The Science of Juggling

Studying the ability to toss and catch balls and rings provides insight into human coordination, robotics and mathematics

by Peter J. Beek and Arthur Lewbel

To complete a delivery of munitions, a 148-pound man must traverse a high, creaking bridge that can support only 150 pounds. The problem is, he has three, one-pound cannonballs and time for only one trip across. The solution to this old riddle is that the man juggled the cannonballs while crossing. In reality, juggling would not have helped, for catching a tossed cannonball would exert a force on the bridge that would exceed the weight limit. The courier would in fact end up at the bottom of the gorge.

Though not practical in this case, juggling definitely has uses beyond hobby and entertainment. It is complex enough to have interesting properties and simple enough to allow the modeling of these properties. Thus, it provides a context in which to examine other, more complex fields. Three in particular have benefited.

One is the study of human movement and the coordination of the limbs. Another is robotics and the construction of juggling machines, which provides a good test bed for developing and applying principles of real-time mechanical control. The third is mathematics; juggling patterns have surprising numerical properties.

Juggling is an ancient tradition—the earliest known depiction is Egyptian, in the 15th Beni Hassan tomb of an unknown prince from the Middle Kingdom period of about 1994 to 1781 B.C.; however, the first scientific study we know of did not appear until 1903. At that time, Edgar James Swift published an article in the *American Journal of Psychology* documenting the rate at which some students learned to toss two balls in one hand. By the 1940s, early computers were being used to calculate the trajectories of thrown objects, and the International Jugglers Association was founded. The 1950s and 1960s saw a few scattered applications, mostly successors to Swift's work, which used juggling as a task to compare general methods of learning sensorimotor skills.

Finally, in the 1970s, juggling began to be studied on its own merits, as evidenced by events at the Massachusetts Institute of Technology. There, Claude E. Shannon created his juggling machines and formulated his juggling theorem, which set forth the relation between the position of the balls and the action of the hands. Seymour A. Papert and other researchers at Project MAC (which later became M.I.T.'s Artificial Intelligence Laboratory) investigated how people master the art of juggling, and the M.I.T. juggling club, one of the oldest organizations devoted to amateur juggling still in existence, was established. The 1980s witnessed the rise of the mathematics of juggling, as several workers developed a special kind of notation to summarize juggling patterns [*see box on page 94*].

With three balls, most neophytes attempt to juggle in the shower pattern (around in a circle), although the cascade pattern—in which the hands alternate throwing balls to each other, resulting in a figure eight—is far easier. It often takes just hours or days to learn to juggle three balls. But learning times can be weeks or months for four balls, months or years for five, as practitioners refine their sense of touch and toss.

The world record for the greatest number of objects juggled (where each object is thrown and caught at least once, known as a flash) is 12 rings, 11 balls or eight clubs. The types of juggled objects, or props, affect the number because they vary in correct orientation, in the difficulty of holding and throwing them and in the margin of error to avoid collisions.

Limits on the Learning Curve

The obvious physical constraints that affect mastery and limit the number of objects juggled arise from gravity—more specifically, Newtonian mechanics. Each ball must be thrown sufficiently high to allow the juggler time to deal with the other balls. The time that a ball spends in flight is proportional to the square root of the height of the throw. The need for either speed or height increases rapidly with the number of objects juggled.

Then there is human imperfection, leading to errors of both space and time. Juggling low leaves little room to avoid collisions and hence requires catching and throwing quickly, which can cause mistakes. Throwing higher provides more time to either avoid or correct mistakes but also amplifies any error. For throws of only a few meters, a devi-
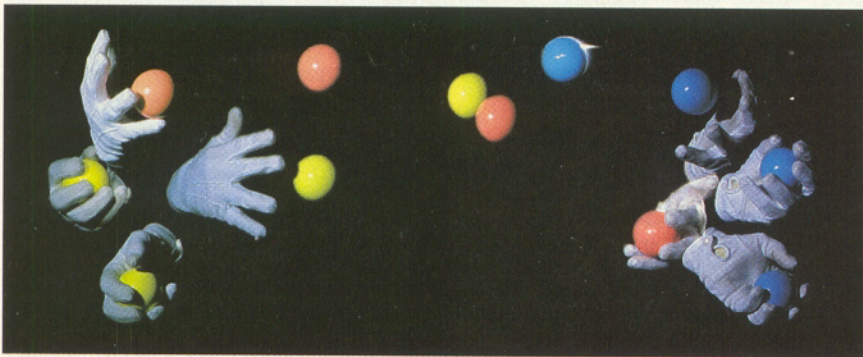
EXPERT JUGGLING relies more on the sense of touch and less on sight than does novice juggling. Hence, a professional can manage to juggle common patterns blindfolded for several minutes, from a simple exchange of balls (*opposite page*) to the four-ball fountain (*top left corner*) and the three-ball cascade (*above*).

EARLIEST DEPICTION OF JUGGLING known shows skillful Egyptian women on the 15th Beni Hassan tomb of an unknown prince from the Middle Kingdom period of about 1994 to 1781 B.C.

ation of just two or three degrees in the toss can cause an error in the landing location of 30 centimeters or more.

The temporal constraints on juggling are elegantly summarized by Shannon's theorem. It defines relations that must exist among the times that the hands are empty or full and the time each ball spends in the air. In other words, increasing the number of balls leaves less room to vary the speed of the juggle. If one were to juggle many balls to a certain height, the theorem indicates that even the smallest variation in tossing speed would destroy the pattern.

How jugglers coordinate their limbs to move rhythmically and at the same frequency within these constraints has become a primary focus in the study of human movement. Researchers have borrowed concepts from the mathematical theory of coupled oscillators [see "Coupled Oscillators and Biological Synchronization," by Steven H. Strogatz and Ian Stewart; SCIENTIFIC AMERICAN, December 1993].

The key phenomenon in coupled oscillation is synchronization: the tendency of two limbs to move at the same frequency. The particular type of coordination displayed by juggling hands depends on the juggling pattern. In the cascade, for instance, the crossing of the balls between the hands demands that one hand catches at the same rate that the other hand throws. The hands also take turns: one hand catches a ball after the other has thrown one.

The fountain pattern, in contrast, can be stably performed in two ways: by throwing (and catching) the balls simultaneously with both hands (in sync) or by throwing a ball with one hand and catching one with the other at the same time (out of sync). Theoretically, one can perform the fountain with different frequencies for the two hands, but that coordination is difficult because of the tendency of the limbs to synchronize.

Defining the physical and temporal constraints is one aspect of juggling analysis. A realistic model must also incorporate at least three other complicating factors. First, the oscillation of a juggling hand is not uniform, because the hand is filled with a ball during part of its trajectory and empty during the remaining part. Second, the movements of both hands are affected by the phys-

The Mathematics of Juggling

One useful method that many jugglers rely on to summarize patterns is site-swap notation, an idea invented independently around 1985 by Paul Klimek of the University of California at Santa Cruz, Bruce Tiemann of the California Institute of Technology and Michael Day of the University of Cambridge. Site swaps are a compact notation representing the order in which props are thrown and caught in each cycle of the juggle, assuming throws happen on beats that are equally spaced in time.

To see how it works, consider the basic three-ball cascade. The first ball is tossed at time periods 0, 3, 6..., the second at times 1, 4, 7..., and the third ball at times 2, 5, 8.... Site-swap notation uses the time between tosses to characterize the pattern. In the cascade, the time between throws of any ball is three beats, so its site swap is 33333..., or just 3 for short. The notation for the three-ball shower (first ball 0, 5, 6, 11, 12..., second ball 1, 2, 7, 8, 13..., third ball 3, 4, 9, 10, 15....) consists of two digits, 51, where the 5 refers to the duration of the high toss and the 1 to the time needed to pass the ball from one hand to the other on the lower part of the arc. Other three-ball site swaps are 441, 45141, 531 and 504 (a 0 represents a rest, where no catch or toss is made).

The easiest way to unpack a site swap to discover how the balls are actually tossed is to draw a diagram of semicircles on a numbered time line. The even-numbered points on the line correspond to throws from the right hand, the odd-numbered points to throws from the left.

As an example, consider the pattern 531. Write the numbers 5, 3 and 1 a few times in a row, each digit under the next point in the number line starting at point 0 [see top illustration at right]. The number below point 0 is 5, so starting there, draw a semicircle five units in diameter to point 5, representing a throw that is high enough to spend five time units (beats) in the air. The number below point 5 is a 1, so draw a semicircle

of diameter 1 from point 5 to 6. Point 6 has a 5 under it, so the next semicircle is from point 6 to 11. You have now traced out the path in time of the first ball, which happens to be the same as the first ball in the three-ball shower pattern 51 described above. Repeat the process starting at times 1 and 2, respectively, to trace out the path of the remaining two balls. The result is that the first and third balls both move in shower patterns but in opposite directions, and the second ball weaves between the two showers in a cascade rhythm. Leaving out this middle ball results in the neat and simple two-ball site swap 501.

Not all sequences of numbers can be translated into legitimate juggling patterns. For example, the sequence 21 leads to both balls landing simultaneously in the same hand (although more complicated variants of site-swap notation permit more than one ball to be caught or thrown at the same time, a feat jugglers call multiplexing).

Site-swap notation has led to the invention of some patterns that are gaining popularity because they look good in performances, such as 441, or because they are helpful in mastering other routines, such as the four-ball pattern 5551 as a prelude to learning to cascade five balls. Several computer programs exist that can animate arbitrary site swaps and identify legitimate ones. Such software enables jugglers to see what a pattern looks like before attempting it or allows them simply to gaze at humanly impossible tricks.
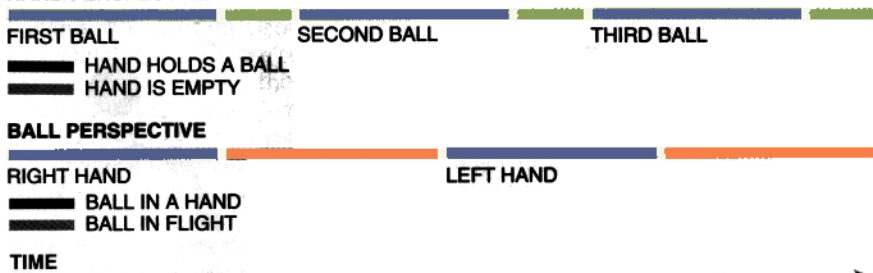
The strings of numbers that result in legitimate patterns have unexpected mathematical properties. For instance, the number of balls needed for a particular pattern equals the numerical average of the numbers in the site-swap sequence. Thus, the pattern 45141 requires $(4+5+1+4+1)/5$, or three balls. The number of legitimate site swaps that are n digits long using b (or fewer) balls is exactly b raised to the n power. Despite its simplicity, the formula was surprisingly difficult to prove.

ical demands of accurate throwing and catching. Third, the timing between the hands is based on a combination of vision, feel and memory.

These three factors render juggling patterns intrinsically variable: however solid a run, no two throws and no two catches are exactly the same. Analyzing this changeability provides useful clues about the general strategy of jugglers to produce a solid pattern that minimizes breakdown.

Variables associated with throwing (angle of release, release velocity, location of throws, height of throws) are those most tightly controlled: jugglers attempt to throw the balls as consistently as possible, the timing of which must obey Shannon's theorem. Given a height, a crucial measure of the rate of juggling is the so-called dwell ratio. It is defined as the fraction of time that a hand holds on to a ball between two catches (or throws). In general, if the dwell ratio is large, the probability of collisions in the air will be small. That is because the hand cradles the ball for a relatively long time and hence has the opportunity to throw accurately. If the dwell ratio

HAND PERSPECTIVE

FIRST BALL   SECOND BALL   THIRD BALL

▬ HAND HOLDS A BALL
▬ HAND IS EMPTY

BALL PERSPECTIVE

RIGHT HAND   LEFT HAND

▬ BALL IN A HAND
▬ BALL IN FLIGHT

TIME →

JUGGLING THEOREM proposed by Claude E. Shannon of the Massachusetts Institute of Technology is schematically represented for the three-ball cascade. The exact equation is $(F+D)H = (V+D)N$, where F is the time a ball spends in the air, D is the time a ball spends in a hand, V is the time a hand is vacant, N is the number of balls juggled, and H is the number of hands. The theorem is proved by following one complete cycle of the juggle from the point of view of the hand and of the ball and then equating the two.
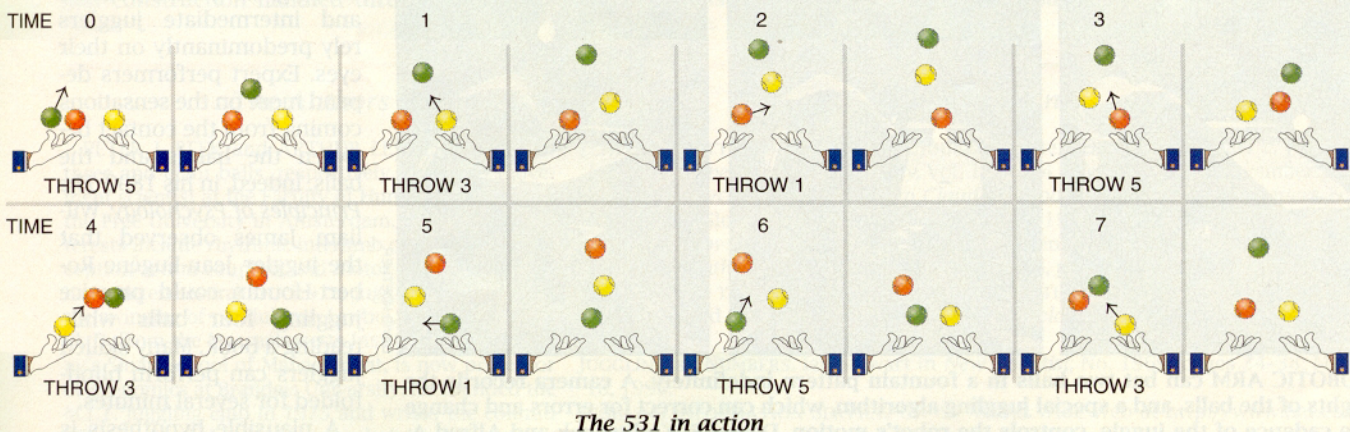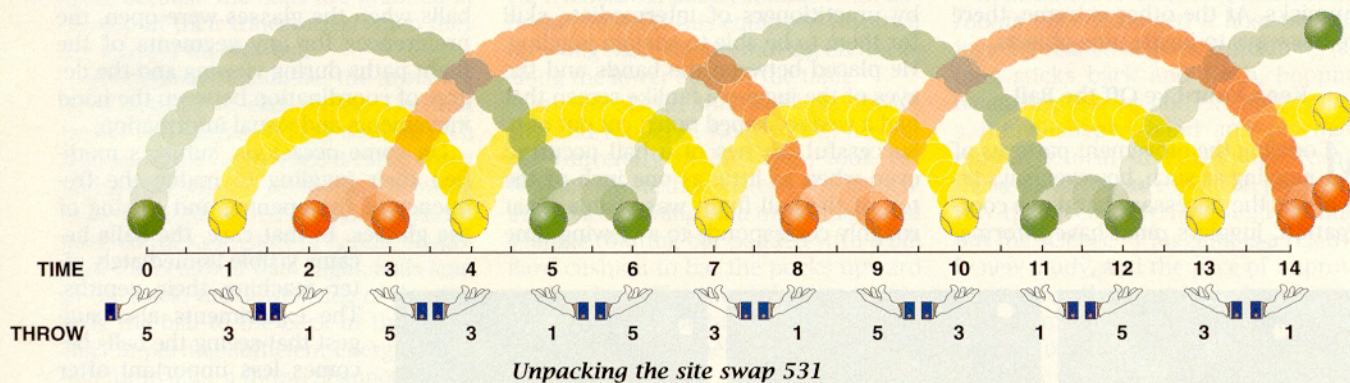
is small, the number of balls in the air averaged over time is large, which is favorable for making corrections, because the hands have more time to reposition themselves.

Novice jugglers will opt for larger dwell ratios to emphasize accurate tosses. More proficient jugglers tend toward smaller values, especially when juggling
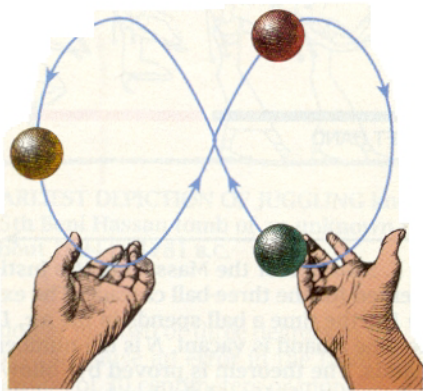
three balls, because of a greater flexibility to shift the pattern. Measurements by one of us (Beek) demonstrate that the ratio attained in cascade juggling changes roughly between 0.5 and 0.8, with ratios close to $3/4$, $2/3$ and $5/8$ most commonly observed. That is, in a three-ball cascade, the balls may spend up to twice as much time in the air as they

Site-swap theory does not come close to describing completely all possible juggling feats, because it is concerned only with the order in which balls are thrown and caught. It ignores all othe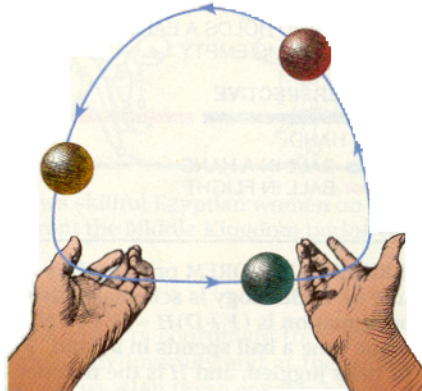r aspects of juggling, such as the location and style of throws and catches. Many of the most popular juggling tricks to learn, such as throwing balls from under the leg or behind the back, are done as part of a regular cascade and so have the same site-swap notation.

TIME	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
THROW	5	3	1	5	3	1	5	3	1	5	3	1	5	3	1

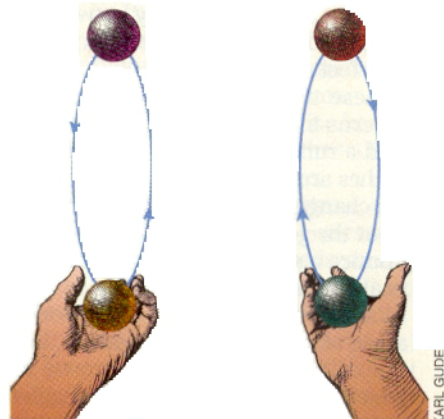*Unpacking the site swap 531*

The 531 in action

Common Juggling Patterns

THREE-BALL CASCADE

THREE-BALL SHOWER

FOUR-BALL FOUNTAIN ("IN SYNC")

KARL GUDE

do in the hands. Such a range suggests jugglers strike a balance between the conflicting demands of stability and flexibility, correcting for external perturbations and errors. Moreover, the tendency toward dwell ratios that are simple fractions subtly illustrates a human tendency to seek rhythmic solutions to physical tasks.

Juggling more than three balls leaves less room to vary the ratio, because the balls have to be thrown higher and, hence, more accurately. This fact greatly limits what jugglers can do. Juggling three objects allows ample opportunity for modification, adaptation, tricks and gimmicks. At the other extreme, there are few ways to juggle nine objects.

Keep Your Eye Off the Ball

Modeling the movement patterns of juggling as such, however, says little about the necessary hand-eye coordination jugglers must have. Informa-

tion about the motions of both the hands and the balls. There are few contexts in which the coaching advice to "Keep your eye on the ball" makes as little sense as it does in juggling. Attention must shift from one ball to the next, so that a juggler sees only a part of each ball's flight.

Which part is most informative and visually attended to? "Look at the highest point" and "Throw the next ball when the previous one reaches the top" are common teaching instructions. As a graduate student at M.I.T. in 1974, Howard A. Austin investigated how large a region around the zenith had to be seen by practitioners of intermediate skill for them to be able to sustain juggling. He placed between the hands and the eyes of the juggler a fanlike screen that had a wedge-shaped notch cut out of it. Successful catches of a ball occurred even when as little as one inch of the top of the ball flight was visible. That roughly corresponds to a viewing time

of 50 milliseconds, implying that briefly glimpsing the zeniths of the ball flights was sufficient to maintain a juggle.

In 1994 Tony A. M. van Santvoord of the Free University in Amsterdam examined the connection between hand movements and ball viewing in more detail. He had intermediate-level jugglers perform a three-ball cascade while wearing liquid-crystal glasses, which opened and closed at preset intervals and thus permitted only intermittent sightings of the balls. From the relation between the motion of the balls in the air and the rhythm defined by the glasses, one could deduce the location of the balls when the glasses were open, the preferences for any segments of the flight paths during viewing and the degree of coordination between the hand movements and visual information.

On some occasions, subjects modified their juggling to match the frequency of the opening and closing of the glasses. In that case, the balls became visible immediately after reaching their zeniths. The experiments also suggest that seeing the balls becomes less important after training. In general, novice and intermediate jugglers rely predominantly on their eyes. Expert performers depend more on the sensations coming from the contact between the hands and the balls. Indeed, in his 1890 *The Principles of Psychology,* William James observed that the juggler Jean-Eugène Robert-Houdin could practice juggling four balls while reading a book. Many skilled jugglers can perform blindfolded for several minutes.

A plausible hypothesis is that viewing the moving ball
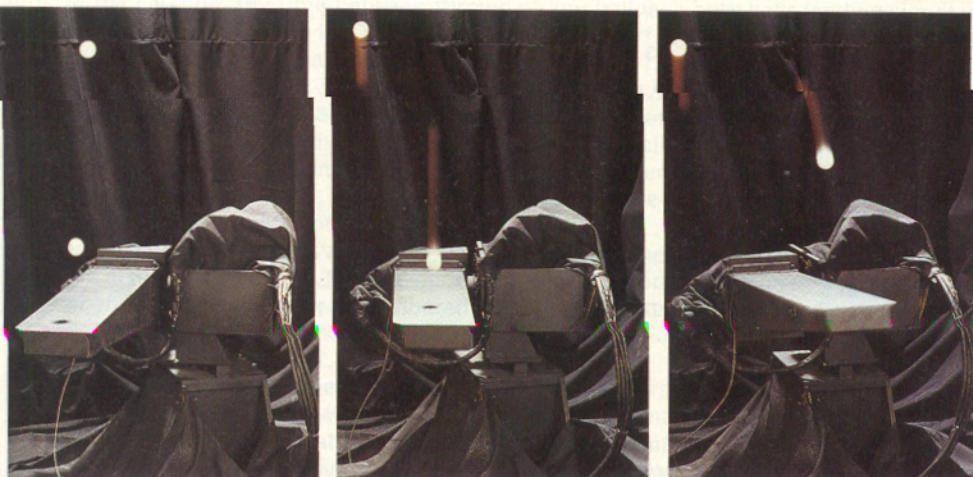
ROBOTIC ARM can bat two balls in a fountain pattern indefinitely. A camera records the flights of the balls, and a special juggling algorithm, which can correct for errors and change the cadence of the juggle, controls the robot's motion. Daniel E. Koditschek and Alfred A. Rizzi of the University of Michigan built the apparatus.

DAVID KOETHER

gradually calibrates the sense of touch in the course of learning. An expert immediately detects a slight deviation in the desired angle of release or in the energy imparted to the ball, whereas a novice has to see the effect of mistakes in the flight trajectories. As a consequence, the corrections made by an expert are often handled with little disturbance to the integrity of the pattern. Fixes made by less proficient jugglers, in contrast, often disrupt the overall stability of the performance.

Robots That Juggle

Insights into human juggling have led researchers to try to duplicate the feat with robots. Such machines would serve as a basis for more sophisticated automatons. Indeed, juggling has many of the same aspects as ordinary life, such as driving an automobile on a busy street, catching a fly ball on a windy day or walking about in a cluttered room. All these tasks require accurately anticipating events about to unfold so as to organize current actions.

Shannon pioneered juggling robotics, constructing a bounce-juggling machine in the 1970s from an Erector set. In it, small steel balls are bounced off a tightly stretched drum, making a satisfying "thunk" with each hit. Bounce juggling is easier to accomplish than is toss juggling because the balls are grabbed at the top of their trajectories, when they are moving the slowest.

In Shannon's machine, the arms are fixed relative to each other. The unit moves in a simple rocking motion, each side making a catch when it rocks down and a toss when it rocks up. Throwing errors are corrected by having short, grooved tracks in place of hands. Caught near the zenith of their flight, balls land in the track; the downswing of the arm rolls the ball to the back of the track, thus imparting sufficient energy to the ball for making a throw. Shannon's original construction handled three balls,

although Christopher G. Atkeson and Stefan K. Schaal of the Georgia Institute of Technology have since constructed a five-ball machine along the same lines.

Although the bounce-juggling robots are fiendishly clever, a robot that can toss-juggle a three-ball cascade and actively correct mistakes has yet to be built. Some progress, however, has been achieved. Machines that can catch, bat and paddle balls into the air have been crafted. Engineers have also built robots that juggle in two dimensions. In the

PROFESSIONAL JUGGLER Tony Duncan, whose skilled hands appear in the opening photographs, also handles pins.

1980s Marc D. Donner of the IBM Thomas J. Watson Research Center used a tilted, frictionless plane, similar to an air-hockey table. It was equipped with two throwing mechanisms moving on tracks along the lower edge of the table.

In 1989 Martin Bühler of Yale University and Daniel E. Koditschek, now at the University of Michigan, took the work a step further. Instead of a launching device running on a track, they used a single rotating bar padded with a billiard cushion to bat the pucks upward on the plane. To control the bar so as to achieve a periodic juggle, the researchers relied on the help of the so-called mirror algorithm.

This concept essentially combines

two ideas. The first is to translate (or "mirror") the continuous trajectory of the puck into an on-line reference trajectory for controlling the motion of the robot (via a carefully chosen nonlinear function). The advantage of this mirror algorithm is that it avoids the need to have perfect information about the state of the puck at impact, which is difficult to obtain in reality. The second idea, to stabilize the vertical motion of the puck, analyzes the energy of the ball to see if it matches the ideal energy from a perfect throw. Thus, the program registers the position of the puck, calculates the reference mirror trajectory, as well as the puck's actual and desired energy, and works out when and how hard the puck must be hit. With an extended version of the mirror algorithm, the robot can also perform a kind of two-dimensional, two-puck, one-hand juggle. It bats the pucks straight up, alternately with the left and the right part of the pivoting bar, in two separate columns.

Watching the mirror algorithm in action is spooky. If you perturb one of the pucks, the robot arm will make some jerky movements that look completely unnatural to human jugglers but result in a magically rapid return to a smooth juggle. The mirror algorithm cleverly controls batting, but it does not extend to the more difficult problem of juggling with controlled catches.

In addition to bouncing and batting, robots have managed a host of other juggling-related activities, including tapping sticks back and forth, hopping, balancing, tossing and catching balls in a funnel-shaped hand and playing a modified form of Ping-Pong. Despite these advances, no robot can juggle in a way that seems even passingly human. But the science of juggling is a relatively new study, and the pace of improvement over the past two decades has been remarkable. It may not be too long before we ask, How did the robot cross the creaking bridge holding three cannonballs?

The Authors

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Further Reading

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