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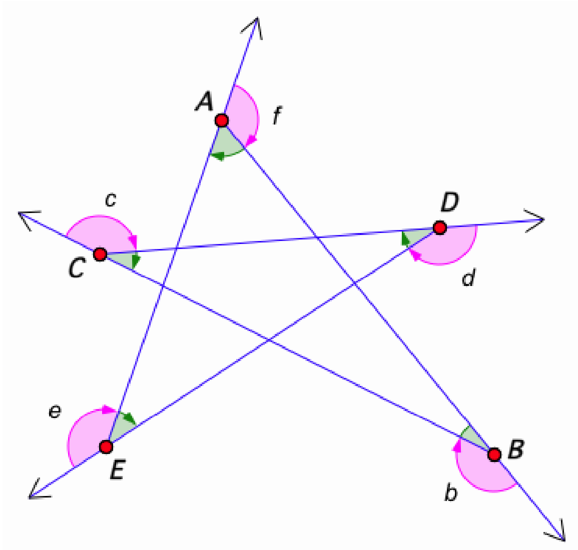
Stars: A Second Look

Star polygons as presented by Winicki-Landman (1999) certainly provide an excellent opportunity for students for investigating, conjecturing, refuting and explaining (proving). However, it could also be insightful to alternatively explain (prove) the results in terms of the *exterior angles* of the star polygons. It is also likely that students who have had a strong experiential background in LOGO (Turtle Geometry) would find this approach quite natural and easy (see Activity 1 in De Villiers, 2011).

For example, consider the star pentagon shown below. Imagine that one is a turtle (like in turtle geometry) starting from A, then walking along the perimeter from A to B, turning through the exterior angle b, then from B to C, turning through exterior angle c, etc. When one returns to A, turning through the exterior angle a, one again **faces in the same direction** one started off from. The total turning undergone is therefore two full revolutions (use a pen or pencil and consider the sum of the clockwise turns at each of the vertices), therefore: $a + b + c + d + e = 720°$.



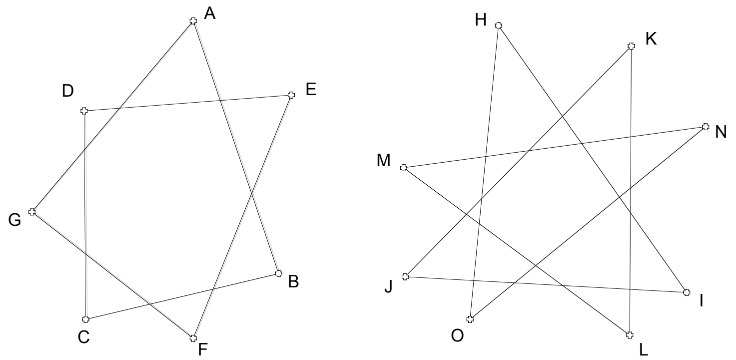
Since $a = 180° - \angle A$, etc., the interior angle sum can now easily be determined as follows:

$$(180° - \angle A) + (180° - \angle B) + (180° - \angle C) + (180° - \angle D) + (180° - \angle E) = 720°$$
$$\Leftrightarrow \angle A + \angle B + \angle C + \angle D + \angle E = 5 \bullet 180° - 4 \bullet 180° = 180°$$

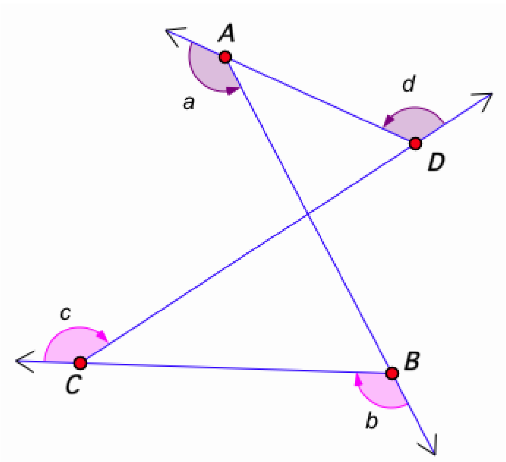
The value of this approach is that it is almost immediately generalizable to *any* closed polygon (of which star polygons are only a special case) as follows. In general, after walking completely around the perimeter of the polygon, one is facing in the same direction one has started off from, and therefore the *total turning* (sum of all the turning angles) must be a multiple of 360º, i.e. k.360º where k = 0; 1; 2; 3; etc. (This intuitively obvious result is called the Turtle Closed Path Theorem by Abelson & DiSessa, 1986, who also give a formal proof). The sum of the interior angles is now simply the difference between n.180º and the sum of the turning angles where n is the number of vertices, for example:

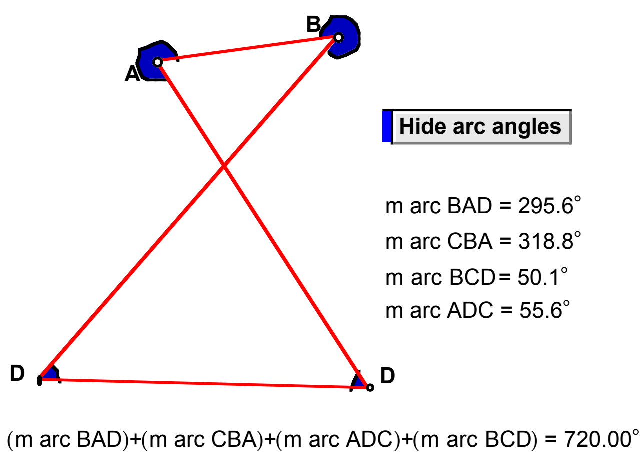
$$S = n \bullet 180° - k \bullet 360°$$
$$= 180°(n - 2k)$$

Note that k represents the total number of full revolutions of 360º one undergoes as one walks around the figure and turning at each vertex. Using this formula one can now easily determine the interior angle sums of the septagon and octagon shown below. In the first case, the value of k is 2 and in the second it is 3. The respective interior angle sums are therefore 540º and 360º.



This formula, however, becomes even more useful if one is working with more complicated closed polygons, such as a crossed quadrilateral as shown below. In this case, the total number of full revolutions k = 0, since one first undergoes two clockwise turns at B and C, but these are cancelled out by the two anti-clockwise turns at D and A respectively. The interior angle sum of the crossed quadrilateral is therefore 180º x 4 = 720º, and two of the "*interior*" angles are now reflexive and lie





"*outside*"! This is usually very surprising to children and adults alike, and often at first want to reject the notion. It is therefore a good example of the Lakatosian heuristic of 'refutation' and 'monster-barring' at an accessible level at school. (This is discussed, illustrated and developed in more detail as a learning activity in De Villiers (1999/2003) where arc measurements are used in *Sketchpad* for the crossed quadrilateral - see last figure above – and as also in De Villiers (2010), '*interior angles*' are formally defined in terms of the concept of '*directed angles*' so as to extend it in a consistent way to the interior angle sum of a crossed quadrilateral, and crossed polygons in general.)

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