Math 582
Intro to Set Theory
Lecture 31

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April 6, 2009

Introduction: Cofinalities

☞ The cofinality of a limit ordinal γ is the length of the shortest sequence of smaller ordinals, $\langle \alpha_\xi \mid \xi < \lambda \rangle$, which converges to γ: $\sup\langle \alpha_\xi \mid \xi < \lambda \rangle = \gamma$.

☞ This material is from Section 9.2 of Hrbacek and Jech.

Cardinal exponentiation

☞ When λ is infinite and $2 \leq \kappa \leq 2^\lambda$ then

$$\kappa^\lambda = 2^\lambda.$$

☞ There are further complexities computing κ^λ when $\kappa > \lambda$. For example, if we assume GCH then

$$(\aleph_{27})^{\aleph_0} = \aleph_{27} \quad (\aleph_{\omega_1})^{\aleph_0} = \aleph_{\omega_1} \quad \text{but } (\aleph_\omega)^{\aleph_0} = \aleph_{\omega+1}$$

The property that explains these differences is that $\aleph_\omega$ has countable cofinality.

Cofinality defined

Definition

If γ is any limit ordinal then the cofinality of λ is

$$\text{cf}(\gamma) = \min\{\text{type}(X) \mid X \subseteq \gamma \,\&\, \sup X = \gamma\}.$$

γ is regular iff $\text{cf}(\gamma) = \gamma$, and otherwise γ is singular.

- For any limit ordinal γ, $\text{cf}(\gamma) \geq \omega$: if $X \subseteq \gamma$ and X is finite then X has a maximal member, so that $\sup X < \gamma$.
- $\text{cf}(\omega) = \omega$,
- $\text{cf}(\omega^2) = \omega$: consider $X = \{\omega \cdot n \mid n < \omega\}$,
- $\text{cf}(\aleph_\omega) = \omega$: consider $X = \{\aleph_n \mid n < \omega\}$.
- ω is regular; $\omega^2, \aleph_\omega$ are singular.

Computing cofinality

As an aid to computing cofinalities:

① If $A \subseteq \gamma$ and $\sup(A) = \gamma$ then $\text{cf}(\gamma) = \text{cf}(\text{type}(A))$.

② $\text{cf}(\text{cf}(\gamma)) = \text{cf}(\gamma)$, so $\text{cf}(\gamma)$ is regular.

③ If γ is regular, then γ is a cardinal. So, a cofinality is a cardinal.
Note: H+J define regular ordinal to be a cardinal.

④ $\omega \leq \text{cf}(\gamma) \leq |\gamma| \leq \gamma$.

⑤ When α is zero or a successor, $\aleph_\alpha$ is regular.

⑥ When α is a limit, $\text{cf}(\aleph_\alpha) = \text{cf}(\alpha)$

Examples

- $\aleph_2$ is regular (by ⑤)
- $\text{cf}(\alpha + \beta) = \text{cf}(\beta)$ (let $A = \{\alpha + \xi \mid \xi < \beta\}$ and apply ①),
- Every limit ordinal below ω_2 has either cofinality ω (such as ω^ω or $\omega_1 + \omega$) or cofinality ω_1 (such as $\omega_1 + \omega_1$).
- $\text{cf}(\aleph_{\omega_1}) = \omega_1$ (by ⑥), so $\aleph_{\omega_1}$ is singular
- Recall the following construction of a cardinal which is a fixed-point of the $\aleph$-functions:

$$\beta_0 = 0 \qquad \beta_{n+1} = \aleph^+_{\beta_n}$$

Let $\beta = \sup_{n<\omega} \beta_n$. Then $\beta = \aleph_\beta$, and β is a singular cardinal of cofinality ω.
(Lecture 26, slide 15, or H+J, Lemma 9.2.6.)

Proofs

① cf(γ) = cf(type(A)), for any unbounded $A \subseteq \gamma$.

☞ Let $A \subseteq \gamma$ be unbounded in γ and α = type(A) via the (order-preserving) enumeration $\langle a_\xi \mid \xi < \alpha \rangle$ of A.

Note that α is a limit ordinal since A is unbounded in the limit γ.

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cf$(\gamma) \leq$ cf(α). Any cofinal sequence of α is one of γ.

☞ Let $Y \subseteq \alpha$ be unbounded in α of type cf(α).

Then $\langle a_\xi \mid \xi \in Y \rangle$ is unbounded in A.

✓ Therefore, cf$(\gamma) \leq$ type(Y) = cf(α).

Proofs

① Let $A \subseteq \gamma$ be unbounded in γ and α = type(A) via the (order-preserving) enumeration $\langle a_\xi \mid \xi < \alpha \rangle$ of A.

Note that α is a limit ordinal since A is unbounded in the limit γ.

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cf$(\alpha) \leq$ cf(γ). When $B = \langle b_\xi \mid \xi < \text{cf}(\gamma) \rangle$ is unbounded in γ, define $A' \subseteq A$ unbounded in A by

$$A' = \langle a'_\xi \in A \mid a'_\xi \text{ least in } A \text{ with } a'_\xi > b_\xi \rangle.$$

Let $Y \subseteq \alpha$ by the indices of A' in A:

$$Y = \{\eta \mid \exists \xi \, a_\eta = a'_\xi\}.$$

Since Y is a cofinal sequence in α,

$$\text{cf}(\alpha) \leq \text{type}(Y) \leq \text{type}(B) = \text{cf}(\gamma).$$

Proofs

② $\mathrm{cf}(\mathrm{cf}(\gamma)) = \mathrm{cf}(\gamma)$.

☞ Let $A \subseteq \gamma$ be unbounded with $\mathrm{type}(A) = \mathrm{cf}(\gamma)$.
By ①

$$\mathrm{cf}(\gamma) \;=\; \mathrm{cf}(\mathrm{type}(A)) \;=\; \mathrm{cf}(\mathrm{cf}(\gamma))$$

Cofinalities

Proofs

③ If γ is regular, then γ is a cardinal.

☞ Let $\kappa = |\gamma|$ and show that $\kappa \geq \mathrm{cf}(\gamma)$. It follows that $\kappa = \mathrm{cf}(\gamma)$:
Since γ is regular, $\quad \gamma \;=\; \mathrm{cf}(\gamma) \;\leq\; \kappa \;=\; |\gamma| \;\leq\; \gamma$.

☞ Fix $f : \kappa \twoheadrightarrow \gamma$ and define $A \subseteq \kappa$ by

$$A = \{\eta < \kappa \,\big|\, \forall \xi \left(\xi < \eta \;\rightarrow\; f(\xi) < f(\eta)\right)\}$$

Note that $f[A]$ is unbounded in γ.
Since f is increasing in A,

$$\mathrm{type}(A) = \mathrm{type}(f[A])$$

Thus by ①

$$\mathrm{cf}(\gamma) \leq \mathrm{type}(f[A]) = \mathrm{type}(A) \leq \kappa.$$

Proofs

④ $\omega \leq \mathrm{cf}(\gamma) \leq |\gamma| \leq \gamma$.

$\omega \leq \mathrm{cf}(\gamma)$ and $|\gamma| \leq \gamma$ are from the definition. $\mathrm{cf}(\gamma)$ is regular by ② so a cardinal by ③; thus, $\mathrm{cf}(\gamma) \leq |\gamma|$.

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⑤ When α is zero or a successor, $\aleph_\alpha$ is regular

$\aleph_0$ is regular by ④. Suppose $A \subseteq \aleph_{\beta+1}$ and $|A| \leq \aleph_\beta$; then, A is bounded in $\omega_{\beta+1}$: since each $x \in A$ satisfies $|x| \leq \aleph_\beta$ we have:

$$|\sup(A)| = |\bigcup A| \leq \sum_{x \in A} |x| \leq \aleph_\beta \cdot \aleph_\beta = \aleph_\beta < \aleph_{\beta+1}$$

So, A cannot be a cofinal sequence in $\aleph_{\beta+1}$.

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⑥ When α is a limit, $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.

Apply ① with $A = \{\aleph_\xi \mid \xi < \alpha\}$.

✓ This completes all six proofs.

Alternative characterization of cofinality

We give a useful alternative characterization of cofinality:

Theorem

For every infinite cardinal κ, $\mathrm{cf}(\kappa)$ is the least ordinal α such that there is a sequence of cardinals $\langle \kappa_\xi \mid \xi < \alpha \rangle$ with

$$\sum_{\xi < \alpha} \kappa_\xi = \kappa.$$

Proof of characterization

Let κ be an infinite cardinal and α the least ordinal such that there is a sequence of cardinals $\langle \kappa_\xi \mid \xi < \alpha \rangle$ with

$$\sum_{\xi<\alpha} \kappa_\xi = \kappa.$$

$\alpha \leq \mathrm{cf}(\kappa)$. Let $\zeta = \mathrm{cf}(\kappa)$; then, there is an increasing sequence of ordinals $\langle \alpha_\xi \mid \xi < \zeta \rangle$ with $\kappa = \sup \langle \alpha_\xi \mid \xi < \zeta \rangle$.

Let $\kappa_\xi = |\alpha_\xi|$, so

$$\kappa \leq \sum_{\xi<\zeta} \kappa_\xi \leq \zeta \cdot \kappa = \kappa.$$

Cofinalities

Proof of characterization

$\mathrm{cf}(\kappa) \leq \alpha$. If $\alpha \geq \kappa$, then $\mathrm{cf}(\kappa) \leq \kappa \leq \alpha$. So, suppose $\alpha < \kappa$.

Choose a sequence of cardinals $\langle \kappa_\xi \mid \xi < \alpha \rangle$ with

$$\sum_{\xi<\alpha} \kappa_\xi = \kappa.$$

We will show $\sup \langle \kappa_\xi \mid \xi < \alpha \rangle = \kappa$.

☞ Suppose $\langle \kappa_\xi \mid \xi < \alpha \rangle$ is **not** cofinal in κ; fix an ordinal $\rho < \kappa$ which bounds these cardinals: $\kappa_\xi < \rho$ for all $\xi < \alpha$. Then

$$\sum_{\xi<\alpha} \kappa_\xi \leq |\rho| \cdot |\alpha| < \kappa \ \lightning.$$

Thus, $\sup \langle \kappa_\xi \mid \xi < \alpha \rangle = \kappa$ and so $\mathrm{cf}(\kappa) \leq \alpha$.

Characterization of regular and singular

☞ $\aleph_\omega$ is singular and also the sum of $< \aleph_\omega$ cardinals of size $< \aleph_\omega$:

$$\sum_{n<\omega} \aleph_n = \aleph_\omega$$

This is distinctive of singular cardinals:

Corollary

Let θ be any infinite cardinal.

(a) *θ is regular iff for any family of cardinals $\langle \kappa_\xi \mid \xi < \gamma \rangle$ with $\kappa_\xi < \theta$ (for each $\xi < \gamma$) and $\gamma < \theta$ then $\sum_{\xi<\gamma} \kappa_\xi < \theta$.*

(b) *θ is singular iff there is an increasing family of cardinals $\langle \kappa_\xi \mid \xi < \gamma \rangle$ with $\kappa_\xi < \theta$ (for each $\xi < \gamma$) and $\gamma < \theta$ but $\sum_{\xi<\gamma} \kappa_\xi = \theta$.*

Weakly Inaccessible Cardinals

☞ Every successor cardinal, $\aleph_{\alpha+1}$, is regular. Which limit cardinals are regular? These will have to be cardinals with the property $\kappa = \aleph_\kappa$.

Definition

A regular limit cardinal is called a weakly inaccessible cardinal.

☞ We know how to produce fixed points of the $\aleph$ function (see slide 9), but this construction produces singular cardinals. In fact, if κ is weakly inaccessible it must be the κth fixed point of the aleph function!

$$\left| \{\beta < \kappa \mid \beta = \aleph_\beta\} \right| = \kappa$$

(It is actually much bigger than this.)

☞ It is impossible to prove the weakly inaccessible cardinals exist in ZFC. The statement that *There exists weakly inaccessible cardinals* is our first example of a statement of large cardinal existence.