

The Gibbs-Wilbraham Phenomenon: An Episode in Fourier Analysis

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Introduction

What is called the GIBBS phenomenon or GIBBS's phenomenon deals with "overshoot" in the convergence of the partial sums of certain FOURIER series in the neighborhood of a discontinuity of the function being expanded. The integral

$$(1) \quad \int_0^{\pi} \frac{\sin(t)}{t} dt = 1.8519370 \dots^1$$

plays an essential rôle in computing the amount of this overshoot. While teaching a course in the theory of functions of a real variable, E. HEWITT found the value 1.71... listed for the integral (1) in HARDY & ROGOSINSKI [27], page 36. This anomaly, as well as others encountered in the literature, led us to a study of the GIBBS phenomenon and its history. In the course of this study we uncovered a maze of forgotten results, interesting and difficult generalizations, faulty constants, and some details about the GIBBS phenomenon that have escaped the attention of many writers on the subject. Despite the familiarity of our theme, we therefore entertain a hope that readers of the *Archive* will find some interest in a discussion of this corner of FOURIER analysis.

The paper is divided into three Parts. In Part I, we examine GIBBS's phenomenon in some detail. In Part II, we take up its curious history and describe briefly some of its congeners. In Part III, we offer some conclusions.

The computations given in this paper were carried out on two computers: a Hewlett-Packard 9810 and a Univac 1110. The graphs (barring the simplest) were drawn by a Hewlett-Packard 9862A plotter. All finite decimal expansions are truncated decimal expansions.

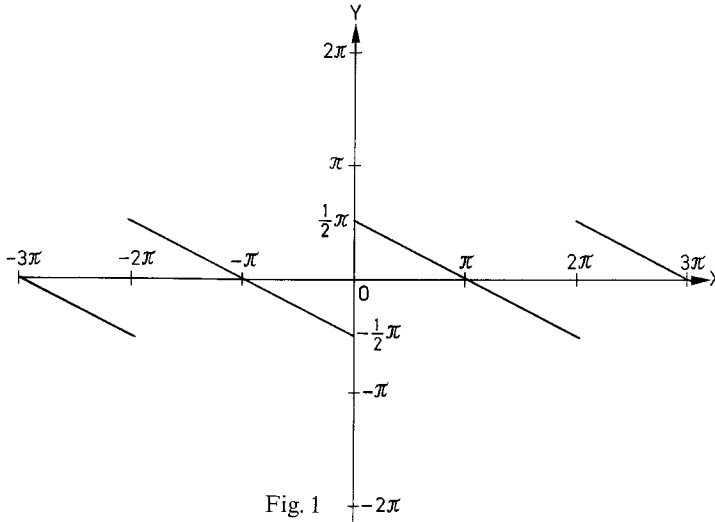
It is a pleasure to record our indebtedness to GERALD B. FOLLAND, THOMAS L. HANKINS, EINAR HILLE, and STEPHEN P. KEELER, who have made valuable suggestions to us.

¹ See for example [1], p. 244, Table 5.3, which was kindly drawn to our attention by Mr. STEPHEN P. KEELER. We have also repeated the computation.

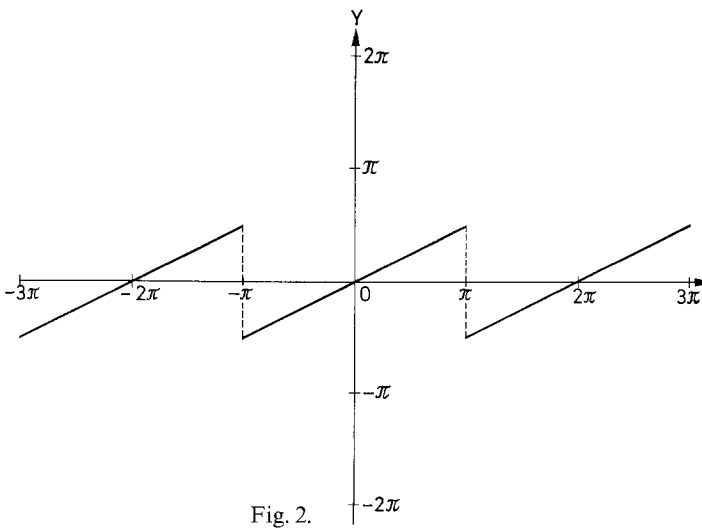
Part I. The Gibbs Phenomenon

For well over 200 years mathematicians have known that certain discontinuous functions can be represented as sums of infinite series of sines and cosines. A wealth of such representations were found by EULER. Today we know the most widely used class of these series as *Fourier series*. EULER, writing during the period 1755-1772, naturally did not use this phrase. Here are three of the expansions discovered by EULER:

$$(2) \quad \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{1}{2}(\pi - x) \quad (0 < x < 2\pi);$$



$$(3) \quad \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(kx)}{k} = \frac{1}{2}x \quad (-\pi < x < \pi);$$



$$(4) \quad \sum_{k=0}^{\infty} (-1)^k \frac{\cos((2k+1)x)}{2k+1} = \begin{cases} \frac{1}{4}\pi & (|x| < \frac{1}{2}\pi) \\ -\frac{1}{4}\pi & (\frac{1}{2}\pi < |x| < \pi). \end{cases}$$

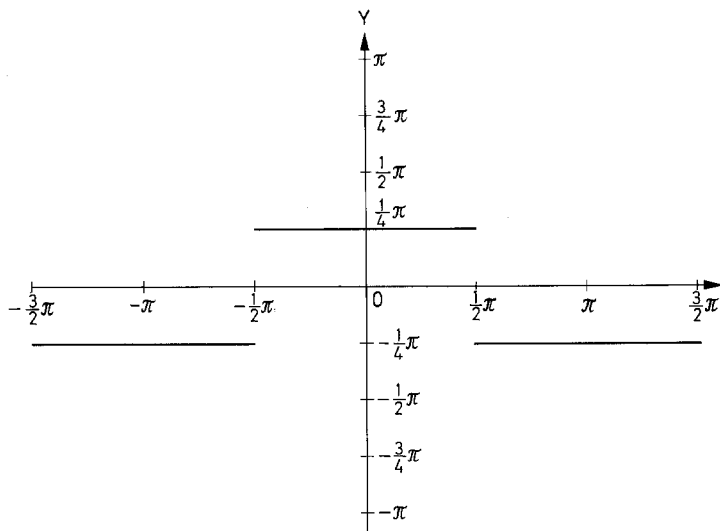


Fig. 3.

Expansions (2) and (3) date from the year 1755, (4) from 1772. (These citations are taken from BURKHARDT's monumental treatise [7], pages 857, 858, and 933.) We list these three expansions because all of them have been minutely studied: (4) by WILBRAHAM in 1848 and by CARSLAW in 1917; (3) by GIBBS in 1899; and (2) by GRONWALL in 1912. The partial sums of these series are naturally continuous functions, while the sums of the series are functions with discontinuities. The partial sums of these three series (and of many other Fourier series) show the same curious behavior near discontinuities of the sum. Following GRONWALL [24], we take up first the series (2). GRONWALL's paper, published in 1912, is a masterpiece of clarity and is a pleasure to read. His treatment of GIBBS's phenomenon is strictly elementary, requiring nothing beyond the most elementary calculus. In our opinion, it is the most elegant treatment extant of the matter.

For an arbitrary positive integer n , consider the n^{th} partial sum $\phi_n(x)$ of the series (2):

$$(5) \quad \phi_n(x) = \sum_{k=1}^n \frac{\sin(kx)}{k}.$$

It is plain that $\phi_n(x) = -\phi_n(-x) = -\phi_n(2\pi - x)$ and that $\phi_n(0) = \phi_n(\pi) = 0$. Thus to describe completely the behavior of the function ϕ_n , we need only consider its behavior in the open interval $]0, \pi[$. We first identify the points in $]0, \pi[$ where ϕ_n assumes either local maxima or local minima. Differentiating (5), we use standard trigonometric identities to write

$$\begin{aligned}
 \phi'_n(x) &= \sum_{k=1}^n \cos(kx) \\
 &= \frac{1}{2 \sin(\frac{1}{2}x)} \sum_{k=1}^n 2 \sin(\frac{1}{2}x) \cos(kx) \\
 &= \frac{1}{2 \sin(\frac{1}{2}x)} \sum_{k=1}^n \sin(\frac{1}{2}x + kx) + \sin(\frac{1}{2}x - kx) \\
 (6) \quad &= \frac{1}{2 \sin(\frac{1}{2}x)} \left[\sum_{k=1}^n \sin((k + \frac{1}{2})x) - \sum_{k=1}^n \sin((k - \frac{1}{2})x) \right] \\
 &= \frac{1}{2 \sin(\frac{1}{2}x)} \left[\sum_{k=1}^n \sin((k + \frac{1}{2})x) - \sum_{k=0}^{n-1} \sin((k + \frac{1}{2})x) \right] \\
 &= \frac{\sin((n + \frac{1}{2})x) - \sin(\frac{1}{2}x)}{2 \sin(\frac{1}{2}x)} \\
 &= \frac{\sin(\frac{1}{2}nx) \cos(\frac{1}{2}(n+1)x)}{\sin(\frac{1}{2}x)}.
 \end{aligned}$$

The first and last lines of (6) show that the zeros of ϕ'_n in $]0, \pi[$ are at

$$(7) \quad \frac{1}{n+1}\pi < \frac{2}{n}\pi < \frac{3}{n+1}\pi < \dots < \frac{2l}{n}\pi < \frac{2l+1}{n+1}\pi < \dots < \frac{n-2}{n}\pi < \frac{n-1}{n+1}\pi$$

for n even and at

$$(8) \quad \frac{1}{n+1}\pi < \frac{2}{n}\pi < \frac{3}{n+1}\pi < \dots < \frac{2l}{n}\pi < \frac{2l+1}{n+1}\pi < \dots < \frac{n-1}{n}\pi < \frac{n}{n+1}\pi$$

for n odd. Note that $n-1 = 2[\frac{1}{2}(n-1)] + 1$ for n even and that $n = 2[\frac{1}{2}(n-1)] + 1$ for n odd. Thus the last entry in both (7) and (8) is

$$\frac{2[\frac{1}{2}(n-1)] + 1}{n+1}\pi,$$

for both even and odd n .

The points listed in (7) and (8) are alternatively maxima and minima: none of them is a point of inflection. To see this, again use the last line of (6). At each point $\frac{2l+1}{n+1}\pi$, the function $\cos(\frac{1}{2}(n+1)x)$ changes sign while the function $\sin(\frac{1}{2}nx)$ is of constant sign throughout the interval $]\frac{2l}{n}\pi, \frac{2(l+1)}{n}\pi[$. Therefore the derivative $\phi'_n(x)$ changes its sign at each of the points $\frac{2l+1}{n+1}\pi$ listed in (7) and (8). For similar reasons the derivative $\phi'_n(x)$ changes its sign at every point $\frac{2l}{n}\pi$ listed in (7) and (8). All of these points are therefore extreme points. Since

² We follow common practice by writing $[t]$ for the greatest integer not exceeding the real number t .

$\phi'_n(x)$ is positive in the interval $]0, \frac{1}{n+1} \pi[$, the point $\frac{1}{n+1} \pi$ is a maximum for the function ϕ_n . Since the maxima and minima alternate, it follows that all of the points $\frac{2l+1}{n+1} \pi$ are maxima and that all of the points $\frac{2l}{n} \pi$ are minima.

One can also compute $\phi''_n(x)$ from the sixth line of (6):

$$(9) \quad \phi''_n(x) = \frac{n \sin((n+1)x) - (n+1) \sin(nx)}{4 \sin^2(\frac{1}{2}x)} \quad (0 < x < \pi).$$

One sees at once from (9) that the zeros of $\phi''_n(x)$ in the interval $]0, \pi[$ lie strictly between the zeros of $\phi'_n(x)$. Consider $n=1000$ as an example. The five smallest positive zeros of $\phi''_{1000}(x)$ are

$$(10) \quad 0.004491164, 0.007721392, 0.010898673, 0.014059166, 0.017212151,³$$

for which we have

$$(11) \quad \frac{1}{1001} \pi < 0.004491164 < \frac{1}{500} \pi < 0.007721392 < \frac{3}{1001} \pi < 0.010898673 < \frac{1}{250} \pi < 0.014059166 < \frac{5}{1001} \pi < 0.017212151 < \frac{3}{500} \pi.$$

in conformity with the above.

Thus the function ϕ_n has $\frac{1}{2}n = [\frac{1}{2}(n-1)] + 1$ maxima $\frac{2l+1}{n+1} \pi$ for n even and $\frac{1}{2}(n+1) = [\frac{1}{2}(n-1)] + 1$ maxima $\frac{2l+1}{n+1} \pi$ for n odd. The number of minima $\frac{2l}{n} \pi$ is one fewer in both cases.

We will now prove four theorems due to GRONWALL [24], showing how ϕ_n behaves at its l^{th} maxima and minima.

Theorem A. For $0 \leq l \leq [\frac{1}{2}(n-1)]$, we have

$$(12) \quad \phi_{n+1} \left(\frac{2l+1}{n+2} \pi \right) > \phi_n \left(\frac{2l+1}{n+1} \pi \right)$$

and for $1 \leq l \leq [\frac{1}{2}(n-1)]$, we have

$$(13) \quad \phi_{n+1} \left(\frac{2l}{n+1} \pi \right) > \phi_n \left(\frac{2l}{n} \pi \right).$$

Proof. The inequalities

$$\frac{2l+1}{n+2} \pi < \frac{2l+1}{n+1} \pi < \frac{2l+2}{n+1} \pi$$

³ In tabulations of computed numbers, we take the shortcut of writing $t=0.9873121$, for example, to mean that $0.9873121 \leq t < 0.9873122$.

are evident. The function ϕ_{n+1} has a maximum at $\frac{2l+1}{n+2}\pi$ and its next minimum at $\frac{2l+2}{n+1}\pi$, and so it decreases in this interval. We thus have

$$\begin{aligned} \phi_{n+1}\left(\frac{2l+1}{n+2}\pi\right) &> \phi_{n+1}\left(\frac{2l+1}{n+1}\pi\right) = \phi_n\left(\frac{2l+1}{n+1}\pi\right) + \frac{1}{n+1} \sin\left((n+1)\frac{2l+1}{n+1}\pi\right) \\ &= \phi_n\left(\frac{2l+1}{n+1}\pi\right). \end{aligned}$$

This is inequality (12). We also have

$$\frac{2l-1}{n+1}\pi < \frac{2l}{n+1}\pi < \frac{2l}{n}\pi;$$

the left and right ends of this expression are respectively a maximum of ϕ_n and the succeeding minimum of ϕ_n , so that

$$\phi_{n+1}\left(\frac{2l}{n+1}\pi\right) = \phi_n\left(\frac{2l}{n+1}\pi\right) > \phi_n\left(\frac{2l}{n}\pi\right).$$

This is (13). \square

Theorem B.⁴ *The inequality*

$$(14) \quad \phi_n(x) > 0$$

holds for all positive integers n and all x in the interval $]0, \pi[$.

Proof. We estimate ϕ_n at its minima $\frac{2l}{n}\pi$ ($2l < n$). By inequality (13), we have

$$(15) \quad \phi_n\left(\frac{2l}{n}\pi\right) > \phi_{n-1}\left(\frac{2l}{n-1}\pi\right) > \dots > \phi_{2l+1}\left(\frac{2l}{2l+1}\pi\right).$$

We compute the right end of (15):

$$\begin{aligned} \phi_{2l+1}\left(\frac{2l}{2l+1}\pi\right) &= \sum_{k=1}^{2l+1} \frac{1}{k} \sin\left(\frac{k2l}{2l+1}\pi\right) \\ (16) \quad &= \sum_{k=1}^{2l} \frac{1}{k} (-1)^{k+1} \sin\left(k\pi - \frac{k2l}{2l+1}\pi\right) \\ &= \sum_{k=1}^{2l} (-1)^{k+1} \frac{1}{k} \sin\left(\frac{k}{2l+1}\pi\right). \end{aligned}$$

Since the function $t \rightarrow \frac{\sin(t)}{t}$ is strictly decreasing in the interval $[0, \pi]$, the last line of (16) is positive, and so (14) holds. \square

⁴ This theorem, with the proof given here, appears in GRONWALL's paper [24]. It was also proved almost simultaneously by JACKSON [29]. Several other writers have returned to this fact and have found many interesting analogues: FEJÉR [21]; LANDAU [31] (his proof in six lines is also given by ZYGMUND [49], Ch. II, Theorem (9.4), p. 62); TURÁN [41]; HYLÉN-CAVALIUS [28].

Theorem C. For all integers l such that $0 \leq l \leq [\frac{1}{2}(n-1)] - 1$, we have

$$(17) \quad \phi_n \left(\frac{2l+1}{n+1} \pi \right) > \phi_n \left(\frac{2l+3}{n+1} \pi \right).$$

Proof. While elementary, this proof is a little longer than the proof of Theorem A. Again, we follow GRONWALL [24]. We first carry out some trigonometric manipulations beginning with the last line of (6):

$$\begin{aligned}
 & \phi'_n(x) - \phi'_n \left(x + \frac{2}{n+1} \pi \right) \\
 &= \frac{\sin(\frac{1}{2}nx) \cos(\frac{1}{2}(n+1)x)}{\sin(\frac{1}{2}x)} - \frac{\sin \left(\frac{1}{2}nx + \pi - \frac{1}{n+1} \pi \right) \cos(\frac{1}{2}(n+1)x + \pi)}{\sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right)} \\
 &= \frac{\sin(\frac{1}{2}nx) \cos(\frac{1}{2}(n+1)x)}{\sin(\frac{1}{2}x)} - \frac{\sin \left(\frac{1}{2}nx - \frac{1}{n+1} \pi \right) \cos(\frac{1}{2}(n+1)x)}{\sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right)} \\
 &= \frac{\cos(\frac{1}{2}(n+1)x)}{\sin(\frac{1}{2}x) \sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right)} \left[\sin(\frac{1}{2}nx) \sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right) \right. \\
 & \quad \left. - \sin(\frac{1}{2}x) \sin \left(\frac{1}{2}nx - \frac{1}{n+1} \pi \right) \right] \\
 &= \frac{\cos(\frac{1}{2}(n+1)x)}{\sin(\frac{1}{2}x) \sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right)} \left[\sin(\frac{1}{2}nx) \sin(\frac{1}{2}x) \cos \left(\frac{1}{n+1} \pi \right) \right. \\
 & \quad \left. + \sin(\frac{1}{2}nx) \cos(\frac{1}{2}x) \sin \left(\frac{1}{n+1} \pi \right) \right. \\
 & \quad \left. - \sin(\frac{1}{2}x) \sin(\frac{1}{2}nx) \cos \left(\frac{1}{n+1} \pi \right) \right. \\
 & \quad \left. + \sin(\frac{1}{2}x) \cos(\frac{1}{2}nx) \sin \left(\frac{1}{n+1} \pi \right) \right] \\
 &= \frac{\cos(\frac{1}{2}(n+1)x)}{\sin(\frac{1}{2}x) \sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right)} \left[\sin \left(\frac{1}{n+1} \pi \right) \sin(\frac{1}{2}(n+1)x) \right] \\
 &= \frac{\sin((n+1)x) \sin \left(\frac{1}{n+1} \pi \right)}{2 \sin(\frac{1}{2}x) \sin \left(\frac{1}{2}x + \frac{1}{n+1} \pi \right)} \\
 &= \frac{\sin \left(\frac{1}{n+1} \pi \right) \sin((n+1)x)}{\cos \left(\frac{1}{n+1} \pi \right) - \cos \left(x + \frac{1}{n+1} \pi \right)}.
 \end{aligned}$$

(18)

For l and n as above and a continuous real variable t , write

$$(19) \quad \psi_{\pm}(t) = \phi_n \left(\frac{(2l+2)\pi \pm t}{n+1} \right)$$

and

$$(20) \quad \omega(t) = \psi_{-}(t) - \psi_{-}(t-2\pi) - \psi_{+}(t) + \psi_{+}(t+2\pi).$$

The identity (18) and some routine calculations, which we omit, yield the relation

$$(21) \quad \frac{d\omega}{dt}(t) = \frac{\sin\left(\frac{1}{n+1}\pi\right)\sin(t)}{n+1} \cdot \left\{ \frac{1}{\cos\left(\frac{1}{n+1}\pi\right) - \cos\left(\frac{(2l+3)\pi-t}{n+1}\right)} - \frac{1}{\cos\left(\frac{1}{n+1}\pi\right) - \cos\left(\frac{(2l+3)\pi+t}{n+1}\right)} \right\}.$$

For n even, the condition $l \leq [\frac{1}{2}(n-1)] - 1$ implies that $2l+3 \leq n-1$; for n odd, this condition implies that $2l+3 \leq n$. In either case, we have

$$0 < \frac{2l+3}{n+1} < \frac{2l+4}{n+1} \leq 1,$$

and for $0 < t < \pi$,

$$\pi \geq \frac{2l+4}{n+1}\pi > \frac{(2l+3)\pi+t}{n+1} > \frac{(2l+3)\pi-t}{n+1} > \frac{2l+2}{n+1}\pi > \frac{1}{n+1}\pi > 0.$$

It follows that

$$\cos\left(\frac{1}{n+1}\pi\right) > \cos\left(\frac{(2l+3)\pi-t}{n+1}\right) > \cos\left(\frac{(2l+3)\pi+t}{n+1}\right),$$

and so the quantity $\{\dots\}$ appearing in (21) is positive for all t such that $0 < t < \pi$. Therefore the function $\omega(t)$ is strictly increasing for $0 \leq t \leq \pi$. Since $\omega(0) = 0$, we infer that $\omega(\pi)$ is positive. Referring to the definitions (19) and (20), we find that the inequality $\omega(\pi) > 0$ is equivalent to

$$(22) \quad \phi_n\left(\frac{2l+3}{n+1}\pi\right) - \phi_n\left(\frac{2l+5}{n+1}\pi\right) < \phi_n\left(\frac{2l+1}{n+1}\pi\right) - \phi_n\left(\frac{2l+3}{n+1}\pi\right);$$

note that (22) holds for all integers l such that $2l+3 \leq n-1$ (if n is even) or $2l+3 \leq n$ (if n is odd). For $2l+3 = n-1$, we have

$$\phi_n\left(\frac{2l+5}{n+1}\pi\right) = \phi_n(\pi) = 0.$$

For $2l+3=n$, the relations

$$1 < \frac{2l+5}{n+1} = \frac{n+2}{n+1} < 2$$

obtain, so that

$$\phi_n \left(\frac{n+2}{n+1} \pi \right) = \phi_n \left(\pi + \frac{1}{n+1} \pi \right) = -\phi_n \left(-\pi - \frac{1}{n+1} \pi \right) = -\phi_n \left(\pi - \frac{1}{n+1} \pi \right).$$

By Theorem B, $\phi_n \left(\frac{n+2}{n+1} \pi \right)$ is negative. Applying (22) with $2l+3=n-1$ (the case of even n), we find

$$(23_e) \quad 0 < \phi_n \left(\frac{n-1}{n+1} \pi \right) < \phi_n \left(\frac{n-3}{n+1} \pi \right) - \phi_n \left(\frac{n-1}{n+1} \pi \right).$$

Applying (22) with $2l+3=n$ (the case of odd n), we find

$$(23_o) \quad 0 < \phi_n \left(\frac{n}{n+1} \pi \right) < \phi_n \left(\frac{n}{n+1} \pi \right) - \phi_n \left(\frac{n+2}{n+1} \pi \right) < \phi_n \left(\frac{n-2}{n+1} \pi \right) - \phi_n \left(\frac{n}{n+1} \pi \right).$$

Evident induction arguments, using (23_e) and (22) for even n and (23_o) and (22) for odd n , now give (17) for all values of l . \square

We thus have a good grasp on the maxima of $\phi_n(x)$, and some grasp, although imperfect, on the minima of $\phi_n(x)$.

Theorem D. For every positive integer s , the sequence $\left(\phi_n \left(\frac{2s-1}{n+1} \pi \right) \right)_{n=1}^{\infty}$ is ultimately increasing and has as its limit the number $\int_0^{(2s-1)\pi} \frac{\sin(t)}{t} dt$. The sequence $\left(\phi_n \left(\frac{2s}{n} \pi \right) \right)_{n=1}^{\infty}$ is ultimately increasing and has as its limit the number $\int_0^{2s\pi} \frac{\sin(t)}{t} dt$.

Proof. Theorem A shows that $\phi_n \left(\frac{2s-1}{n+1} \pi \right)$, increases with n for $n > 2s-1$.

We may write

$$(24) \quad \begin{aligned} \phi_n \left(\frac{2s-1}{n+1} \pi \right) &= \sum_{k=1}^n \frac{1}{k} \sin \left(\frac{k(2s-1)}{n+1} \pi \right) \\ &= \left(\sum_{k=1}^n \frac{1}{k(2s-1)\pi} \sin \left(\frac{k(2s-1)}{n+1} \pi \right) \right) \frac{2s-1}{n+1} \pi. \end{aligned}$$

The second line of (24) differs by the amount $\frac{2s-1}{n+1} \pi$ from a RIEMANN sum for the integral $\int_0^{(2s-1)\pi} \frac{\sin(t)}{t} dt$ (the integrand at 0 being defined as 1) and so we

have $\lim_{n \rightarrow \infty} \phi_n \left(\frac{2s-1}{n+1} \pi \right) = \int_0^{(2s-1)\pi} \frac{\sin(t)}{t} dt$. The argument for the sequence $\left(\phi_n \left(\frac{2s}{n} \pi \right) \right)_{n=1}^{\infty}$ is similar. \square

We next recall some facts about the integrals

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt \quad (0 < x < \infty).$$

The integrand $\frac{\sin(t)}{t}$ is graphed in Figure 4.

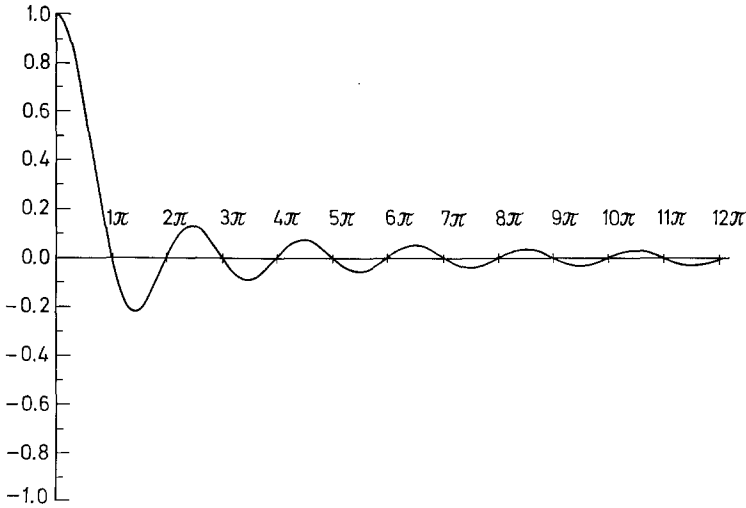


Fig. 4.

From the sign of the integrand, it is evident that $\text{Si}(x)$ increases as x runs from 0 to π , decreases as x runs from π to 2π , and so on. It is also a classical fact that

$$(25) \quad \lim_{x \rightarrow \infty} \text{Si}(x) = \frac{1}{2} \pi.$$

We pause to prove (25). We go back to the formulas (6). Integrating the first and 6th lines of (6), we find

$$\begin{aligned} 0 &= \phi_n(\pi) - \phi_n(0) \\ &= \int_0^\pi \phi_n'(t) dt \\ &= \int_0^\pi \left[\frac{\sin\left((n+\frac{1}{2})t\right)}{2 \sin\left(\frac{1}{2}t\right)} - \frac{1}{2} \right] dt, \end{aligned}$$

so that

$$(26) \quad \int_0^\pi \frac{\sin\left((n+\frac{1}{2})t\right)}{2 \sin\left(\frac{1}{2}t\right)} dt = \frac{1}{2} \pi.$$

We now look at

$$(27) \quad I_n = \int_0^{\pi} \sin\left((n + \frac{1}{2})t\right) \left[\frac{1}{2 \sin(\frac{1}{2}t)} - \frac{1}{t} \right] dt.$$

We write

$$\begin{aligned} \kappa(t) &= \frac{1}{2 \sin(\frac{1}{2}t)} - \frac{1}{t} \\ &= \frac{1}{2} \left(\frac{1}{\sin(u)} - \frac{1}{u} \right) \\ &= \frac{1}{2} \lambda(u), \end{aligned}$$

where $u = \frac{1}{2}t$ runs from 0 to $\frac{1}{2}\pi$. Since

$$(28) \quad \sin(u) < u \quad \text{for } u > 0,$$

$\lambda(u)$ is positive. Write

$$\lambda(u) = \frac{u - \sin(u)}{u \sin(u)}.$$

We wish to show that $\lim_{u \downarrow 0} \lambda(u) = 0$ and that λ is an increasing function in the interval $[0, \frac{1}{2}\pi]$. We use an argument kindly suggested to us by Professor EINAR HILLE. Integrating (28) three times, we find for all $u > 0$ that

$$(29) \quad 1 - \frac{1}{2}u^2 < \cos(u),$$

$$(30) \quad u - \frac{1}{6}u^3 < \sin(u),$$

$$(31) \quad \cos(u) < 1 - \frac{1}{2}u^2 + \frac{1}{24}u^4.$$

From (30) we find

$$\lambda(u) = \frac{u - \sin(u)}{u \sin(u)} < \frac{\frac{1}{6}u^3}{u^2 - \frac{1}{6}u^4} = \frac{\frac{1}{6}u}{1 - \frac{1}{6}u^2}.$$

It follows that

$$(32) \quad \lim_{u \downarrow 0} \lambda(u) = 0.$$

Now look at

$$(33) \quad \lambda'(u) = \frac{\sin^2(u) - u^2 \cos(u)}{u^2 \sin^2(u)}.$$

The inequalities (30) and (31) show that

$$\begin{aligned} \sin^2(u) - u^2 \cos(u) &> (u - \frac{1}{6}u^3)^2 - u^2(1 - \frac{1}{2}u^2 + \frac{1}{24}u^4) \\ &= \frac{1}{6}u^4 - \frac{1}{72}u^6 \\ &> 0 \end{aligned}$$

if $0 < u < 2 \cdot 3^{\frac{1}{2}}$. Since $\frac{1}{2}\pi < 2 \cdot 3^{\frac{1}{2}}$, (33) shows that $\lambda'(u) > 0$ for $0 < u \leq \frac{1}{2}\pi$. Therefore the function λ is strictly increasing in $[0, \frac{1}{2}\pi]$. We have shown that the integrand $\kappa(t)$ in (27) is increasing in $[0, \pi]$ and by (32) $\kappa(t)$ has limit zero at zero.

Now apply the second mean value theorem for integrals to the integral appearing in (27). According to this theorem, there is a number ξ such that $0 < \xi < \pi$ for which we have

$$\begin{aligned} I_n &= \kappa(0) \int_0^{\xi} \sin\left(\left(n + \frac{1}{2}\right)t\right) dt + \kappa(\pi) \int_{\xi}^{\pi} \sin\left(\left(n + \frac{1}{2}\right)t\right) dt \\ (34) \quad &= \left(\frac{1}{2} - \frac{1}{\pi}\right) \int_{\xi}^{\pi} \sin\left(\left(n + \frac{1}{2}\right)t\right) dt \\ &= \left(\frac{1}{2} - \frac{1}{\pi}\right) \frac{1}{n + \frac{1}{2}} \cos\left(\left(n + \frac{1}{2}\right)\xi\right). \end{aligned}$$

It follows that

$$(35) \quad |I_n| \leq \frac{1 - \frac{2}{\pi}}{2n + 1}.$$

Since

$$\int_0^{\pi} \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{t} dt = \int_0^{(n + \frac{1}{2})\pi} \frac{\sin(t)}{t} dt,$$

we may combine (26), (27), and (34) to infer that

$$\left| \int_0^{(n + \frac{1}{2})\pi} \frac{\sin(t)}{t} dt - \frac{\pi}{2} \right| \leq \frac{1 - \frac{2}{\pi}}{2n + 1},$$

from which the equality

$$\lim_{T \rightarrow \infty} \text{Si}(T) = \lim_{T \rightarrow \infty} \int_0^T \frac{\sin(t)}{t} dt = \frac{1}{2}\pi$$

follows. We present a graph of $\text{Si}(T)$ in Figure 5 below.

It is simple now to show that

$$(36) \quad \lim_{n \rightarrow \infty} \phi_n(x) = \frac{1}{2}(\pi - x) \quad (0 < x < 2\pi).$$

Using (6) again, we write

$$\begin{aligned} \phi_n(x) &= \int_0^x \phi_n'(t) dt \\ (37) \quad &= \int_0^x \left(\frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{2 \sin\left(\frac{1}{2}t\right)} - \frac{1}{2} \right) dt \\ &= \int_0^x \sin\left(\left(n + \frac{1}{2}\right)t\right) \left[\frac{1}{2 \sin\left(\frac{1}{2}t\right)} - \frac{1}{2} \right] dt + \int_0^x \frac{\sin\left(\left(n + \frac{1}{2}\right)t\right)}{t} dt - \frac{1}{2}x \\ &= I_n(x) + \int_0^{(n + \frac{1}{2})x} \frac{\sin(t)}{t} dt - \frac{1}{2}x. \end{aligned}$$

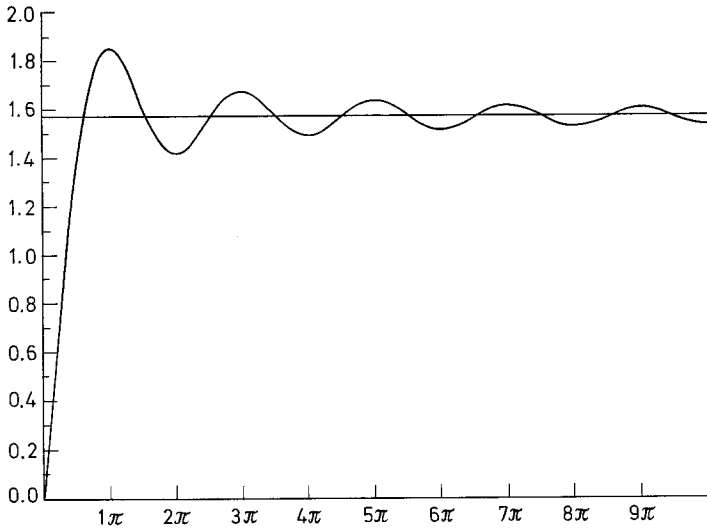


Fig. 5.

(The I_n of (34) is $I_n(\pi)$.) The integral computations (34) and (35) obviously hold with I_n replaced by $I_n(x)$: only the right side of (35) must be multiplied by the factor 2. Here we must restrict x to lie in the interval $[0, \pi]$ so that we know that $\kappa(x) \leq \kappa(\pi)$. For every positive number η , the convergence of

$$\int_0^{(n+\frac{1}{2})x} \frac{\sin(t)}{t} dt$$

to $\frac{1}{2}\pi$ is obviously uniform for all x such that $x \geq \eta$. Thus (37) implies (36) for all x such that $0 < x \leq \pi$, the convergence being uniform in every interval $[\eta, \pi]$, no matter how small the positive number η may be. Since $\phi_n(x) = -\phi_n(2\pi - x)$ and $\frac{1}{2}(\pi - x) = -\frac{1}{2}(\pi - (2\pi - x))$, (31) holds for all x such that $0 < x < 2\pi$, and the convergence in (31) is uniform in every interval $[\eta, 2\pi - \eta]$.

We are now ready to describe completely the convergence of $\phi_n(x)$ to its sum $\frac{1}{2}(\pi - x)$ in the interval $]0, \pi[$. At the first maximum, $\frac{\pi}{n+1}$, we find

$$\lim_{n \rightarrow \infty} \phi_n\left(\frac{\pi}{n+1}\right) = \text{Si}(\pi) = 1.8519370 = 1.1789797\left(\frac{1}{2}\pi\right).$$

That is, the sums overshoot the line $\frac{1}{2}(\pi - x)$ by a factor of 1.1789797. At the first minimum, $\frac{2}{n}\pi$, we find

$$\lim_{n \rightarrow \infty} \phi_n\left(\frac{2}{n}\pi\right) = \text{Si}(2\pi) = 1.4181516 = 0.9028233\left(\frac{1}{2}\pi\right).^5$$

⁵ See for example [1], p. 244, Table 5.3. We have repeated the computation.

That is, the sums *undershoot* at the first minimum by a factor of 0.9028233. We list these factors for the first five maxima and minima, writing

$$\frac{2}{\pi} \lim_{n \rightarrow \infty} \phi_n \left(\frac{2l-1}{n+1} \pi \right) = W(l), \quad \frac{2}{\pi} \lim_{n \rightarrow \infty} \phi_n \left(\frac{2l}{n} \pi \right) = w(l).$$

Table 1.

l	1	2	3	4	5
$W(l)$	1.1789797	1.0661864	1.0402143	1.0288319	1.0224603
$w(l)$	0.9028233	0.9499393	0.9664104	0.9747484	0.9797763

The convergence of $\frac{2}{\pi} \phi_n \left(\frac{2l-1}{n+1} \pi \right)$ to $W(l)$ and of $\frac{2}{\pi} \phi_n \left(\frac{2l}{n} \pi \right)$ to $w(l)$ is not very rapid. In view of the slowness of convergence of many familiar series (see the interesting article of R. P. BOAS on this theme [3]), this convergence may be

Table 2

$n \rightarrow$	5	10	25	100	1000	5000	10,000	25,000	100,000
1 st max	1.007675	1.086692	1.140271	1.169062	1.177980	1.178779	1.178879	1.178939	1.178969
1 st min	0.516461	0.706175	0.823357	0.882856	0.900823	0.902423	0.902623	0.902743	0.902803
2 nd max	0.551737	0.789279	0.950060	1.036434	1.063188	1.065586	1.065886	1.066066	1.066156
2 nd min	0.179736	0.556780	0.791010	0.910006	0.945940	0.949139	0.949539	0.949779	0.949899
3 rd max	0.180681	0.578542	0.846666	0.990627	1.035218	1.039214	1.039714	1.040014	1.040164
3 rd min	***	0.377044	0.728025	0.906510	0.960411	0.965210	0.965810	0.966170	0.966350
4 th max	***	0.382127	0.757854	0.959410	1.021837	1.027432	1.028132	1.028552	1.028761
4 th min	***	0.189713	0.656918	0.894881	0.966749	0.973148	0.973948	0.974428	0.974668
5 th max	***	0.190277	0.674042	0.933204	1.013467	1.020660	1.021560	1.022100	1.022370
5 th min	***	0	0.582515	0.879943	0.969778	0.977776	0.978776	0.979376	0.979676

regarded as not pathologically slow. Results for the first five extreme values are given in Table 2, where we tabulate the values of $\frac{2}{\pi}\phi_n$ for the indicated values of n at the first five maxima and minima of these functions. The reader may wish to compare the extreme right column of Table 2 with the limiting values tabulated in Table 1.

The behavior of ϕ_n near 0 for large values of n is thus fairly complicated. The sums oscillate rapidly both *above* and *below* the values $\frac{1}{2}(\pi-x)$. Graphs of ϕ_{10} , ϕ_{25} , ϕ_{200} , and ϕ_{1000} appear in Figures 6-9. Figure 10 exhibits a greatly enlarged version of the graph of ϕ_{1000} near 0.

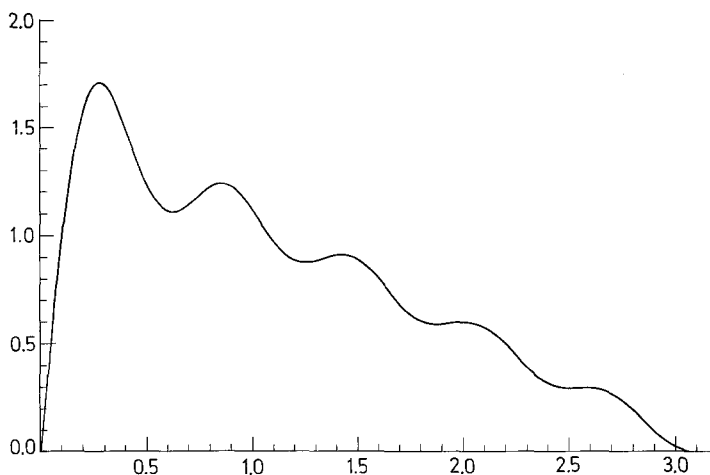


Fig. 6

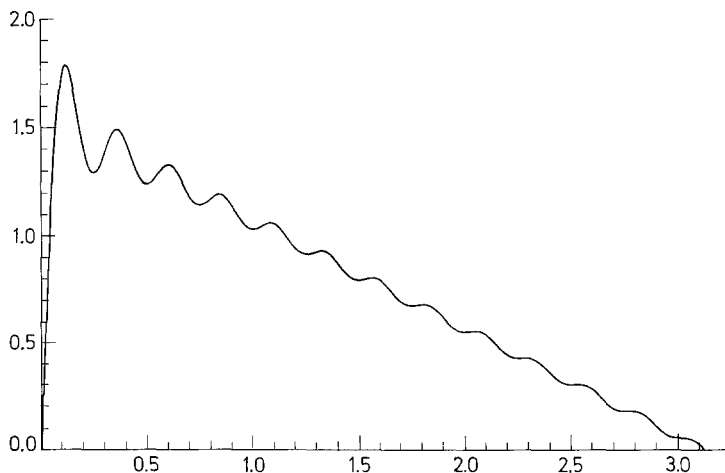


Fig. 7

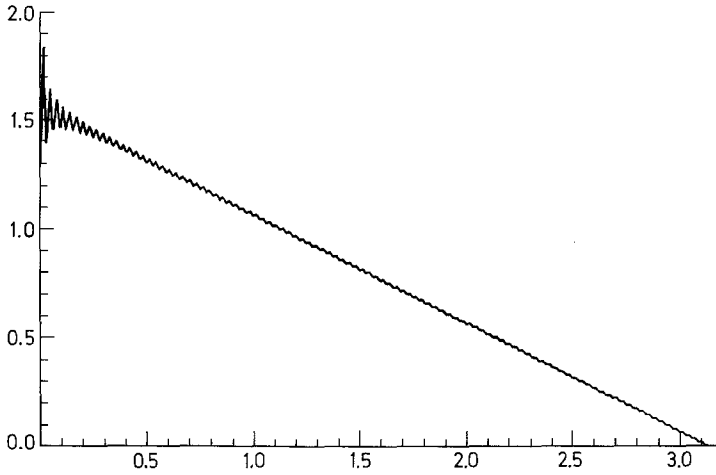


Fig. 8

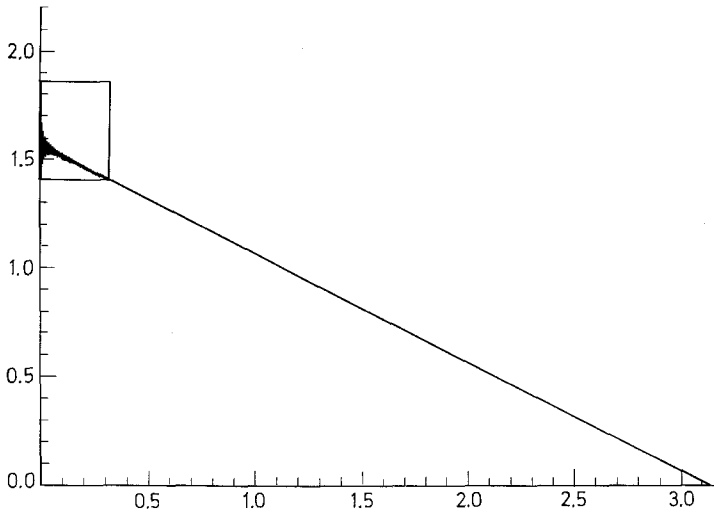


Fig. 9

The graphs show that the minimum values of ϕ_{200} and ϕ_{1000} increase for the first few minima and then decrease. This phenomenon was identified by GRONWALL in [24], Satz 5. He proved the following remarkable fact.

Theorem E. For $n \leq 42$, the minimum values of ϕ_n in the interval $]0, \pi[$ form a decreasing sequence. For $n \geq 43$, there is an integer m_0 such that

$$\phi_n \left(\frac{2l}{n} \pi \right) < \phi_n \left(\frac{2l+2}{n} \pi \right) \quad \text{for } l=1, 2, \dots, m_0-1,$$

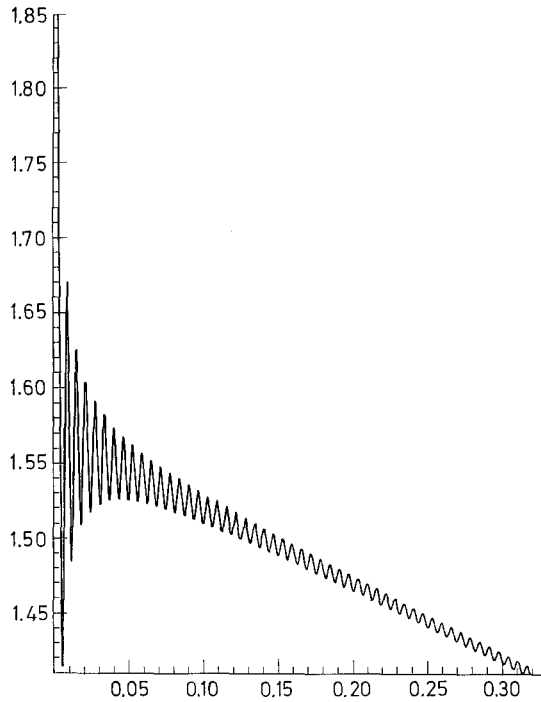


Fig. 10

and

$$\phi_n \left(\frac{2l}{n} \pi \right) > \phi_n \left(\frac{2l+2}{n} \pi \right) \quad \text{for } l = m_0, m_0 + 1, \dots, \left[\frac{n-1}{2} \right] - 1.$$

The number m_0 is either $\left[\frac{(2n)^{\frac{1}{2}}}{2\pi} \right]$ or $\left[\frac{(2n)^{\frac{1}{2}}}{2\pi} \right] + 1$.

We shall not here give a proof of Theorem E, despite its interest, for lack of space.

Since $\frac{(2000)^{\frac{1}{2}}}{2\pi} = 7.11 \dots$, the number m_0 for $n=2000$ is either 7 or 8, which agrees with a visual estimate made from Figure 10. Since $\frac{(200)^{\frac{1}{2}}}{2\pi} = 2.25 \dots$, the number m_0 for $n=200$ is either 2 or 3; this is corroborated by Figure 8. Figures 6 and 7 show decreasing minimum values, also in agreement with Theorem E.

The name *Gibbs's phenomenon* is usually attached to the fact that $W(1) = 1.1787797 \dots$, and to similar behavior for other Fourier series near discontinuities of the function being expanded. As we have seen, the behavior of $\phi_n(x)$ near 0 is actually much more complicated than just overshooting.

The graceful form of the graph of $\phi_{1000}(x)$ near zero, as shown in Figure 10, leads us to the following query. Are there functions $\beta_+ > 0$ and $\beta_- < 0$ on $[0, \infty[$ such that the maxima (minima) of ϕ_n lie very close to

$$(38) \quad \frac{1}{2}(\pi - x) + \beta_+(nx) \quad \left(\frac{1}{2}(\pi - x) + \beta_-(nx) \right)$$

for very large n and very small x ? The answer is affirmative. First use (37), (35), and an easy calculation to show that

$$(39) \quad \left| (\phi_n(x) - \frac{1}{2}(\pi - x)) - (\text{Si}(nx) - \frac{1}{2}\pi) \right| < \frac{2 - \frac{4}{\pi}}{2n + 1} + \log \left(1 + \frac{1}{2n} \right),$$

an estimate valid for all x such that $0 \leq x \leq \pi$. Next note the inequality

$$(40) \quad \int_a^{a+2\pi} \frac{\sin(t)}{t} dt > \int_b^{b+2\pi} \frac{\sin(t)}{t} dt,$$

which holds for all a and b such that $0 \leq a < b$. To check (40), write the left side as

$$\int_0^\pi \sin(t) \left(\frac{1}{t+a} - \frac{1}{t+\pi+a} \right) dt;$$

write the right side similarly and compare.

Now look at Figure 5. From (40) we see that the sequence $(\text{Si}((2l-1)\pi) - \frac{1}{2}\pi)_{l=1}^\infty$ is convex and that the sequence $(\text{Si}(2l\pi) - \frac{1}{2}\pi)_{l=1}^\infty$ is concave. Let β_+ be a concave strictly decreasing differentiable function on $[0, \infty[$ such that $\beta_+(2k\pi) = \text{Si}((2k+1)\pi) - \frac{1}{2}\pi$ for $k=0, 1, 2, \dots$. Let β_- be a convex strictly increasing differentiable function on $[0, \infty[$ such that $\beta_-(2k\pi) = \text{Si}(2(k+1)\pi) - \frac{1}{2}\pi$ for $k=0, 1, 2, \dots$. Possible choices for β_+ and β_- are sketched in Figure 11. It is clear from (39) that by compressing the X -axis, that is, by writing $\beta_+(nx)$ and $\beta_-(nx)$, we obtain (38).

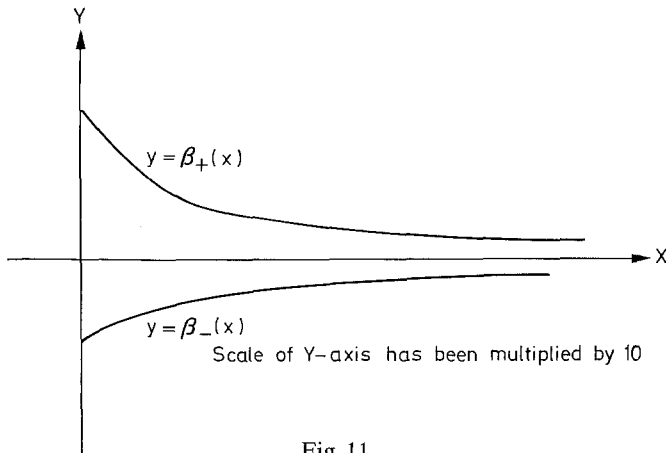


Fig. 11

All of the foregoing applies *mutatis mutandis* to the series (3) and (4). The derivatives of the partial sums are sums of cosines or sines, which can be summed in closed form as in (6), and the same behavior near discontinuities of the sum occurs as for the series (2). The series (4) behaves in some ways more simply than (2). For example, the minimum values of each partial sum form a strictly increasing sequence, for minima assumed in the interval $]-\frac{1}{2}\pi, 0[$, and the maximum values form a strictly decreasing sequence. See CARSLAW [8] for a discussion.

Part II. The History of Gibbs's Phenomenon

As noted in Part I, the infinite series (2) goes back to EULER. The first study known to us of overshoot and undershoot in the neighborhood of discontinuities of the sums of Fourier series was carried out not by GIBBS, but by HENRY WILBRAHAM. WILBRAHAM was an A.B. of Trinity College, Cambridge, who published an article on this topic in the year 1848 [44]. WILBRAHAM dealt with the series (4), citing FRANCIS NEWMAN [39] for the first part of his analysis. (NEWMAN does not refer to overshoot or undershoot, and indeed does not seem to have understood the real problem.) WILBRAHAM clearly grasped both the overshoot and undershoot phenomena (which as noted above occur with the series (4) just as with (2)). We reproduce as Figure 12 a lithograph from his paper. We cannot determine whether or not he has a factor $W(1)$ in mind, although he does refer to the integral $\text{Si}(\pi)$. We have recomputed three of his graphs, giving the results in Figures 13–15. His Figures 3 and 4 are our Figures 13 and 14, respectively. The reader will see that WILBRAHAM's Figures 3 and 4 are excellent. (Virtually identical graphs appear in CARSLAW [8], page 198.) WILBRAHAM's Figure 2, although it plainly shows overshoot and undershoot, is seriously in error, as our Figure 15 shows. Since WILBRAHAM's graphs were computed manually, he may be forgiven this lapse.

In 1874, DU BOIS-REYMOND [6] published an analysis of the behavior of Fourier series and Fourier integrals near points of discontinuity of the function being expanded. He came close to identifying overshoot and undershoot, but missed doing so because (in our notation) he considered the limit of $\phi_n(x)$ as both $n \rightarrow \infty$ and $x \rightarrow 0$. Apparently DU BOIS-REYMOND had some *a priori* notion of how the series ought to behave, and simply made his computations to fit this notion. All of his integrals are meticulously computed; only the conclusions are erroneous.

In 1898, MICHELSON & STRATTON [37] published a description of a mechanical machine called a harmonic analyser, which produced graphs of finite trigonometric series with terms up to $\cos(80x)$ and $\sin(80x)$. They published along with this account a number of graphs produced by their machine, including many handsome special curves and also graphs of the partial sums of the series (2), (3), and (4) out to 80 terms. Apparently the machine was not accurate enough to show GIBBS's phenomenon clearly. We reproduce as Figures 16 and 17 two of their plates. Figure 16 shows partial sums of $\frac{1}{2}$ of the series (4). Compare with our Figure 18 to see the limitations of their harmonic analyser.

Fig. 1.

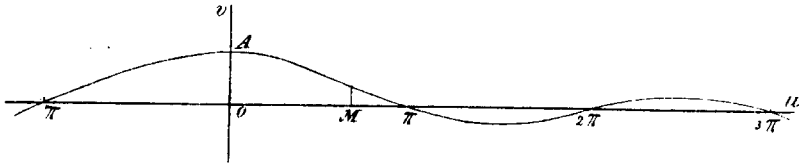


Fig. 2.

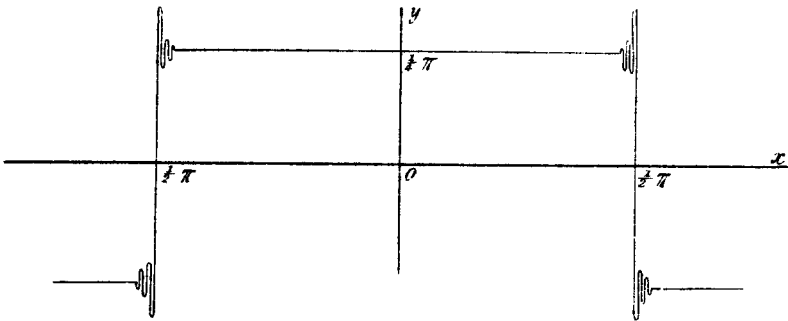


Fig. 3.

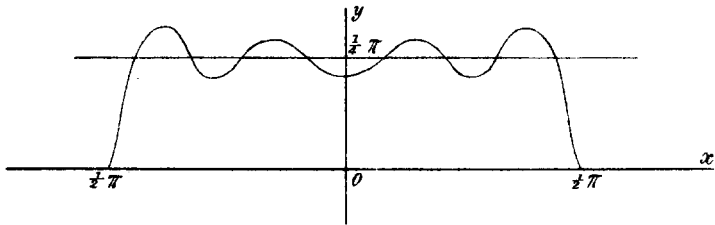


Fig. 4.

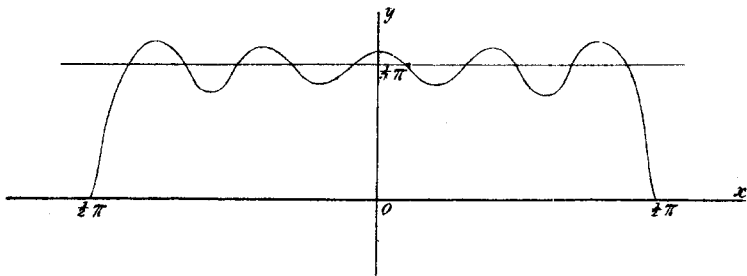


Fig. 12

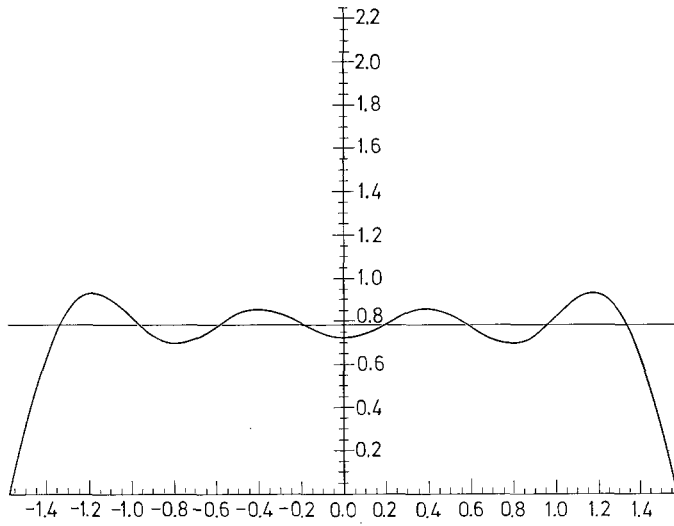


Fig. 13

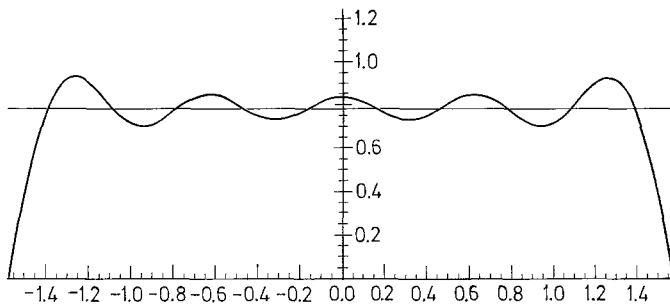


Fig. 14

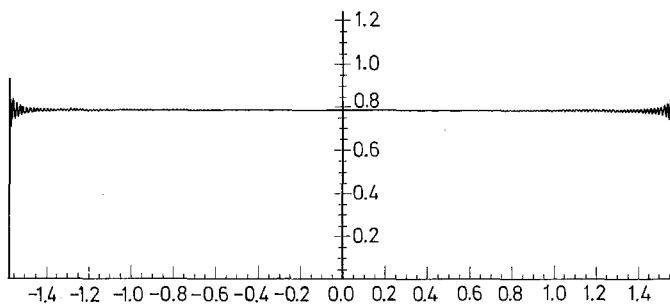


Fig. 15

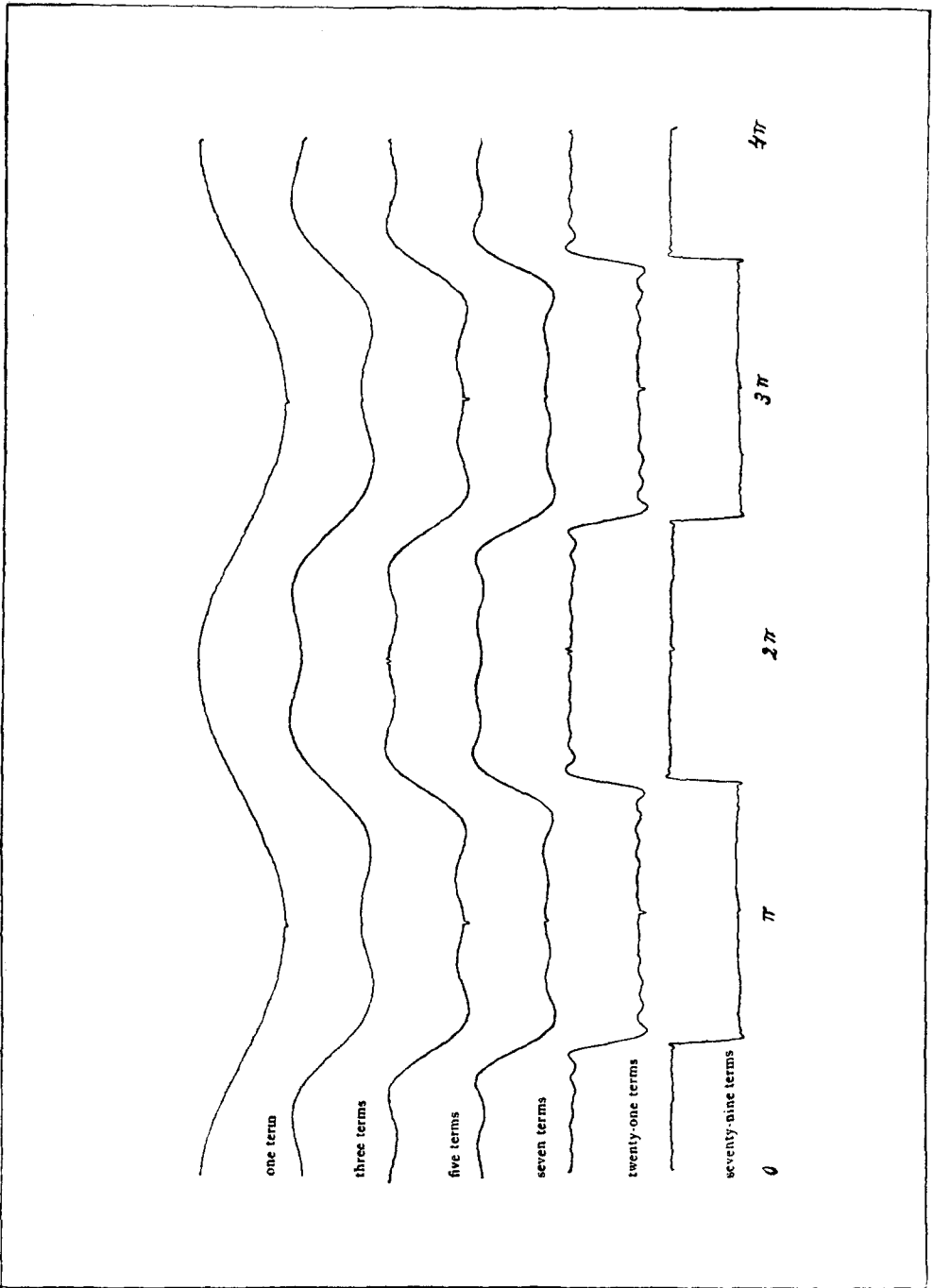


Fig. 16

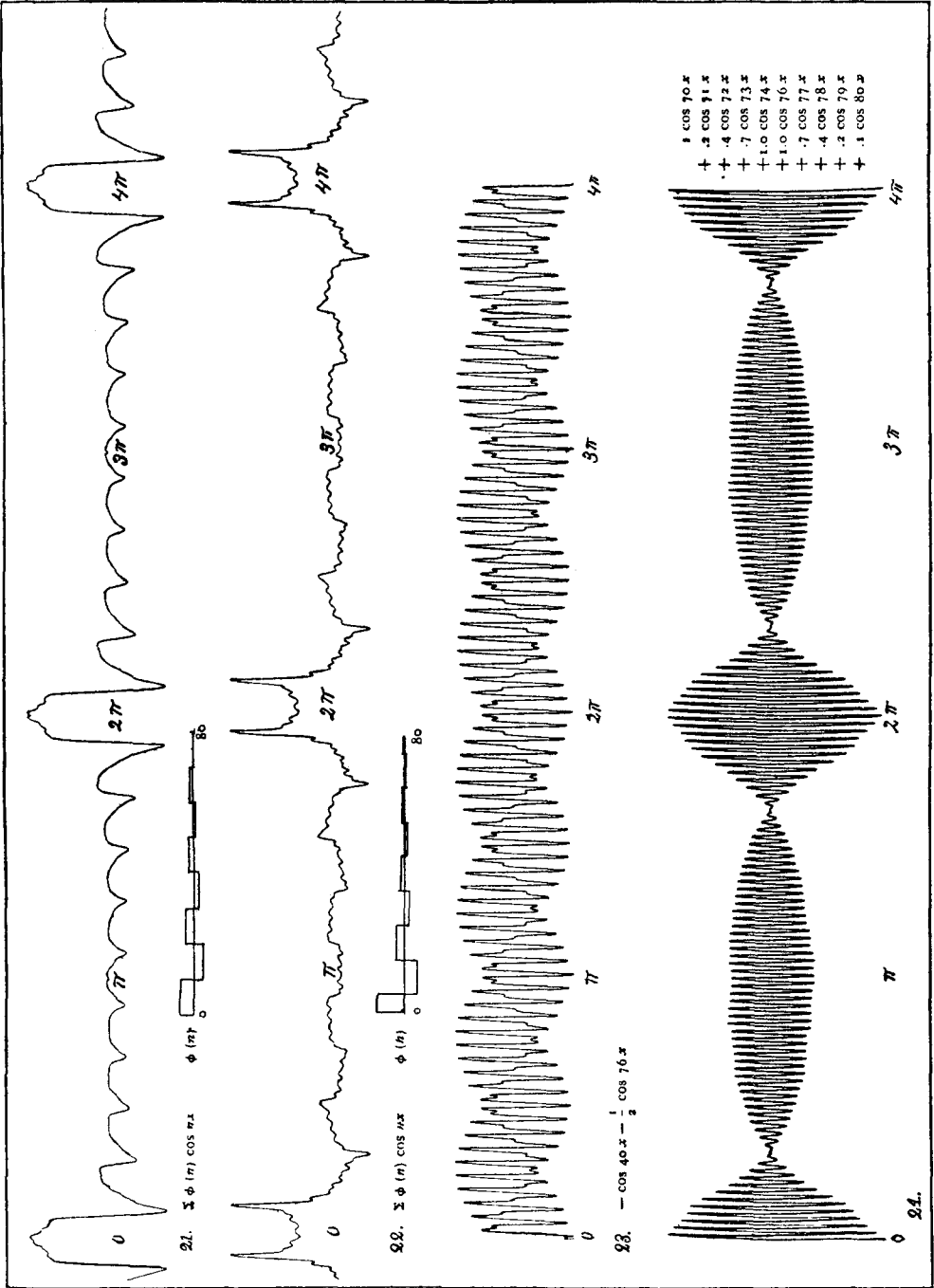


Fig. 17

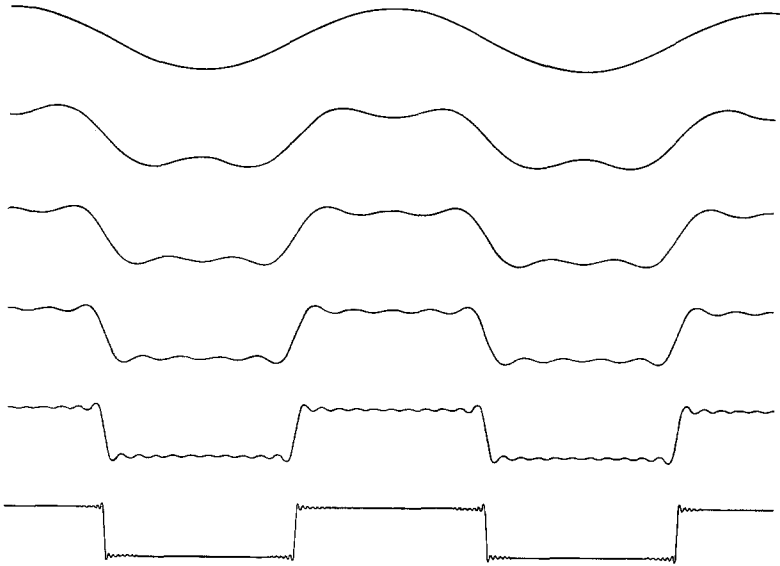


Fig. 18

Perhaps inspired by his work with the harmonic analyser, MICHELSON published a letter to *Nature* [35] in the same year, 1898, dealing with the series (3) and containing a criticism of the assertion that the series converges to $\frac{1}{2}x$ in the interval $-\pi < x < \pi$. This letter, as we shall see, was perhaps the original cause for GIBBS's interest in the matter. At any rate, MICHELSON was taken severely to task by LOVE in the very next issue of *Nature* [32]: one marvels at a golden age in which scientific publication took place within a week. LOVE's strictures on MICHELSON's comments were valid, but were stated very brusquely. MICHELSON essayed a mild, if still unclear, rejoinder to LOVE in December 1898 [36]. From the fact that 28 days elapsed between MICHELSON's writing in Chicago and the publication of his letter in London, one concludes that transatlantic mails in the days of coal-fired steamships were on a par with those of the jet age.

We now come to GIBBS's contributions to the subject, which seem to have been impelled by sympathy with MICHELSON. In a letter to *Nature* of December 29, 1898 [22], printed directly after MICHELSON [36], GIBBS mildly reproved LOVE and also clearly explained the difference between the function $\lim_{n \rightarrow \infty} \sum_{k=1}^n (-1)^{k+1} \frac{\sin(kx)}{k}$ and the curve consisting of slanting line segments connected with vertical line segments (these last shown dashed in Figure 2) to which he thought that the graphs of the functions $y = \sum_{k=1}^n (-1)^{k+1} \frac{\sin(kx)}{k}$ converge. As GIBBS wrote in [23], there can be a difference between the limit of the graphs and the graph of the limit.

The very same page of *Nature* bears a second letter from LOVE expatiating on GIBBS's letter and offering a grudging (or so it seems to us) apology to MICHELSON.

GIBBS, however, was not finished with the matter. In a second letter to *Nature* [23], published less than four months after his first, GIBBS refers to "a careless error" and "an unfortunate blunder" in [22]. He goes on to describe exactly the limit curve of the graphs of the functions $y = 2 \sum_{k=1}^n (-1)^{k+1} \frac{\sin(kx)}{k}$.

He describes this limit curve as being the curve sketched in Figure 19, the lengths of the vertical line segments being $4\text{Si}(\pi)$. This is exactly correct, of course, as we showed in Part I in our study of the series (2). GIBBS was clearly unaware of WILBRAHAM's paper of 1848. It is of some small interest that GIBBS gave no hint of a proof.

There were two final letters to *Nature* about the nature of the convergence of the series (3). POINCARÉ himself wrote a vigorous defense of MICHELSON [40] (which was submitted for publication by MICHELSON). POINCARÉ did not mention overshoot or undershoot, and his letter contains also a curious slip: he writes that $\int_0^\infty \frac{\sin(t)}{t} dt = \frac{1}{4}\pi$. LOVE [34] wrote a final rejoinder to POINCARÉ, and the pages of *Nature* disclose no more of this *scandalum magnatum*.

GIBBS's phenomenon received a thorough treatment seven years after the appearance of GIBBS's second letter, in a long, scholarly paper by BÔCHER [4], published in 1906. BÔCHER also introduced the term "GIBBS's phenomenon". He studied the function

$$\frac{1}{2}(\pi - x) - \phi_n(x) = \frac{1}{2}\pi - \int_0^x \frac{\sin((n + \frac{1}{2})t)}{2 \sin(\frac{1}{2}t)} dt,$$

and carried out an analysis of its behavior similar to, but not so detailed as, the analysis we set down in Part I for the functions ϕ_n . BÔCHER gave a complete proof of GIBBS's assertion, but paid scant attention to the undershooting of ϕ_n near zero. He also greatly extended GIBBS's assertion, as follows.

Theorem F. *Let f be a real-valued function on the real line \mathbb{R} with period 2π , and suppose that f and its derivative f' are both continuous except for a finite number of finite jump discontinuities in the interval $[0, 2\pi]$. Let $S_n(x)$ be the n^{th} partial sum of the FOURIER series of the function f , computed at the point x . The graphs of the functions $y = S_n(x)$ converge to curves as sketched in Figure 20: in this figure, a is a generic point of discontinuity of f . The vertical segments are of length $\frac{2}{\pi} \text{Si}(\pi) |f(a+0) - f(a-0)|$ and are centered at $\frac{1}{2}(f(a+0) + f(a-0))$.*

To prove BÔCHER's theorem from Theorem D, we need only to subtract out the discontinuity of f . Write $\phi(x)$ for the function $\sum_{k=1}^\infty \frac{1}{k} \sin(kx)$. Consider the function

$$f^*(x) = f(x) - \frac{f(a+0) - f(a-0)}{\pi} \phi(x - a).$$

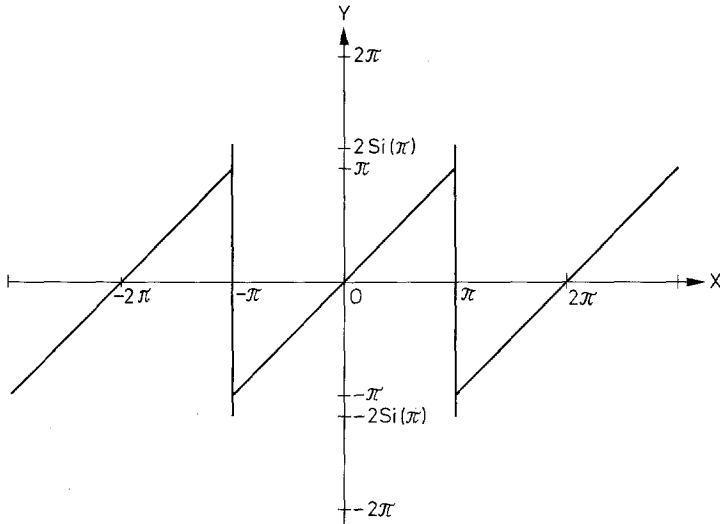


Fig. 19

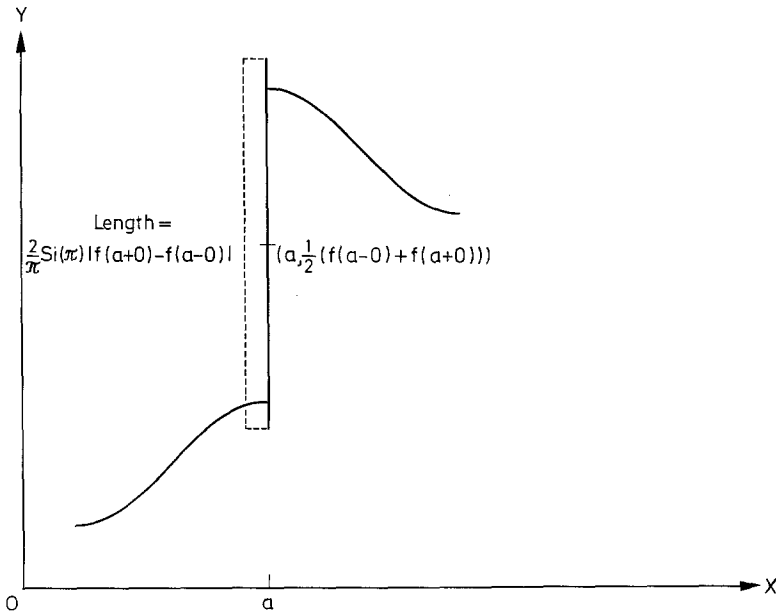


Fig. 20

It is clear that f^* is continuous in an interval containing a and that $f^*(a) = \frac{1}{2}(f(a-0) + f(a+0))$. The partial sums $S_n^*(x)$ of the FOURIER series of f^* converge uniformly to f^* in an interval containing a . (See for example ZYGMUND [49], Chapter II, Theorem (8.6), p. 58.) Thus the overshoot (and undershoot!) of the partial sums $S_n(x)$ are governed completely by the overshoot and

undershoot of the sums $\phi_n(x-a) = \sum_{k=1}^n \frac{\sin(k(x-a))}{k}$. We have

$$(33) \quad S_n(x) = S_n^*(x) + \frac{f(a+0) - f(a-0)}{\pi} \phi_n(x-a).$$

Applying Theorem D, we see from (33) that

$$(34) \quad \lim_{n \rightarrow \infty} S_n \left(a + \frac{\pi}{n+1} \right) = \frac{1}{2} (f(a-0) + f(a+0)) + \frac{\text{Si}(\pi)}{\pi} (f(a+0) - f(a-0)).$$

Since $S_n^*(x)$ converges uniformly in an interval containing a , the right side of (34) is the largest possible limit for sequences $S_n(t_n)$ such that $t_n \downarrow a$. Similarly we see that

$$(35) \quad \lim_{n \rightarrow \infty} S_n \left(a - \frac{\pi}{n+1} \right) = \frac{1}{2} (f(a-0) + f(a+0)) - \frac{\text{Si}(\pi)}{\pi} (f(a+0) - f(a-0)).$$

Theorem F follows from (34) and (35).

In 1911, DUNHAM JACKSON [29] published a study of the functions $\phi_n(x)$, proving that $\phi_n \left(\frac{\pi}{n+1} \right)$ is the largest value assumed by ϕ_n , that $\lim_{n \rightarrow \infty} \phi_n \left(\frac{\pi}{n+1} \right) = \text{Si}(\pi)$, and that $\phi_n(x) > 0$ for $0 < x < \pi$. He made no mention of GIBBS's phenomenon as such, or of the undershoot in the convergence of $\phi_n(x)$ to $\frac{1}{2}(\pi - x)$ near the point 0.

GRONWALL's paper [24], on which much of Part I is based, appeared in 1912, obviously independent of JACKSON [29]. (The dates of submittal are 12 March 1911 for [29] and 12 April 1911 for [24].) GRONWALL cites BÔCHER [4], albeit only in a footnote, and improves somewhat on BÔCHER's Theorem F, with the hypothesis that f be of finite variation and have only a finite number of discontinuities in any period interval.

In 1913, FEJÉR [19] took up the problem of determining $f(a+0) - f(a-0)$ from its Fourier series: a problem which, as he rightly stated, belongs to "the circle of the theorems that are related to the 'Gibbs' phenomenon'". His results, while interesting, are not properly part of this historical sketch. He studies the functions $\phi_n(x)$ and cites both GRONWALL [24] and JACKSON [29] in complimentary terms, but does not refer at all to BÔCHER.

BÔCHER quickly responded to both GRONWALL and FEJÉR, publishing the somewhat polemical article [5] in 1914. In this note, he describes his first acquaintance with GIBBS's letter [23] (EDWIN BIDWELL WILSON⁶ drew it to his attention) and remarks that RUNGE had also described GIBBS's phenomenon for a particular Fourier series. BÔCHER is mostly concerned in [5], however, with his priority for Theorem F, with criticizing GRONWALL, which he does with considerable spleen, and with explaining that FEJÉR's result of 1913 [19] follows from BÔCHER's paper of 1906 [4].

⁶ The last doctoral student of J. W. GIBBS.

The editors of *Journal für die reine und angewandte Mathematik* furnished FEJÉR with proof sheets of BÔCHER's paper [5], eliciting from FEJÉR a rejoinder to BÔCHER that appeared in the same issue of the journal and directly after [5]; this is item [20] in our bibliography. FEJÉR remarks that the editors had kindly furnished him with proof sheets of [5] and notes that BÔCHER was able, after the publication of [19], to derive some of the results of [19] from [4]. He goes on to challenge BÔCHER's other claims, and in fact politely calls BÔCHER a liar. For those interested in how a gentleman did this in 1914, we quote (translated from the German). "I could therefore with pleasure verify that *after* the publications of Herr T. H. Gronwall (1912) and myself (1913), certain questions can in fact today be handled with the greatest ease, for which however in the year 1906 every trace of a hint was lacking." (See the footnote on pages 48-49 of [20].)

All this time, from 1899 to 1914, WILBRAHAM's paper of 1848 apparently lay forgotten. In 1914, BURKHARDT's great Encyklopädie article [7] on the history of trigonometric series and integrals to 1850 appeared. BURKHARDT seems to have missed nothing in his study of the literature, and on page 1049 he describes in detail what WILBRAHAM had done two thirds of a century before.

In 1917, H. S. CARSLAW published a paper [8] repeating part of BÔCHER's and GRONWALL's work, evidently still in ignorance of BURKHARDT and WILBRAHAM, since he writes "and it is most remarkable that its (GIBBS's phenomenon) occurrence in Fourier's Series remained undiscovered till so recent a date."

In this paper, CARSLAW treats the functions $\psi_n(x) = 2 \sum_{k=1}^n \frac{\sin((2k-1)x)}{2k-1}$, which converge to $\frac{1}{2}\pi$ in $]0, \pi[$ and to $-\frac{1}{2}\pi$ in $] -\pi, 0[$. He repeats for the functions $\psi_n(x)$ GRONWALL's analysis of the functions $\phi_n(x)$. The functions ψ_n are somewhat simpler in fact than the functions ϕ_n . For example, for each fixed n , the maximum values of ψ_n decrease monotonically in the interval $]0, \frac{1}{2}\pi[$ and the minimum values of ψ_n increase monotonically in the same interval. Theorems A, B, and D are proved for the functions ψ_n just as in GRONWALL [24]. CARSLAW also proves Theorem F in the same generality as GRONWALL, using the functions ψ_n to eliminate the discontinuity.

Two historical notes published in 1925, by C. N. MOORE [38] and CARSLAW [9], point out in print that WILBRAHAM's work antedated GIBBS's by a half century. CARSLAW concludes that the term "GIBBS's phenomenon" is justified but that one should recognize WILBRAHAM's priority. So far as we know, no writers have done so between 1925 and 1972 (DYM & MCKEAN [18]).

GIBBS's phenomenon has been studied for expansions in BESSEL functions by WILTON [45], [46] and by COOKE [12], [14], [15]. HERMANN WEYL [42], [43] took up GIBBS's phenomenon for expansions in spherical harmonics of functions on the 2-sphere $x^2 + y^2 + z^2 = 1$. His results are beautiful but also very complicated, and cannot be described here. The papers contain several very interesting figures, showing the curious and complicated modes of convergence in the neighborhood of singularities. These papers would undoubtedly yield much of interest for expansions in other special functions if properly studied.

GIBBS's phenomenon appears for certain CESÀRO summation methods for certain Fourier series. This interesting fact was discovered by CRAMÉR [16].

Contributions were made by CARSLAW [10], COOKE [13], and finally by GRONWALL [25]. GRONWALL settled completely the question of what summation methods produce GIBBS's phenomenon, in an extraordinary piece of computation. It was one of his last papers. ZYGMUND [49], Chapter III, Section 11, pp. 110–112 gives CRAMÉR's version of the result.

Finally we must mention the confusion that has appeared—some of it very recent—over the size of the overshoot in GIBBS's phenomenon. As proved by BÔCHER (see Theorem F), the absolute value of the overshoot (above or below)

the right limit $f(a+0)$ is $\frac{1}{\pi} \text{Si}(\pi) |f(a+0) - f(a-0)|$. A number of writers have mistaken the number $\frac{1}{\pi} \text{Si}(\pi)$ for the number $W(1) = \frac{2}{\pi} \text{Si}(\pi)$ (Table 1), which as

we know is the supremum of the maximum values of $\frac{2}{\pi} \phi_n(x)$ in its entire domain.

Also, for some unaccountable reason, a number of writers have set down false values for $\text{Si}(\pi)$. In KNOPP [30], published in 1924, we find $W(1)$ listed on page 380 as 1.08940..., although KNOPP cites both GIBBS [23] and GRONWALL [24], where $W(1)$ appears correctly. The English translation of KNOPP [30], published in 1928, repeats the error. The value of $W(1)$ is correctly given as 1.17898... in the second English edition (1948).

In 1928, ZALCWASSER [47] listed $W(1)$ as 1.089....

ZYGMUND [48], published in 1935, lists $\frac{2}{\pi} \text{Si}(\pi)$ as 1.089490... (p. 180), the same value as in KNOPP *loc. cit.* p. 380. The second edition of ZYGMUND's great treatise [49], published in 1958, gives $\frac{2}{\pi} \text{Si}(\pi)$ as 1.179..., correct with rounding (p. 61).

HARDY & ROGOSINSKI [27], published in 1950, list $\text{Si}(\pi)$ as 1.71... on p. 36 (corrected in later reprintings), although the same authors several years earlier [26] listed $\frac{1}{\pi} \text{Si}(\pi)$ as 0.58..., which is correct.

HYLTÉN-CAVALLIUS [28] has given some interesting geometric methods for estimating finite trigonometric series, among them the function $\phi_n(x)$. He remarks in a footnote to page 15 that a number of authors have set down incorrect values of $W(1)$. Like us, he offers no explanation. His value for $W(1)$ is correct.

BARI [2], published in 1961, gives $W(1)$ as 1.17... on p. 126.

DAVIS [17], published in 1963, describes GIBBS's phenomenon on pp. 115–118. He estimates $\phi_n\left(\frac{\pi}{20}\right)$ for a few values of n , up to $n=32$, and states that the overshoot “would tend” to 8.9490...% of π , which is correct. He gives no proof of this.

DYM & MCKEAN [18], published in 1972, treat GIBBS's phenomenon on pp. 43–46. They list $\frac{2}{\pi} \text{Si}(\pi)$ as 1.089490+. Although they cite GIBBS [23], WILBRAHAM [44], and CARSLAW [9] and [11], they somehow did not get the integral correct.

Part III. Conclusions

As a coda should be, this Part of our essay is short. GIBBS's phenomenon, while not a fundamental part of mathematics, displays *in parvo* a number of central features of the development of mathematics. We find forgotten pioneers. We encounter shocking disputes over priority. We study brilliant achievements, some (like WEYL [42] and [43] and GRONWALL [25]) never properly appreciated. We discover a remarkable succession of blunders, which could hardly have arisen save through copying from predecessors without checking.

In short, GIBBS's phenomenon and its history offer ample evidence that mathematics, for all of its majesty and austere exactitude, is carried on by humans.

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