# Nets and Filters

Notations and terms not defined here are as in [3]. **Ex** denotes either an example or an exercise. The choice is usually up to you the reader, depending on the amount of work you wish to do. Those which direct or expect you to verify something, however, should be done.

Most of the first three sections is adapted from [2]. The proof at the end of Section 3 is taken from [3].

## 1 Filters

**Definition 1** *A* filter *is a non-empty collection $\mathscr{F}$ of subsets of a topological space $X$ such that:*

1. $\varnothing \notin \mathscr{F}$;
2. *if* $A \in \mathscr{F}$ *and* $B \supseteq A$, *then* $B \in \mathscr{F}$;
3. *if* $A \in \mathscr{F}$ *and* $B \in \mathscr{F}$, *then* $A \cap B \in \mathscr{F}$.

*A maximal filter is also called an* ultrafilter.

**Ex** The set of all neighborhoods of a point $x \in X$ is a filter $\mathscr{N}_x$ called the *neighborhood filter* of $x$.

**Ex** The set of all sets in $X$ containing $x \in X$ is an ultrafilter $\mathscr{U}_x$.

**Definition 2** *A filter $\mathscr{F}$ is said to* converge *to $x \in X$, denoted by $\mathscr{F} \to x$, if and only if every neighborhood of $x$ belongs to $\mathscr{F}$.*

Defining a subfilter in the obvious way, this is equivalent to saying that $\mathscr{N}_x$ is a subfilter of $\mathscr{F}$.

**Ex** $\mathscr{N}_x$ and $\mathscr{U}_x$ both converge to $x$.

**Definition 3** *If $x \in \bar{F}$ for every $F \in \mathscr{F}$, we call $x$ a* cluster point *or an* accumulation point *of $\mathscr{F}$.*

**Proposition 4** *Any collection of sets $\mathscr{F}'$ satisfying the finite intersection property* (FIP) *is contained in an ultrafilter $\mathscr{U}$.*

**Proof:** Partially order by inclusion the set of all collections satisfying FIP which contain $\mathscr{F}'$. Then each chain has its union as an upper bound, so (by Zorn's Lemma) there exists a maximal element $\mathscr{U}$. We claim that $\mathscr{U}$ is a filter, hence an ultrafilter as it is maximal. Indeed, the first and third properties of a filter are true by definition, so we need consider only the second. But this follows from maximality. □

**Proposition 5** *A filter $\mathscr{U}$ is maximal if and only if every set $A$ that intersects every member of $\mathscr{U}$ (nonemptily) belongs to $\mathscr{U}$.*

The proof is left as an exercise. (Hint: use maximality each way.)

**Proposition 6** *If $\mathscr{F} \to x$, then $x$ is a cluster point of $\mathscr{F}$. Conversely, if $x$ is a cluster point of an ultrafilter $\mathscr{U}$, then $\mathscr{U} \to x$.*

**Proof:** If $\mathscr{F} \to x$, then $\mathscr{N}_x \subseteq \mathscr{F}$ by Definition 2. If $A \in \mathscr{F}$, then $A \cap N \neq \varnothing$ for every $N \in \mathscr{N}_x$ by property 3 of Definition 1. Thus $x \in \bar{A}$.

Conversely, if $x$ is a cluster point of an ultrafilter $\mathscr{U}$, then it follows from Definition 3 that each neighborhood $N$ of $x$ intersects (nonemptily) every member of $\mathscr{U}$. By the previous proposition, $N \in \mathscr{U}$. Therefore $\mathscr{N}_x \subseteq \mathscr{U}$ and $\mathscr{U} \to x$. □

As in analysis, it is often more convenient to work with inequalities than with equations.

**Definition 7** *A* filterbase *of a filter $\mathscr{F}$ is a subcollection $\mathscr{B}$ such that for every $F \in \mathscr{F}$ there exists $B \in \mathscr{B}$ with $B \subseteq F$.*

**Ex** Such a $\mathscr{B}$ satisfies

1. $\varnothing \notin \mathscr{B}$;
2. for every $B_1, B_2 \in \mathscr{B}$ there exists $B_3 \in \mathscr{B}$ with $B_3 \subseteq B_1 \cap B_2$.

**Ex** Conversely, given any collection $\mathscr{B}$ satisfying the two properties in the preceding **Ex**,

$$\mathscr{F} = \{F : F \supseteq B \text{ for some } B \in \mathscr{B}\}$$

is a filter with $\mathscr{B}$ as a filterbase. We say that $\mathscr{B}$ *generates* $\mathscr{F}$.

Thus we define a *filterbase* in general to be any collection $\mathscr{B}$ satisfying the two preceding properties, and may refer to the filter $\mathscr{F}$ that it generates.

**Definition 8** *A filterbase $\mathscr{B}$ in $X$ is said to* converge *to $x \in X$ if and only if for every neighborhood $N \in \mathscr{N}_x$, there exists $B \in \mathscr{B}$ with $B \subseteq N$. We write $\mathscr{B} \to x$ here also.*

**Ex** If $\mathscr{B}$ is a filterbase of $\mathscr{F}$, then $\mathscr{B} \to x$ if and only if $\mathscr{F} \to x$.

**Ex** Let $\mathscr{B} = \{(n, \infty) : n = 1, 2, 3 \ldots\}$ in $\mathbb{E}^1$. Then $\mathscr{B}$ is not a filter but is a filterbase of a filter $\mathscr{F}$. Show that neither $\mathscr{B}$ nor $\mathscr{F}$ converge.

**Ex** Let $p, q \in \mathbb{E}^2$ be distinct points and define $\mathscr{F}$ to be the set of all subsets of $\mathbb{E}^2$ which contain both $p$ and $q$. Show that $\mathscr{F}$ is a filter, $p$ and $q$ are cluster points of $\mathscr{F}$, but $\mathscr{F}$ does not converge.

## 2 Nets

**Definition 1** *A* directed set *is a set $\mathscr{D}$ with a relation $\leq$ on it satisfying*

1. *$\leq$ is reflexive;*
2. *$\leq$ is transitive;*
3. *$\leq$ is* directed: *for all $a, b \in \mathscr{D}$ there exists $c \in \mathscr{D}$ such that $a \leq c$ and $b \leq c$.*

**Definition 2** *A* net *in a set $X$ is a map $\lambda : \mathscr{D} \to X$. If $X$ is a topological space, we say that the net $\lambda$* converges *to $x \in X$ and write $\lambda \to x$ if and only if for every neighborhood $U$ of $x$ some* tail *$\Lambda_d = \{\lambda(c) : d \leq c \in \mathscr{D}\} \subseteq U$.*

Now let $\lambda : \mathscr{D} \to X$ be a net in $X$ with directed set $\mathscr{D}$.

**Definition 3** *We say that a net $\lambda$ is* eventually *in $A \subseteq X$ if and only if $A$ contains some tail of $\lambda$. We say that a net $\lambda$ is* frequently *in $A$ if and only if for every $d \in \mathscr{D}$ there exists $c \geq d$ such that $\lambda(c) \in A$.*

**Ex** $\lambda \to x \in X$ if and only if $\lambda$ is eventually in every neighborhood of $x$.

**Definition 4** *We say that $x$ is a* cluster point *or an* accumulation point *of a net $\lambda$ in $X$ if and only if $\lambda$ is frequently in every neighborhood of $x$.*

**Definition 5** *A net is called* maximal *if and only if for every $A \subseteq X$, it is eventually in either $A$ or $X - A$. Maximal nets may also be called* ultranets.

Nets are not as completely straightforward a generalization of sequences as one might wish. Consider the obvious naïve definition of a subnet $\lambda'$ of a net $\lambda : \mathscr{D} \to X$ as the restriction of $\lambda$ to a subset $\mathscr{D}'$ of $\mathscr{D}$.

**Ex** Let $\mathscr{D} = \{(n, m) : n, m = 1, 2, 3, \ldots\} \subseteq \mathbb{N} \times \mathbb{N}$. Define $(n, m) \leq (n', m')$ if and only if $n \leq n'$ and $m \leq m'$. Then $\mathscr{D}$ becomes a directed set. Consider the net

$$\lambda : \mathscr{D} \longrightarrow \mathbb{E}^2 : (n, m) \longmapsto \left(\frac{1}{n}, \frac{1}{m}\right).$$

Show $\lambda \to (0, 0)$. For $\mathscr{D}' = \{(n, 1)\} \subseteq \mathscr{D}$, show that the subnet $\lambda' = \lambda|_{\mathscr{D}'}$ neither converges to $(0, 0)$ nor has it as a cluster point.

Thus something more may be required of a subnet.

**Definition 6** *A subset $\mathscr{D}'$ of a directed set $\mathscr{D}$ is called* cofinal *in $\mathscr{D}$ if and only if for every $d \in \mathscr{D}$ there exists $d' \in \mathscr{D}'$ with $d \leq d'$. Given such a cofinal subset, we may refer to the* cofinal subnet *$\lambda' = \lambda|_{\mathscr{D}'}$ of any net $\lambda : \mathscr{D} \to X$.*

**Ex** $x$ is a cluster point of $\lambda$ if some cofinal subnet of $\lambda$ converges to $x$.

**Ex** Let $\lambda : \mathscr{D} \to X$ be a net and assume $\lambda \to x$. If $\mathscr{D}'$ is cofinal in $\mathscr{D}$, then the cofinal subnet $\lambda' \to x$ also.

But not all sequential properties carry over even so.

**Ex** Let $\mathscr{D}$ be the directed set of all countable ordinals with the usual ordering. For each $d \in \mathscr{D}$ there is a unique sequence $d_1 < d_2 < \cdots < d_n = d$ such that $d_1$ is a limit ordinal and each $d_{i+1}$ is the immediate successor of $d_i$; *i.e.*, $d_{i+1} = d_i + 1$. Define a net $\lambda : \mathscr{D} \to \mathbb{E}^1 : d \mapsto 1/n$ for the unique $n$ just determined. Show that 0 is a cluster point of $\lambda$, but that no cofinal subnet converges to 0.

This example shows that a more sophisticated notion of subnet is required, roughly something more than just cofinality. See [1, p. 70] for the general definition, and p. 77 for another counterexample due to R. Arens.

## 3 Convergence

It turns out that the concepts of convergence defined using nets and filters are equivalent. We shall now establish this rigorously.

Let $\mathscr{F}$ be a filter and let $\mathscr{D}$ be a set that is bijective with $\mathscr{F}$. We shall call $\mathscr{D}$ an *index set* for $\mathscr{F}$ and denote the bijective correspondance by subscript labeling: $\mathscr{F} = \{F_d : d \in \mathscr{D}\}$.

**Ex** Show that $\mathscr{D}$ becomes a directed set when partially ordered by containment: $c \leq d$ if and only if $F_c \supseteq F_d$.

In this case we speak of an *indexed* filter.

**Definition 1** *Let $\mathscr{F}$ be an indexed filter in $X$ with index set $\mathscr{D}$. Any net $\lambda : \mathscr{D} \to X$ with $\lambda(d) \in F_d$ is called a* derived net *of $\mathscr{F}$.*

**Definition 2** *Let $\lambda$ be a net in $X$ with directed set $\mathscr{D}$. Then*

$$\mathscr{F} = \{F \subseteq X : \lambda \textit{ is eventually in } F\}$$

*is called the* derived filter *of $\lambda$.*

**Ex** Verify that this $\mathscr{F}$ is indeed a filter.

**Theorem 3** *A filter $\mathscr{F}$ in $X$ converges to $x \in X$ if and only if every derived net $\lambda$ does.*

**Proof:** Assume $\mathscr{F} \to x$ and index $\mathscr{F}$ with an index set $\mathscr{D}$. If $N$ is any neighborhood of $x$, then $N = F_d \in \mathscr{F}$. Now, if $c \geq d$ then $F_c \subseteq F_d$, so $\lambda(c) \in F_c \subseteq N$ and $\lambda$ is eventually in $N$. Thus $\lambda \to x$.

Conversely, if $\mathscr{F} \not\to x$ then there exists some neighborhood $N$ of $x$ such that $F_d \neq N$ for every $F_d \in \mathscr{F}$. Choose any net $\lambda$ with $\lambda(d) \in F_d - N$ for each $d \in \mathscr{D}$. Then $\lambda$ is a derived net of $\mathscr{F}$, and does not converge to $x$. $\square$

**Theorem 4** *A net $\lambda : \mathscr{D} \to X$ converges to $x \in X$ if and only if the derived filter $\mathscr{F}$ does.*

**Proof:** If $\lambda \to x$, then $\lambda$ is eventually in each neighborhood $N$ of $x$. Thus each $N$ belongs to the derived filter $\mathscr{F}$, so $\mathscr{N}_x \subseteq \mathscr{F}$ and $\mathscr{F} \to x$.

Conversely, if the derived filter $\mathscr{F} \to x$ then each neighborhood $N$ of $x$ belongs to $\mathscr{F}$, so $\lambda$ is eventually in $N$. Therefore $\lambda \to x$. $\square$

Finally, we show that closure in any topological space is completely determined by convergence of nets or filters.

**Theorem 5** *Given a topological space $X$ and a subset $A$ of $X$, a point $x$ is in $\bar{A}$ if and only if there is a net (equivalently, a filter) in $A$ converging to $x$.*

**Proof:** By the two preceding results, it suffices to consider nets. Considering $V = X - \bar{A}$, it will suffice to prove that $V$ is open if and only if no limit of a convergent net $\lambda$ in $X - V = \bar{A}$ lies in $V$.

If $V$ is open, then $V$ is a neighborhood of each $x \in V$. Thus no tail of $\lambda$ is contained in $V$, so $\lambda$ cannot converge to any $x \in V$.

Conversely, assume that for every neighborhood $U$ of some $x \in V$ there exists a point of the net denoted by $\lambda_U \in U - V$. Then $\lambda \to x$ but $\lambda$ was assumed to be a net in $X - V$; contradiction. Therefore there exists some neighborhood $U$ of $x$ with $U - V = \varnothing$, whence $U \subseteq V$ so $V$ is a neighborhood of $x$ for every $x \in V$ and $V$ is open. $\square$

**Ex** A function $f : X \to Y$ between topological spaces is continuous if and only if for every net $\lambda \to x$ in $X$, $f(\lambda) \to f(x)$ in $Y$.

## Additional exercises

1. List all the filters in Sierpinski space and their limits.
2. The image of a filter is a filter in the image; *i. e.*, if $f : X \to Y$ and $\mathscr{F}$ is a filter in $X$, then $f(\mathscr{F}) = \{f(F) : F \in \mathscr{F}\}$ is a filter in $\operatorname{im} f \subseteq Y$. When is convergence preserved?
3. A filter $\mathscr{F} \to x$ if and only if $x$ is a cluster point of every filter of which $\mathscr{F}$ is a subfilter.
4. A filter $\mathscr{F}$ in a topological space $X$ is maximal if and only if, for every $A \subseteq X$, either $A$ or $X - A$ belongs to $\mathscr{F}$.
5. Any derived net of an ultrafilter is maximal.
6. The derived filter of a maximal net is maximal.
7. Discuss the relationship between cluster points of filters and of nets.
8. An ultrafilter converges to each of its cluster points.

9. Show that the set of all partitions of a given set $X$ has a natural ordering making it a directed set. In particular, Riemann integrability is given by convergence of a net in $\mathbb{R}$.

10. $A \subseteq X$ is a neighborhood of $x \in X$ if and only if $A$ belongs to every filter converging to $x$.

11. Let $A \subseteq X$ and set $B = \{x \in X :$ there is a filter $\mathscr{F}$ in $X$ with $A \in \mathscr{F} \to x\}$. For any filter $\mathscr{F}'$ with $B \in \mathscr{F}' \to y$, we have $y \in B$.

12. Let $X$ be a set and assume that convergence of filters in $X$ is defined so that:

    (a) if $\{x\} \in \mathscr{F}$, then $\mathscr{F} \to x$;

    (b) if $\mathscr{F} \to x$ and a filter $\mathscr{F}' \supseteq \mathscr{F}$, then $\mathscr{F}' \to x$;

    (c) the condition of exercise 11 is satisfied.

    Now define a set $N \subseteq X$ to be a neighborhood of $x$ if and only if the condition of exercise 10 is satisfied. Show that this defines a topology on $X$.

## References

[1] J. L. Kelley, *General Topology.* Princeton: D. Van Nostrand, 1955.

[2] J.-I. Nagata, *Modern General Topology.* Amsterdam: North-Holland, 1968.

[3] A. J. Sieradski, *An Introduction to Topology and Homotopy.* Boston: PWS-Kent, 1992.