REDUCED MHD EQUATIONS FOR LOW ASPECT RATIO PLASMAS

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1. Introduction

Reduced descriptions of the MHD equations have a number of attractive features for theoretical and numerical calculations [1, 2, 3]. The goal of these descriptions is a reduced set of equations which embody the most salient physics of MHD stability properties in magnetized, toroidal plasmas. These reduced models eliminate the fast time-scale magnetosonic waves, which significantly constrain the computational speed of solving the full MHD equations but do not significantly contribute to instabilities. Strauss [1] introduced these models by reducing the MHD equations using the inverse aspect ratio of the torus as the expansion parameter. Hazeltine and Meiss [2] furthered the basic physics understanding of reduced MHD by giving a derivation using $k_\parallel/k_\perp$ as the expansion parameter, which was introduced as a means to eliminate the fast time scale associated with motions perpendicular to the magnetic field. Despite the success in heuristically explaining the fundamental physics of reduced MHD equations, the equations derived by Hazeltine and Meiss do not exhibit energy conservation or a divergence-free magnetic field to all orders. The goal of the present work is to derive a set of reduced equations that do not have these inadequacies and therefore are more suitable for nonlinear numerical simulations.

2. Fundamentals of reduced MHD equations

The ordering used is designed to look at modes whose wavelengths are small compared to the minor radius a (equilibrium scale length) and to the parallel wavelength, i.e, $\lambda_\perp/\lambda_\parallel \sim \epsilon$ and $\lambda_\perp/a \sim \epsilon$ where $\epsilon \ll 1$. The wavelength ordering is defined for directions relative to a large-scale magnetic field.

To formally obtain the wavelength ordering described, we order first-order quantities as $Q_1 = Q_1\left(\vec{x}_\perp/\epsilon, \vec{x}_\parallel, t_\perp/\epsilon, t_\parallel\right)$ where the notation $\vec{x}_\perp(\vec{x}_\parallel)$ denotes spatial dependence of perturbed quantities in the direction perpendicular (parallel) to the zeroth-order magnetic field, and $t_\perp$ and $t_\parallel$ are the time scales associated with their respective spatial scales. This ordering give $\vec{\nabla} Q_1 = \left(\frac{1}{\epsilon}\vec{\nabla}_\perp + \vec{\nabla}_\parallel\right) Q_1(\vec{x}_\perp, \vec{x}_\parallel)$, where $\vec{\nabla}_\parallel \equiv \hat{\mathbf{b}_0}(\hat{\mathbf{b}_0}\cdot\vec{\nabla})$ and $\vec{\nabla}_\perp = \vec{\nabla} - \vec{\nabla}_\parallel$. The expansion parameter is given by the anisotropy of the perturbed response; therefore the zeroth-order quantities are ordered simply as $Q_0 = Q_0\left(\vec{x}, t\right)$.

The MHD variables, $\rho, p, \vec{B}, \vec{V}, \Pi$, and Π_e, which are the plasma density, pressure, magnetic field, flow velocity field, total stress tensor, and electron stress tensor respectively, are ordered as described above. Note that no assumptions are made on the zeroth-order quantities *a priori* (i.e., it is not assumed that they satisfy the usual MHD equilibrium force balance). This derivation most significantly differs from previous derivations of reduced equations by explicitly retaining the zeroth-order and perpendicular time scales. They are kept here to elucidate the motions on these time scales as well as the desired $t_\parallel$ time scale. The stress tensor Π is the sum of both the ion and electron contributions and is ordered ϵ. The resistivity η and the electron stress tensor Π_e are ordered ϵ^2.

We apply this ordering to the MHD equations including the anisotropic stress tensor, but neglect heat flows. Taking $\mu_0 = 1$ we obtain in the lowest order

$$\begin{gathered} -\frac{\partial \rho_0}{\partial t} = \frac{\partial \rho_1}{\partial t_\perp} + \rho_0 \vec{\nabla}_\perp \cdot \vec{V}_1, \quad -\frac{\partial p_0}{\partial t} = \frac{\partial p_1}{\partial t_\perp} + \gamma p_0 \vec{\nabla}_\perp \cdot \vec{V}_1 \\ -\frac{\partial \vec{B}_0}{\partial t} = \frac{\partial \vec{B}_1}{\partial t_\perp} + \vec{B}_0 \vec{\nabla}_\perp \cdot \vec{V}_1, \\ -\vec{\nabla}(p_0 + B_0^2/2) + \left(\vec{B}_0 \cdot \vec{\nabla}\right) \vec{B}_0 = \rho_0 \frac{\partial \vec{V}_1}{\partial t_\perp} + \vec{\nabla}_\perp \left(p_1 + \vec{B}_0 \cdot \vec{B}_1\right) \end{gathered} \tag{1}$$

We assume that the zeroth-order quantities do not vary on the perpendicular time scale and assume wave-like solutions for the first-order quantities: $\rho_1, p_1, \vec{V}_1, \vec{B}_1 \sim e^{i\left(\vec{k_\perp} \cdot \vec{x}_\perp - \omega t_\perp\right)}$. A perpendicular time scale average is introduced as $\langle Q \rangle_{t_\perp} = \frac{1}{T_\perp} \int_0^{T_\perp} Q \, dt_\perp$ Using this averaging operator on Eq. (1) one can derive the longer time scale behavior of these leading-order equations. After taking this average, the terms on the right-hand side vanish and leave the left-hand side of Eq. (1), which are the exact MHD equilibrium equations.

The perturbed parts of Eqs. (1) can be seen to be the the leading order equations for fast magnetosonic waves justifying the wave-like behavior assumed. To eliminate the fast, perpendicular time scale in the equations (i.e., to obtain $\partial Q_1/\partial t_\perp = 0$), we choose

$$\vec{\nabla}_\perp \cdot \vec{V}_1 = O(\epsilon), \quad p_1 + \vec{B}_0 \cdot \vec{B}_1 = O(\epsilon) \tag{2}$$

as constraints on our equations, which must be satisfied to order ϵ. These constraints form the basis of the MHD equations reduction, and explicitly show the reduced MHD assumption corresponds to fast magnetosonic waves equilibrating to the ideal MHD equilibrium.

In the next order, we will have fast magnetosonic waves for the second-order quantities. Proceeding as before, we can eliminate these motions and are left with averages of first-order quantities over the fast ($t_\perp$) time scale. The remaining equations, which describe evolution on the $t_\parallel$ time-scale, will have 8 variables. Equations (2) and $\vec{\nabla} \cdot \vec{B} = 0$ introduce 3 constraints which leaves 5 fundamental variables. We now wish to derive 5 equations to evolve scalar variables on the $t_\parallel$ time-scale, which satisfy our constraints.

3. Derivation of reduced MHD equations

Before beginning the derivation of these equations, we note that we will be keeping lower-order terms in the derivation in order to satisfy energy conservation. For simplicity, the terms that are kept will not be shown explicitly.

Proceeding as described, the pressure equation becomes

$$\frac{dp_1}{dt_\parallel} + (\vec{V}_1 \cdot \vec{\nabla}) p_0 + \gamma p_T (\vec{\nabla}_\parallel \cdot \vec{V}_1) = (\gamma - 1) \left[\eta \left(J_{T_\parallel}^2 + \frac{\mid \vec{\nabla} p_T \mid^2}{B_0^2} \right) - \Pi : \vec{\nabla} \vec{V}_1 + \Pi_e : \vec{\nabla} \frac{\vec{J}}{ne} \right]. \tag{3}$$

Here, we have defined

$$\frac{d}{dt_{\|}}=\frac{\partial}{\partial t_{\|}}+(\vec{V}_1\cdot\vec{\nabla}),\quad \hat{\mathbf{b}}_T=\hat{\mathbf{b}}_0+\hat{\mathbf{b}}_1=\frac{\vec{B}_0}{B_0}+\frac{\vec{B}_1}{B_0},\quad p_T=p_0+p_1 \tag{4}$$

The density equation when ordered is similar to Eq. 3 in form:

$$\frac{d\rho_1}{dt_{\|}}+\rho_T\vec{\nabla}_{\|}\cdot\vec{V}_1=0 \tag{5}$$

Taking the parallel component of the momentum equation and ordering gives

$$\rho_T\frac{dV_{\|}}{dt_{\|}}=-\hat{\mathbf{b}}_0\cdot\vec{\nabla}p_1-\hat{\mathbf{b}}_1\cdot\vec{\nabla}p_T-\hat{\mathbf{b}}_0\cdot\vec{\nabla}\cdot\Pi \tag{6}$$

where $V_{\|}=\vec{V}_1\cdot\hat{\mathbf{b}}_0$.

To derive equations for the perpendicular components of the magnetic induction equation and the momentum equation, it is easier to recast the ordering process in terms of the electrostatic and magnetic potentials. These are given by $\Phi=\epsilon^2\phi$ and $\vec{A}=\vec{A}_0+\epsilon^2\vec{A}_2$ such that the electric and magnetic fields are $\vec{E}=\epsilon\left(-\vec{\nabla}_{\perp}\phi\right)+\epsilon^2\left(-\vec{\nabla}_{\|}\phi-\partial\vec{A}_2/\partial t_{\|}+\vec{E}^A\right)$ and $\vec{B}=\vec{\nabla}\times\vec{A}_0+\epsilon\vec{\nabla}_{\perp}\times\vec{A}_2$, where $\vec{E}^A$ is the applied electric field which is ordered to be consistent with the equilibrium Ohm's law.

We now look at projections of Ohm's Law. The first and second-order Ohm's Law perpendicular to $\vec{B}_0$ allows us to write

$$\vec{V}_1=\frac{\vec{B}_0\times\vec{\nabla}\phi}{B_0^2}+V_{\|}\hat{\mathbf{b}}_0-\frac{B_{\|_1}\hat{\mathbf{b}}_0\times\vec{\nabla}\phi}{B_0^2}+V_{\|}\hat{\mathbf{b}}_1+\eta\frac{\vec{\nabla}p_T}{B_0^2}-\frac{\vec{B}_0\times\frac{1}{ne}\vec{\nabla}\cdot\Pi_e}{B_0^2}. \tag{7}$$

The last four terms are lower-order, but are kept to satisfy energy conservation. Note that this equation satisfies our constraint $\vec{\nabla}_{\perp}\cdot\vec{V}_1=O(\epsilon)$ in Eq. (2). The component of Ohm's Law along $\vec{B}_0$ is written as

$$\frac{\partial\Psi}{\partial t_{\|}}-\hat{\mathbf{b}}_T\cdot\vec{\nabla}\phi=\eta\tilde{J}_{\|}-\frac{1}{ne}\hat{\mathbf{b}}_0\cdot\vec{\nabla}\cdot\Pi_e \tag{8}$$

where $J_{T_{\|}}=J_{\|_0}+\tilde{J}_{\|}=\vec{\nabla}\times\vec{B}_0+\vec{\nabla}_{\perp}\times\vec{B}_1$, $\Psi=-A_{\|}\equiv-\vec{A}_2\cdot\hat{\mathbf{b}}_0$ and $\eta J_{\|_0}$ has been cancelled with $\vec{E}^A\cdot\hat{\mathbf{b}}_0+\hat{\mathbf{b}}_0/(ne)\cdot\vec{\nabla}\cdot\Pi_{e0}$. Here, the perturbed current has been denoted with a tilde rather than a subscript 1 because it is of order unity.

In ordering the momentum equation, fast magnetosonic waves appeared in the lowest order. To eliminate these motions, we derive a vorticity equation from the quasineutrality condition, $\vec{\nabla}\cdot\vec{J}=0$. Using the momentum equation to find $\vec{J}_{\perp}$, we obtain

$$\begin{aligned}\vec{\nabla}\cdot\left(\frac{\rho_T}{B_0}\frac{d}{dt_{\|}}\frac{\vec{\nabla}\phi}{B_0}\right)&=(\vec{B}_T\cdot\vec{\nabla})\frac{\tilde{J}_{\|}}{B_0}+(\vec{B}_1\cdot\vec{\nabla})\frac{J_{\|0}}{B_0}\\&+\vec{\nabla}\cdot\frac{\vec{B}_0\times\vec{\nabla}p_1}{B_0^2}+\vec{\nabla}\cdot\frac{p_1}{B_0^2}\frac{\vec{B}_0\times\vec{\nabla}p_T}{B_0^2}+\vec{\nabla}\cdot\frac{\vec{B}_0}{B_0^2}\times\vec{\nabla}\cdot\Pi,\end{aligned} \tag{9}$$

where $\vec{B}_T=\vec{B}_0+\vec{B}_1$ and $\rho_T=\rho_0+\rho_1$.

Up to this point, $\vec{B}_1$ has only been defined by $\vec{B}_1 = \vec{\nabla}_\perp \times \vec{A}_2$. To make the perturbed magnetic field manifestly divergence free to all orders, it is necessary to keep a lower order term so that $\vec{B}_1 = \vec{\nabla} \times \vec{A}_2$. Expressing $\vec{A}_2$ in terms of two new scalar variables ψ (poloidal flux) and χ (toroidal flux), $\vec{A}_2 = -\psi\vec{\nabla}\zeta - \chi\vec{\nabla}\Theta$, will allow a closed set of equations if ψ, χ can be related to the variables we are evolving. Applying $\hat{\mathbf{b}}_0\cdot$ to the relation for $\vec{A}_2$, gives one equation $\Psi = (\mathcal{J}^{-1}/B_0)(q\psi + \chi)$, where $\mathcal{J}$ is the Jacobian of the straight-field-line magnetic flux coordinates ψ_0, Θ, ζ. These coordinates are based upon the axisymmetric equilibrium magnetic field $\vec{B}_0 = \left(\vec{\nabla}\zeta - q\vec{\nabla}\Theta\right) \times \vec{\nabla}\psi_0$, where ψ_0 is the poloidal flux, Θ and ζ are the poloidal and toroidal angles respectively, and $q = q(\psi_0)$ is the safety factor. The other needed equation comes from the constraint required to eliminate fast magnetosonic waves, $p_1 = -\vec{B}_0 \cdot \vec{B}_1$. To order ϵ, this is

$$\frac{\partial\psi}{\partial\psi_0} = -\frac{p_1}{B_0^2} + \frac{I}{B_0^2}\frac{\partial}{\partial\psi_0}(\Psi B_0) - \frac{qg^{\psi\Theta}}{IB_0^2}\frac{\partial}{\partial\zeta}(\Psi B_0) \tag{10}$$

where the toroidal flux function is $I = RB_{toroidal}$ and $g^{\psi\Theta}$ is the off-diagonal metric element.

Energy conservation is obtained by multiplying Eq. (9) by $-\phi$, Eq. (6) by $V_\parallel$, Eq. (8) by $J_{T_\parallel}$, adding the result to a manipulated from of Eq. (3), and then integrating over all space. It is also necessary when forming the energy integral, to keep only the leading order terms to $\tilde{J}_\parallel$: $\tilde{J}_\parallel = \nabla^2\Psi + O(\epsilon)$. With this definition of $\tilde{J}_\parallel$, one can form a divergence term that will cancel when integrated over all space. The (nonlinear) integral that is conserved is

$$\int d^3x \left(\frac{\rho_T V_\parallel^2}{2} + \frac{\rho_T \mid \vec{\nabla}\phi \mid^2}{2B_0^2} + \frac{\mid \vec{\nabla}\Psi \mid^2}{2} - J_{\parallel 0}\Psi + \frac{p_1}{\gamma - 1}\right). \tag{11}$$

This integral avoids the nonstandard conserved energy of the original reduced MHD derivation [1].

Our reduced MHD equations are Eqs. (3),(5), (6), (8), and (9). Because the aspect-ratio of the plasma is not used as an expansion parameter, these equations are valid for low-aspect ratio plasmas. It is also possible to self-consistently incorporate equilibrium flow profiles and to introduce neoclassical effects such as poloidal flow damping, polarization current enhancement, and bootstrap currents [3], which not included in resistive MHD. Further, linear layer physics calculations of these equations show that one can reproduce the same linear stability criterion as given in Glasser, Greene, and Johnson [4], who used the full MHD model.

References

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