	2	2
box	4	1
Jack's goods		
pen	10	1
toy	4	1
knife	6	2
hat	2	2

The graph for this bargaining situation is included as an illustration (Figure 2). It turns out to be a convex polygon in which the point where the product of the utility gains is maximized is at a vertex and where there is but one corresponding anticipation. This is:

Bill gives Jack: book, whip, ball, and bat,
Jack gives Bill: pen, toy, and knife.

When the bargainers have a common medium of exchange the problem may take on an especially simple form. In many cases the money equiva-

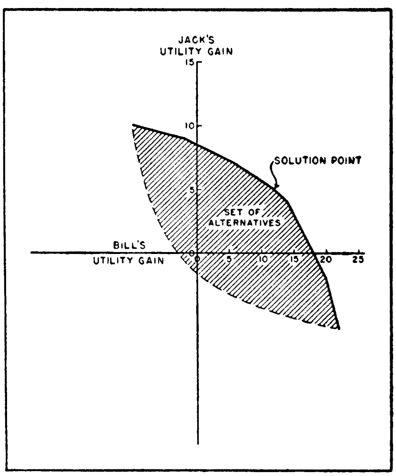

FIGURE 2

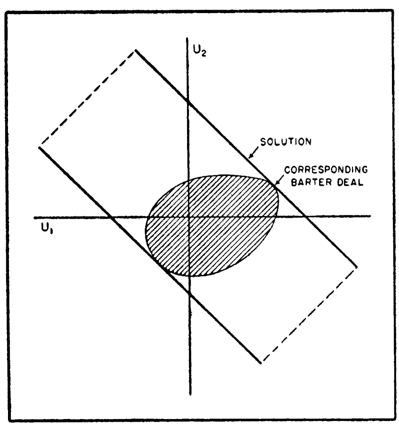

FIGURE 3

FIGURE 2—The solution point is on a rectangular hyperbola lying in the first quadrant and touching the set of alternatives at but one point.

FIGURE 3—The inner area represents the bargains possible without the use of money. The area between parallel lines represents the possibilities allowing the use of money. Utility and gain measured by money are here equated for small amounts of money. The solution must be formed using a barter-type bargain for which $u_1 + u_2$ is at a maximum and using also an exchange of money.

lent of a good will serve as a satisfactory approximate utility function. (By the money equivalent is meant the amount of money which is just as desirable as the good to the individual with whom we are concerned.) This occurs when the utility of an amount of money is approximately a linear function of the amount in the range of amounts concerned in the situation. When we may use a common medium of exchange for the utility function for each individual the set of points in the graph is such that that portion of it in the first quadrant forms an isosceles right triangle. Hence the solution has each bargainer getting the same money profit (see Figure 3).

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