VLADIMIR IGOREVICH ARNOLD
AND
THE INVENTION OF SYMPLECTIC TOPOLOGY

by

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Abstract. In 1965, with a *Comptes rendus* note of Vladimir Arnold, a new discipline, symplectic topology, was born. In 1986, its (remarkable) first steps were reported by Vladimir Arnold himself. In the meantime...

First step: a definition (1986)

First steps in symplectic topology, this was the (English) title of a 1986 paper [**14**] of Vladimir Igorevich Arnold. Like any good mathematical paper, this one started with a definition:

> By symplectic topology, I mean the discipline having the same relation to ordinary topology as the theory of Hamiltonian dynamical systems has to the general theory of dynamical systems.

And, to make things clearer, the author added:

> The correspondence here is similar to that between real and complex geometry.

Well... this was Arnold's style. A definition by analogy (an analogy I am not sure I understand clearly). Nobody could accuse him of formalism or, worse, of Bourbakism.

However, this paper was, is, "stimulating" (as the reviewer in *Math. Reviews* would write [(1)]). Its first part (after the provocative introduction), entitled "Is there such a thing as symplectic topology?", even contains a proof of the "existence of symplectic topology" (hence the answer to the question is yes), that the author attributed to Gromov in [**50**] (as he notes, Eliashberg also contributed to the statement, see below):

Theorem. *If the limit of a uniformly ($\mathcal{C}^0$) converging sequence of symplectomorphisms is a diffeomorphism, then it is symplectic.*

1. This one was Jean-Claude Sikorav.

No geometer would contest that such a statement is indeed a proof: this is a theorem about the behavior of symplectic diffeomorphisms with respect to the $\mathcal{C}^0$-topology; the terms of the sequence are defined via their first derivatives while the convergence is in the $\mathcal{C}^0$-topology. This indeed belongs to symplectic topology. Hence the latter is not empty.

But, whatever the credit Arnold decided to give to Gromov and Eliashberg in this article, symplectic topology existed twenty years before Gromov's seminal paper [**50**] appeared: symplectic topology has an official birthdate, and this is October 27$^{\text{th}}$, 1965.

In this paper, I plan to sketch a picture of how symplectic topology grew, in the hands of Arnold, his students, and followers, between his two papers [**3**] of 1965 and [**14**] of 1986.

Acknowledgment. I thank Bob Stanton and Marcus Slupinski for their help with the translation of the adjectives in footnote 17.

Many thanks to Alan Weinstein and Karen Vogtmann, who were so kind to send me recollections and information and also to Alan, for allowing me to publish an excerpt of a letter Arnold had sent to him.

I am very grateful to Mihai Damian, Leonid Polterovich and Marc Chaperon, who kindly agreed to read preliminary versions of this paper, for their friendly comments and suggestions.

The last sentence in this paper was inspired by [**61**].

October 27$^{\text{th}}$ 1965

This is the day when a short paper by Vladimir Arnold (so the author's name was spelled, see Figure 1), *Sur une propriété topologique des applications globalement canoniques de la mécanique classique*, was presented to the Paris Academy of sciences by Academician Jean Leray and became the *Comptes rendus* note [**3**].

TOPOLOGIE DIFFÉRENTIELLE. — *Sur une propriété topologique des applications globalement canoniques de la mécanique classique.* Note (*) de M. VLADIMIR ARNOLD, présentée par M. Jean Leray.

On utilise les inégalités de M. Morse, concernant le nombre de points critiques d'une fonction sur une variété, afin de trouver les solutions périodiques des problèmes de la mécanique.

FIGURE 1. A birth announcement (title and abstract of [**3**])

The so-called "applications globalement canoniques" would become symplectomorphisms, the topology was already in the title. Here are the statements of this note (my translation):

Theorem 1. *The tori T and AT have at least 2^n intersection points (counted with multiplicities) assuming that AT is given by*

$$p = p(q) \qquad \left|\frac{\partial p}{\partial q}\right| < \infty. \tag{7}$$

Here T is the zero section $p = 0$ in the "toric annulus" $\Omega = T^n \times B^n$ (with coordinates (q,p)) and the mapping $A : \Omega \to \Omega$ is globally canonical, namely, it is homotopic to the identity and satisfies

$$\oint_\gamma p\,dq = \oint_{A\gamma} p\,dq, \qquad (p\,dq = p_1\,dq_1 + \cdots + p_n\,dq_n)$$

for any closed curve (possibly not nullhomologous) γ.

Hence, Theorem 1 asserts that the image of the zero section in $T^n \times B^n$ under a certain type of transformations should intersect the zero section itself. We shall come back to this later. The second statement concerned fixed points. To this also we shall come back.

Theorem 2. *Let A be a globally canonical mapping, close enough to A_0. The mapping A^N has at least 2^n fixed points (counted with multiplicities) in a neighborhood of the torus $p = p_0$.*

Here, A_0 has the form $(q,p) \mapsto (q + \omega(p), p)$, for a map $\omega : B^n \to \mathbf{R}^n$ such that $\det\left|\frac{\partial \omega}{\partial p}\right| \not\equiv 0$, so that there exist $p_0 \in B^n$ and integers $m_1, \ldots, m_n, N$ with

$$\omega_1(p_0) = \frac{2\pi m_1}{N}, \ldots, \omega_n(p_0) = \frac{2\pi m_n}{N}$$

(this defining the p_0 and the N in the statement).

Remark A. *Replacing in the proofs the theory of M. Morse by that of L.A. Lusternik and L.G. Schnirelman, we obtain, in Theorem 1, $(n+1)$ geometrically different intersection points of T and AT. One could wonder whether there exist $(n+1)$ intersection points of T and AT for the globally canonical homeomorphisms A?*

Remark B. *The existence of infinitely many periodic orbits near a generic elliptic orbit follows from Theorem 2 (extension of Birkhoff's Theorem to $n > 1$).*

Remark C. *It is plausible that Theorem 1 is still true without the assumption (7), if A is a diffeomorphism*[(2)]*. From the proof, several "recurrence theorems" would follow.*

2. If A is not a diffeomorphism, counter-examples can be constructed with $n = 1$. Note of V.I. Arnold.

Remark D. *It also seems plausible that Poincaré's last theorem can be extended as follows:*

Let $A : \Omega \to \Omega$ $(\Omega = B^r \times T^n;\ B^n = \{p, |p| \leq 1\};\ T^n = \{q \bmod 2\pi\})$ *be a canonical diffeomorphism such that, for any* $q \in T^n$ *the spheres* $S^{n-1}(q) = \partial B^n \times q$ *and* $AS^{n-1}(q)$ *are linked in* $\partial B^n \times \mathbf{R}^n$ ($\mathbf{R}^n$ *being the universal cover of* T^n). *Then* A *has at least* 2^n *fixed points in* Ω *(counted with multiplicities).*

Remark C is the statement that will become "Arnold's conjecture". The question in Remark A will also be part of this conjecture. Note that, twenty years after, when he wrote [**14**], Arnold mentioned that the statement in Remark D had still not been proved.

Before I comment more on the statements and their descendants, let me go back to one of their ancestors, the so-called last geometric theorem of Poincaré.

A theorem of geometry, 1912

On March 7th 1912, Henri Poincaré finished writing a paper and sent it to the *Rendiconti di Circolo matematico di Palermo.* It was accepted at the meeting of the Mathematical circle which took place three days later (adunanza del 10 marzo 1912), together, *e.g.* with papers of Francesco Severi and Paul Lévy, and it was printed in May [(3)] as [**58**]. In this paper, Poincaré stated what he called "un théorème de géométrie". Before that, he apologized for publishing a result

– that he would have liked to be true, because he had applications (to celestial mechanics) for it,

– that he believed to be true, because he was able to prove some special cases of it

but that he could not prove. Here is this statement (my translation). Poincaré denotes by x and y (mod 2π for the latter) the polar coordinates of a point. He considers an annulus $a \leq x \leq b$ and a transformation T of this annulus $(x, y) \mapsto (X(x, y), Y(x, y))$.

First condition. As T transforms the annulus into itself, it preserves the two boundary circles $x = a$ and $x = b$. [He then explains that T moves one of the circle in a direction and the other in the opposite one. I shall (anachronistically) call this the twist condition.]

Second condition. The transformation preserves the area, or, more generally, it admits a positive integral invariant, that is, there exists a positive function $f(x, y)$, so that

$$\iint f(x,y)\,dx\,dy = \iint f(X,Y)\,dX\,dY,$$

the two integrals being relative to any area and its transform.

3. All of this was very fast, including the mail from Paris to Palermo (recall that there was no air-mail and that Palermo was already on an island). All the dates given here can be found on the printed journal. For some reason (which I was unable to understand), they were cut out in Poincaré's complete works, even the date he probably wrote himself at the end of his paper.

> *If these two conditions are satisfied, I say that there will always exist in the interior of the annulus two points that are not modified by the transformation.*

Clearly, the two conditions are necessary: there exists

– maps preserving the area without fixed points, a rotation for instance, but it does not satisfy the twist condition,

– twist maps without fixed points, *e.g.* (4) $(x, y) \mapsto (x^2, x + y - \pi)$, but it does not preserve the area.

Notice also that there exist twist maps preserving the area with exactly two fixed points, like the one evoked by Figure 2. The picture show a part of an infinite strip. The diffeomorphism is the flow of the vector field drawn. It descends to the quotient (by the integral horizontal translation) annulus where it has two fixed points.

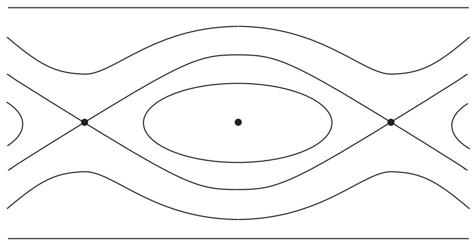

FIGURE 2. A twist map with two fixed points

Such area preserving maps of the annulus arose as Poincaré sections for Hamiltonian systems with two degrees of freedom—namely, in dimension 4—and their fixed points would correspond to periodic orbits. Needless to say: celestial mechanicians love periodic orbits. Hence the Poincaré problem.

Let me add that, in the introduction of his paper, Poincaré wrote that he had thought of letting the problem mature for a few years and then of coming back to it more successfully, but that, at his age, he could not be sure. He was actually only 58, but he died, unexpectedly, four months later, on July 17$^{\text{th}}$.

On October 26$^{\text{th}}$, the same year, George David Birkhoff presented a proof of this theorem to the American mathematical society, and his paper *Proof of Poincaré's geometric theorem* was published in the Transactions of this society **[30]**. Birkhoff considered himself as a student (and even as the last student) of Poincaré. He and Jacques Hadamard were probably the two mathematicians who knew Poincaré's work best. Although this was not as easy as it is nowadays, Birkhoff would go very often to Paris and lecture at Hadamard's Seminar, on Poincaré's theorems, during the 1920's and 1930's. The main reference in his paper was a previous paper of him **[29]**, published, in French, by the French mathematical society. No wonder that his proof of Poincaré's theorem was translated and republished, in French, as "Démonstration du dernier théorème de géométrie de Poincaré (5)" **[31]**.

4. I copied this example from **[54]**.
5. "Dernier", which means last, was not in the American title. Also, the translation kept the original phrasing "théorème de géométrie" rather than "théorème géométrique", as in English.

Note that, using a degree argument (that Poincaré attributed to Kronecker), the existence of one fixed point implies that there are two of them... except that they could coincide. It is not absolutely clear that the original proof of Birkhoff gave the existence of two geometrically distinct fixed points. This is why he himself came back to this theorem later. See his paper **[32]** and his book **[33]** (6).

For modern symplectic readers: there is a proof of the existence of one fixed point in **[54]**, which can be completed with **[36]**.

Chapter VI of Birkhoff's book is devoted to the application of Poincaré's geometric theorem. It starts as follows:

> Poincaré's last geometric theorem and modifications thereof (7) yield an additional instrument for establishing the existence of periodic motions. Up to the present time no proper generalization of this theorem to higher dimensions has been found, so that its application remains limited to dynamical systems with two degrees of freedom.

At that time, the symplectic nature of Hamilton's equations still needed some clarification. Now we know that the good generalization of "preserving the area" is not "preserving the volume". And Arnold was (one of) the mathematicians who taught us that. A Hamiltonian flow, namely a solution $(q(t), p(t))$ of Hamilton's equations

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

preserves the symplectic form

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$$

and not only the volume form

$$dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n = \frac{\omega^{\wedge n}}{n!}.$$

This is written in $\mathbf{R}^n \times \mathbf{R}^n$, but could also be understood in $T^n \times \mathbf{R}^n$ (if H is periodic in q), which is the same as $T^\star T^n$, hence can be generalized to $T^\star V$ (which has a "$p\,dq$" and thus also a "$dp \wedge dq$" form), and to any symplectic manifold W. To a function $H : W \to \mathbf{R}$, the symplectic form ω associates a vector field (the Hamiltonian vector field) X_H by $dH = i_{X_H}\omega$ and thus a flow (the Hamiltonian flow) which preserves ω since

$$\mathcal{L}_{X_H}\omega = di_{X_H}\omega = d\,dH = 0.$$

6. Note that, in the preface Marston Morse wrote for the 1966 edition of this 1927 book, he insisted on the relationship between Birkhoff's work on periodic orbits and "the work of Moser, Arnold and others on stability".

7. See my paper, An extension of Poincaré's last geometric theorem, Acta Mathematica, vol.47 (1926). Note of G.D. Birkhoff.

Back to Arnold and his golden sixties

In 1965, although he was a young man of 28, Arnold was not a beginner. Ten years before, he had contributed (with his master Kolmogorov, as he would say) to Hilbert's thirteenth problem. Then he had worked on stability and had already proved the theorem on invariant tori that would soon be known, first as "Kolomogorov-Arnold-Moser", and later as "KAM". This was what he lectured on when he came to Paris at the Spring of 1965, as the book **[16]** (8) shows (the "KAM" statement is Theorem 21.11 and there is a proof in Appendix 33). He had already published, for instance, the big paper **[1]** (9), about which the reviewer of *Math. Reviews* (10) wrote:

> It is to be hoped that this remarkable paper and exceptional work helps to arouse the interest of more mathematicians in this subject.

This might have been the first appearance of the famous cat of Arnold, and of a figure such as Figure 3 (11).

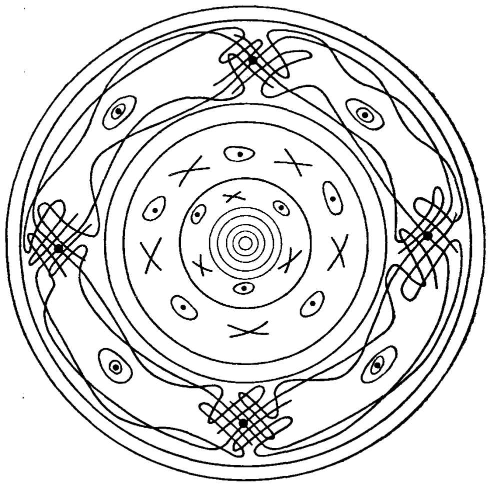

FIGURE 3. More fixed points... after **[1]**

Of course, KAM theorem was also the main topic of the half-an-hour talk Arnold gave at the ICM in Moscow in 1966, Проблема устойчивости и эргодические свойстба классических динамических систем (12) **[4]**. However, there was a short section with the statements of (and reference to) the note **[3]**.

8. Soon translated in English as **[17]**.
9. This was also very fast: the translation in English in *Russian mathematical surveys* would arrive in the libraries less than one year after the publication of the Russian original.
10. This one was Jürgen Moser.
11. Note that Figure 3 contains a 5-fold covering and a 3-fold covering of the map in Figure 2.
12. A stability problem and ergodic properties of classical dynamical systems

Problems of present day mathematics, 1974

In May 1974, the American mathematical society had a Symposium on developments arising from Hilbert problems. The organizers also intended to make another list of problems—for the present day. Arnold sent a problem (if I understand well, the problems were collected by Jean Dieudonné and edited by Felix Browder), which appeared in a list of "Problems for present day mathematics" in the book [**35**]. This is Problem XX, on page 66:

> **XX. Fixed points of symplectic diffeomorphisms (V. Arnold).** The problem goes back to the "last geometric theorem" of Poincaré. The simplest case is the following problem: *Does every symplectic diffeomorphism of a 2-dimensional torus, which is homologous to the identity, have a fixed point?*
>
> A symplectic diffeomorphism is a diffeomorphism which preserves a nondegenerate closed 2-form (the area in the 2-dimensional case). It is homologous to the identity iff it belongs to the commutator subgroup of the group of symplectic diffeomorphisms homotopical to the identity. With coordinates, such a diffeomorphism is given by $x \to x + f(x)$, where x is a point of the plane and f is periodic. It is symplectic iff the Jacobian $\det(D(x + f(x))/Dx)$ is identically 1, and it is homologous to the identity iff the mean value of f is 0.
>
> The "last geometric theorem" of Poincaré (proved by G. D. Birkhoff) deals with a circular ring. The existence of 2 geometrically different fixed points for symplectic diffeomorphisms of the 2-sphere is also proved (A. Shnirelman, N. Nikishin). In the general case, one may conjecture that the number of fixed points is bounded from below by the number of critical points of a function (both algebraically and geometrically).

The AMS book appeared two years later, in 1976. Notice that the "simplest" question is asked in dimension 2, but that the general case, at the very end of the text, seems to refer to an arbitrary symplectic manifold. The complicated definition of "homologous to the identity" given shows that Arnold was indeed thinking of a general symplectic manifold. Note that, according to a theorem Augustin Banyaga [**25**] would prove in 1980, and that Arnold would quote in [**14**] and in 1986, these are the Hamiltonian diffeomorphisms.

Also note there was already a proof available, and this was for the S^2 case: Arnold was working... and his students were working too. The very first symplectic fixed point theorem (after [**3**]) was that of N.A. Nikishin [**57**]—note that, although published in 1974, the paper was submitted to the journal as soon as November 1972:

> ***Theorem.*** *A diffeomorphism of S^2 which preserves the area has at least two geometrically distinct fixed points.*

Namely, at least as many as a function has critical points. The proof was not very hard: Nikishin proved that the index of a fixed point of such a diffeomorphism should be ≤ 1. But the Lefschetz number is 2. □

Arnold was working. For instance on singularity (or catastrophe) theory. One of the people he met in Paris in Spring 1965 was René Thom (this we know at least from [**16**] and from [**59**]), whose seminar he attended. Arnold was working. Starting a seminar on singularity theory in Moscow (13). Lecturing on classical mechanics in 1966–68. And writing up notes. Nikishin, in [**57**], quotes Arnold's *Lectures on classical mechanics*, dated 1968. They would become a famous book...

Mathematical methods in classical mechanics, 1974 (our golden seventies)

In 1974, the Soviet publishing house Наука published Arnold's Математические методы классической механики [**8**].

At that time, a wicked bureaucracy had decided not to allow Arnold to travel abroad anymore. However, his book was soon translated to French and published, in Moscow, by the foreign language Soviet publishing house Мир, Mir, and [**9**] was available in France, at a very low price, in 1976.

A few personal remarks. In the seventies, the only math books we could afford, we Parisian students, were the Mir books. We would go quite often to their bookstore *la Librairie du Globe* rue de Buci to fetch the new books (whatever they were). The Soviet translation program was devoted helping French-speaking developing countries, not French students. So what?

The word "translation" was already used at least seven times in this text. A French mathematician publishing a paper in French in an Italian journal, an American mathematician writing papers in French and in English, a Russian one writing in Russian and in French. Before I leave the language question, let me comment on that. When I visited Arnold in Moscow in the Fall of 1986, he told me that he preferred to speak French than English, so we used to discuss in French. Of course, he asked me to lecture in English, because of his students. So I spoke English... but, he would interrupt quite often to ask a question or make a comment (well, this was Arnold's seminar, you know (14)), and, of course (?) he would do it in French, then I would answer (or not), and he would translate and comment in Russian, for his students (15). And of course, I would try to understand the comment: I knew perfectly well that he was explaining things I was talking about but did not quite understand (16). Arnold's fast, intricate and subtle questions (17), plus two foreign languages at the same time—hard work!

⋆

13. Let me mention here the beautiful little book [**13**] he wrote on this subject for a general audience in the eighties.
14. If you don't know, look at [**59**].
15. Again, you should read [**59**].
16. In any case, you should read [**59**].
17. Let me quote what I wrote at the very moment I learned his death in a short online paper [**23**]: he was charming, provocative, brilliant, cultured, funny, caustic sometimes even wicked, adorable, quick, lively, incisive, yes, all this together.

There and then (I mean in **[9]** and in 1976), we discovered, after the Newtonian and the Lagrangian mechanics, the third part of the book, Hamiltonian mechanics (18), symplectic manifolds and action-angle variables, notably. So, mechanics was, after all, geometry! Good news! And you could put so much mathematics in a series of so-called "appendices".

The symplectic community

Two years later, Springer published a translation in English, by Karen Vogtmann and Alan Weinstein (19) **[10]**. In a letter to Alan Weinstein, Arnold complained:

> There is something wrong with the occidental scientific books editions: the prices are awful. e.g. my undergraduate ordinary differential equations textbook (20) costs here 0,67 rbls ($\sim$ 1/30 the price of a pair of boots), and 40 000 exemplairs where sold in few months, so it is impossible to buy it at Moscow at present; the MIT Press translation by Silverman price was perhaps more than 20$ and the result – 650 sales the 1 year.
>
> Now the 17 000 exemplaires of the "mechanics" disappeared here at few days, the price being rbls 1,10. I think the right price for the translation must be less than 1$, then the students will buy it.

As Weinstein pointed out in his answer, books were unsubsidized in the U.S. economy. And, as it could be added, scientific publishers were not non-profitmaking organizations.

The English translation appeared. This time, this was no longer a short *Comptes rendus* note in French, a cheap translation made in the Soviet Union for developing countries or a paper in Russian. You (or your library) had to pay to read it. For instance, Helmut Hofer **[52]** would remember:

> As a student I read Arnold's wonderful book Mathematical Methods of Classical Mechanics.

After the AMS volume **[35]** and the Springer book **[10]**, nobody in the West could ignore Arnold's question! It was more or less at the same time that Gromov emigrated (21), first to the States, then to Paris. Thirteen years after, things started to become serious (22).

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18. Nothing is perfect. One thing I never understood and never dared to ask, is why there is a Lagrangian but no Hamiltonian treatment of the spinning top in this book.

19. It seems that the idea was Jerry Marsden's. The translation was made by Karen Vogtmann and edited by Alan Weinstein, who knew the domain and its lexicon better.

20. This one was **[5, 6, 7]**, before becoming **[15]**.

21. Mikhail Gromov's paper **[49]** (at ICM Nice 1970), where the h-principle for Lagrangian immersions was announced, should also be mentioned.

22. *Math. Reviews* waited until May 1979 to publish a review of the 1974 Russian edition. The reviewer was very enthusiastic, so enthusiastic that he added a very elegant remark:

> The reader should be aware that the reviewer participated in the English translation of the work under review, and so has been prejudiced in favor of the book by the pleasure which that project provided.

This one was Alan Weinstein.

In Appendix 9 of [**10**], one can read:

> Thus we come to the following generalization of Poincaré's theorem:
>
> ***Theorem.*** *Every symplectic diffeomorphism of a compact symplectic manifold, homologous to the identity, has at least as many fixed points as a smooth function on this manifold has critical points (at least if this diffeomorphism is not too far from the identity).*

Quoting Hofer again [**52**]:

> The symplectic community has been trying since 1965 to remove the parenthetical (23) part of the statement. After tough times from 1965 to 1982, an enormously fruitful period started with the Conley-Zehnder theorem in 1982–83.

It is not absolutely clear to me that there existed a symplectic community in the "tough times from 1965 to 1982". I may be wrong, so I will not insist on the precise date, but I would say that the "symplectic topology community" was born around 1982. So far, I have mainly mentioned Arnold (24) (and the Soviet Union). But there were indeed mathematicians working on celestial mechanics and stability questions elsewhere. The names of Marston Morse (who had been a student of Birkhoff) and Jürgen Moser have already been written in this paper. That of Michel Herman should be added. This would be connected to KAM rather than to actual symplectic geometry (25). Working on periodic orbits in the States and in the 1970's, Alan Weinstein not only solved problems [**63, 64**], but wrote a series of lectures [**62**], on symplectic geometry, which have also been quite useful. If I were to qualify all this activity in only two words, I would probably say "variational methods".

Well, another side of the story I have told so far, which also starts with the Poincaré-Birkhoff theorem and also ends with Weinstein's lecture notes, but is quite different—and complementary—is given in [**54**, p. 2].

There were some connections. Of course the name of Alan Weinstein must be repeated here. I should add that what we did not learn in [**9**], we learned it in [**62**].

However, it is around the Arnold conjecture (as it was named since then) that a community began to aggregate, and, if we needed a birthdate for this community, I would agree with Hofer and suggest March 1983, when Charles Conley and Eduard Zehnder sent their paper [**40**] to *Inventiones mathematicae.* This was soon reviewed by Marc Chaperon for the Bourbaki Seminar in Paris [**37**] (26). In this "report", Chaperon added a few personal (and new) ideas and results, in particular, he proved the non-displacement property for tori. At the same time, Daniel Bennequin [**26**] had succeeded in attacking the contact side of the story... and Gromov developed solutions of an elliptic operator, pseudo-holomorphic curves—the powerful new tool.

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23. The French translation has no parenthesis, only a comma.
24. and Gromov
25. Not taking Moser's homotopy method [**55**] (see also [**62**]) into account.
26. Replacing Fourier series by a broken geodesics idea, Chaperon himself soon gave a more elementary proof in [**39**], which is the basis of the proof given in [**54**].

Symplectic geometry/topology. I am not sure I can date the locution "symplectic-topology".

I shall not take sides in the question "what is symplectic topology/what is symplectic geometry?". For instance, where should I put the symplectic reduction process [**53**]? And the glorious convexity theorem of Atiyah, Gullemin and Sternberg [**19, 51**], which appeared more or less at the same time as [**40**]? In geometry? But topologists use it a lot... And what about deformation quantization, which originated—in the Soviet Union and in the seventies—in Berezin's work [**28**]?

Let me just say that Arnold was a geometer in the widest possible sense of the word, and that he was very fast to make connections between different fields.

One of Arnold's important symplectic texts was published shortly before the "first steps" of [**14**]. Written in collaboration with the young Sasha Givental, it was still called "Symplectic geometry" [**18**]. This was in 1985. The Soviet Union was still publishing cheap books, in this case volumes of an "Encyclopedia" (27). This is probably the best place to look at if you want to see the global idea Arnold had on the subject "symplectic geology-or-topometry". Note first that this is part of a series called "Dynamical systems". And then, let me make a list:

Well... integrable systems with the so-called Liouville Theorem (and the invariant tori some of which survive perturbations in KAM theory), Lagrangian and Legendrian submanifolds, caustics and wavefronts (and through generating functions, singularity theory, catastrophes and versal deformations), real algebraic geometry, the Maslov class (which he had defined in [**2**] (28) and which is related to Fourier integral operators), Lagrange and Legendre cobordisms (this turned out to be symplectic algebraic topology (29)), generating functions, and, yes, fixed points of symplectic diffeomorphisms.

Lagrangian submanifolds, statements of Arnold's conjecture

A Lagrangian in a symplectic manifold is a submanifold of the maximal possible dimension (which is half the dimension of the symplectic manifold) on which the symplectic form vanishes.

Sections of a cotangent bundle and fixed points. For instance, the zero section in a cotangent bundle $T^\star V$ is Lagrangian. Also the graph of a 1-form on V is Lagrangian if and only if this 1-form is closed. Notice, in connection with Theorem 1 in Arnold's note [**3**] (here page 3), that the graph of an exact 1-form df intersects the zero section precisely at the critical points of f.

Let us now consider a Hamiltonian diffeomorphism φ of $T^\star V$, that is, a diffeomorphism generated by a Hamiltonian vector field X_H. A version of the Arnold conjecture would be:

27. And this became one of the most expensive Springer series in the 1990's.

28. The contents of [**2**] would deserve a whole paper... Note that the adjectives Lagrangian, Legendrian, in the sense used in symplectic geometry, were invented in [**2**].

29. Allow me to mention that this was the way I entered symplectic geometry. See [**44**] and [**20, 21, 22**].

Conjecture. *The Lagrangian submanifold $\varphi(V)$ intersects the zero section V of $T^\star V$ at at least as many points as a function on V has critical points.*

Suppose that φ is $\mathcal{C}^1$-close to the identity. Then $\varphi(V)$ is a section of $T^\star V$. The fact that φ is symplectic implies that $\varphi(V)$ is Lagrangian and hence, the graph of a closed 1-form; the fact that φ is Hamiltonian implies that this is the graph of the differential of a function. Hence the result in this case. Note that the nondegenerate case, that is, when $\varphi(V)$ is transverse to V, is the case where the function is a Morse function. With the Morse inequalities, this leads to the weak (although nontrivial) form of the conjecture: the number intersection points is not less than the sum of the Betti numbers of V.

Now, according to a theorem of Weinstein [**62**], a tubular neighborhood of any Lagrangian submanifold L in any symplectic manifold is isomorphic (as a symplectic manifold) to a tubular neighborhood of the zero section in $T^\star L$. Generalizations of the statement above follow...

Graphs of symplectic diffeomorphisms. Another important class of examples is the following. Denote by W a manifold endowed with a symplectic form ω. Let $\varphi : W \to W$ be any map. Now, $W \times W$, endowed with $\omega \oplus -\omega$, is a symplectic manifold, and the graph of φ is a submanifold therein. Clearly, this is a Lagrangian submanifold if and only if $\varphi^\star\omega = \omega$, that is, if and only if φ is a symplectic diffeomorphism. And the intersection points of the graph with the diagonal are the fixed points of φ. Hence Lagrangian intersections are related to fixed points of symplectic diffeomorphisms.

Conjecture. *A Hamiltonian diffeomorphism of a compact symplectic manifold W has at least as many fixed points as a function on W has critical points.*

Generating functions

A connection between symplectic geometry and catastrophe theory is via generating functions. Remember that, if S is a function, the graph of dS is a Lagrangian submanifold of the cotangent bundle. Together with symplectic reduction, this has the following generalization (see [**62**]). Let $S : V \times \mathbf{R}^k \to \mathbf{R}$ be a function, so that the graph of dS is a Lagrangian submanifold in $T^\star(V \times \mathbf{R}^k)$. If this is transversal to the coisotropic submanifold $T^\star V \times \mathbf{R}^k$, the symplectic reduction process ensures that the projection

$$\operatorname{graph}(dS) \cap (T^\star V \times \mathbf{R}^k) \longrightarrow T^\star V$$

is a Lagrangian immersion. In coordinates $(q, a) \in V \times \mathbf{R}^k$, this is to say that, if

$$\Sigma_S = \left\{(q, a) \in V \times \mathbf{R}^k \mid \frac{\partial S}{\partial a} = 0\right\}$$

is a submanifold, then

$$\begin{aligned} \Sigma_S &\longrightarrow T^\star V \\ (q, a) &\longmapsto \left(q, \frac{\partial S}{\partial q}\right) \end{aligned}$$

is a Lagrangian immersion. For instance (with $V = \mathbf{R}^n$ and $k = 1$), if we start from

$$\begin{aligned} S : \mathbf{R}^n \times \mathbf{R} &\longrightarrow \mathbf{R} \\ (q,a) &\longmapsto a\,\|q\|^2 + \frac{a^3}{3} - a \end{aligned}$$

then

$$\Sigma_S = \left\{(q,a) \mid \|q\|^2 + a^2 = 1\right\} = S^n$$

is an n-sphere and

$$\begin{aligned} S^n &\longrightarrow \mathbf{R}^n \times \mathbf{R}^n = T^\star \mathbf{R}^n \\ (q,a) &\longmapsto (q, 2aq) \end{aligned}$$

is a Lagrangian immersion. Note that it has a double point ($q = 0$, $a = \pm 1$): this is a Lagrangian version of the "Whitney immersion".

Caustics and wave fronts. The geometric version of a wave front is as follows. Start with $L \subset T^\star V$, a Lagrangian in a cotangent bundle (it may be only immersed) and look at the projection $L \to T^\star V \to V$. Using "canonical" coordinates (q,p), we are just forgetting the p. The caustic is the singular locus in the projection.

Now comes the contact structure. We rather look at the jet space $J^1(V;\mathbf{R})$, that is, $T^\star V \times \mathbf{R}$, with the 1-form $dz - p\,dq$. As the 2-form $dp \wedge dq$ vanishes on L, the 1-form $p\,dq$ is closed, hence (up to a covering) it is exact, $p\,dq = df$ and, well, now we can "draw" L in $V \times \mathbf{R}$, namely in codimension 1 rather than n.

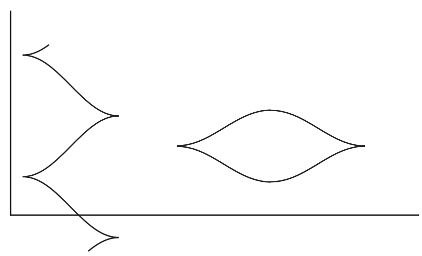

FIGURE 4. Two wave fronts

For instance if S is a generating function

$$\begin{aligned} \Sigma_S &\longrightarrow V \times \mathbf{R} \\ (q,a) &\longmapsto (q, S(q,a)) \end{aligned}$$

is the wave front of the Lagrangian immersion defined by S.

The pictures in Figure 4 represent (in coordinates (q,z)) a round circle and a figure eight (in coordinates (q,p)), the latter being the one-dimensional version of the Whitney immersion. Of course, only exact Lagrangians give closed wavefronts. Note also that any picture like the ones on Figures 4 or 5 would allow you to reconstruct a Lagrangian. Namely: knowing z and q, you get p by $dz = pdq$. For instance, to the two points with the same abscissa and horizontal tangents on the “smile” (right of Figure 4) correspond to the double point of the Whitney immersion.

Of course, this is related to the propagation, of light, say, this is related to evolvents, and to what Arnold calls “Singularities of ray systems” [**12**] and Daniel Bennequin the “Mystic caustic” [**27**].

So what? Well, this allowed Givental to construct examples of Lagrangian embeddings in $\mathbf{R}^4$ of all the surfaces which could have one, just by drawing them [**48**] in $\mathbf{R}^3$ ($4 = 2n \Rightarrow 3 = n+1$) (and leaving the Klein bottle case to posterity (30)).

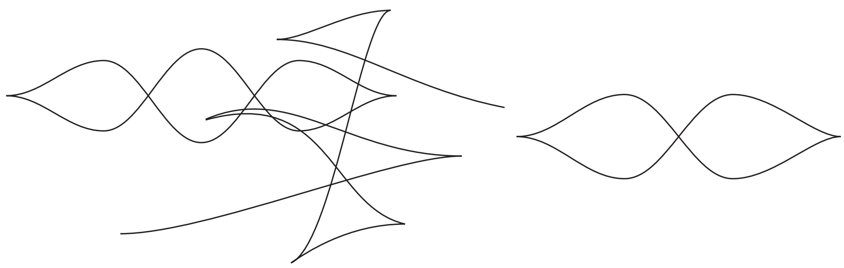

Figure 5

This also allowed Eliashberg to prove the Arnold conjecture for surfaces (31)—at the same time as Floer dit it. Eliashberg even had a proof [**41**] of the “existence theorem” of symplectic topology stated at the beginning of this article (see also [**42**]) using a decomposition of wave fronts.

Crossbows... The last wave front drawn (right of Figure 5) represents an exact Lagrangian immersion of the circle with two double points, which is regularly homotopic to the standard embedding (exactness meaning that the total area enclosed by this curve is zero). It appeared in Arnold’s papers on Lagrangian cobordisms [**11**]: this is the generator of the cobordism group in dimension 1. Arnold calls it “the crossbow”. Which reminds me of something Stein is supposed to have told Remmert in 1953 when he learned the use Cartan and Serre made of sheaves and their cohomology to solve problems in complex analysis: “The French have tanks. We only have bows and arrows” [**34**].

30. See [**56**].

31. Note that Nikishin’s article [**57**] quoted in [**35**] more or less disappeared from the literature. The statement and a (different) proof were given in [**37**] without any reference. A few years later the conjecture for $\mathbf{CP}^n$ was announced by Fortune and Weinstein [**47**] then published by Fortune [**46**] with no mention that the $\mathbf{CP}^1$-case was already known. Even in [**14**] the S^2-case is mentioned as an analogous of Poincaré’s geometric theorem, but not in connection with the proof of the conjecture for surfaces (attributed both to Eliashberg [**43**] and Floer [**45**]).

... and tanks. This time the tank was Floer theory. Well, we were not anymore in 1953. And the war metaphor is not the best possible to speak of the Floer Power...

The starting point was the action functional, like

$$\mathcal{A}_H(x) = \int_0^1 (p\,dq - H_t\,dt)$$

where $x(t)$ is a path and H_t a (time-dependent) Hamiltonian... except that we are on a general symplectic manifold, where $p\,dq$ does not mean anything. Well this can be arranged and replaced by a (closed) action form α_H, defined on a path x and a vector field Y along this path by

$$(\alpha_H)_x(Y) = \int_0^1 \omega_{x(t)}(\dot{x}(t) - X_{H_t}(x(t)), Y(t))\,dt.$$

The critical points are the solutions of the Hamilton equation. Once you have fixed a compatible almost complex structure, the gradient lines connecting the critical points are the solutions of the Floer equation:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad}\, H_t(u) = 0.$$

Note that, when $H_t \equiv 0$, this is just the Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0$$

giving Gromov's pseudo-holomorphic curves.

Taking in his hands both the variational methods (Morse theory) used by Conley and Zehnder and the elliptic operators (pseudo-holomorphic curves) of Gromov, using the "characteristic class entering in quantization conditions" of **[2]**, Andreas Floer built for us a Yellow-Brick-Road to prove the Arnold conjecture in greater and greater generality. (And this is what we [32] did.)

Generating functions (continuation)

From the very description of wave fronts, it is clear that generating functions are a good tool for the study of contact geometry/topology. Note also that there are contact analogues of self-intersections of Lagrangians, namely chords of Legendrian knots.

Much progress has been done, but there is not enough space here to mention all this. The name of another former student of Arnold's, Yuri Chekanov, should be added here.

32. By "we" here, I mean the community. I could also mention that some of us (and here, by "us", I mean the two authors of **[24]**) wrote a textbook to explain all this (a translation to English will be available soon).

Twenty years after... First steps again

Let us go back to the 1986 paper **[14]** we started with. Poincaré's geometric theorem was mentioned in the "Is there such a thing as symplectic topology?" section, but not its possible generalizations, which appeared only in Section 2, where, quoting **[3, 4]** for the statement and **[40]** for the proof, Arnold stated:

> ***Theorem.*** *A symplectomorphism of the torus homologous to the identity has no fewer than four fixed points (taking multiplicities into account) and no fewer than three geometrically distinct fixed points.*

Four was for 2^n, three for $n+1$, hence the torus in the statement was 2-dimensional—this was the case, neither for the conjecture nor for the proof... The "multidimensional generalization" was more than just multidimensional, and for it Arnold quoted the problem in **[35]**... and his comments to the Russian edition of Poincaré's selected works (33), a book I never saw:

> ***Conjecture.*** *A symplectomorphism of a compact manifold, homologous to the identity transformation* (34)*, has at least as many fixed points as a smooth function on the manifold has critical points.*

I think this was the first time the word "conjecture" (in reference to this problem) appeared in a paper by Arnold himself.

And he listed the results obtained so far—a state of the art in 1986. That is, the torus (**[40, 37, 38]**), the surfaces (**[43, 45]**), the complex projective space (**[47]**), (many) Kähler manifolds of negative curvature (**[45, 60]**), diffeomorphisms that are $\mathcal{C}^0$-close to the identity (**[65]**).

Epilogue (2012)

And now, this is 2012. Twenty-six years after the "first steps". Three new appendices have been added to a second (1989) edition of **[10]**. Some, many versions of Arnold's conjecture have been proved. Others are still open. Many powerful techniques have been created, used, improved. Even the crossbows turned out to be very efficient. Helping to solve old problems, the new tools generated new ones.

Vladimir Igorevich died in Paris on June 3rd, 2010.

Symplectic topology is not standing still.

33. See Review 52#5337 on *Math. Reviews.* Already in 1972, it was possible to publish double translations without checking the signification. The title of our favorite Poincaré paper **[58]** became there "A certain theorem of geometry".

34. Joined by a one-parameter family of symplectomorphisms with single valued (but time-dependent) Hamiltonians. Note of V.I. Arnold.

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